S. Negri
E. Pavlović

Proof-Theoretic Analysis of the Logics of Agency: The Deliberative STIT

Abstract. A sequent calculus methodology for systems of agency based on branching-time frames with agents and choices is proposed, starting with a complete and cut-free system for multi-agent deliberative STIT; the methodology allows a transparent justification of the rules, good structural properties, analyticity, direct completeness and decidability proofs.

Keywords: BT + AC frames, Labelled sequent calculus, Deliberative STIT.

1. Introduction

STIT (*seeing to it that*) modalities have played a pivotal role in the formal studies of the logic of agency. As emphasized in Belnap's et al. [2], they have a unifying role as they can give formal meaning to expressions of various linguistic forms, such as the indicative, imperative, and subjunctive, e.g. in sentences as

- *Alice prepares her slides before leaving to the conference.*
- *Alice, prepare your slides before leaving to the conference!*
- *Alice should have prepared her slides before leaving to the conference.*

They can be either positive, referring to an action, or negative, denoting the absence of an action (doing otherwise, avoid doing, preventing, refraining, etc.) when intentionality is involved.

As one of the above example shows, STIT modalities can be also be counterfactual modalities (could have done, might have done, should have done); they may occur in the scope of deontic modalities, in the form of obligation, prohibition, permission (to do something) and interact with temporal modalities when the time of their evaluation may refer to a time different from the time of action, as in the duty to apologize or the duty to admonish.

STIT modalities are traditionally defined upon indeterministic frames—a semantics that builds upon a combination of Prior–Thomason–Kripke

Presented by **Heinrich Wansing**; *Received* January 7, 2019

Studia Logica (2021) 109: 473–507
https://doi.org/10.1007/s11225-020-09914-1

branching-time semantics and Kaplan's indexical semantics—enriched with agency. The temporal structure for branching time (BT) is given by trees with forward branching time, corresponding to indeterminacy of the future, but no backward branching, corresponding to uniqueness of the past. Moments are ordered by a partial order, reflecting the temporal relation, and maximal chains of moments are called histories. The trees are enriched by agent's choice (AC), a partition relative to an agent at a given moment of all histories passing through that moment.

In such frames, formulas are evaluated at moment/history pairs. The reason for doing so becomes clear by thinking of the evaluation of a *Will* sentence for a proposition that becomes true for certain turns of events, i.e. along some histories, but not along others. With branching time, such a modality does not have a well-defined truth value referred to moments only. As for the STIT-modality, the evaluation critically depends on the distribution of the evaluation on the frame. A formula stating that an agent α *sees to it that* A (written, in Belnap's notation, as $[\alpha\ stit : A]$) holds at the moment m of a history h if (i) A holds in all histories choice-equivalent to h for the agent α, but (ii) doesn't hold in at least one history of which m is part of. In simple terms, an agent sees to it that A if their choice brings about those histories where A holds, but nonetheless it could have been otherwise (i.e. an agent can't bring about something that would have happened anyway).

While the semantics for STIT modalities and logics built upon them is well-established, their proof theory has been largely restricted to axiomatic systems (starting with [21] and [2]) with just a few exceptions, namely a treatment of the logic of multi-agent *deliberative* STIT through labelled tableaux in [19] that builds upon Belnap's original semantics, and of the related *logic of imagination* in [15][1] that exploits a newly defined neighbourhood semantics, introduced in [20].

As for the meta-theoretical properties of STIT logics, as for other logics, completeness is usually established through the method of canonical models for axiomatic systems and through exhaustive proof search for tableaux [15]. Decidability, on the other hand, has been achieved through filtration methods [1,21].

Our aim in this work is to lay down the bases for the development of systems of deduction that cover the STIT modalities presented by Belnap et al. [2] in a way that respects all the desiderata of good proof systems, in

[1] See also the simplified calculus in [16] which includes a detailed proof of soundness and completess, with proofs of some of the assumptions that were implicit in [15].

particular to achieve a direct proof of decidability though a bound on proof search in a suitable analytic proof system.

That this cannot be an easy task is already clear from the fact that one cannot reason with STIT modalities using classical truth-valued logic. In fact, STIT modalities are not in general truth-functional: If A is (always) true, then $[\alpha\ stit : A]$ is false (since determinism prevents agentiveness), but if A is (sometimes) false, then $[\alpha\ stit : A]$ can be either true or false, depending on the distribution of A in the branching frame of possibilities.

Here the method of labelled sequent calculi developed since [7] comes to rescue: relatively complex truth conditions can be transformed into rules with the help of auxiliary modalities, as in the treatment of Lewis' counterfactuals [14], and additional properties for the characteristic frame conditions are expressed as sequent calculus rules following [11,12]. The result is a G3-style labelled sequent calculus which is shown to possess all the desired structural properties of a good proof system, including invertibility of the rules and admissibility of contraction and cut.

Moreover, we demonstrate multiple applications of the system. We prove the impossibility of delegation of tasks among independent agents, as well as the treatment of refraining from [2] and [18]. Finally, we demonstrate the meta-theoretical properties of our system, namely soundness, completeness and decidability via a bounded proof search.

2. Basic Definitions

The STIT modality that we will examine in this paper is the *deliberative* STIT, or DSTIT. The language of DSTIT is (presented here in the notation we will utilize in this paper, with $\circ$ standing for Booleans as standard):

$$A ::= \bot \mid p \mid A \circ A \mid \Box^i A \mid \mathcal{S}A \mid \mathcal{P}A \mid \mathcal{D}^i A$$

We will start by recalling the basic definitions of the semantics of DSTIT.

Definition 2.1. A pair $(T, \leqslant)$ is called a *branching temporal frame* if T is a non-empty set (of moments in time) and $\leqslant$ is a reflexive and transitive relation on T (a preorder) that in addition satisfies the following properties:

- *historical connectedness* $\forall m_1, m_2 \exists m(m \leqslant m_1 \wedge m \leqslant m_2)$
- *no backward branching*

$$\forall m_1, m_2, m(m_1 \leqslant m \wedge m_2 \leqslant m \rightarrow m_1 \leqslant m_2 \vee m_2 \leqslant m_1)$$

A *history* in T is a maximal chain of moments (in T) linearly ordered by $\leqslant$. The set of *histories passing through moment* m, H_m, is defined as $\{h \mid h \text{ is a history and } m \in h\}$.

Definition 2.2. Let $(T, \leqslant)$ be a branching temporal frame. A *DSTIT frame* is a structure $(T, \leqslant, Ag, Ch)$ where Ag is a non-empty set (of agents) and Ch is a function sending any agent/moment-pair (i, m) to a partition of H_m (the histories choice-equivalent for i at m) satisfying the property of *no choice between undivided histories* (here $m < m' \equiv m \leqslant m' \,\&\, m \neq m'$):

$$(\forall i \in Ag)(\forall H \in Ch(i,m))\forall h, h'[(h \in H \wedge \exists m'(m < m' \wedge m' \in h \cap h')) \\ \rightarrow h' \in H]$$

The definition states that if two histories are undivided at a given moment m, i.e. there is a moment successive to m that (still) belongs to both histories, then they are choice-equivalent for any agent at that given moment. We shall denote with $\sim^i_m$ the equivalence relation among histories for agent j at moment m. With this notation, an equivalent formulation of the principle of no choice between undivided histories is

$$\exists m'(m < m' \,\&\, m' \in h \cap h') \rightarrow h \sim^i_m h'$$

If $h \in H_m$, let $Ch^i_m(h)$ be the element of the partition $Ch(i, m)$ that contains h.

In the presence of multiple agents, an additional condition on a DSTIT frame is *independence of agents*, wherein for any finite number k of mutually distinct agents (with the notation $\mathit{Diff}(i_1, \ldots, i_k)$ precisely defined below) and histories h_j, there is a history which is, for each agent i_j, choice-equivalent to h_j :

$$(\forall i_1 \ldots i_k)(\forall h_1 \ldots h_k)[\mathit{Diff}(i_1, \ldots, i_k) \rightarrow \bigcap Ch^{i_j}_m(h_j) \neq \emptyset],\ 1 \leq j \leq k$$

Definition 2.3. Given a *DSTIT* frame $(T, \leqslant, Ag, Ch)$, a *DSTIT model* is a structure $\mathcal{M} \equiv (T, \leqslant, Ag, Ch, \mathcal{V})$ where $\mathcal{V}$ is a valuation of atomic formulas into sets of moment/history-pairs, called *points*. The valuation is extended inductively to DSTIT-formulas as follows: $(m, h) \Vdash [i\ dstit : A]$ iff

(i) $\forall h' \in Ch^m_i(h).(m, h') \Vdash A$

(ii) $\exists h' \in H_m.(m, h') \nVdash A$.

A formula A is said to be *satisfiable* in this semantics iff there exists a DSTIT model $\mathcal{M} \equiv (T, \leqslant, Ag, Ch, \mathcal{V})$ and a point (m, h) such that $\mathcal{M}, (m, h) \Vdash A$. A formula A is *valid* if it is true at any point in any DSTIT model.

We shall denote points by m/h rather than by (m, h). The truth condition for the deliberative STIT, using as above the equivalence relation rather than the choice function, can be rewritten as follows:

$$m/h \Vdash \mathcal{D}^i A \equiv \forall h'(h' \sim^i_m h \to m/h' \Vdash A) \wedge \exists h''(m \in h'' \wedge m/h'' \Vdash \neg A)$$

The definition can be factorized (in the sense of [9]) for the first part using a necessity operator (called in the literature *cstit*) with the truth condition

$$m/h \Vdash \Box^i A \equiv \forall h'(h' \sim^i_m h \to m/h' \Vdash A)$$

2.1. Rules of G3DSTIT

We shall show in a step-by-step fashion how to obtain a sequent calculus for the logic of DSTIT on the basis of its semantics. First, the above truth conditions are ready to be turned into inference rules; for the $\Box^i$-modality, these have just the form of the standard rules for a necessity modality, here with an accessibility relation given by the equivalence relation $\sim^i_m$:

$$\frac{h' \sim^i_m h, \Gamma \Rightarrow \Delta, m/h' : A}{\Gamma \Rightarrow \Delta, m/h : \Box^i A} \; R\Box^i, \; h' \mathit{fresh}$$

$$\frac{h' \sim^i_m h, m/h : \Box^i A, m/h' : A, \Gamma \Rightarrow \Delta}{h' \sim^i_m h, m/h : \Box^i A, \Gamma \Rightarrow \Delta} \; L\Box^i$$

Observe that the rules above display the distinction between *mobile* and *immobile* parameters made in [2]. In terms of rules the distinction corresponds to the distinction between *dynamic* and *static* rules made in [6]. Static rules derive from semantic truth conditions that are verified locally, with the available parameters, whereas dynamic rules derive from truth conditions that call for additional parameters arbitrarily linked to the current ones by some relations. Such arbitrary parameters are syntactically distinguished from the immobile ones by the syntactic condition of *freshness*. The distinction occurs at the level of application of the rule. Once such parameters have been introduced they become static. So it is better to attach this distinction to rules rather than to variables.

It will be useful to consider two more modalities, *settled true* and *possible*, both agent-independent, with the following truth conditions (we slightly depart here from the notation in the literature, $\mathit{Sett} : A$, since the colon is part of the specific syntax of labelled sequent calculi, so it might be confusing to use it in another context):

$$m/h \Vdash \mathcal{S}A \equiv \forall h'(m \in h' \to m/h' \Vdash A)$$
$$m/h \Vdash \mathcal{P}A \equiv \exists h'(m \in h' \,\&\, m/h' \Vdash A)$$

The rules follow the pattern of the rules for alethic modalities:

$$\frac{m \in h', \Gamma \Rightarrow \Delta, m/h' : A}{\Gamma \Rightarrow \Delta, m/h : \mathcal{S}A} \; R\mathcal{S},\ h'\ fresh$$

$$\frac{m \in h', m/h' : A, m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}{m \in h', m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta} \; L\mathcal{S}$$

$$\frac{m \in h', m/h' : A, \Gamma \Rightarrow \Delta}{m/h : \mathcal{P}A, \Gamma \Rightarrow \Delta} \; L\mathcal{P},\ h'\ fresh$$

$$\frac{m \in h', \Gamma \Rightarrow \Delta, m/h : \mathcal{P}A, m/h' : A}{m \in h', \Gamma \Rightarrow \Delta, m/h : \mathcal{P}A} \; R\mathcal{P}$$

These will then allow us to factorize the second part of the definition above as:

$$m/h \Vdash \neg\mathcal{S}A \equiv \exists h''(m \in h'' \wedge m/h'' \Vdash \neg A)$$

The rules for DSTIT are then found as follows: first we rewrite the truth conditions as rules using the modalities already defined (we indicate these rules with a bar since they are not in the final form)

$$\frac{\Gamma \Rightarrow \Delta, m/h : \Box^i A \quad \Gamma \Rightarrow \Delta, m/h : \neg\mathcal{S}A}{\Gamma \Rightarrow \Delta, m/h : \mathcal{D}^i A} \; \overline{R\mathcal{D}^i}$$

$$\frac{m/h : \Box^i A, m/h : \neg\mathcal{S}A, \Gamma \Rightarrow \Delta}{m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta} \; \overline{L\mathcal{D}^i},\ h'\ fresh$$

Second, we eliminate the negation by the use of the classical symmetry of the calculus to obtain the final version:

$$\frac{\Gamma \Rightarrow \Delta, m/h : \Box^i A \quad m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, m/h : \mathcal{D}^i A} \; R\mathcal{D}^i$$

$$\frac{m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{S}A}{m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta} \; L\mathcal{D}^i$$

2.1.1. Rules for Relational Atoms In addition to the rules for the STIT modalities, we have to make explicit the rules that correspond to the properties of the equivalence relation between histories and equality of agents. As usual, an equivalence relation can be given by just two rules, reflexivity and Euclidean transitivity. Additionally, we have a rule of replacement of equals. The rule is restricted to atomic formulas to guarantee the structural properties of the calculus, and its general form is shown to be admissible.

