Fast superconvergent solvers for weakly singular Hammerstein equations

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Abstract. This article investigates discrete versions of the projection-type and modified projection-type methods for solving Hammerstein integral equations with a weakly singular kernel. The approximating operator utilized is either the orthogonal projection or an interpolatory projection onto a space of piecewise polynomials of degree $\leq r-1$ with respect to a graded partition of $[0,1]$. The study reveals that the proposed methods achieve optimal rates of convergence, demonstrating that the numerical quadrature used to estimate the integrals maintains the same rates of convergence as the continuous methods. The theoretical findings are supported by numerical experiments.

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1. Introduction

We consider the following *Hammerstein* integral equation

$$x - Tx = f, \tag{1.1}$$

where T is a compact operator defined on $\mathbb{X} = C[0,1]$ by

$$(Tx)(s) = \int_0^1 \kappa(s,t)\psi(t,x(t))dt, \quad s \in [0,1]. \tag{1.2}$$

The kernel $\kappa(s,t)$ is of weakly singular type

$$\kappa(s,t) = m(s,t)g_\alpha|s-t|, \tag{1.3}$$

with

$$g_\alpha|s-t| = \begin{cases} |s-t|^{\alpha-1}, & \text{for} \quad 0<\alpha<1, \\ \log(|s-t|), & \text{for} \quad \alpha = 1, \end{cases} \tag{1.4}$$

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and $m(s,t) \in C([0,1] \times [0,1])$, f and ψ are known functions with $\psi(t,u)$ nonlinear in u and x is the function to be determined. The solution of *Hammerstein* equations (1.1) is what we call the fixed point of the operator T. In this framework, authors in [5, 27] examined the existence and uniqueness of the solution of equation (1.1). The most important method for analyzing the solvability theory for such equations is the *Banach fixed-point theorem* (see e.g.,[6, 7, 8, 21, 23] and the references contained in them).

Previous research has explored various numerical methods using piecewise polynomials for solving equation (1.1). One of these methods is product integration, which is particularly useful for solving (1.1) with a weakly singular kernels. In a study conducted by Kaneko et al. [16] both product integration and collocation methods were employed to solve *Hammerstein* equations with weakly singular kernels, and they discovered certain properties of the approximate solutions that demonstrate superconvergence. In another study by Hakk and Pedas [14], they established that the collocation solution exhibits superconvergence at the interpolation points if the appropriate interpolation points are selected. The notion of a graded mesh was first presented by Rice [22]. Schneider [25] then applied this concept by using nonlinear spline approximation to create a graded mesh, and evaluated the convergence rates of approximate solutions in different discrete methods. The discrete modified collocation method was examined in [13] as an alternative approach to solve equation (1.1), which contains kernels that are weakly singular.

The method known as *Galerkin* is examined in relation to piecewise polynomials using orthogonal projection. The study in [12, 17] established that the iterated *Galerkin* method attains superconvergence for *Hammerstein* equations with both smooth and weakly singular kernels. Additionally, Golberg and Chen [9, 10] compiled a comprehensive analysis of the discrete *Galerkin* methods for integral equations. Recently, the authors in [3] proposed a modified *Galerkin*-type method for weakly singular kernels that is based on the superconvergent version of the Kumar and Sloan method [20], which is commonly known as the collocation-type method in literature due to its initial definition using an interpolatory operator. In this paper, the method will be referred to as a discrete *Galerkin*-type method when a *Hyperinterpolation* projection is used, or a discrete projection-type method when the projection type is not specified.

The aim of this paper is to investigate the discrete projection-type and discrete modified projection-type methods for approximate solution of (1.1) with a weakly singular kernel using piecewise polynomial basis functions . It is worth mentioning that the continuous method is proposed in [3] for *Hammerstein* integral equations. Our research shows that when dealing with weakly singular kernels, the iterated version of the discrete modified projection-type method has a better convergence rate compared to the projection-type and modified projection-type methods. Additionally, the proposed method achieves convergence rates that are consistent with those of the continuous methods.

Regarding computational analysis, the research literature lacks sufficient computational results for finding an approximate solution of (1). To fill this gap, we present some computational results for integral equations with weakly singular properties. Since we used a graded mesh, we had to compute the weights required for the product integration method carefully. To do this, we followed the approach outlined in a previous work by Atkinson [4].

The contents of the paper are as follows. Section 2 presents an overview of relevant results and establishes the necessary background. Numerical results of the discrete projection-type and modified projection-type methods for the integral operator with both an algebraic singularity and a logarithmic singularity are defined in Section 3. In Section 4, we give the convergence orders of the proposed method and its iterated version. Numerical validation is given in Section 5.

2. Background and results

For a fixed $s \in [0,1]$, define the kernel $\kappa_s(t) \equiv \kappa(s,t)$ for $t \in [0,1]$ to be the s section of κ. We assume that

$$M \equiv \sup_{s\in[0,1]} \int_0^1 |\kappa(s,t)|dt < \infty \quad \text{and} \quad \lim_{s\to\sigma} \|\kappa_s - \kappa_\sigma\|_\infty = 0, \quad \sigma \in [0,1].$$

If $x \in C[0,1]$, then from Lemma 2.3 of Kaneko et al. [15], $Tx \in C^{(0,\alpha)}[0,1]$. Here $C^{(0,\alpha)}[0,1]$ denotes the class of α-Hölder continuous functions defined on $[0,1]$.

$$C^{(0,\alpha)}[0,1] = \left\{ g \in C[0,1], \sup_{0\le x,\zeta\le 1} \frac{|g(x)-g(\zeta)|}{|x-\zeta|^\alpha} < \infty \right\}$$

for $0 < \alpha < 1$, and

$$C^{(0,1)}[0,1] = \left\{ g \in C[0,1], \sup_{0\le x,\zeta\le 1} \frac{|g(x)-g(\zeta)|}{|x-\zeta| log|B/(x-\zeta)|} < \infty \right\},$$

for some $B > 1$. We assume that $\psi(.,x(.)) \in C[0,1]$ and $\partial\psi/\partial u(.,x(.)) \in C[0,1]$. Then, the operator T is Fréchet differentiable and its Fréchet derivative at x is the linear operator

$$(T'(x)g)(s) = \int_0^1 \kappa(s,t)\frac{\partial\psi}{\partial u}(t,x(t))g(t)dt \quad g \in C[0,1].$$

Let x_0 be an isolated solution of (1.1) such that $[\min_{s\in[0,1]} x_0(s), \max_{s\in[0,1]} x_0(s)] \subset \mathbb{R}$. For $\delta_0 > 0$, let

$$\mathcal{B}(x_0,\delta_0) = \{x \in \mathbb{X} : \|x_0 - x\|_\infty < \delta_0\}.$$

The partial derivative $\partial\psi/\partial u$ of ψ with respect to the second variable exists and is *Lipschitz* continuous in a neighborhood of x_0, that is, there exists a constant $\delta_1 > 0$ such that

$$\left|\frac{\partial\psi}{\partial u}(t, x_0) - \frac{\partial\psi}{\partial u}(t, x)\right| \leq \delta_1 |x_0 - x|, \quad x \in \mathcal{B}(x_0, \delta_0).$$

If $f \in C[0,1]$, then from Theorem 2.4 in [15], the *Hammerstein* equation (1.1) has a unique solution $x_0 \in C[0,1]$.
Let S be a finite set in $[0,1]$ and define the function $\omega_S(t) = \inf\{|s-t| : s \in S\}$. In the language of Rice [22], a function x is said to be of Type(α, r, S), for $-1 < \alpha < 0$ if

$$|x^{(r)}(t)| \leq c|\omega_S(t)|^{\alpha - r}, \quad t \notin S.$$

and for $\alpha > 0$, if the above condition holds and $x \in Lip(\alpha)$, where

$$Lip(\alpha) = \{x : |x(s) - x(t)| \leq c|s-t|^{\alpha}, \quad s, t \in [0,1]\}.$$

According to [15], if f is of Type$(\beta, k, \{0,1\})$, then a solution of (1.1) with the kernel defined by (1.3) is of Type$(\gamma, k, \{0,1\})$, where $\gamma = \min\{\alpha, \beta\}$.
For $x, y \in L^p[0,1]$ and $p \in [1, \infty]$. The inner product is defined by

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \text{ and norm is } \quad \|x\|_{L^p} = \left(\int_0^1 x(t)^p dt\right)^{\frac{1}{p}}.$$

We denote by $W_p^m[0,1]$ the Sobolev space of functions g whose m^{th} generalized derivative $g^{(m)}$ belongs to $L^p[0,1]$. The norm in the space $W_p^m[0,1]$ is defined as

$$\|g\|_{W_p^m} = \sum_{k=0}^{m} \|g^{(k)}\|_p.$$

Let $q = \frac{r}{\gamma}$ be the index of singularity for some integer $r \geq 1$. Based on Rice's concept of graded mesh [22], we examine the partition of $[0,1]$ defined as

$$t_i = \begin{cases} \dfrac{1}{2}\left(\dfrac{2i}{n}\right)^q, & 0 \leq i \leq \frac{n}{2}, \\ 1 - t_{n-i}, & \frac{n}{2} \leq i \leq n. \end{cases} \tag{2.1}$$

Define $0 \leq \zeta_1 < \zeta_2 < \ldots < \zeta_r \leq 1$. The nr collocation points are chosen as

$$t_{ij} = t_i + \zeta_j(t_i - t_{i-1}), \quad 1 \leq i \leq n, \qquad 1 \leq j \leq r. \tag{2.2}$$

In order that

$$t_{i-1} \leq t_{i1} < t_{i2} < \ldots < t_{ir} \leq t_i, \quad 1 \leq i \leq n.$$

Put $I_i = [t_{i-1}, t_i]$, we denote by $\mathbb{X}_n = \mathcal{S}_r^{\nu}(\Pi_n)$ the space of piecewise polynomials of order r and ν continuous derivatives, $(-1 \leq \nu \leq r-2)$ with knots at Π_n, that is

$$\mathbb{X}_n = \{\vartheta \in C^{\nu}[0,1] : \vartheta|_{I_i} \in \mathcal{P}_r, 1 \leq i \leq n\}$$

where $\mathcal{P}_{r-1}$ denotes the space of polynomials of degree at most $r-1$. Here $\nu = 0$ corresponds to the case of continuous piecewise polynomials. If $\nu =$

-1 there is no continuity requirements at the breakpoints. Note that the $\dim(\mathbb{X}_n) = nr - (n-1)(\nu - 1)$.

Lemma 2.1. *(Schneider [25]) Let $\kappa(s,t)$ be a kernel of the form (1.3), then for each $s \in [0,1]$, there exists $u \in \mathcal{S}_r^{\nu}(\Pi_n)$ such that*

$$\|\kappa_s - u\|_{L^1} = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1, \\ O(n^{-1}\log n), & \alpha = 1. \end{cases} \tag{2.3}$$

One of the goals of this paper is to develop Gauss-type numerical quadratures for the singular integrals. To introduce the discrete methods, we consider a quadrature formula defined by

$$\int_0^1 \rho(x) f(x) dx \tag{2.4}$$

where $f \in Type(\alpha, 2r, \{0\})$ with $\alpha > 0$ and ρ a weight function that is positive on $(0,1)$. Let $q = 2r + 1/\alpha$ and a partition

$$(\Pi_n): \quad t_0 = n^{-q}, \quad t_i = i^q t_1, \quad i = 1, 2, \ldots, n. \tag{2.5}$$

A particular case of the quadrature scheme which will be called Gauss-Legendre type quadrature developed in [18] is concerned when the weight function $\rho(x) = 1$ on each subinterval $[t_{i-1}, t_i]$

$$\int_{t_{i-1}}^{t_i} f(x) dx \simeq \sum_{j=1}^{r} \omega_{ij} f(t_{ij}), \tag{2.6}$$

where $0 \leq t_{ij} \leq 1$ are quadrature points and $\omega_{ij} > 0$ the corresponding weights. This formula has degree of precision $2r$ on each subinterval. We assume for $x \in C[0,1]$ it holds

$$\left| \int_{t_{i-1}}^{t_i} f(x) dx - \sum_{j=1}^{r} \omega_{ij} f(t_{ij}) \right| = O(n^{-2r-1}). \tag{2.7}$$

Currently, we use

$$\int_{t_0}^1 f(x) dx = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(x) dx, \tag{2.8}$$

to approximate (2.4). In order to evaluate (2.8), we can consider the following composite integration rule over all of $[0,1]$. As indicated in Theorem 3.1 of [18], for any $f \in Type(\alpha, 2r, \{0\})$, the error estimate has the following form

$$\left| \int_0^1 f(x) dx - \sum_{i=1}^{n} \sum_{j=1}^{r} \omega_{ij} f(t_{ij}) \right| = O(n^{-2r}). \tag{2.9}$$

Using the above numerical integration method, we define the discrete inner product as

$$\langle f, g \rangle_n = \sum_{i=1}^{n} \sum_{j=1}^{r} \omega_{ij} f(t_{ij}) g(t_{ij}), \qquad g \in C[0,1]. \tag{2.10}$$

Then, we define a numerical approximation to the integral operator in (4.1) by

$$(T_n x)(s) = \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{ij}\kappa_s(t_{ij})\psi(t_{ij}, x(t_{ij})), \quad s \in [0,1]. \tag{2.11}$$

Assume that $x_0 \in C[0,1]$ and that $\psi \in C[0,1]$. By virtue of estimate (2.9), we find

$$\|T(x_0) - T_n(x_0)\|_\infty = O(n^{-2r}). \tag{2.12}$$

The Fréchet derivative of T_n is given by

$$(T_n^{'}(x_0)g)(s) = \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{ij}\kappa_s(t_{ij})\psi_1(t_{ij})g(t_{ij}), \quad g \in C[0,1],$$

where $\psi_1(t) \equiv \frac{\partial\psi}{\partial u}(t, x_0(t))$. The uniformly boundedness of $T_n'(x_0)$ follows from

$$\begin{aligned}\|T_n'(x_0)g\|_\infty &\le \sup_{s\in[0,1]} \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{ij}\,|\kappa_s(t_{ij})|\,|\psi_1(t_{ij})|\,|g(t_{ij})| \\ &\le \overline{M}\Psi_1\|g\|_\infty,\end{aligned} \tag{2.13}$$

where $\overline{M} \equiv \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{ij}\,|\kappa_s(t_{ij})|$, $\Psi_1 \equiv \sup_{t\in[0,1]} |\psi_1(t)|$, this implies $\|T_n'(x_0)\|_\infty \le \overline{M}\Psi_1$.