This follows the general treatment of equality as detailed in [13]. Observe that in general At denotes either a relational formula, say of the form $i = j$, or $h' \sim^i_m h$, or a labelled atomic formula.

$$\frac{i = i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl}_{=} \qquad \frac{j = k, i = j, i = k, \Gamma \Rightarrow \Delta}{i = j, i = k, \Gamma \Rightarrow \Delta}\ \mathit{Etrans}_{=}$$

$$\frac{i = j, At(i), At(j), \Gamma \Rightarrow \Delta}{i = j, At(i), \Gamma \Rightarrow \Delta}\ \mathit{Repl}_{At}$$

$$\frac{h \sim^i_m h, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl}_{\sim^i_m} \quad \frac{h_2 \sim^i_m h_3, h_1 \sim^i_m h_2, h_1 \sim^i_m h_3, \Gamma \Rightarrow \Delta}{h_1 \sim^i_m h_2, h_1 \sim^i_m h_3, \Gamma \Rightarrow \Delta}\ \mathit{Etrans}_{\sim^i_m}$$

Given the two types of relational formulas, history membership $m \in h$ and choice equivalence $h \sim^i_m h'$, it will be useful to indicate their relationship. To this end we use the rule WD, which corresponds to $h \sim^i_m h' \to m \in h$ (the rule for the other part, $h \sim^i_m h' \to m \in h'$, need not be added because of symmetry of $\sim^i_m$):

$$\frac{m \in h, h \sim^i_m h', \Gamma \Rightarrow \Delta}{h \sim^i_m h', \Gamma \Rightarrow \Delta}\ WD$$

The usefulness of this rule can be seen in the derivation of $\mathcal{S}A \supset \Box^i A$. The semantics of $\mathcal{S}$ and $\Box^i$ clearly indicate that this should hold (intuitively, what holds in every point, for any history h such that $m \in h$, also holds in all those histories that are choice-equivalent for i). However, since $\mathcal{S}$ and $\Box$ rules use different relational formulas, we can derive this valid formula only if we connect them (proof in the next section under A3).

The axiom of independence of agents requires a predicate that expresses distinctness of any finite number of agents. This is defined as pairwise inequality ranging over all pairs of non-identical agents:

$$\mathit{Diff}(i_1, \ldots, i_k) \equiv \&_{1 \leq l < m \leq k} \neg\, i_l = i_m$$

Correspondingly, we have the rule (to be precise, a rule for every k)

$$\frac{\{\neg\, i_l = i_m\}_{1 \leq l < m \leq k}, \Gamma \Rightarrow \Delta}{\mathit{Diff}(i_1, \ldots, i_k), \Gamma \Rightarrow \Delta}$$

One is tempted to apply the negation rule (or $L\supset$ if we prefer defined negation) and obtain the equivalent rule

$$\frac{\Gamma \Rightarrow \Delta, \{i_l = i_m\}_{1 \leq l < m \leq k}}{\mathit{Diff}(i_1, \ldots, i_k), \Gamma \Rightarrow \Delta}\ \mathit{Diff}_k$$

Although appealing, this choice has a disadvantage: in order to frame our treatment within the approach of "axioms-as-rules," we need to formulate rules where all the material from the extra-logical language, i.e. relations such as equalities, occur all on the left of sequents. This is an essential assumption in the proof of cut elimination in the presence of extensions with extra-logical rules, and a deviation from this path would bring undesired consequences (the most undesired for a sequent calculus practitioner being the loss of cut elimination!). There is a way out, that consists in adding a new primitive, together with the rules that correspond to its definition (a definitional extension, in the sense of Skolem). The new primitive is of course inequality between agents, $i \neq i$, and the defining condition is

$$i \neq j \supset\subset \neg i = j$$

For reasons explained at length in [4], it is enough to consider only the *semidefinitional* extension, that corresponds to taking only the left-to-right side of the above equivalence. This is good news because this is the one that falls within the scheme of regular rules, whereas the other falls outside the pattern of geometric implications and thus cannot receive a similarly good proof-theoretic treatment. The rule that corresponds to $i \neq j \supset \neg i = j$ is the zero-premiss rule derived from the equivalent $i \neq j \,\&\, i = j \supset \bot$, so in conclusion the rules for the *Diff* operator are

$$\frac{\{i_l \neq i_m\}_{1 \leq l\, <\, m \leq k}, \Gamma \Rightarrow \Delta}{\mathit{Diff}\,(i_1, \ldots, i_k), \Gamma \Rightarrow \Delta}\ \mathit{Diff}_k \qquad \frac{}{i \neq j, i = j, \Gamma \Rightarrow \Delta} \neq$$

The axiom of independence of agents states that for every choice of k different agents at moment m, there is a history that is compatible with all the individual choices. As a rule, it is formulated in the following way, by representing the k choices through representative histories; the existing history compatible with all the choices appears as a fresh variable, linked to all the given histories through all the individual equivalences at moment m:

$$\frac{h \sim_m^{i_1} h_1, \ldots, h \sim_m^{i_k} h_k, \mathit{Diff}\,(i_1, \ldots, i_k), m \in h_1, \ldots, m \in h_k, \Gamma \Rightarrow \Delta}{\mathit{Diff}\,(i_1, \ldots, i_k), m \in h_1, \ldots, m \in h_k, \Gamma \Rightarrow \Delta}\ \overline{\mathit{Ind}_k},\ h\,\mathit{fresh}$$

As we will see in a moment, this rule enables us to derive the *independence of agents axioms*,

$$AIA_k : \mathcal{P}(\Box^1 A_1) \& \ldots \& \mathcal{P}(\Box^k A_k) \supset \mathcal{P}(\Box^1 A_1 \& \ldots \& \Box^k A_k).$$

Finally, since outside of rules for relational atoms the choice-equivalence relation features only in the rules with parametrized modalities (note that the relation is itself agent-relative), we can limit the rule further to its final, parametrized version, for $1 \leq n \leq k$:

$$\frac{h \sim_m^{i_1} h_1, \ldots, h \sim_m^{i_n} h_n, \mathit{Diff}(i_1, \ldots, i_k), m \in h_1, \ldots, m \in h_n, h_1 \sim_m^{i_1} h'_1, \ldots, h_n \sim_m^{i_n} h'_n, \Gamma \Rightarrow \Delta}{\mathit{Diff}(i_1, \ldots, i_k), m \in h_1, \ldots, m \in h_n, h_1 \sim_m^{i_1} h'_1, \ldots, h_n \sim_m^{i_n} h'_n, \Gamma \Rightarrow \Delta} \; \mathit{Ind}_k,\ h\,\mathit{fresh}$$

It is easy to check, using results from Section 4, that in the presence of $Refl_{\sim_m^i}$ the systems containing either of these two rules are deductively equivalent. The choice here is made with an eye towards the demonstration of meta-theoretical properties.

2.1.2. Rules for the Relation $\leq$ The principle of *no choice between undivided histories*, $\exists m'(m < m' \,\&\, m' \in h \cap h') \to h \sim_m^j h'$ can be formulated, using first-order logic, in universal form as

$$\forall m'(m < m' \,\&\, m' \in h \cap h' \to h \sim_m^j h')$$

which in turn can be formulated as a rule that follows the regular rule scheme

$$\frac{h \sim_m^j h', m < m', m' \in h, m' \in h', \Gamma \Rightarrow \Delta}{m < m', m' \in h, m' \in h', \Gamma \Rightarrow \Delta} \; NC$$

The principle of *historical connectedness*, $\forall m_1, m_2 \exists m(m \leqslant m_1 \wedge m \leqslant m_2)$ is formulated as a rule

$$\frac{m' \leq m_1, m' \leq m_2, m' \in h_1, m' \in h_2, m_1 \in h_1, m_2 \in h_2, \Gamma \Rightarrow \Delta}{m_1 \in h_1, m_2 \in h_2, \Gamma \Rightarrow \Delta} \; HC$$

The principle of *no backward branching*, $\forall m_1, m_2, m((m_1 \leqslant m \wedge m_2 \leqslant m) \to (m_1 \leqslant m_2 \vee m_2 \leqslant m_1))$, is formulated as a rule

$$\frac{m_1 \leq m_2, m_1 \leq m, m_2 \leq m, \Gamma \Rightarrow \Delta \quad m_2 \leq m_1, m_1 \leq m, m_2 \leq m, \Gamma \Rightarrow \Delta}{m_1 \leq m, m_2 \leq m, \Gamma \Rightarrow \Delta} \; NBB$$

Finally, we require that $\leq$ is a preorder:

$$\frac{m \leq m, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; \mathit{Refl}_{\leq} \qquad \frac{m_1 \leq m_3, m_1 \leq m_2, m_2 \leq m_3, \Gamma \Rightarrow \Delta}{m_1 \leq m_2, m_2 \leq m_3, \Gamma \Rightarrow \Delta} \; \mathit{Etrans}_{\leq}$$

Observe that all the logical rules, when applied root-first, may modify only histories, and the moment of evaluation remains unchanged. It follows that the rules NC, HC and NBB, $Refl_{\leq}$ and $Etrans_{\leq}$ give a conservative extension, so we will omit them from our calculus.

Initial sequents: $m/h : p, \Gamma \Rightarrow \Delta, m/h : p \qquad m/h : \bot, \Gamma \Rightarrow \Delta$

Propositional rules: Standard.

Modal rules:

$$\frac{h' \sim^i_m h, m/h : \Box^i A, m/h' : A, \Gamma \Rightarrow \Delta}{h' \sim^i_m h, m/h : \Box^i A, \Gamma \Rightarrow \Delta}\ L\Box^i \qquad \frac{h' \sim^i_m h, \Gamma \Rightarrow \Delta, m/h' : A}{\Gamma \Rightarrow \Delta, m/h : \Box^i A}\ R\Box^i$$

$$\frac{m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{S}A}{m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta}\ L\mathcal{D}^i$$

$$\frac{\Gamma \Rightarrow \Delta, m/h : \Box^i A \quad m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, m/h : \mathcal{D}^i A}\ R\mathcal{D}^i$$

$$\frac{m \in h', m/h' : A, m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}{m \in h', m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}\ L\mathcal{S} \qquad \frac{m \in h', \Gamma \Rightarrow \Delta, m/h' : A}{\Gamma \Rightarrow \Delta, m/h : \mathcal{S}A}\ R\mathcal{S}$$

$$\frac{m \in h', m/h' : A, \Gamma \Rightarrow \Delta}{m/h : \mathcal{P}A, \Gamma \Rightarrow \Delta}\ L\mathcal{P} \qquad \frac{m \in h', \Gamma \Rightarrow \Delta, m/h : \mathcal{P}A, m/h' : A}{m \in h', \Gamma \Rightarrow \Delta, m/h : \mathcal{P}A}\ R\mathcal{P}$$

$$\frac{i = i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl}_= \qquad \frac{j = k, i = j, i = k, \Gamma \Rightarrow \Delta}{i = j, i = k, \Gamma \Rightarrow \Delta}\ \mathit{Etrans}_=$$

$$\frac{i = j, At(i), At(j), \Gamma \Rightarrow \Delta}{i = j, At(i), \Gamma \Rightarrow \Delta}\ \mathit{Repl}_{At} \qquad \frac{h \sim^i_m h, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl}_{\sim^i_m}$$

$$\frac{h_2 \sim^i_m h_3, h_1 \sim^i_m h_2, h_1 \sim^i_m h_3, \Gamma \Rightarrow \Delta}{h_1 \sim^i_m h_2, h_1 \sim^i_m h_3, \Gamma \Rightarrow \Delta}\ \mathit{Etrans}_{\sim^i_m}$$

$$\frac{m \in h, h \sim^i_m h', \Gamma \Rightarrow \Delta}{h \sim^i_m h', \Gamma \Rightarrow \Delta}\ WD$$

$$\frac{\{i_l \neq i_m\}_{1 \leq l < m \leq k}, \Gamma \Rightarrow \Delta}{\mathit{Diff}(i_1, \ldots, i_k), \Gamma \Rightarrow \Delta}\ \mathit{Diff}_k \qquad \frac{}{i \neq j, i = j, \Gamma \Rightarrow \Delta}\ \neq$$

$$\frac{h \sim^{i_1}_m h_1, \ldots, h \sim^{i_n}_m h_n, \mathit{Diff}(i_1, \ldots, i_k), m \in h_1, \ldots, m \in h_n, h_1 \sim^{i_1}_m h'_1, \ldots, h_n \sim^{i_n}_m h'_n, \Gamma \Rightarrow \Delta}{\mathit{Diff}(i_1, \ldots, i_k), m \in h_1, \ldots, m \in h_n, h_1 \sim^{i_1}_m h'_1, \ldots, h_n \sim^{i_n}_m h'_n, \Gamma \Rightarrow \Delta}\ \mathit{Ind}_k$$

Application conditions: h' is fresh in R$\Box^i$,R$\mathcal{S}$, L$\mathcal{P}$ and Ind_k.