If $\psi_1 \in C[0,1]$, then from (2.9)

$$\|T'(x_0) - T_n^{'}(x_0)\|_\infty = O(n^{-2r}). \tag{2.14}$$

The operator T_n' is *Lipschitz* continuous in a neighborhood $\mathcal{B}(x_0, \delta_0)$ of x_0, that is, there exists a constant $\delta_2 > 0$ independent of n such that

$$\|T_n^{'}(x) - T_n^{'}(x_0)\|_\infty \le \delta_2\|x - x_0\|_\infty, \quad x \in \mathcal{B}(x_0, \delta_0). \tag{2.15}$$

For the rest of paper, we define two types of projections from $\mathbb{X}$ to $\mathbb{X}_n$.

•**Discrete orthogonal projection operator:** Namely *Hyperinterpolation* operator $\pi_n^G x :\ L^2[0,1] \to \mathbb{X}_n$ is defined by

$$(\pi_n^G x)(s) = \sum_{i=1}^{n_r} \langle x, \varphi_i\rangle_n \varphi_i(s), \tag{2.16}$$

where $\{\varphi_1, \varphi_2, \ldots, \varphi_{n_r}\}$ is an orthonormal basis for $\mathbb{X}_n$.

• **Interpolatory projection operator:** For $x \in C[0,1]$, let $\pi_n^C x : C[0,1] \to \mathbb{X}_n$ be the interpolatory operator defined by

$$\begin{aligned}(\pi_n^C x)(s) &= \sum_{i=1}^{n_r} x(\tau_i)\ell_i(s), \qquad s \in [0,1], \\ (\pi_n^C x)(\tau_i) &= x(\tau_i), \qquad 1 \le i \le n_r,\end{aligned} \tag{2.17}$$

where the collocation points are

$$\{\tau_i : \quad i = 1, 2, ..., n_r\} = \{t_{ij} = t_i + \zeta_j(t_i - t_{i-1}), \quad 1 \leq i \leq n, \quad 1 \leq j \leq r\},$$

and $\{\ell_i : \quad i = 1, 2, ..., n_r\}$ is the Lagrange basis of $\mathbb{X}_n$.
For notational convenience from now on we write $\pi_n \equiv \pi_n^G$ or π_n^C. In both cases π_n converge to the identity operator pointwise on $C[0,1]$. From Atkinson et al. in [11], π_n can be extended to $L^\infty[0,1]$ and

$$p = \sup_n \|\pi_n\| < C. \tag{2.18}$$

In this paper, C will denotes a generic constant independent of n.

3. The numerical methods

The projection-type method involves the approximation of the function $z(t) = \psi(t, x(t))$ using a polynomial $z_n = \pi_n z$ of degree $\leq n$. The approximating equation of (1.1) is

$$x_n^S - T_n^S x_n^S = f, \quad x_n^S \in \mathbb{X} \tag{3.1}$$

The operator T_n^S, which is defined in [20], is a nonlinear operator given by

$$(T_n^S x)(s) = \int_0^1 \kappa(s,t)\pi_n z(t)dt, \qquad s \in [0,1]. \tag{3.2}$$

The linear operator that corresponds to the Fréchet derivative of T_n^S is expressed as

$$((T_n^S)'(x)g)(s) = \int_0^1 \kappa(s,t)\pi_n \frac{\partial z}{\partial u}(t)g(t)dt.$$

In [3], a modified projection-type method is introduced to achieve a more precise approximation solution than x_n^S

$$x_n^M - T_n^M x_n^M = f, \quad x_n^M \in \mathbb{X} \tag{3.3}$$

where

$$T_n^M = \pi_n T + T_n^S - \pi_n T_n^S. \tag{3.4}$$

is a superconvergent operator with a finite rank. We define the solution obtained through the iterated modified projection-type method as

$$\widetilde{x}_n^M = T x_n^M + f. \tag{3.5}$$

The discrete projection-type method for equation (1.1) is seeking an approximate solution $\overline{x}_n^S$ to x_0 such that

$$\overline{x}_n^S - \overline{T_n^S}\,\overline{x}_n^S = f, \quad \overline{x}_n^S \in \mathbb{X}. \tag{3.6}$$

where $\overline{T_n^S}$ is the discrete nonlinear operator given by

$$(\overline{T_n^S} x)(s) = \langle \kappa_s, \pi_n z \rangle_n, \quad s \in [0,1]. \tag{3.7}$$

To obtain an approximation solution that is more accurate than $\overline{x}_n^S$, we can use the discrete modified projection-type method, which is expressed as follows

$$\overline{x}_n^M - \overline{T_n^M}\,\overline{x}_n^M = f, \quad \overline{x}_n^M \in \mathbb{X} \tag{3.8}$$

and

$$\overline{T_n^M} = \pi_n T_n + \overline{T_n^S} - \pi_n \overline{T_n^S}, \tag{3.9}$$

while the discrete iterated modified projection-type solution is defined by

$$\widehat{x}_n^M = T_n \overline{x}_n^M + f. \tag{3.10}$$

Implementation details. Let π_n^G be the *Hyperinterpolation* operator defined by (2.16). Then, the associated operator $\overline{T_n^S}$ defined by (3.7) is given by

$$\overline{T_n^S}x(s) = \sum_{l=1}^{n_r} \overline{k_l}(s)\langle z, \varphi_l\rangle_n, \qquad s \in [0,1], \tag{3.11}$$

we denote by $\overline{k}_l(s) = \langle \kappa_s, \varphi_l\rangle_n$ the numerically computed value of (2.10) using the Gauss legendre-type quadrature. From (3.6), we observe that $\overline{x}_n^S$ has the following form

$$\overline{x}_n^S(s) = f(s) + \sum_{l=1}^{n_r} a_l \overline{k_l}(s).$$

The coefficients $\{a_i, i = 1, \ldots, n_r\}$, are obtained by substituting $\overline{x}_n^S$ in (3.6). Hence we obtain the following system of nonlinear equations of size n.

$$a_i - \sum_{j=1}^{n}\sum_{k=1}^{r} \omega_{jk}\psi\Big(t_{jk}, f(t_{jk}) + \sum_{l=1}^{n_r} a_l \overline{k_l}(t_{jk})\Big)\varphi_i(t_{jk}) = 0, \quad 1 \le i \le n_r. \tag{3.12}$$

From equation (3.8) we can show that the approximate solution $\overline{x}_n^M$ has the following form

$$\overline{x}_n^M(s) = f(s) + \sum_{i=1}^{n_r} a_i\varphi_i(s) + \sum_{j=1}^{n_r} b_j \overline{k_j}(s), \tag{3.13}$$

where the coefficients $\{a_i, b_i, \quad i = 1, ..., n_r\}$ are obtained by substituting $\overline{x}_n^M$ from equation (3.13) into equation (3.8) then, we successively have

$$\pi_n^G \mathcal{K}_n \overline{x}_n^M = \sum_{i=1}^{n_r} \langle \mathcal{K}_n \overline{x}_n^M, \varphi_i\rangle_n \varphi_i = \sum_{i=1}^{n_r}\Big\{\langle \sum_{j=1}^{n}\sum_{k=1}^{r} \omega_{jk} z(t_{jk}), \varphi_i\rangle_n\Big\}\varphi_i,$$

$$\overline{T_n^S}\,\overline{x}_n^M = \langle \kappa_s, \varphi_i\rangle_n \langle z, \varphi_i\rangle_n = \sum_{i=1}^{n_r} \overline{k_i}(s) \sum_{j=1}^{n}\sum_{k=1}^{r} \omega_{jk} z(t_{jk})\varphi_i(t_{jk}),$$

$$\pi_n^G \overline{T_n^S}\,\overline{x}_n^M = \sum_{i=1}^{n_r} \langle \overline{T_n^S}\,\overline{x}_n^M, \varphi_i\rangle_n \varphi_i = \sum_{i=1}^{n_r}\Big\{\sum_{j=1}^{n_r} \langle z, \varphi_j\rangle_n \langle \overline{k_j}, \varphi_i\rangle_n\Big\}\varphi_i,$$

where

$$z(t) = \psi\Big(t, f(t) + \sum_{i=0}^{n_r} a_i \varphi_i(t) + \sum_{l=0}^{n_r} b_l \overline{k_l}(t)\Big).$$

Except for some very specific situations, the family of functions $\{\varphi_i, k_j\}$ are linearly independent, therefore we can identify the coefficients of φ_i and k_j respectively. Then for $i, j = 1, \dots, n$ we obtain the nonlinear system of size $2n$,

$$\begin{cases} a_i &= \langle \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{r} \omega_{jk} z(t_{jk}), \varphi_i \rangle_n - \sum\limits_{j=1}^{n_r} \langle \overline{k_j}, \varphi_i \rangle_n, \\ b_j &= \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{r} \omega_{jk} z(t_{jk}) \varphi_j(t_{jk}). \end{cases}$$

Let π_n^C be the interpolatory operator defined by (2.17). Then, $\overline{T_n^S}$ can be written as

$$\overline{T_n^S} x(s) = \sum_{j=1}^{n_r} \overline{\omega}_j(s) z(\tau_j), \qquad s \in [0,1], \tag{3.14}$$

where $\overline{\omega}_j(s) = \langle \kappa_s, \ell_j \rangle_n$. It is easy to see from equation (3.6) that the approximate solution $\overline{x}_n^S$ is given by

$$\overline{x}_n^S(s) = f(s) + \sum_{j=1}^{n_r} a_j \overline{\omega}_j(s).$$

Equivalently, we obtain the system of nonlinear equations defined by

$$a_i - \psi\left(\tau_i, f(\tau_i) + \sum_{j=1}^{n_r} a_j \overline{\omega}_j(\tau_i)\right) = 0, \quad 1 \le i \le n_r. \tag{3.15}$$

For the interpolatory projection given by (2.17), we apply π_n^C and $(I - \pi_n^C)$ to equation (3.8), to obtain

$$\pi_n^C \overline{x}_n^M - \pi_n^C T_n \overline{x}_n^M = \pi_n^C f, \tag{3.16}$$

$$(I - \pi_n^C)\overline{x}_n^M - (I - \pi_n^C)\overline{T_n^S}\overline{x}_n^M = (I - \pi_n^C) f. \tag{3.17}$$

By writing

$$T_n \overline{x}_n^M = T_n (I - \pi_n^C)\overline{x}_n^M + T_n \pi_n^C \overline{x}_n^M, \tag{3.18}$$

and replacing $(I - \pi_n^C)\overline{x}_n^M$ by its expression from equation (3.17), $T_n \overline{x}_n^M$ becomes

$$T_n \overline{x}_n^M = T_n\big((I - \pi_n^C)\overline{T_n^S}\overline{x}_n^M + \pi_n^C \overline{x}_n^M + (I - \pi_n^C) f\big). \tag{3.19}$$

Now, by replacing $T_n \overline{x}_n^M$ in equation (3.16), we obtain

$$\pi_n^C \overline{x}_n^M - \pi_n^C T_n\big((I - \pi_n^C)\overline{T_n^S}\overline{x}_n^M + \pi_n^C \overline{x}_n^M + (I - \pi_n^C) f\big) = \pi_n^C f, \tag{3.20}$$

and then for $i = 1, ..., n_r$, we have

$$\overline{x}_n^M(\tau_i) - T_n\big((I - \pi_n^C)\overline{T_n^S}\overline{x}_n^M + \pi_n^C \overline{x}_n^M + (I - \pi_n^C) f\big)(\tau_i) = f(\tau_i). \tag{3.21}$$

Now using the expressions of the operators π_n^C, T_n and $\overline{T_n^S}$, we obtain the following nonlinear system of size n_r

$$a_i - \sum_{j=1}^{n}\sum_{k=1}^{r}\omega_{jk}\kappa(\tau_i,t_{ij})\psi\Big(t_{jk},\sum_{i=1}^{n_r}(a_i-f_i)\ell_i(t_{jk})+\sum_{i=1}^{n_r}\overline{\omega}_i(t_{jk})\psi(\tau_i,a_i)$$
$$-\sum_{i=1}^{n_r}\sum_{l=1}^{n_r}\overline{\omega}_l(\tau_i)\psi(\tau_l,a_l)\ell_i(t_{jk})\Big)=f_i,$$

where $f_i := f(t_i)$ and $\{a_i = \overline{x}_n^M(\tau_i), i = 1, 2, \ldots, n\}$ are the unknowns. From (3.17), the approximate solution is given by

$$\begin{aligned}\overline{x}_n^M &= \pi_n^C\overline{x}_n^M + (I-\pi_n^C)\overline{T_n^S}\overline{x}_n^M + (I-\pi_n^C)f,\\ &= f + \sum_{i=1}^{n_r}(a_i-f_i)\ell_i + \sum_{i=1}^{n_r}\overline{\omega}_i(.)\psi(\tau_i,a_i) - \sum_{i=1}^{n_r}\sum_{l=1}^{n_r}\overline{\omega}_l(\tau_i)\psi(\tau_l,a_l)\ell_i.\end{aligned} \tag{3.22}$$

When using a *Hyperinterpolation* operator, equations (3.8) and (3.10) result in discrete modified *Galerkin*-type and iterated modified *Galerkin*-type methods. On the other hand, if an interpolatory projection operator is used instead of π_n, equations (3.8) and (3.10) produce discrete modified collocation-type and iterated modified collocation-type methods, respectively.

4. Convergence rates

The main results of this section can be established by making use of the following lemma.

Lemma 4.1. *(Ahues et al.* [1]*) Let $\mathbb{X}$ be a Banach space and A, A_n be bounded linear operators on $\mathbb{X}$. If $\|A_n - A\| \to 0$, as $n \to \infty$ and $(I-A)^{-1}$ exists, then for n large enough $(I - A_n)^{-1}$ exists and is uniformly bounded on $\mathbb{X}$.*

In the lemma that follows, we demonstrate the invertibility of the linear operators $(I - (\overline{T_n^S})'(x_0))^{-1}$.