Figure 1. **G3DSTIT**

The full system of rules of **G3DSTIT** is presented in Figure 1, with the propositional rules as in [7], but with labels of the form m/h. The terminology used is standard—we call the multisets Γ and Δ in a sequent its *context*, the remaining formula(s) in the lower sequent *principal* and those in the upper sequent(s) *active*. In any rule that requires h fresh, h is called its *eigenvariable*.

3. Axioms

The treatment of DSTIT in [2] is via an axiomatic system. In this section we will show that all the axioms presented therein are derivable in our system, or more precisely

PROPOSITION 3.1. *The axioms given in* [2] *are derivable in the calculus, namely for every axiom A the sequent $\Rightarrow m/h : A$ is derivable in* ***G3DSTIT***.

A1:

(i) $\mathcal{S}(A \supset B) \supset (\mathcal{S}A \supset \mathcal{S}B)$

(ii) $\mathcal{S}A \supset A$

(iii) $\mathcal{P}A \supset \mathcal{S}\mathcal{P}A$

(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{m \in h', m/h' : A, m/h' : A \supset B, m/h : \mathcal{S}A, m/h : \mathcal{S}(A \supset B) \Rightarrow m/h' : B}{m \in h', m/h' : A, m/h : \mathcal{S}A, m/h : \mathcal{S}(A \supset B) \Rightarrow m/h' : B}\, L\mathcal{S}}{m \in h', m/h : \mathcal{S}A, m/h : \mathcal{S}(A \supset B) \Rightarrow m/h' : B}\, L\mathcal{S}}{m/h : \mathcal{S}A, m/h : \mathcal{S}(A \supset B) \Rightarrow m/h : \mathcal{S}B}\, R\mathcal{S}}{m/h : \mathcal{S}(A \supset B) \Rightarrow m/h : \mathcal{S}A \supset \mathcal{S}B}\, R\supset}{\Rightarrow m/h : \mathcal{S}(A \supset B) \supset (\mathcal{S}A \supset \mathcal{S}B)}\, R\supset$$

(ii)

$$\dfrac{\dfrac{m \in h, m/h : \mathcal{S}A, m/h : A \Rightarrow m/h : A}{m/h : \mathcal{S}A \Rightarrow m/h : A}\, R\mathcal{S}, WD}{\Rightarrow m/h : \mathcal{S}A \supset A}\, R\supset$$

(iii)

$$\dfrac{\dfrac{\dfrac{\dfrac{m \in h', m \in h'', m/h'' : A \Rightarrow m/h' : \mathcal{P}A, m/h'' : A}{m \in h', m \in h'', m/h'' : A \Rightarrow m/h' : \mathcal{P}A}\, R\mathcal{P}}{m \in h', m/h : \mathcal{P}A \Rightarrow m/h' : \mathcal{P}A}\, L\mathcal{P}}{m/h : \mathcal{P}A \Rightarrow m/h : \mathcal{S}\mathcal{P}A}\, R\mathcal{S}}{\Rightarrow m/h : \mathcal{P}A \supset \mathcal{S}\mathcal{P}A}\, R\supset$$

Next we have the axioms for the modality $\Box^i$:

A2:

(i) $\Box^i(A \supset B) \supset (\Box^i A \supset \Box^i B)$

(ii) $\Box^i A \supset A$

(iii) $\neg\Box^i A \supset \Box^i\neg\Box^i A$

(i) The first is shown as the corresponding axiom for $\mathcal{S}$, so we move on to the second and third axiom, where the rules for the equivalence relation come into play. (ii)

$$\cfrac{\cfrac{\cfrac{h \sim^i_m h, m/h : A, m/h : \Box^i A \Rightarrow m/h : A}{h \sim^i_m h, m/h : \Box^i A \Rightarrow m/h : A}\ L\Box^i}{m/h : \Box^i A \Rightarrow m/h : A}\ \mathit{Refl}_{\sim^i_m}}{\Rightarrow m/h : \Box^i A \supset A}\ R\supset$$

(iii)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{h' \sim^i_m h'', h \sim^i_m h', h \sim^i_m h'', m/h'' : A, m/h' : \Box^i A \Rightarrow m/h'' : A}{h' \sim^i_m h'', h \sim^i_m h', h \sim^i_m h'', m/h' : \Box^i A \Rightarrow m/h'' : A}\ \Box^i}{h \sim^i_m h', h \sim^i_m h'', m/h' : \Box^i A \Rightarrow m/h'' : A}\ \mathit{Etrans}_{\sim^i_m}}{h \sim^i_m h', m/h' : \Box^i A \Rightarrow m/h : \Box^i A}\ R\Box^i}{h \sim^i_m h', m/h : \neg\Box^i A \Rightarrow m/h' : \neg\Box^i A}\ R\neg,\ L\neg}{\cfrac{m/h : \neg\Box^i A \Rightarrow m/h : \Box^i\neg\Box^i A}{\Rightarrow m/h : \neg\Box^i A \supset \Box^i\neg\Box^i A}\ R\supset}\ R\Box^i$$

A3: $\mathcal{S}A \supset \Box^i A$

$$\cfrac{\cfrac{\cfrac{\cfrac{m \in h', h' \sim^i_m h, m/h : \mathcal{S}A, m/h' : A \Rightarrow m/h' : A}{m \in h', h' \sim^i_m h, m/h : \mathcal{S}A \Rightarrow m/h' : A}\ L\mathcal{S}}{h' \sim^i_m h, m/h : \mathcal{S}A \Rightarrow m/h' : A}\ WD}{m/h : \mathcal{S}A \Rightarrow m/h : \Box^i A}\ R\Box^i}{\Rightarrow m/h : \mathcal{S}A \supset \Box^i A}\ R\supset$$

This axiom takes $\Box^i$ and $\mathcal{S}$ as primitive, and defines $\mathcal{D}^i$ in terms of those two as:

$$\mathcal{D}^i A \supset\subset \neg\mathcal{S}A \ \&\ \Box^i A$$

We prove each direction of the definition in turn:

(1)

$$\dfrac{\dfrac{\dfrac{\dfrac{m/h' : A, m \in h', m/h : \mathcal{S}A, m/h : \Box^i A \Rightarrow m/h' : A}{m \in h', m/h : \mathcal{S}A, m/h : \Box^i A \Rightarrow m/h' : A}\, L\mathcal{S}}{m/h : \mathcal{S}A, m/h : \Box^i A \Rightarrow m/h : \mathcal{S}A}\, R\mathcal{S}}{m/h : \mathcal{S}A, m/h : \mathcal{D}^i A \Rightarrow}\, L\mathcal{D}^i}{m/h : \mathcal{D}^i A \Rightarrow m/h : \neg\mathcal{S}A}\, R\neg$$

$$\dfrac{(1) \qquad \dfrac{\dfrac{\dfrac{h \sim_m^i h', m/h : \Box^i A, m/h' : A \Rightarrow m/h' : A, m/h : \mathcal{S}A}{h \sim_m^i h', m/h : \Box^i A \Rightarrow m/h' : A, m/h : \mathcal{S}A}\, L\Box^i}{m/h : \Box^i A \Rightarrow m/h : \Box^i A, m/h : \mathcal{S}A}\, R\Box^i}{m/h : \mathcal{D}^i A \Rightarrow m/h : \Box^i A}\, L\mathcal{D}^i}{m/h : \mathcal{D}^i A \Rightarrow m/h : \neg\mathcal{S}A \;\&\; \Box^i A}\, R\&$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ m \in h_1, m/h : \Box^i A \Rightarrow m/h : \Box^i A, m/h_1 : A\end{matrix} \qquad \dfrac{m/h_1 : A, m/h : \mathcal{S}A, m \in h_1, m/h : \Box^i A \Rightarrow m/h_1 : A}{m/h : \mathcal{S}A, m \in h_1, m/h : \Box^i A \Rightarrow m/h_1 : A}\, L\mathcal{S}}{m \in h_1, m/h : \Box^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : A}\, R\mathcal{D}^i}{m/h : \Box^i A \Rightarrow m/h : \mathcal{D}^i A, m/h : \mathcal{S}A}\, R\mathcal{S}}{m/h : \neg\mathcal{S}A, m/h : \Box^i A \Rightarrow m/h : \mathcal{D}^i A}\, L\neg}{m/h : \neg\mathcal{S}A \;\&\; \Box^i A \Rightarrow m/h : \mathcal{D}^i A}\, L\&$$

In fact, any two of the three modalities could be taken as primitive with the third defined. This is straightforward to prove and therefore omitted here.

A4: Equality between agents is reflexive, symmetric, and transitive.

As emphasized in [7], in order to obtain the properties of the relational part as derivable sequents, one would have to add initial sequents of the form, say, $i = j, \Gamma \Rightarrow \Delta, i = j$. However, this is not needed: because of the form of the rules, we get all the consequences of having a reflexive, symmetric, and transitive relation. This is a general property of the formulation of axioms as rules (see also [13]).

A5: If $i = j$, $A \supset A(i/j)$

See Proposition 4.3.

AIA$_k$: If *Diff* $(i_1, \ldots, i_k)$, then

$$\mathcal{P}\Box^{i_1} A_1 \;\&\; \ldots \;\&\; \mathcal{P}\Box^{i_k} A_k \supset \mathcal{P}(\Box^{i_1} A_1 \;\&\; \ldots \;\&\; \Box^{i_k} A_k)$$

For simplicity, we prove **AIA**$_2$. The generalization to k agents is straightforward.

$$
\begin{array}{cl}
(1) & \\
m/h_4 : A_1, h_4 \sim^{i_1}_m h_1, h_4 \sim^{i_1}_m h_3, \ldots, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow \ldots, m/h_4 : A_1 & \\
\hline
h_4 \sim^{i_1}_m h_1, h_4 \sim^{i_1}_m h_3, h_3 \sim^{i_1}_m h_1, \ldots, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow \ldots, m/h_4 : A_1 & L\Box^1 \\
\hline
h_4 \sim^{i_1}_m h_3, h_3 \sim^{i_1}_m h_1, \ldots, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow \ldots, m/h_4 : A_1 & ETrans_{\sim^{i_1}_m} \\
\hline
h_3 \sim^{i_1}_m h_1, \ldots, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow \ldots, m/h_3 : \Box^{i_1} A_1 & R\Box^1
\end{array}
$$

$$
\begin{array}{cl}
(1) \qquad\qquad\qquad (2) & \\
\hline
m \in h_3, h_3 \sim^{i_1}_m h_1, \ldots, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow \ldots, m/h_3 : \Box^{i_1} A_1 \& \Box^{i_2} A_2 & R\& \\
\hline
m \in h_3, h_3 \sim^{i_1}_m h_1, \ldots, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow m/h : \mathcal{P}(\Box^{i_1} A_1 \& \Box^{i_2} A_2) & R\mathcal{P} \\
\hline
h_3 \sim^{i_1}_m h_1, h_3 \sim^{i_2}_m h_2, h_1 \sim^{i_1}_m h_1, h_2 \sim^{i_2}_m h_2, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow m/h : \mathcal{P}(\Box^{i_1} A_1 \& \Box^{i_2} A_2) & WD \\
\hline
h_1 \sim^{i_1}_m h_1, h_2 \sim^{i_2}_m h_2, \mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow m/h : \mathcal{P}(\Box^{i_1} A_1 \& \Box^{i_2} A_2) & Ind_2 \\
\hline
\mathit{Diff}(i_1,i_2), m/h_1 : \Box^{i_1} A_1, m/h_2 : \Box^{i_2} A_2 \Rightarrow m/h : \mathcal{P}(\Box^{i_1} A_1 \& \Box^{i_2} A_2) & Ref_{\sim^{i_1}_m}, Ref_{\sim^{i_2}_m} \\
\hline
\mathit{Diff}(i_1,i_2), m/h : \mathcal{P}\Box^{i_1} A_1, m/h : \mathcal{P}\Box^{i_2} A_2 \Rightarrow m/h : \mathcal{P}(\Box^{i_1} A_1 \& \Box^{i_2} A_2) & L\mathcal{P}, L\mathcal{P}
\end{array}
$$

The derivation indicated by (2) is similar to the corresponding derivation of the left premiss of $R\&$ and therefore omitted.

4. Structural Properties

All the structural properties are easily established and their verification follows the general pattern for labelled sequent calculi, i.e. the properties are established in this order:

- Derivability of initial sequents of the form $m/h : A, \Gamma \Rightarrow \Delta, m/h : A$ where A is an arbitrary formula in the STIT language.
- Height-preserving substitution on moments/histories/agents.
- Height-preserving admissibility of weakening.
- Height-preserving invertibility of all the rules.
- Height-preserving admissibility of contraction.
- Admissibility of cut.

All these results are established in a rather routine way, with the proviso of defining a suitable notion of weight of formulas. The weight reflects the way in which we have unfolded the rule of the STIT operator using additional modalities, and guarantees that each time we use a rule for such modalities the weight of active formulas is less than the weight of principal formulas. We just illustrate the central case in the cut elimination procedure, but before that give the definition of weight of formulas:

Definition 4.1. The *weight of a labelled formula* $m/h : A$ in the STIT language is given by the weight of A, $\mathtt{w}(A)$, and is defined inductively as follows:

- $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$,

– $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ$ conjunction, disjunction, or implication,
– $\mathtt{w}(\Box^i A) = \mathtt{w}(\mathcal{S}A) = \mathtt{w}(\mathcal{P}A) = \mathtt{w}(A) + 1$,
– $\mathtt{w}(\mathcal{D}^i A) = \mathtt{w}(A) + 2$.