Lemma 4.2. *Suppose that $x_0 \in C[0,1]$ is a unique solution of (1.1) and that 1 is not an eigenvalue of $T'(x_0)$. Let $\kappa(s,t)$ be a kernel of the form (1.3). Then for n large enough, the operators $(I - (\overline{T_n^S})'(x_0))^{-1}$ exists and are uniformly bounded, i.e., there exists a constant $C_1 > 0$ independent of n such that*

$$\|(I-(\overline{T_n^S})'(x_0))^{-1}\|_\infty \le C_1. \tag{4.1}$$

Proof. Note that

$$\|T'(x_0)-(\overline{T_n^S})'(x_0)\|_\infty \le \|T'(x_0)-(T_n^S)'(x_0)\|_\infty + \|(T_n^S)'(x_0)-(\overline{T_n^S})'(x_0)\|_\infty,$$

it follows that

$$\max\{\|T'(x_0)-(T_n^S)'(x_0)\|_\infty, \|(T_n^S)'(x_0)-(\overline{T_n^S})'(x_0)\|_\infty\} \to 0 \quad \text{as} \quad n\to\infty.$$

Note that there exits n_0 such that for all $n \in \mathbb{N}$, $n \geq n_0$, then from the integration rule (2.9), we obtain $\|(T_n^S)'(x_0) - (\overline{T_n^S})'(x_0)\|_\infty \to 0 \quad \text{as} \quad n \to \infty$. Let $g \in C[0,1]$, we can write

$$\begin{aligned}\left|[T'(x_0) - (T_n^S)'(x_0)]g(s)\right| &\leq \sup_{0\leq s\leq 1}\int_0^1 \left|\kappa(s,t)\frac{\partial}{\partial u}[(z_0 - \pi_n z_0)(t)]g(t)\right| dt,\\ &\leq \sup_{0\leq s\leq 1}\int_0^1 |\kappa_s(t)(\psi_1(t) - \pi_n\psi_1(t))|\,|g(t)|dt.\end{aligned}$$

Then

$$\begin{aligned}\|[T'(x_0) - (T_n^S)'(x_0)]g\|_\infty &\leq \|\kappa_s(\psi_1 - \pi_n\psi_1)\|_\infty\|g\|_\infty,\\ &\leq M\|\psi_1 - \pi_n\psi_1\|_\infty\|g\|_\infty.\end{aligned} \tag{4.2}$$

Since $\psi_1 \in C[0,1]$, we have $\|\psi_1 - \pi_n\psi_1\|_\infty \to 0$ as $n \to \infty$ which implies that $(T_n^S)'(x_0) \to T'(x_0)$ pointwise in $C[0,1]$ as $n \to \infty$. Then again,

$$\begin{aligned}\|(T_n^S)'(x_0)g\|_\infty &\leq \sup_{0\leq s\leq 1}\int_0^1 |\kappa(s,t)|\left|\pi_n\frac{\partial z}{\partial u}(t)g(t)\right| dt,\\ &\leq Mp\|\psi_1 g\|_\infty,\\ &\leq Mp\Psi_1\|g\|_\infty,\end{aligned}$$

and by using the *Hölder* inequality,

$$\begin{aligned}&\left|((T_n^S)'(x_0)g)(s) - (T_n^S)'(x_0)g)(\sigma)\right| =\\ &\sup_{0\leq s\leq 1}\int_0^1 \left|\kappa(s,t)\frac{\partial}{\partial u}\pi_n z(t)g(t) - \kappa(\sigma,t)\frac{\partial}{\partial u}\pi_n z(t)g(t)\right| dt,\\ &\leq \|\kappa_s - \kappa_\sigma\|_{L^1}\|\pi_n\psi_1 g\|_\infty,\\ &\leq p\Psi_1\|\kappa_s - \kappa_\sigma\|_{L^1}\|g\|_\infty.\end{aligned} \tag{4.3}$$

This shows that $\{(\overline{T_n^S})'(x_0)\}$ is collectively compact. As a result of the theory of collectively compact operators, the operators $(I - (\overline{T_n^S})'(x_0))^{-1}$ exists and are uniformly bounded, for some sufficiently large n. □

The following result can be proven in the same manner as in Theorem 2 in [26].

Theorem 4.3. *Suppose that $x_0 \in C[0,1]$ is a unique solution of (1.1) and $\overline{x}_n^S$ be the unique solution of (3.6) in the sphere $\mathcal{B}(x_0,\delta_0)$. Let $\kappa(s,t)$ be a kernel of the form (1.3) and f be of Type$(\beta, r, \{0,1\})$. Assume that for $r = 0,1$, $\psi \in C^{(0,1)}([0,1]\times(-\infty,\infty))$ and for $r \geq 2, \psi \in C^{r-1}([0,1]\times(-\infty,\infty))$. If 1 is not an eigenvalue of the compact linear operator $T'(x_0)$. For a sufficiently large n, we have*

$$\|x_0 - \overline{x}_n^S\|_\infty \leq C_1\left\|T(x_0) - \overline{T_n^S}(x_0)\right\|_\infty. \tag{4.4}$$

Under the hypothesis of Lemma 4.2, it becomes straightforward to prove the invertibility of the operators $(I - (\overline{T_n^M})'(x_0))^{-1}$.

Lemma 4.4. *Suppose that $x_0 \in C[0,1]$ is a unique solution of (1.1) and that* 1 *is not an eigenvalue of $T'(x_0)$. Let $\kappa(s,t)$ be a kernel of the form (1.3). Then for n large enough, the operators $(I-(\overline{T_n^M})'(x_0))^{-1}$ exists and are uniformly bounded, i.e., there exists a constant $C_2>0$ such that*

$$\|(I-(\overline{T_n^M})'(x_0))^{-1}\|_\infty \leq C_2 < \infty. \tag{4.5}$$

Proof. By writing

$$T'(x_0)-(\overline{T_n^M})'(x_0)=\pi_n(T'(x_0)-T_n'(x_0))+(I-\pi_n)(T'(x_0)-(\overline{T_n^S})'(x_0)).$$

Since the estimate (2.8) is convergent on $C[0,1]$, π_n is uniformly bounded and converges to the identity operator pointwise, additionally $\{T_n'(x_0)\}$ is collectively compact and is pointwise convergent on $C[0,1]$. Choose $n \geq n_0$

$$\max\{\|T'(x_0)-T_n'(x_0)\|_\infty, \|T'(x_0)-(\overline{T_n^S})'(x_0)\|_\infty\} \to 0 \quad \text{as} \quad n\to\infty.$$

For each $g \in C[0,1]$, it follows from estimate (2.18) and (4.2) that

$$\left|(I-\pi_n)(T'(x_0)-(\overline{T_n^S})'(x_0))g(s)\right| \leq (1+\|\pi_n\|_\infty)\|(T'(x_0)-(\overline{T_n^S})'(x_0))g\|_\infty.$$

Then by applying (4.2), we obtain $\|(T'(x_0)-(\overline{T_n^M})'(x_0)\|_\infty \longrightarrow 0 \quad \text{as} \quad n\to\infty$. Therefore (4.5) is a consequence of Lemma 4.1. □

From the above results, we remark that the operator $(\overline{T_n^M})'$ is *Lipschitz* continuous in a neighborhood $\mathcal{B}(x_0,\delta_0)$ of x_0, that is, there exists a constant $\delta_3>0$ independent of n such that

$$\|(\overline{T_n^M})'(x)-(\overline{T_n^M})'(x_0)\|_\infty \leq \delta_3\|x-x_0\|_\infty, \quad x\in\mathcal{B}(x_0,\delta_0). \tag{4.6}$$

The succeeding theorem presents the error of approximation for the discrete modified projection-type method and its iterated version.