LEMMA 4.2. *Sequents of the form $m/h : A, \Gamma \Rightarrow \Delta, m/h : A$ are derivable in* **G3DSTIT** *for an arbitrary formula A in the language.*

PROOF. Routine proof by induction on $\mathtt{w}(A)$. The base cases are given by initial sequents/conclusion of $L\bot$. For the inductive cases, we show one which is specific to the STIT extension, namely the case of $\mathcal{D}^i A$. We have the following derivation

$$\dfrac{\dfrac{m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \Box^i A, m/h : \mathcal{S}A \quad m/h : \mathcal{S}A, m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{S}A}{m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{D}^i A, m/h : \mathcal{S}A}\ R\mathcal{D}^i}{m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{D}^i A}\ L\mathcal{D}^i$$

where the topsequents are derivable by induction hypothesis because both $\mathtt{w}(A)$ and $\mathtt{w}(\Box^i A)$ are less than $\mathtt{w}(\mathcal{D}^i A)$. The other cases ($\Box^i A$, $\mathcal{S}A$, $\mathcal{P}A$) are shown in a similar way. ■

Height is defined routinely, with derivability with height bounded by n noted as $\vdash_n$.

Substitution of labels is likewise defined routinely. In addition to moment and history labels we have labels for agents. Even if the latter are not labels in the usual sense, but rather parameters, we can extend to them the property of height-preserving substitution.

PROPOSITION 4.3. *1. If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(m'/m) \Rightarrow \Delta(m'/m)$;*

2. If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(h'/h) \Rightarrow \Delta(h'/h)$;

3. If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(j/i) \Rightarrow \Delta(j/i)$.

PROOF. All statements are proved by induction on the height of the derivation. The base cases with height zero (initial sequents, conclusion of $L\bot$) are immediate. For the inductive steps, we consider the last step in a derivation of height $n+1$ and distinguish two cases: in the first case the last step is a rules with eigenvariable and the label to be substituted is the same as the eigenvariable. In this case we have to first apply inductive hypothesis to the premiss of the rule to replace the eigenvariable with a new fresh variable, we then apply inductive hypothesis with the substitution to be made, and finally the rule. To give some flesh to this abstract description, suppose the sequent is $\Gamma \Rightarrow \Delta, m/h : \Box^i A$, the last rule $R\Box^i$ with premiss $h' \sim_m^i h, \Gamma \Rightarrow \Delta, m/h' : A$, and the substitution h'/h. We have by inductive hypothesis a derivation of height n of $h'' \sim_m^i h, \Gamma \Rightarrow \Delta, m/h'' : A$ where h'' is the new fresh variable, then by inductive hypothesis again we get $h'' \sim_m^i h', \Gamma(h'/h) \Rightarrow \Delta(h'/h), m/h'' : A$, and by $R\Box^i$, $\Gamma(h'/h) \Rightarrow \Delta(h'/h), m/h' : \Box^i A$ in $n+1$ steps.

In all other cases (rule without eigenvariable or label to be substituted different from the eigenvariable) we just apply the inductive hypothesis to the premiss(es) of the rule and then the rule. ■

Lemma 4.4. *Weakening is height-preserving admissible:*

1. *If* $\vdash_n \Gamma \Rightarrow \Delta$*, then* $\vdash_n m/h : A, \Gamma \Rightarrow \Delta$
2. *If* $\vdash_n \Gamma \Rightarrow \Delta$*, then* $\vdash_n \Gamma \Rightarrow \Delta, m/h : A$
3. *If* $\vdash_n \Gamma \Rightarrow \Delta$*, then* $\vdash_n B, \Gamma \Rightarrow \Delta$
4. *If* $\vdash_n \Gamma \Rightarrow \Delta$*, then* $\vdash_n \Gamma \Rightarrow \Delta, B$

Where B is a relational formula, of the form $i = j$, $h \sim_m^i h'$, $m \in h$, $m < m'$, or $\mathit{Diff}(i_1, \ldots, i_k)$.

Proof. By induction on the height of the derivation, using Proposition 4.3 to deal with eigenvariables. ■

Lemma 4.5. *The rules of* **G3DSTIT** *are height-preserving invertible.*

Proof. Routine by induction on the height of the derivation. Standard for propositional rules.

For rules $L\Box^i$ and $L\mathcal{S}$ the proof is like that of $L\Box$, for $R\Box^i$, and $R\mathcal{S}$ the proof is like that of $R\Box$, and for $L\mathcal{P}$ and $R\mathcal{P}$ like $L\Diamond$ and $R\Diamond$, respectively, of [7].

For $Refl_=$, $ETrans_=$, $Repl_{At}$, $Refl_{\sim_m^i}$, $ETrans_{\sim_m^i}$, WD and Ind_k, it follows from Lemma 4.4. Simple for Diff_k.

For $L\mathcal{D}^i$, if $n = 0$, then $m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta$ is an initial sequent, and then so is $m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{S}A$. If $n > 0$, then if the final rule $\mathcal{R}$ is some other than $L\mathcal{D}^i$, we obtain $m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta$ from some $m/h : \mathcal{D}^i A, \Gamma' \Rightarrow \Delta'$ ($m/h : \mathcal{D}^i A, \Gamma'' \Rightarrow \Delta''$) of height $n-1$. We possibly apply the Proposition 4.3, then the inductive hypothesis, and then $\mathcal{R}$ to obtain the desired sequent with height n. We do the same if R is $L\mathcal{D}^i$ with the principal formula in Γ, and if it is $L\mathcal{D}^i$ with $m/h : \mathcal{D}^i A$ principal, the upper sequent of $\mathcal{R}$ is already the desired sequent. Similar for $R\mathcal{D}^i$. ■

Lemma 4.6. *Contraction is height-preserving admissible in* **G3DSTIT**.

Proof. Routine, using case analysis on the last rule used to derive the premiss of contraction and Lemma 4.5. ■

LEMMA 4.7. *Cut is admissible in* **G3DSTIT**.

PROOF. The interesting case here is when the cut formula is a $\mathcal{D}^i$-formula principal in both premisses of *Cut*. The instance of *Cut* is then:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, m/h : \Box^i A \quad m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, m/h : \mathcal{D}^i A}\, R\mathcal{D}^i \quad \dfrac{m/h : \Box^i A, \Gamma \Rightarrow \Delta, m/h : \mathcal{S}A}{m/h : \mathcal{D}^i A, \Gamma \Rightarrow \Delta}\, L\mathcal{D}^i}{\Gamma \Rightarrow \Delta}\, Cut$$

This is transformed into:

$$\dfrac{\Gamma \Rightarrow \Delta, m/h : \Box^i A \quad \dfrac{m/h : \Box^i : A, \Gamma \Rightarrow \Delta, m/h : \mathcal{S}A \quad m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta}{m/h : \Box^i : A, \Gamma \Rightarrow \Delta}\, Cut_1}{\Gamma \Rightarrow \Delta}\, Cut_2$$

where both Cut_1 and Cut_2 are of lesser weight. All other cases are routine. ■

5. Applications

As already discussed in the introduction, STIT modalities have a number of uses. In this section we examine several applications of the DSTIT modality in order to demonstrate the usefulness of the labelled approach.

5.1. Impossibility of Delegation

It is clear that we can treat nested STIT modalities and that agents can be different for each of the nested modalities, that is, we have individual multiple agency. So one can ask, for example, whether it is possible that an agent sees to it that another agent sees to it that A, as in the delegation of a task. In [2, p. 274], a semantic argument is given to show that this is impossible for the achievement STIT. We can give a proof-theoretic argument to show that this holds also for the deliberative STIT. We show that assuming $m/h : \mathcal{D}^{i_1}\mathcal{D}^{i_2}A$ leads to a contradiction (independently on the form of A). This is expressed in our system as a derivation for the sequent (throughout i_1 and i_2 are assumed different)

$$m_1/h : \mathcal{D}^{i_1}(\mathcal{D}^{i_2}A) \Rightarrow$$

Before giving the derivation we prove a useful lemma:

LEMMA 5.1. *The following sequent is derivable in* ***G3DSTIT***:

$$h_1 \sim^{i_1}_m h_2, m/h_1 : \mathcal{D}^{i_1}A \Rightarrow m/h_2 : \mathcal{D}^{i_1}A.$$

Proof. By the following derivation (where the topsequents are derivable by Lemma 4.2):

(1)

$$\dfrac{\dfrac{m/h_3 : A, m \in h_3, \ldots, m/h_2 : \mathcal{S}A \Rightarrow m/h_3 : A \ldots}{m \in h_3, \ldots, m/h_2 : \mathcal{S}A \Rightarrow m/h_3 : A \ldots}\ L\mathcal{S}}{\ldots m/h_2 : \mathcal{S}A \Rightarrow m/h_1 : \mathcal{S}A \ldots}\ R\mathcal{S}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{h_1 \sim^{i_1}_m h_4, \ldots, m/h_4 : A, m/h_1 : \Box^{i_1} A \Rightarrow m/h_4 : A, m/h_1 : \mathcal{S}A}{h_1 \sim^{i_1}_m h_4, \ldots, m/h_1 : \Box^{i_1} A \Rightarrow m/h_4 : A, m/h_1 : \mathcal{S}A}\ L\Box^{i_1}}{h_4 \sim^{i_1}_m h_2, h_1 \sim^{i_1}_m h_2, m/h_1 : \Box^{i_1} A \Rightarrow m/h_4 : A, m/h_1 : \mathcal{S}A}\ Etrans_{\sim^{i_1}_m}}{h_1 \sim^{i_1}_m h_2, m/h_1 : \Box^{i_1} A \Rightarrow m/h_2 : \Box^{i_1} A, m/h_1 : \mathcal{S}A}\ R\Box^{i_1} \quad (1)}{h_1 \sim^{i_1}_m h_2, m/h_1 : \Box^{i_1} A \Rightarrow m/h_2 : \mathcal{D}^{i_1} A, m/h_1 : \mathcal{S}A}\ R\mathcal{D}^{i_1}}{h_1 \sim^{i_1}_m h_2, m/h_1 : \mathcal{D}^{i_1} A \Rightarrow m/h_2 : \mathcal{D}^{i_1} A}\ L\mathcal{D}^{i_1}$$

■

We are now ready to prove impossibility of delegation:

Theorem 5.2. *The following sequent is derivable in* ***G3DSTIT****:*

$$m_1/h : \mathcal{D}^{i_1}(\mathcal{D}^{i_2} A) \Rightarrow$$

Proof. The topsequent is derivable by Lemma 5.1. The derivation then proceeds as follows:

(1)

$$\dfrac{\dfrac{\dfrac{\ldots, m/h_5 : \mathcal{D}^{i_2} A, h_5 \sim^{i_1}_m h_1, h_5 \sim^{i_2}_m h_2, h_4 \sim^{i_2}_m h_2, h_1 \sim^{i_1}_m h_1, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, \ldots \Rightarrow m/h_2 : \mathcal{D}^{i_2} A, m/h_4 : A, \ldots}{\ldots, h_5 \sim^{i_1}_m h_1, h_5 \sim^{i_2}_m h_2, h_4 \sim^{i_2}_m h_2, h_1 \sim^{i_1}_m h_1, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, \ldots \Rightarrow m/h_2 : \mathcal{D}^{i_2} A, m/h_4 : A, \ldots}\ L\Box^{i_1}}{\ldots, h_4 \sim^{i_2}_m h_2, h_1 \sim^{i_1}_m h_1, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, m \in h_1, \ldots \Rightarrow m/h_2 : \mathcal{D}^{i_2} A, m/h_4 : A, \ldots}\ Ind_2}{\ldots, h_1 \sim^{i_1}_m h_1, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, \ldots \Rightarrow m/h_2 : \mathcal{D}^{i_2} A, m/h_2 : \Box^{i_2} A, \ldots}\ R\Box^{i_2}, WD$$

(2)

$$\dfrac{m/h_3 : A, m/h_2 : \mathcal{S}A, m \in h_3, h_1 \sim^{i_1}_m h_1, m \in h_2, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, m/h_1 : \Box^{i_2} A, m/h_3 : A \Rightarrow m/h_3 : A}{m/h_2 : \mathcal{S}A, m \in h_3, h_1 \sim^{i_1}_m h_1, m \in h_2, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, m/h_1 : \Box^{i_2} A \Rightarrow m/h_3 : A}\ L\mathcal{S}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(1) \qquad\qquad (2)}{m \in h_3, h_1 \sim^{i_1}_m h_1, m \in h_2, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, m/h_1 : \Box^{i_2} A \Rightarrow m/h_2 : \mathcal{D}^{i_2} A, m/h_3 : A}\ R\mathcal{D}^{i_2}}{h_1 \sim^{i_1}_m h_1, m \in h_2, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A, m/h_1 : \mathcal{D}^{i_2} A \Rightarrow m/h_2 : \mathcal{D}^{i_2} A}\ L\mathcal{D}^{i_2}, R\mathcal{S}}{m \in h_2, m/h_1 : \Box^{i_1}\mathcal{D}^{i_2} A \Rightarrow m/h_2 : \mathcal{D}^{i_2} A}\ L\Box^{i_1}, Refl_{\sim^{i_1}_m}}{m/h_1 : \mathcal{D}^{i_1}\mathcal{D}^{i_2} A \Rightarrow}\ L\mathcal{D}^{i_1}, R\mathcal{S}$$