Theorem 4.5. *Suppose that $x_0 \in C[0,1]$ is a unique solution of (3.8) and $\overline{x}_n^M$ and $\widehat{x}_n^M$ be the approximate solution defined by (3.8) and (3.10) respectively. Let $\kappa(s,t)$ be a kernel of the form (1.3) and f be of Type$(\beta,r,\{0,1\})$. Assume that for $r=0,1$, $\psi_1 \in C^{(0,1)}([0,1]\times(-\infty,\infty))$ and for $r\geq 2, \psi_1 \in C^{r-1}([0,1]\times(-\infty,\infty))$. If 1 is not an eigenvalue of the compact linear operator $T'(x_0)$. For a sufficiently large n, we have*

$$\|x_0-\overline{x}_n^M\|_\infty \leq Cn^{-2r}+C_2\|(I-\pi_n)(T(x_0)-\overline{T_n^S}(x_0))\|_\infty. \tag{4.7}$$

Additionally

$$\begin{aligned}\|x_0-\widehat{x}_n^M\|_\infty \leq C\|x_0-\overline{x}_n^M\|_\infty^2 &+ C_2\|T_n^{'}(x_0)(T(x_0)-\overline{T_n^M}(x_0))\|_\infty \\ &+\|T(x_0)-T_n(x_0)\|_\infty.\end{aligned} \tag{4.8}$$

Proof. The proof of (4.7) is a simply application of Theorem 2 in [26]. Then

$$\|x_0-\overline{x}_n^M\|_\infty \leq C_2\|T(x_0)-\overline{T_n^M}(x_0)\|_\infty. \tag{4.9}$$

Now, we show that the estimate $\|T(x_0) - \overline{T_n^M}(x_0)\|\infty$ can be expressed in the following manner

$$\begin{aligned}\|T(x_0) - \overline{T_n^M}(x_0)\|_\infty &= \|\pi_n(T(x_0) - T_n(x_0)) + (I - \pi_n)(T(x_0) - \overline{T_n^S}(x_0))\|_\infty \\ &\le Cn^{-2r} + \|(I - \pi_n)(T(x_0) - \overline{T_n^S}(x_0))\|_\infty.\end{aligned} \tag{4.10}$$

Hence, (4.7) follows by adding the above estimate in (4.9). Recognizing that the second term on right hand side of (4.10) can also be bound by

$$\begin{aligned}\|(I - \pi_n)(T(x_0) - \overline{T_n^S}(x_0))\|_\infty \le (1 + p)\{&\|T(x_0) - T_n^S(x_0)\|_\infty \\ &+ \|T_n^S(x_0) - \overline{T_n^S}(x_0)\|_\infty\}.\end{aligned} \tag{4.11}$$

Note that from (1.1) and (3.10) we have

$$\begin{aligned}x_0 - \widehat{x}_n^M &= T(x_0) - T_n(\overline{x}_n^M), \\ &= T_n(x_0) - T_n(\overline{x}_n^M) + T(x_0) - T_n(x_0).\end{aligned} \tag{4.12}$$

To prove (4.12) in infinity norm we use a mean value theorem and (2.15). Therefore, for some $0 < \theta < 1$, we get

$$\begin{aligned}\|T_n(x_0) - T_n(\overline{x}_n^M)\|_\infty &= \|T_n'(x_0 + \theta(x_0 - \overline{x}_n^M))(x_0 - \overline{x}_n^M)\|_\infty, \\ &= \|[T_n'(x_0 + \theta(x_0 - \overline{x}_n^M)) - T_n'(x_0) + T_n'(x_0)][x_0 - \overline{x}_n^M]\|_\infty, \\ &\le \overline{M}\theta\delta_2\|x_0 - \overline{x}_n^M\|_\infty^2 + \|T_n^{'}(x_0)(x_0 - \overline{x}_n^M)\|_\infty.\end{aligned}$$

By applying the *Lipschitz's* continuity of $T_n^{'}$ and taking the norm on both sides of the above equation. Then (4.12) can be written as

$$\begin{aligned}\|x_0 - \widehat{x}_n^M\|_\infty &\le \|T_n(x_0) - T_n(\overline{x}_n^M)\|_\infty + \|T(x_0) - T_n(x_0)\|_\infty, \\ &\le \overline{M}\theta\delta_2\|x_0 - \overline{x}_n^M\|_\infty^2 + \|T_n^{'}(x_0)(x_0 - \overline{x}_n^M)\|_\infty + \|T(x_0) - T_n(x_0)\|_\infty.\end{aligned} \tag{4.13}$$

In order to evaluate the second term of the estimate (4.13), we write

$$\begin{aligned}(I - (\overline{T_n^M})'(x_0))(x_0 - \overline{x}_n^M) = T(x_0) - \overline{T_n^M}(x_0) - (\overline{T_n^M})'(x_0)(x_0 - \overline{x}_n^M) \\ + \overline{T_n^M}(x_0) - \overline{T_n^M}(\overline{x}_n^M).\end{aligned}$$

Applying $T_n^{'}(x_0)$ to both sides and using the mean value theorem, it follows that

$$\begin{aligned}&T_n^{'}(x_0)(x_0 - \overline{x}_n^M) = T_n^{'}(x_0)(I - (\overline{T_n^M})'(x_0))^{-1}[T(x_0) - \overline{T_n^M}(x_0) \\ &- (\overline{T_n^M})'(x_0)(x_0 - \overline{x}_n^M) + \overline{T_n^M}(x_0) - \overline{T_n^M}(\overline{x}_n^M)] \\ &= T_n^{'}(x_0)(I - (\overline{T_n^M})'(x_0))^{-1}[T(x_0) - \overline{T_n^M}(x_0)] + T_n^{'}(x_0)(I - (\overline{T_n^M})^{'}(x_0))^{-1} \\ &[(\overline{T_n^M})^{'}(x_0 + \theta(x_0 - \overline{x}_n^M)) - (\overline{T_n^M})^{'}(x_0)](x_0 - \overline{x}_n^M),\end{aligned}$$

where $0 < \theta < 1$. Now from estimates (4.5) and (4.6) one has

$$\|T_n^{'}(x_0)(x_0 - \overline{x}_n^M)\|_\infty \le C_2\|T_n^{'}(x_0)(T(x_0) - \overline{T_n^M}(x_0))\|_\infty + C_2\overline{M}\theta\delta_3\|x_0 - \overline{x}_n^M\|_\infty^2.$$

By Combining (4.13) with the above estimate, we get

$$\begin{aligned}\|x_0-\widehat{x}_n^M\|_\infty \leq & C\|x_0-\overline{x}_n^M\|_\infty^2 + C_2\|T_n^{'}(x_0)(T(x_0)-\overline{T_n^M}(x_0))\|_\infty \\ & + \|T(x_0)-T_n(x_0)\|_\infty,\end{aligned}$$

with $C=\overline{M}\theta(\delta_2+C_2\delta_3)$. This completes the proof. $\square$

Since π_n is the projection operator defined using the nonuniform breakpoints (2.1), the rate of convergence of the proposed methods is closely linked to the smoothness of z_0. Based on the regularity result of z_0 obtained in [15], we can derive the following lemma.