■

In addition to adapting the result to deliberative STIT, we can likewise show it is impossible to *prevent* somebody from doing something (i.e. make them not do it):

$$m/h : \mathcal{D}^a \neg \mathcal{D}^b A \Rightarrow$$

PROOF.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \ldots, h^i \sim^b_m h', \ldots, m/h' : \Box^b A \Rightarrow \ldots, m/h^i : \Box^b A\end{matrix} \quad \dfrac{m/h'' : A, \ldots \Rightarrow m/h'' : A, \ldots}{m/h^i : \mathcal{S}A, m \in h'', \ldots \Rightarrow m/h'' : A, \ldots}\, L\mathcal{S}}{h^i \sim^a_m h, h^i \sim^b_m h', \ldots, m \in h', h'', m/h : \Box^a \neg \mathcal{D}^b A, m/h' : \Box^b A \Rightarrow m/h'' : A, m/h^i : \mathcal{D}^b A}\, R\mathcal{D}^b}{h^i \sim^a_m h, h^i \sim^b_m h', \ldots, m \in h', h'', m/h : \Box^a \neg \mathcal{D}^b A, m/h^i : \neg \mathcal{D}^b A, m/h' : \Box^b A \Rightarrow m/h'' : A}\, L\neg}{h^i \sim^a_m h, h^i \sim^b_m h', \ldots, m \in h', h'', m/h : \Box^a \neg \mathcal{D}^b A, m/h' : \Box^b A \Rightarrow m/h'' : A}\, L\Box^a}{h \sim^a_m h, h' \sim^b_m h', m \in h', h'', m/h : \Box^a \neg \mathcal{D}^b A, m/h' : \Box^b A \Rightarrow m/h'' : A}\, Ind_2}{h \sim^a_m h, h' \sim^b_m h', m \in h', m/h : \Box^a \neg \mathcal{D}^b A, m/h' : \mathcal{D}^b A \Rightarrow}\, L\mathcal{D}^b, R\mathcal{S}}{h \sim^a_m h, h' \sim^b_m h', m \in h', m/h : \Box^a \neg \mathcal{D}^b A \Rightarrow m/h' : \neg \mathcal{D}^b A}\, R\neg}{m \in h', m/h : \Box^a \neg \mathcal{D}^b 4A \Rightarrow m/h' : \neg \mathcal{D}^b A}\, Refl_{\sim^{a,b}_m}}{m/h : \mathcal{D}^a \neg \mathcal{D}^b A \Rightarrow}\, L\mathcal{D}^a, R\mathcal{S}$$

■

5.2. Refraining

A good starting point in the discussion of refraining appears in G.H. von Wright [18], and the account of events therein. There von Wright treats *events* as ordered pairs of states of affairs, the first, *initial state* temporally preceding the second, *end-state*, and the event itself a transition from the former to the latter. In von Wright's (somewhat cumbersome) notation, an event is written as $p\,T\,q$—a transformation from the initial p state to the end q state [18, pp. 28–29].

An act, then, is the bringing about of an event by an agent, written as $d\,(p\,T\,q)$. An accurate, if, as von Wright notes, clumsy way to express, say, $d(\sim p\,T\,p)$ is to say it is the "doing so that p" [18, pp. 42–43]. The link to the phrase see-to-it-that should be clear here. The connection of this consideration to STIT is reinforced by the conditions for doing $d(\sim p\,T\,p)$ that p does not happen "*independently of the action of the agent*" [18, p. 43]. The counter (or could-have-been-otherwise) condition is a crucial feature of STIT.

The "correlative" of doing is to refrain from doing something (von Wright uses the term 'forbear'). This, however, is not simply not doing an action. Rather, to forbear p, written as $f(\sim p\,T\,p)$, is to be able to do it, but not do it [18, p. 45], so in our notation it would be understood as:

$$Ref^{\,i} A \equiv_{def.} \mathcal{P}\mathcal{D}^i A \;\&\; \neg \mathcal{D}^i A$$

Both acts and forbearances are a mode of action [18, p. 48], and it is likewise the condition of forbearance that it does not come about independently of an agent [18, p. 46]. Doing and forbearing are closely correlated but separate in von Wright's analysis.

In [2, pp. 42-43] the authors find that refraining can be more thoroughly analysed by using, unlike von Wright, embedded modalities. Then noting that refraining itself is a mode of doing (something the first definition doesn't state), the definition becomes

$$Ref^{\,i} A \equiv_{def.} \mathcal{D}^i \neg \mathcal{D}^i A$$

We can show that the two accounts are equivalent (cf. [2, p. 438]), i.e.

$$\mathcal{D}^i \neg \mathcal{D}^i A \equiv \mathcal{P}\mathcal{D}^i A \,\&\, \neg \mathcal{D}^i A$$

in our system by proving:

PROPOSITION 5.3. *The following sequents are derivable in* **G3DSTIT***:*

(a) $m/h : \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{P}\mathcal{D}^i A \,\&\, \neg \mathcal{D}^i A$*;*

(b) $m/h : \mathcal{P}\mathcal{D}^i A \,\&\, \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i \neg \mathcal{D}^i A$*.*

PROOF. (a) We have the following derivations:

(1)

$$\cfrac{\cfrac{\cfrac{m/h_1 : \mathcal{D}^i A, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{P}\mathcal{D}^i A, m/h_1 : \mathcal{D}^i A}{m \in h_1, m/h : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{P}\mathcal{D}^i A, m/h_1 : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i A}\; R\neg}{m \in h_1, m/h : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{P}\mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i A}\; R\mathcal{P}}{m/h : \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{P}\mathcal{D}^i A}\; L\mathcal{D}^i, R\mathcal{S}$$

(2)

$$\cfrac{\cfrac{\cfrac{h \sim^i_m h, m/h : \Box^i \neg \mathcal{D}^i A, m/h : \neg \mathcal{D}^i A \Rightarrow m/h : \neg \mathcal{D}^i A, m/h : \mathcal{S}\neg \mathcal{D}^i A}{h \sim^i_m h, m/h : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \neg \mathcal{D}^i A, m/h : \mathcal{S}\neg \mathcal{D}^i A}\; L\mathcal{D}^i}{m/h : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \neg \mathcal{D}^i A, m/h : \mathcal{S}\neg \mathcal{D}^i A}\; Refl_{\sim^i_m}}{m/h : \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \neg \mathcal{D}^i A}\; L\mathcal{D}^i$$

and the compound derivation:

$$\frac{(1) \qquad\qquad (2)}{m/h : \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{P}\mathcal{D}^i A \,\&\, \neg \mathcal{D}^i A}\; R\&$$

(b) We have the following derivation:

(1)

$$\cfrac{\cfrac{m/h : \mathcal{S}\neg \mathcal{D}^i A, m \in h_1, m/h_1 : \mathcal{D}^i A, m/h : \neg \mathcal{D}^i A \Rightarrow m/h_1 : \mathcal{D}^i A}{m/h_1 : \neg \mathcal{D}^i A, m/h : \mathcal{S}\neg \mathcal{D}^i A, m \in h_1, m/h_1 : \mathcal{D}^i A, m/h : \neg \mathcal{D}^i A \Rightarrow}\; L\neg}{m/h : \mathcal{S}\neg \mathcal{D}^i A, m \in h_1, m/h_1 : \mathcal{D}^i A, m/h : \neg \mathcal{D}^i A \Rightarrow}\; L\mathcal{S}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{h_2 \sim^i_m h, m \in h_1, m/h_1 : \mathcal{D}^i A, m/h_2 : \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A}\ \textit{Lemma 5.1}}{h_2 \sim^i_m h, m \in h_1, m/h_1 : \mathcal{D}^i A, m/h : \neg\mathcal{D}^i A \Rightarrow m/h_2 : \neg\mathcal{D}^i A}\ L\neg, R\neg}{m \in h_1, m/h_1 : \mathcal{D}^i A, m/h : \neg\mathcal{D}^i A \Rightarrow m/h : \Box^i \neg\mathcal{D}^i A}\ R\Box^i \qquad (1)}{m \in h_1, m/h_1 : \mathcal{D}^i A, m/h : \neg\mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i \neg\mathcal{D}^i A}\ R\mathcal{D}^i}{m/h : \mathcal{P}\mathcal{D}^i A, m/h : \neg\mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i \neg\mathcal{D}^i A}\ L\mathcal{P}}{m/h : \mathcal{P}\mathcal{D}^i A \,\&\, \neg\mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i \neg\mathcal{D}^i A}\ L\&$$

■

Under this interpretation, it holds for DSTIT that doing is equivalent to refraining from refraining [2, p. 50, 439]:

$$\text{(Refref)}: \mathcal{D}^i A \equiv \mathcal{D}^i \neg\mathcal{D}^i \neg\mathcal{D}^i A$$

We can likewise show that this holds for DSTIT.

Proposition 5.4. *This equivalence, meaning the sequents in both directions, holds in G3DSTIT.*

$$m/h : \mathcal{D}^i A \Leftrightarrow m/h : \mathcal{D}^i \neg\mathcal{D}^i \neg\mathcal{D}^i A$$

Proof. We start with the proof of the direction left-to-right. For legibility we write $m \in h, \ldots, m \in h'$ as $m \in h, \ldots, h'$.

(1)

$$\dfrac{\dfrac{\dfrac{m \in h, h_1, \ldots, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h : \Box^i A \qquad \dfrac{m/h_1 : A, m/h : \mathcal{S}A, m \in h, h_1, \ldots, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A}{m/h : \mathcal{S}A, m \in h, h_1, \ldots, m/h : \Box^i A \Rightarrow m/h_1 : A}\ L\mathcal{S}}{m \in h, h_1, \ldots, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h : \mathcal{D}^i A}\ R\mathcal{D}^i}{m \in h, h_1 \ldots, m/h : \neg\mathcal{D}^i A, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A}\ L\neg}{m \in h, h_1, m/h_1 : \mathcal{S}\neg\mathcal{D}^i A, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A}\ L\mathcal{S}$$

(2)

$$h_2 \sim^i_m h_1, m/h_2 : \Box^i A, \ldots \Rightarrow m/h_1 : A$$
$$\vdots$$
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{h_2 \sim^i_m h_1, m \in h, h_1, h_5, m/h : \Box^i A, m/h_2 : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h_1 : \mathcal{D}^i \neg\mathcal{D}^i A, m/h_5 : A}{h_2 \sim^i_m h_1, m \in h, h_1, m/h : \Box^i A, m/h_2 : \mathcal{D}^i A, \ldots \Rightarrow m/h_1 : A, m/h_1 : \mathcal{D}^i \neg\mathcal{D}^i A}\ L\mathcal{D}^i, R\mathcal{S}}{h_2 \sim^i_m h_1, m \in h, h_1, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h_1 : \mathcal{D}^i \neg\mathcal{D}^i A, m/h_2 : \neg\mathcal{D}^i A}\ R\neg}{m \in h, h_1, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h_1 : \mathcal{D}^i \neg\mathcal{D}^i A, m/h_1 : \Box^i \neg\mathcal{D}^i A}\ R\Box^i \qquad (1)}{m \in h, h_1, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h_1 : \mathcal{D}^i \neg\mathcal{D}^i A}\ R\mathcal{D}^i}{m \in h_1, m/h : \Box^i A, \ldots \Rightarrow m/h_1 : A, m/h_1 : \mathcal{D}^i \neg\mathcal{D}^i A}\ WD}{m \in h_1, m/h : \Box^i A, \ldots, m/h_1 : \neg\mathcal{D}^i \neg\mathcal{D}^i A \Rightarrow m/h_1 : A}\ L\neg}{m \in h_1, m/h : \Box^i A, m/h : \mathcal{S}\neg\mathcal{D}^i \neg\mathcal{D}^i A \Rightarrow m/h_1 : A}\ L\mathcal{S}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{m \in h_1, h_3, h_4, \ldots, m/h : \Box^i A, \ldots \Rightarrow \ldots, m/h : \Box^i A \qquad m \in h_1, h_3, h_4, \ldots, m/h_1 : A, \ldots \Rightarrow m/h_1 : A, \ldots}{h_2 \sim^i_m h, m \in h_1, h_3, h_4, m/h : \Box^i A, m/h_2 : \Box^i \neg\mathcal{D}^i A, m/h_3 : \Box^i A \Rightarrow m/h_1 : A, m/h : \mathcal{D}^i A, m/h_4 : A}\ R\mathcal{D}^i}{h_2 \sim^i_m h, m \in h_1, h_3, m/h : \Box^i A, m/h_2 : \Box^i \neg\mathcal{D}^i A, m/h_3 : \mathcal{D}^i A \Rightarrow m/h_1 : A, m/h : \mathcal{D}^i A}\ L\mathcal{D}^i, R\mathcal{S}}{h_2 \sim^i_m h, m \in h_1, h_3, m/h : \Box^i A, m/h_2 : \Box^i \neg\mathcal{D}^i A, m/h : \neg\mathcal{D}^i A \Rightarrow m/h_1 : A, m/h_3 : \neg\mathcal{D}^i A}\ L\neg, R\neg}{h_2 \sim^i_m h, m \in h_1, h_3, m/h : \Box^i A, m/h_2 : \Box^i \neg\mathcal{D}^i A \Rightarrow m/h_1 : A, m/h_3 : \neg\mathcal{D}^i A}\ L\Box^i}{h_2 \sim^i_m h, m \in h_1, m/h : \Box^i A, m/h_2 : \mathcal{D}^i \neg\mathcal{D}^i A \Rightarrow m/h_1 : A}\ L\mathcal{D}^i, R\mathcal{S}}{h_2 \sim^i_m h, m \in h_1, m/h : \Box^i A \Rightarrow m/h_1 : A, m/h_2 : \neg\mathcal{D}^i \neg\mathcal{D}^i A}\ R\neg}{m \in h_1, m/h : \Box^i A \Rightarrow m/h_1 : A, m/h : \Box^i \neg\mathcal{D}^i \neg\mathcal{D}^i A}\ R\Box^i \qquad (2)}{m \in h_1, m/h : \Box^i A \Rightarrow m/h_1 : A, m/h : \mathcal{D}^i \neg\mathcal{D}^i \neg\mathcal{D}^i A}\ R\mathcal{D}^i}{m/h : \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i \neg\mathcal{D}^i \neg\mathcal{D}^i A}\ L\mathcal{D}^i, R\mathcal{S}$$

We now demonstrate the right-to-left direction of the equivalence.