Lemma 4.6. *(Rice* [22]*) Let $z_0(t)=\psi(t,x_0(t))$ be a function of class Type$(\alpha,r,\{0,1\})$. Then*

$$\|(I-\pi_n)z_0\|_\infty = O(n^{-r}) \tag{4.14}$$

upon choosing $q=\frac{r}{\alpha}$ or $q=\frac{r}{1-\epsilon}$ for any $\epsilon\in(0,1)$ in the logarithmic case.

4.1. Discrete Galerkin-type and modified Galerkin-type methods

In this subsection, we demonstrate the outcomes concerning the convergence rate of the *Hyperinterpolation* projection.

Theorem 4.7. *Let x_0 be an isolated solution of (1.1) and $\overline{x}_n^S$ be the unique solution of (3.6) in the sphere $\mathcal{B}(x_0,\delta_0)$. We assume that the conditions in Theorem (4.3) are satisfied with $r\geq 1$. Also assume that x_0 is of Type$(\alpha,r,\{0,1\})$ for $\alpha>0$ or x_0 is of Type$(\alpha-\epsilon,r,\{0,1\})$ for any $\epsilon\in(0,1)$ in the logarithmic case. Then*

$$\|x_0-\overline{x}_n^S\|_\infty = \begin{cases} O(n^{-r-\alpha}), & 0<\alpha<1, \\ O(n^{-r-1}\log n), & \alpha=1. \end{cases} \tag{4.15}$$

Proof. According to Theorem 4.3, in order to estimate $\|x_0-\overline{x}_n^S\|_\infty$ it is necessary to estimate $\|T(x_0)-\overline{T_n^S}(x_0)\|_\infty$. Consider

$$\|T(x_0)-\overline{T_n^S}(x_0)\|_\infty \leq \|T(x_0)-T_n^S(x_0)\|_\infty + \|T_n^S(x_0)-\overline{T_n^S}(x_0)\|_\infty. \tag{4.16}$$

Since $z_0\in Type(\alpha,r,\{0,1\})$ or $z_0\in Type(\alpha-\epsilon,r,\{0,1\})$ and $\langle u,(I-\pi_n^G)z_0\rangle=0$, for any $u\in\mathcal{S}_r^\nu(\Pi_n)$. Then it follows from the *Hölder* inequality, Lemma (2.1) and estimates (4.14)

$$\begin{aligned}\left\|T(x_0)-T_n^S(x_0)\right\|_\infty &= \sup_{s\in[0,1]}\left|\int_0^1 \kappa(s,t)(I-\pi_n^G)z_0(t)dt\right|, \\ &= \sup_{s\in[0,1]}\left|\langle\kappa_s-u,(I-\pi_n^G)z_0\rangle\right|, \\ &\leq \|\kappa_s-u\|_{L^1}\|(I-\pi_n^G)z_0\|_\infty, \\ &\leq \begin{cases} Cn^{-r-\alpha}, & 0<\alpha<1, \\ Cn^{-r-1}\log n, & \alpha=1. \end{cases}\end{aligned} \tag{4.17}$$

By applying the Banach-Steinhaus theorem [1], it is shown that

$$\sup_{0\le s\le 1}\sum_{i=1}^{n_r}|k_i(s)| \simeq \sup_{0\le s\le 1}\sum_{i=1}^{n_r}\left|\overline{k}_i(s)\right| \le C,$$

where C is a constant independent of n. Next, using equation (3.11) with $k_i(s)=\langle \kappa_s,\varphi_i\rangle$, for the second term in (4.16), we write

$$\begin{aligned}\left|(T_n^S(x_0)-\overline{T_n^S}(x_0))(s)\right| &= \sup_{0\le s\le 1}\left|\sum_{i=1}^{n_r}k_i(s)\langle z_0,\varphi_i\rangle-\sum_{i=1}^{n_r}\overline{k}_i(s)\langle z_0,\varphi_i\rangle_n\right|\\ &\le \sup_{0\le s\le 1}\sum_{i=1}^{n_r}|k_i(s)|\,|\langle z_0,\varphi_i\rangle-\langle z_0,\varphi_i\rangle_n|\\ &\le C\,|\langle z_0,\varphi_i\rangle-\langle z_0,\varphi_i\rangle_n|\,.\end{aligned} \tag{4.18}$$

Hence by (2.7) it follows that

$$\begin{aligned}|\langle z_0,\varphi_i\rangle-\langle z_0,\varphi_i\rangle_n| &\le \sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i} z_0(t)\varphi_i(t)dt-\sum_{j=1}^{r}\omega_{ij}z_0(t_{ij})\varphi_i(t_{ij})\right|\\ &\le Cn^{-2r}.\end{aligned}$$

Then, combining the above estimates with (4.16)- (4.18), the bound (4.15) follows. □

Our next task is to demonstrate a theorem that establishes the rate of convergence of the approximation $\overline{x}_n^M$ and $\widehat{x}_n^M$ to the exact solution x_0.

Theorem 4.8. *Assume that the conditions in Theorem (4.5) are satisfied. Let x_0 be an isolated solution of (1.1) and $x_0\in$ Type $(\alpha,r,\{0,1\})$ for $\alpha>0$ or $x_0\in$ Type $(\alpha-\epsilon,r,\{0,1\})$ for any $\epsilon\in(0,1)$ in the logarithmic case. Let $\overline{x}_n^M$ and $\widehat{x}_n^M$ be the approximate solution defined by (3.8) and (3.10) respectively. For all large n, we have*

$$\|x_0-\overline{x}_n^M\|_\infty=\begin{cases}O(n^{-r-\alpha}), & 0<\alpha<1,\\ O(n^{-r-1}\log n), & \alpha=1.\end{cases} \tag{4.19}$$

In addition, assume that $\psi_1\in$ Type $(\alpha,r,\{0,1\})$ for $\alpha>0$ or $\psi_1\in$ Type $(\alpha-\epsilon,r,\{0,1\})$ for $\alpha=1$, then

$$\|x_0-\widehat{x}_n^M\|_\infty=\begin{cases}O(n^{-r-2\alpha}), & 0<\alpha<1,\\ O(n^{-r-2}(\log n)^2), & \alpha=1.\end{cases} \tag{4.20}$$

Proof. The estimate (4.19) is obtained by combining (4.16)-(4.18) with (4.7). In order to give an approximation error of the iterated discrete modified

Galerkin-type method, we use the second term on right hand side of (4.8) and (4.10)

$$\left|T_n'(x_0)(T(x_0)-\overline{T_n^M}(x_0)(s)\right| \le Cn^{-2r}\left|T_n'(x_0)\right| + \left|T_n'(x_0)(I-\pi_n^G)(T(x_0)-\overline{T_n^S}(x_0))\right| \tag{4.21}$$

By using (2.3) and estimates (4.16)-(4.18) when selecting $q=\frac{r}{\alpha}$ for $\alpha>0$ and $q=\frac{r}{1-\epsilon}$ in the logarithmic case. Then, for all $u\in\mathcal{S}_r^{\nu}(\Pi_n)$

$$\begin{aligned}\|T_n'(x_0)(I-\pi_n^G)(T(x_0)-\overline{T_n^S}(x_0))\|_\infty &\le (1+p)\|\kappa_s\psi_1-u\|_{L^1}\|T(x_0)-\overline{T_n^S}(x_0)\|_\infty\\ &\le \begin{cases} C(1+p)n^{-r-2\alpha}, & 0<\alpha<1,\\ C(1+p)n^{-r-2}(\log n)^2, & \alpha=1.\end{cases}\end{aligned} \tag{4.22}$$