(1)

$$\dfrac{}{h_3 \sim^i_m h, m/h_3 : \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A}\ \textit{Lemma 5.1}$$

$$\vdots$$

$$\dfrac{h_3 \sim^i_m h, h \sim^i_m h, m \in h_1, h_2, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \Box^i \neg \mathcal{D}^i A, m/h_3 : \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_2 : \neg \mathcal{D}^i A}{h_3 \sim^i_m h, h \sim^i_m h, m \in h_1, h_2, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_3 : \neg \mathcal{D}^i A, m/h_2 : \neg \mathcal{D}^i A}\ R\neg$$

$$\dfrac{h_3 \sim^i_m h, h \sim^i_m h, m \in h_1, h_2, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_3 : \neg \mathcal{D}^i A, m/h_2 : \neg \mathcal{D}^i A}{h \sim^i_m h, m \in h_1, h_2, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h : \Box^i \neg \mathcal{D}^i A, m/h_2 : \neg \mathcal{D}^i A}\ R\Box^i$$

$$\dfrac{\ldots, m \in h_1, h_2, \ldots, m/h_2 : \neg \mathcal{D}^i A, \ldots \Rightarrow \ldots, m/h_2 : \neg \mathcal{D}^i A}{\ldots, m \in h_1, h_2, \ldots, m/h : \mathcal{S} \neg \mathcal{D}^i A \Rightarrow \ldots, m/h_2 : \neg \mathcal{D}^i A}\ L\mathcal{S}$$

$$\dfrac{(1) \qquad \ldots, m \in h_1, h_2, \ldots, m/h : \mathcal{S} \neg \mathcal{D}^i A \Rightarrow \ldots, m/h_2 : \neg \mathcal{D}^i A}{h \sim^i_m h, m \in h_1, h_2, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h : \mathcal{D}^i \neg \mathcal{D}^i A, m/h_2 : \neg \mathcal{D}^i A}\ R\mathcal{D}^i$$

$$\dfrac{h \sim^i_m h, m \in h_1, h_2, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \Box^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h : \mathcal{D}^i \neg \mathcal{D}^i A, m/h_2 : \neg \mathcal{D}^i A}{h \sim^i_m h, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h : \mathcal{D}^i \neg \mathcal{D}^i A}\ L\mathcal{D}^i, R\mathcal{S}$$

$$\dfrac{h \sim^i_m h, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h_1 : \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h : \mathcal{D}^i \neg \mathcal{D}^i A}{h \sim^i_m h, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h : \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i \neg \mathcal{D}^i A}\ L\neg, R\neg$$

$$\dfrac{h \sim^i_m h, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A, m/h : \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i \neg \mathcal{D}^i A}{h \sim^i_m h, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i \neg \mathcal{D}^i A}\ L\Box^i$$

$$\dfrac{h \sim^i_m h, m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i \neg \mathcal{D}^i A}{m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i \neg \mathcal{D}^i A}\ Refl_{\sim^i_m}$$

$$\dfrac{m \in h_1, m/h : \Box^i \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A, m/h_1 : \neg \mathcal{D}^i \neg \mathcal{D}^i A}{m/h : \mathcal{D}^i \neg \mathcal{D}^i \neg \mathcal{D}^i A \Rightarrow m/h : \mathcal{D}^i A}\ L\mathcal{D}^i, R\mathcal{S}$$

■

6. Meta-theoretical Properties of DSTIT Logics

In this section we demonstrate the meta-theoretical properties of the system, namely decidability, soundness and completeness, starting with the former. As previously noted, decidability of STIT logics has been achieved using filtration [1,2,21]. Here instead we shall present a direct proof of decidability through a bound on proof search in the given sequent calculus.

6.1. Decidability

To obtain a decidable system we adopt the following axiom. As shown in [2, p. 437], the validity of APC_n in DSTIT frames is equivalent to the requirement that the agent i has at most n choices at any moment, i.e. there are at most n elements of a partition given by $\sim^i_m$.
APC$_n$:

$$\mathcal{P}\Box^i A_1 \ \&\ \mathcal{P}(\neg A_1 \ \&\ \Box^i A_2) \ \&\ \ldots \ \&\ \mathcal{P}(\neg A_1 \ \&\ \ldots \ \&\ \neg A_{n-1} \ \&\ \Box^i A_n) \supset A_1 \vee \ldots \vee A_n$$

The corresponding rule extends our calculus **G3DSTIT** and the resulting calculus will be denoted by **G3Ldm**$_n$:

$$\dfrac{h_1 \sim^i_m h_2, m \in h_1, \ldots, h_{n+1}, \Gamma \Rightarrow \Delta \qquad \ldots \qquad h_n \sim^i_m h_{n+1}, m \in h_1, \ldots, h_{n+1}, \Gamma \Rightarrow \Delta}{m \in h_1, \ldots, m \in h_{n+1}, \Gamma \Rightarrow \Delta} \; Apc_n$$

We can demonstrate that the axiom APC_n is derivable in our system. For simplicity we demonstrate this for the case $n = 2$, but the generalization to any n is straightforward.

LEMMA 6.1. *The sequent*

$$m/h : \mathcal{P}\Box^i A_1 \;\&\; \mathcal{P}(\neg A_1 \;\&\; \Box^i A_2) \Rightarrow m/h : A_1 \vee A_2$$

is derivable in ***G3Ldm***$_2$.

PROOF.

(1)

$$\dfrac{m/h : A_1, h \sim^i_m h_1, m \in h, h_1, h_2, m/h_1 : \Box^i A_1, \ldots \Rightarrow m/h : A_1, m/h : A_2}{h \sim^i_m h_1, m \in h, h_1, h_2, m/h_1 : \Box^i A_1, \ldots \Rightarrow m/h : A_1, m/h : A_2} \; L\Box^i$$

(2)

$$\dfrac{m/h : A_2, h \sim^i_m h_2, m \in h, h_1, h_2, \ldots, m/h_2 : \Box^i A_2 \Rightarrow m/h : A_1, m/h : A_2}{h \sim^i_m h_2, m \in h, h_1, h_2, \ldots, m/h_2 : \Box^i A_2 \Rightarrow m/h : A_1, m/h : A_2} \; L\Box^i$$

(3)

$$\dfrac{m/h_2 : A_1, h_1 \sim^i_m h_2, m \in h, h_1, h_2, m/h_1 : \Box^i A_1, m/h_2 : \neg A_1, \ldots \Rightarrow \ldots}{h_1 \sim^i_m h_2, m \in h, h_1, h_2, m/h_1 : \Box^i A_1, m/h_2 : \neg A_1, \ldots \Rightarrow \ldots} \; L\Box^i$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1) \qquad (2) \qquad (3)}{m \in h, h_1, h_2, m/h_1 : \Box^i A_1, m/h_2 : \neg A_1, m/h_2 : \Box^i A_2 \Rightarrow m/h : A_1, m/h : A_2} \; Apc_2}{m \in h, h_1, h_2, m/h_1 : \Box^i A_1, m/h_2 : \neg A_1 \;\&\; \Box^i A_2 \Rightarrow m/h : A_1, m/h : A_2} \; L\&}{m \in h_1, h_2, m/h_1 : \Box^i A_1, m/h_2 : \neg A_1 \;\&\; \Box^i A_2 \Rightarrow m/h : A_1, m/h : A_2} \; WD}{m/h : \mathcal{P}\Box^i A_1, m/h : \mathcal{P}(\neg A_1 \;\&\; \Box^i A_2) \Rightarrow m/h : A_1, m/h : A_2} \; L\mathcal{P}, L\mathcal{P}}{m/h : \mathcal{P}\Box^i A_1, m/h : \mathcal{P}(\neg A_1 \;\&\; \Box^i A_2) \Rightarrow m/h : A_1 \vee A_2} \; R\vee}{m/h : \mathcal{P}\Box^i A_1 \;\&\; \mathcal{P}(\neg A_1 \;\&\; \Box^i A_2) \Rightarrow m/h : A_1 \vee A_2} \; L\&$$

■

We now provide a decision procedure for **G3Ldm**$_n$ by showing that proof search always terminates in a finite number of steps, adapting the proof from [8]. To begin, we first define several auxiliary concepts, and then the saturation conditions.

To keep track of the new moment/history labels generated during the proof search, we create a graph $\mathcal{T}_h$ defined as follows:

DEFINITION 6.2. Let $\mathcal{B} = \{\Gamma_n \Rightarrow \Delta_n\}$ be a (finite or infinite) branch in proof search for $\Gamma \Rightarrow \Delta$, and let $\Gamma^* = \bigcup \Gamma_n$, $\Delta^* = \bigcup \Delta_n$. For a branch $\mathcal{B}$ of a proof-search tree for $\Rightarrow m/h_0 : A_0$, the graph $\mathcal{T}_h$ is generated from the root and relation $\longmapsto$ as follows:

1. The root of $\mathcal{T}_h$ is m/h_0.
2. If m/h' is a point with a fresh history h' generated by an application of rules R$\Box^i$, R$\mathcal{S}$ or L$\mathcal{P}$, and m/h occurs in the principal formula of the respective rule, then $m/h \longmapsto m/h'$.
3. If m/h' is a point with a fresh history h' generated by an application of rule Ind_k and $m \in h_1 \ldots m \in h_n$ are among the principal formulas of the rule, then $m/h_i \longmapsto m/h'$, $1 \leq i \leq n$. We call m/h' an **independence** point.

DEFINITION 6.3. The saturation conditions for the rules of $\mathbf{G3Ldm}_n$ are:

1. (Init): For all n, there is no $m/h : p$ in $\Gamma_n \cap \Delta_n$; $m/h : \bot$ is not in Γ_n.
2. ($\neq$): For all n, $i = j$ and $i \neq j$ are not both in $\Gamma_n \cap \Delta_n$.
3. (Prop): Standard for propositional rules.
4. (L$\Box^i$): If $h' \sim^i_m h$ and $m/h : \Box^i A$ are in Γ^*, then $m/h' : A$ is also in Γ^*.
5. (R$\Box^i$) If $m/h : \Box^i A$ is in Δ^*, then for some history h', $h' \sim^i_m h$ is in Γ^* and $m/h' : A$ is in Δ^*.
6. (R$\mathcal{D}^i$): If $m/h : \mathcal{D}^i A$ is in Δ^*, then either $m/h : \mathcal{S}A$ is in Γ^* or $m/h : \Box^i A$ is in Δ^*.
7. (L$\mathcal{D}^i$): If $m/h : \mathcal{D}^i A$ is in Γ^*, then $m/h : \Box^i A$ is in Γ^* and $m/h : \mathcal{S}A$ is in Δ^*.
8. (R$\mathcal{S}$): If $m/h : \mathcal{S}A$ is in Δ^*, then for some history h', $m \in h'$ is in Γ^* and $m/h' : A$ is in Δ^*.
9. (L$\mathcal{S}$): If $m \in h'$ and $m/h : \mathcal{S}A$ are in Γ^*, then $m/h' : A$ is also in Γ^*.
10. (R$\mathcal{P}$): If $m \in h'$ is in Γ^* and $m/h : \mathcal{P}A$ is in Δ^*, then $m/h' : A$ is also in Δ^*.
11. (L$\mathcal{P}$): If $m/h : \mathcal{P}A$ is in Γ^*, then for some history h', $m \in h'$ and $m/h' : A$ are also in Γ^*.

12. (Refl$_=$): For every i in $\Gamma^* \cup \Delta^*$, $i = i$ is in Γ^*.
13. (ETrans$_=$): If $i = j$ and $i = k$ are in Γ^*, then $j = k$ is also in Γ^*.
14. (Repl$_{At}$): If $i = j$ and $At(i)$ are in Γ^*, then $At(j)$ is also in Γ^*.
15. (Refl$_{\sim^i_m}$): For any i such that $m/h : \Box^i A$ is in Γ^*, $h \sim^i_m h$ is in Γ^*.
16. (ETrans$_{\sim^i_m}$): If $h \sim^i_m h'$ and $h \sim^i_m h''$ are in Γ^*, then $h' \sim^i_m h''$ is also in Γ^*.
17. (WD): If $h \sim^i_m h'$ or $m/h : A$ is in $\Gamma^* \cup \Delta^*$, then $m \in h$ is in Γ^*.
18. (Diff$_k$): If $\mathit{Diff}(i_1 \dots i_k)$ is in Γ^*, then $\{i_l \neq i_m\}_{1 \leq l < m \leq k}$ are also in Γ^*.
19. (Apc$_n$): If $m \in h_1, \dots, m \in h_{n+1}$ are in Γ^*, then for any a_i in $\Gamma^* \cup \Delta^*$, either $h_1 \sim^{a_i}_m h_2$ or … or $h_n \sim^{a_i}_m h_{n+1}$ is in Γ^*.
20. (Ind$_k$): If $\mathit{Diff}(a_1 \dots a_k)$ is in Γ^*, and for any a_i and a_j, $1 \leq i < j \leq k$, $h_i \sim^i_m h'_i$ and $h_j \sim^j_m h'_j$ are in Γ^*, then for some history h, $h \sim^i_m h_i$ and $h \sim^j_m h_j$ are also in Γ^*.

We call the branch $\mathcal{B}$ saturated w.r.t. an application of a rule if the corresponding condition holds, and saturated simpliciter if it is saturated w.r.t. all the rules.

We can now build (root-first) a proof-search tree for a sequent $\Rightarrow m/h_0 : A_0$ in $\mathbf{G3Ldm}_n$. The building of the tree obeys the following rules:

1. No rule is applied to an initial sequent.
2. Rule $\neq$ is applied to a sequent $\Gamma_i \Rightarrow \Delta_i$ containing $i = j$ and $i \neq j$ in Γ_i.
3. Rule R is not applied to a sequent $\Gamma_i \Rightarrow \Delta_i$ if the branch $\mathcal{B}$ down to $\Rightarrow m/h_0 : A_0$ is saturated w.r.t. R.
4. All rules with no freshness constraint are applied before any rules with a freshness constraint.
5. Whenever a bottom-up application of a rule requires a moment/history label m/h' with a fresh history h', a history that has not occurred anywhere else in the tree is chosen.
6. Rule Ind_k is applied when a branch is saturated w.r.t. every other rule, and then in accordance with the procedure described in Lemma 6.6.

We now show that this proof-search procedure terminates. We do so through several lemmas.

Lemma 6.4. *Any formula of the form $\mathcal{P}A$ occurring labelled in Γ^* or of the form $\mathcal{S}A$ occurring labelled in Δ^* generates at most one fresh point in $\mathcal{T}_h$.*

Proof. It follows straightforwardly from the saturation criterion for L$\mathcal{P}$ that, if that rule had been applied to $m/h : \mathcal{P}A$ (obviously, in Γ^*), then the rule is not applied for any $m/h' : \mathcal{P}A$ (in Γ^*). Same for R$\mathcal{S}$. ■

Lemma 6.5. *For any formula $\Box^i A$ occurring labelled in Δ^*, the application of R$\Box^i$ can only add a finite number of points to each path in $\mathcal{T}_h$.*

Proof. We show for any formula $\Box^i A$ occurring labelled in Δ^*, by Noetherian induction on the length of the branch $\mathcal{B}$, that for any length of the branch rule R$\Box^i$ is applied at most n times. If $m/h : \Box^i A$ occurs in Δ^* and the rule has been applied n times, then there is a history h' such that $m/h' : A$ occurs in Δ^* and (by the saturation criterion for WD) $m \in h'$ occurs in Γ^*, and therefore (by the saturation criterion for Apc_n and possibly $ETrans_{\sim^i_m}$) $h \sim^i_m h'$. Recall that since neither of these rules has a freshness condition, they are applied prior to R$\Box^i$. Therefore, the saturation criterion is met for $m/h : \Box^i A$ and the rule is not applied. ■

Moreover, the tree $\mathcal{T}_h$ is finitely branching and by the subformula property the number of different formulas $\Box^i A$ is finite. Therefore, the number of points generated by the application of rule R$\Box^i$ is finite.

Given Lemmas 6.4 and 6.5, and given the subformula property, it follows that all the formulas in $\mathcal{B}$ can only generate a finite number of points.

What remains to be shown is that only a finite amount of points can be generated from other points via the rule Ind_k. Intuitively, the main idea is that each application of Ind_k generates a new independence point, which inherits all the choice equivalence relations. So, now we need to apply Ind_k to it as well. However, this process will not go on indefinitely—once we have generated three independence points, each of which are connected to the other two with all the inherited choice-equivalence relation, the process stops due to the saturation criteria. Cf. *Etrans*, where once we have three objects connected via an identity relation, each formula satisfies the saturation criterion for the other two and the process likewise terminates.

Lemma 6.6. *Given an arbitrary subset S of Γ_i of some sequent $\Gamma_i \Rightarrow \Delta_i$, such that $S = \{h_l \sim^{a_l}_m h'_l, \ldots, h_n \sim^{a_n}_m h'_n, Diff(a_1, \ldots, a_k)\}$, where $1 \leq l < n \leq k$ and for each a_j from $Diff(a_1, \ldots, a_k)$ there is at most one relational formula $h_j \sim^{a_j}_m h'_j$, we have that S produces at most three applications of Ind_k, each of which generates one new independence point.*

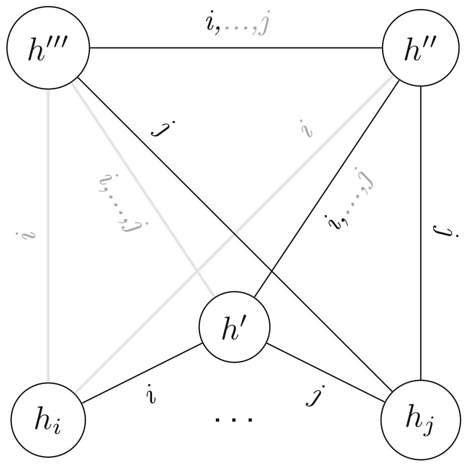

Figure 2. Choice-equivalence relations among histories

PROOF. We will show that the generation of any further points is blocked by the saturation criterion. First note that, by saturation for *WD*, for each distinct $h_j \sim_m^{a_j} h'_j$, there is a relational formula $m \in h_j$ in Γ_i. Therefore, we can apply Ind_k.

Now apply Ind_k to S to create a new independence point m/h'. Apply rule *WD* to obtain $m \in h'$ in Γ_i and let $S' = S[h' \sim_m^{a_l} h_l / h_l \sim_m^{a_l} h'_l]$ for some a_l (which one is irrelevant, but as a convention we can always use the one with lowest l). Applying Ind_k to S' will create a new independence point m/h''. Repeat the same procedure for h'' to obtain h''':

$$\begin{array}{cl}
h''' \sim_m^{a_l} h'', \dots, h''' \sim_m^{a_n} h_n, \dots, \mathit{Diff}(a_1, \dots, a_k), m \in h_l, \dots, h_n, h', h'', \Gamma'_i \Rightarrow \Delta_i & \\ \hline
h'' \sim_m^{a_l} h', \dots, h'' \sim_m^{a_n} h_n, \dots, \mathit{Diff}(a_1, \dots, a_k), m \in h_l, \dots, h_n, h', h'', \Gamma'_i \Rightarrow \Delta_i & Ind_k \\ \hline
h'' \sim_m^{a_l} h', \dots, h'' \sim_m^{a_n} h_n, \dots, \mathit{Diff}(a_1, \dots, a_k), m \in h_l, \dots, h_n, h', \Gamma'_i \Rightarrow \Delta_i & WD \\ \hline
h' \sim_m^{a_l} h_l, \dots, h' \sim_m^{a_n} h_n, \dots, \mathit{Diff}(a_1, \dots, a_k), m \in h_l, \dots, h_n, h', \Gamma'_i \Rightarrow \Delta_i & Ind_k \\ \hline
h' \sim_m^{a_l} h_l, \dots, h' \sim_m^{a_n} h_n, \dots, \mathit{Diff}(a_1, \dots, a_k), m \in h_l, \dots, h_n, \Gamma'_i \Rightarrow \Delta_i & WD \\ \hline
h_l \sim_m^{a_l} h'_l, \dots, h_n \sim_m^{a_n} h'_n, \mathit{Diff}(a_1, \dots, a_k), m \in h_l, \dots, h_n, \Gamma'_i \Rightarrow \Delta_i & Ind_k
\end{array}$$

With these applications of rules, and using *Etrans* for the appropriate agents, we obtain the choice-equivalence relations among histories as illustrated in Figure 2. The choice-equivalence relations created by Ind_k are marked in black, and those created by *Etrans* in gray.

The saturation criterion is now met for any combination of histories in S and independence points. There are three cases to check:

1. two histories from S: the saturation criterion is fulfilled by h' (given *Etrans*, any of the other two independence points likewise fulfil it).
2. a history from S and an independence point: for some h_j from S and agent a_j, and an independence point h_i and an agent a_i, it holds (possibly using *Etrans*) that $h_j \sim_m^{a_j} h'$ and $h_i \sim_m^{a_i} h'$ for any of the other two independence points h', so the saturation criterion is fulfilled.
3. two independence points: for an independence point h_j and agent a_j, and an independence point h_i and an agent a_i, it holds (possibly using *Etrans*) that $h_j \sim_m^{a_j} h'$ and $h_i \sim_m^{a_i} h'$ for the third independence point h', so the saturation criterion is fulfilled. ■

We now have that

LEMMA 6.7. *The graph $\mathcal{T}_h$ is finite.*

And from there

THEOREM 6.8. *Any branch $\mathcal{B}$ of a proof-search for $\Rightarrow m/h : A$ built in accordance with the strategy is finite, therefore the entire proof-search comes to an end in a finite number of steps, and each branch is either closed or saturated.*

6.1.1. Towards Decidability Without APC$_n$ Since APC$_n$ is a fairly strong condition, one might wish to have a system without it which is still decidable. To that end we present here a proof of decidability for a one-agent system without APC$_n$.

The proof is largely the same as the one in the previous section—we need to demonstrate that the graph $\mathcal{T}_h$ is finite. In a one-agent system the rule Ind_k is rendered irrelevant, so we need to demonstrate two lemmas—that the number of points generated by the applications of the rules L$\mathcal{P}$ and R$\mathcal{S}$ is finite, and that any sequence of points which have all been generated by R$\Box^i$ is finite. The former follows from the subformula property and Lemma 6.4, so we now turn towards the latter.

DEFINITION 6.9. Call sequence of points $m/h_1 \dots m/h_n$ in $\mathcal{T}_h$ a *R$\Box^i$-subtree* iff m/h_1 is the lowest element, for every $1 < j \leq n$ it holds that $m/h_1 \longmapsto \dots \longmapsto m/h_j$ and $m/h_2 \dots m/h_n$ have all been created by R$\Box^i$.

LEMMA 6.10. *Every R$\Box^i$-subtree in $\mathcal{T}_h$ is finite.*

PROOF. The tree $\mathcal{T}_h$ is built by applying all possible instances of $ETrans_{\sim_m^i}$ before applying the rule R$\Box^i$. Therefore, for any m/h_j and m/h_k, $1 \leq j < k \leq n$ it holds that $h_j \sim_m^i h_k$. So, for any formula $\Box^i A$ occurring in Δ^*

labelled by any point m/h_j, $1 \leq j \leq n$, the rule R$\Box^i$ can only be applied once to $\Box^i A$ due to the saturation criterion for the said rule. But by subformula property, the number of such formulas is finite. Therefore, the R$\Box^i$-subtree is likewise finite. ∎

Finally, we get

LEMMA 6.11. *The graph $\mathcal{T}_h$ for the single-agent* **G3DSTIT** *(without APC_n) is finite.*

PROOF. The graph $\mathcal{T}_h$ is extended only by the applications of the rules L$\mathcal{P}$, R$\mathcal{S}$ or R$\Box^i$. The first two only generate a finite number of points, and any sequence containing only points generated by R$\Box^i$ is of finite length. Therefore, the entire graph is finite. ∎

6.2. Soundness

Next in line is the proof of soundness of $\mathbf{G3Ldm}_n$. To establish this we will first define the notions of interpretation, valuation and validity:

DEFINITION 6.12. Let $D = (T, \leqslant, Ag, Ch)$ be a DSTIT frame, where Ch maps agent-moment pairs to a set of histories and obeys reflexivity and Euclidean transitivity, an interpretation of agent terms $I(a) \in Ag$, an interpretation of moment terms $I(m) \in T$, and an interpretation of history terms $I(h) \subseteq T$. An interpretation of labels m/h is a moment/history pair (m, h). A valuation $\mathcal{V}$ of atomic formulas assigns to each atomic formulas p a set of moment/history pairs (m, h) in which p holds. We write $(m, h) \in \mathcal{V}(p)$ as $m, h \Vdash p$. The extension of $\mathcal{V}$ to a valuation of arbitrary formula of DSTIT is standard.

DEFINITION 6.13. A sequent $\Gamma \Rightarrow \Delta$ is valid for a valuation and an interpretation in D if for all labelled formulas $m/h : A$ and relational atoms R in Γ, if R holds in D and $I(m/h) \Vdash A$, then for some $m/h' : B$ in Δ, $I(m/h') \Vdash B$. A sequent is valid in D if it is valid under any valuation and interpretation.

THEOREM 6.14. *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in* $\mathbf{G3Ldm}_n$*, then it is valid in every D.*

PROOF. By induction on the derivation of $\Gamma \Rightarrow \Delta$. We illustrate on the example of the rules for $\mathcal{D}^i$.

(i) If $\Gamma \Rightarrow \Delta$ is a conclusion of the rule L$\mathcal{D}^i$, then it follows from the premiss $m/h : \Box^i A, \Gamma' \Rightarrow \Delta, m/h : \mathcal{S}A$. We assume by inductive hypothesis that the premiss is valid. Let I be an arbitrary interpretation such that it validates $m/h : \mathcal{D}^i A$ and all the formulas in Γ'. We want to show that it validates some formula in Δ.

Since I validates $m/h : \mathcal{D}^i A$, it also validates $m/h : \Box^i A$, but not $m/h : \mathcal{S}A$. Clearly, I validates the antecedent of the premiss, and therefore also the succedent. But I does not validate $m/h : \mathcal{S}A$, so it validates some formula in Δ.

(ii) If $\Gamma \Rightarrow \Delta$ is a conclusion of the rule R$\mathcal{D}^i$, then it follows from premisses $\Gamma \Rightarrow \Delta', m/h : \Box^i A$ and $m/h : \mathcal{S}A, \Gamma \Rightarrow \Delta'$. We assume by inductive hypothesis that the premisses are valid. Let I be an arbitrary interpretation such that it validates all the formulas in Γ. We want to show that it validates some formula in Δ' or $m/h : \mathcal{D}^i A$.

Since I validates all the formulas in Γ, it validates some formula in Δ' or $m/h : \Box^i A$. If the former case we are done. In the latter, if I validates $m/h : \mathcal{S}A$ then it again validates some formula in Δ', and otherwise it validates $m/h' : \neg A$ and therefore $m/h : \mathcal{D}^i A$. ■

6.3. Completeness

Finally we prove the completeness of our system. Following up on our proof of decidability, and specifically using the notion of a saturated branch, we now show that

THEOREM 6.15. *The calculus* $\mathbf{G3Ldm}_n$ *is complete with respect to the semantics of* DSTIT *frames.*

PROOF. By generating a countermodel from a saturated branch. Given such $\Gamma^* \cup \Delta^*$ in a search for a proof of the sequent $\Gamma \Rightarrow \Delta$, we generate a DSTIT countermodel $\mathcal{M}$ that makes all the formulas in Γ^* true and all formulas in Δ^* false. The model $\mathcal{M} = (T, \leq, Ag, Ch, \mathcal{V})$, with the frame $(T, \leq, Ag, Ch)$ is defined as follows (intuitively, the set P represents the past of m, given by a linear chain of moments, and the set F its futures, given by multiple linear chains fanning out).

$H_m = \{h \mid m \in h \textit{ occurs in } \Gamma^*\}$

$A = \{m \mid m \textit{ occurs in } \Gamma^* \cup \Delta^*\} = \{m\}$

$P = \{m_i \mid i \geq 1\}$, disjoint from A, ordered by $\leq$ so that if $i \leq j$ then $m_j \leq m_i$.

$F^{h_k} = \{m_i^{h_k} \mid i \geq 1\}$ for every $h_k \in H_m$, where each F^{h_k} is disjoint from any other, as well as P and A, and ordered by $\leq$ so that if $i \leq j$ then $m_i \leq m_j$.

$F = \bigcup F^{h_k}$

$T = P \cup A \cup F$

$h_k = P \cup A \cup F^{h_k}$, ordered by $\leq$ so that $m_i \leq m \leq m_j^{h_k}$ for all $i, j \geq 1$.
$Ag = \{a_i \mid a_i \text{ } occurs \text{ } in \text{ } \Gamma^* \cup \Delta^*\}$
$Ch_m^{a_i} = \{[h]_{\sim_m^{a_i}} \mid h \in H_m\}$ where $[h]_{\sim_m^{a_i}} = \{h' \mid h \sim_m^{a_i} h' \text{ } occurs \text{ } in \text{ } \Gamma^*\}$

For every $m' < m$, for every $a_i \in Ag$, $Ch_{m'}^{a_i} = H_m$.
For every $m < m'$, for every $a_i \in Ag$, $Ch_{m'}^{a_i} = \{h' \mid m' \in h'\} = \{h'\}$.

We write $Ch_m^{a_i}(h)$ for the equivalence class $[h]_{\sim_m^{a_i}}$.
The interpretation I of term τ in $\Gamma^* \cup \Delta^*$ is the appropriate t in T, H_m or Ag. The interpretation of the label m/h, $I(m/h)$, is a pair (m, h) such that $m \in T$ and $h \in H_m$. For legibility below I is left unwritten.
The valuation $\mathcal{V}$ of relational atoms in $\mathcal{M}$ is

$\mathcal{M} \Vdash m \in h$ iff $h \in H_m$,
$\mathcal{M} \Vdash h \sim_m^{a_i} h'$ iff $h' \in [h]_{\sim_m^{a_i}}$ and $[h]_{\sim_m^{a_i}} \in Ch_m^{a_i}$

The valuation for atomic formula p is $\mathcal{M}, (m, h) \Vdash p$ iff $m/h : p \in \Gamma^*$. Remaining valuations are standard. Importantly,

$$\begin{array}{lll} \mathcal{M}, (m,h) \Vdash \Box^{a_i} A \text{ iff} & \forall h' \in Ch_m^{a_i}(h) : \mathcal{M}, (m, h') \Vdash A, & \\ \mathcal{M}, (m,h) \Vdash \mathcal{D}^{a_i} A \text{ iff} & \text{(a) } \forall h' \in Ch_m^{a_i}(h) : \mathcal{M}, (m, h') \Vdash A \text{ and} & \\ & \text{(b) } \exists h'' \in H_m : \mathcal{M}, (m, h'') \nVdash A & \blacksquare \end{array}$$

It is easy to show that

Lemma 6.16. *The frame* $(T, \leq, Ag, Ch)$ *is a DSTIT frame. Specifically, it satisfies no backward branching, historical connectedness and no choice between undivided histories. Moreover, in the presence of multiple agents it likewise satisfies independence of agents.*

Proof. The only branching occurs at m, and none occurs prior, so the *no backward branching* condition is satisfied. Moreover, for any m_1 and m_2 some $m' \in P$ satisfies *historical connectedness*. Next, all histories divide at m and are choice-equivalent for any agent prior to m, so *no choice between undivided histories* holds.

Finally, in the presence of multiple agents, if the moment is m, independence of agents holds by saturation. For any moment in F it holds trivially (equivalence classes are singletons) and for any moment in P equivalence reduces to than in m, so independence of agents is inherited from m. Therefore *independence of agents* is satisfied. $\blacksquare$

Moreover it is now easy to show that

LEMMA 6.17.

1. $\mathcal{M}, (m, h) \Vdash A$ *if* $m/h : A$ *is in* Γ^*.
2. $\mathcal{M}, (m, h) \nVdash A$ *if* $m/h : A$ *is in* Δ^*.

PROOF. By simultaneous induction on the weight of A.
The basic case holds by definition of $\mathcal{V}$. We will illustrate the inductive case on the example of $\Box^i$:

Assume $m/h : \Box^i A$ is in Γ^*. Then, for every m/h' such that $h \sim_m^i h'$, by the saturation criterion and the extension of $\Gamma^* \cup \Delta^*$, $m/h' : A$ is likewise in Γ^*, and by the inductive hypothesis, $\mathcal{M}, (m, h') \Vdash A$. Therefore, $\mathcal{M}, (m, h) \Vdash \Box^i A$.

Assume $m/h : \Box^i A$ is in Δ^*. Then, by the saturation criterion, there is some h' such that $h \sim_m^i h'$ is in Γ^* and $m/h' : A$ is in Δ^*. Therefore, by the inductive hypothesis, $\mathcal{M}, (m, h') \nVdash A$ and therefore $\mathcal{M}, (m, h) \nVdash \Box^i A$. ■

If all branches are closed we have a derivation. Otherwise, by Theorem 6.8, there is a saturated branch and by Lemma 6.17 we have a countermodel. ■

7. Concluding Remarks

In this paper we have presented a number of results concerning the DSTIT modality. First, we have developed a G3-style labelled sequent calculus, using auxiliary modalities to deal with the complex truth conditions and moreover adding frame conditions. The resulting system is shown to have the desired structural properties, most notably admissibility of contraction and cut. Furthermore, we have also shown that it possesses the meta-theoretical properties of soundness, completeness and decidability, using direct proofs. We have likewise shown a number of interesting applications of our system in dealing with notions of delegation and refraining. Looking at a broader picture, we have also established a basis for further uses, especially in dealing with other STIT modalities.

In exploring the deliberative STIT, we have used *cstit* and *settled true* as auxiliary modalities. An exploration of a system without those, along the lines of [21], is left for future research, as are numerous other approaches to stit modalities (e.g. [1]). An important one we have laid the groundwork for here is the other major STIT modality presented by [2], the *achievement* STIT. Here the basic idea is that the present fact that, say, A, has been achieved by the previous choice of an agent. Unlike DSTIT, which deals with multiple histories on a single moment, the achievement STIT makes uses of multiple successive moments. Since we are dealing with multiple moments along histories, their relations come into effect, and therefore the rules for the relation $\leq$, which were presented but not used in this paper, can be made use of. Of course, in addition to the application of previously unused rules, we are still dealing with the branching time models, and we have shown in this paper how well suited the labelled approach is for them.

Acknowledgements. Open access funding provided by University of Helsinki including Helsinki University Central Hospital. The first author presented part of the results contained in this article in the Workshop 'Doxastic Agency and Epistemic Logic' (Bochum, 2017), the organizers of which are gratefully acknowledged. This work was partially supported by the Academy of Finland, research Project No. 1308664.

References

[1] Balbiani, P., A. Herzig, and N. Troquard, Alternative axiomatics and complexity of deliberative STIT theories, *Journal of Philosophical Logic* 37:387–406, 2008.

[2] Belnap, N. D., M. Perloff, and M. Xu, *Facing the Future: Agents and Choices in Our Indeterminist World*, Oxford: Oxford University Press, 2001.

[3] Chellas, B. F., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

[4] Dyckhoff, R., and S. Negri, Geometrization of first-order logic, *The Bulletin of Symbolic Logic* 21:123–163, 2015.

[5] Garg, D., V. Genovese, and S. Negri, Countermodels from sequent calculi in multimodal logics, In D. Galmiche and D. Larchey Wendling (eds.), *Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX2013)*, Lecture Notes in Computer Science, 8123:315–324, 2013.

[6] Girlando, M., S. Negri, N. Olivetti, and V. Risch, Conditional beliefs: from neighbourhood semantics to sequent calculus, *The Review of Symbolic Logic* 11:736-779, 2018.

[7] Negri, S., Proof analysis in modal logic, *Journal of Philosophical Logic* 34:507–544, 2005.

[8] Negri, S., Proofs and countermodels in non-classical logics, *Logica Universalis* 8:25–60, 2014.

[9] Negri, S., Proof theory for non-normal modal logics: The neighbourhood formalism and basic results, *IfCoLog Journal of Logics and their Applications* 4:1241–1286, 2017.

[10] Negri, S., and N. Olivetti, A sequent calculus for preferential conditional logic based on neighbourhood semantics, In H. de Nivelle (ed.), *Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX2015)*, Lecture Notes in Computer Science, 9323:115–134, 2015.

[11] Negri, S., and J. von Plato, Cut elimination in the presence of axioms, *The Bulletin of Symbolic Logic* 4:418-435, 1998.

[12] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge University Press, Cambridge, 2001.

[13] Negri, S., and J. von Plato, *Proof Analysis*, Cambridge University Press, Cambridge, 2011.

[14] Negri, S., and G. Sbardolini, Proof analysis for Lewis counterfactuals, *The Review of Symbolic Logic* 9:44–75, 2016.

[15] Olkhovikov, G. K., and H. Wansing, An axiomatic system and a tableau calculus for STIT imagination logic, *Journal of Philosophical Logic* 47(2):259–279, 2018.

[16] Olkhovikov, G. K., and H. Wansing, Simplified Tableaux for STIT Imagination Logic, *Journal of Philosophical Logic* 2019, https://doi.org/10.1007/s10992-019-09503-1

[17] Orlandelli, E., Proof analysis in deontic logics, In F. Cariani et al. (eds.), *Deontic Logic and Normative Systems (DEON2014)*, Lecture Notes in Artificial Intelligence, 8554:139–148, 2014.

[18] von Wright, G. H., *Norm and Action: A Logical Enquiry*, Routledge & Kegan Paul, Abingdon, 1963.

[19] Wansing, H., Tableaux for multi-agent deliberative-stit logic, In G. Governatori, I.M. Hodkinson, Y. Venema (eds.), *Advances in Modal Logic 6*, 2006, pp. 503–520.

[20] Wansing, H., Remarks on the logic of imagination. A step towards understanding doxastic control through imagination, *Synthese* 194:2843–2861, 2017.

[21] Xu, M., Axioms for deliberative STIT, *Journal of Philosophical Logic* 27:505–552,1998.

S. Negri
Department of Mathematics
University of Genova
Via Dodecaneso 35
16146 Genoa
Italy

S. Negri, E. Pavlović
Department of Philosophy
University of Helsinki
Unioninkatu 40
Helsinki
Finland
sara.negri@unige.it;
sara.negri@helsinki.fi

E. Pavlović
edi.pavlovic@helsinki.fi