Finally, combining estimates (2.12), (4.19), (4.21) and (4.22) with (4.8), we deduce (4.20) in form

$$\|x_0-\widehat{x}_n^M\|_\infty = O(n^{-2r-2\alpha})+O(n^{-r-2\alpha})+O(n^{-2r}), \tag{4.23}$$

if $0<\alpha<1$. This completes the proof. □

Corollary 4.9. *Let $\mathcal{S}_1^{-1}(\Pi_n)$ be the space of piecewise constant functions. We let $\alpha=\frac{1}{2}$, that is $q=2$ and $q>1$ in the logarithmic case, then*

$$\|x_0-\overline{x}_n^S\|_\infty = \begin{cases} O(n^{-1.5}), & \alpha=\frac{1}{2},\\ O(n^{-2}\log n), & \alpha=1\end{cases}$$

In addition, the above error bound and $\|x_0-\overline{x}_n^M\|_\infty$ have almost the same order of convergence, while for the discrete iterated modified Galerkin-type method can be bounded by

$$\|x_0-\widehat{x}_n^M\|_\infty = \begin{cases} O(n^{-2}), & \alpha=1/2,\\ O(n^{-3}(\log n)^2), & \alpha=1.\end{cases}$$

4.2. Discrete collocation-type and modified collocation-type methods

In this subsection, we will exhibit the results that related to the convergence rate of the interpolatory projection.

Theorem 4.10. *Let x_0 be an isolated solution of (1.1) and $\overline{x}_n^S$ be the unique solution of (3.6) in the sphere $\mathcal{B}(x_0,\delta_0)$. We assume that the conditions in Theorem (4.3) are satisfied with $r\ge1$ and that x_0 is of Type$(\alpha,r,\{0,1\})$. Then*

$$\|x_0-\overline{x}_n^S\|_\infty = O(n^{-r}). \tag{4.24}$$

Proof. From estimate (4.4) we have

$$\|x_0-\overline{x}_n^S\|_\infty \le C_1\|T(x_0)-T_n^S(x_0)\|_\infty + C_1\|T_n^S(x_0)-\overline{T_n^S}(x_0)\|_\infty. \tag{4.25}$$

Since z_0 belongs to the class of $Type(\alpha, r, \{0,1\})$ and $\{t_i, i = 0, \dots, n\}$ are selected according to (2.1). Applying (4.14), we obtain

$$\begin{aligned} \|T(x_0) - T_n^S(x_0)\|_\infty &\le \sup_{s\in[0,1]} \int_0^1 |\kappa(s,t)| \left|(I - \pi_n^C)z_0(t)\right| dt, \\ &\le M\|(I - \pi_n^C)z_0\|_\infty, \\ &\le CMn^{-r}. \end{aligned} \tag{4.26}$$

For each $s \in [0,1]$ we let $\omega_i(s) = \langle \kappa_s, \ell_i \rangle$. By using (3.14), we write

$$\begin{aligned} \left|(T_n^S(x_0) - \overline{T_n^S}(x_0))(s)\right| &= \sup_{0\le s\le 1} \left| \sum_{i=1}^{n_r} (\omega_i(s) - \overline{\omega}_i(s)) z_0(t_i) \right| \\ &\le \sup_{0\le s\le 1} \sum_{i=1}^{n_r} |\omega_i(s) - \overline{\omega}_i(s)| \, |z_0(t_i)| \, . \end{aligned}$$

Therefore, by (2.7) we can conclude

$$\begin{aligned} \|T_n^S(x_0) - \overline{T_n^S}(x_0)\|_\infty &\le nr\|\omega_i(s) - \overline{\omega}_i(s)\|_\infty \|z_0\|_\infty \\ &\le Crn^{-2r}\|z_0\|_\infty, \end{aligned} \tag{4.27}$$

□

The following theorem pertains to the overall superconvergence of $\overline{x}_n^S$ to x_0, and it relies heavily on the inequality (4.24).

Theorem 4.11. *Let x_0 be an isolated solution of (1.1) and $\overline{x}_n^S$ be the unique solution of (3.1) in the sphere $\mathcal{B}(x_0, \delta_0)$. We assume that the conditions in Theorem (4.3) are satisfied with $r \ge 1$ and that $M_1 \equiv \int_0^1 \prod_{j=1}^r (\zeta_j - s) ds = 0$ where $\zeta_j, j = \dots, r$ are the points used in (2.2). Also assume that x_0 is of $Type(\beta, r+1, \{0,1\})$ for $\alpha \le \beta \le r+1$ or x_0 is of $Type(\beta - \epsilon, r+1, \{0,1\})$ for any $\epsilon \in (0, \beta)$ in the logarithmic case. Then*

$$\|x_0 - \overline{x}_n^S\|_\infty = \begin{cases} O(n^{-r-\alpha}), & 0 < \alpha < 1, \\ O(n^{-r-1}\log n), & \alpha = 1. \end{cases} \tag{4.28}$$

Proof. Continuing the argument from the previous theorem, $z_0 \in Type(\beta, r+1, \{0,1\})$ or $z_0 \in Type(\beta - \epsilon, r+1, \{0,1\})$. The result of this theorem is always depends on (4.25). However, from Theorem 3 of Schneider [25], we have

$$\left\|T(x_0) - T_n^S(x_0)\right\|_\infty \le \begin{cases} Cn^{-r-\alpha}, & 0 < \alpha < 1, \\ Cn^{-r-1}\log n, & \alpha = 1. \end{cases} \tag{4.29}$$

such that $q = \frac{\alpha+r+1}{\alpha+\beta}$ and $q = \frac{\alpha+r+1}{\alpha+\beta-\epsilon}$ in the logarithmic case used as the graded exponent in (2.1). Hence, the estimate (4.28) is obtained by combining (4.25), (4.27) and (4.29). □

The next theorem establishes the superconvergence of the discrete modified solutions $\overline{x}_n^M$ and $\widehat{x}_n^M$ to the exact solution.

Theorem 4.12. *Assume that the conditions in Theorem (4.5) are satisfied. Let x_0 be an isolated solution of (1.1) and $x_0 \in$ Type $(\alpha, r, \{0,1\})$ for $\alpha > 0$ or $x_0 \in$ Type $(\alpha - \epsilon, r, \{0,1\})$ for any $\epsilon \in (0,1)$ in the logarithmic case. Let $\overline{x}_n^M$ and $\widehat{x}_n^M$ be the approximate solution defined by (3.8) and (3.10) respectively. For all large n, we have*

$$\|x_0 - \overline{x}_n^M\|_\infty = \begin{cases} O(n^{-r-\alpha}), & 0 < \alpha < 1, \\ O(n^{-r-1}\log n), & \alpha = 1. \end{cases} \tag{4.30}$$

In addition, assume that $\psi_1 \in$ Type $(\alpha, r, \{0,1\})$ for $\alpha > 0$ or $\psi_1 \in$ Type $(\alpha - \epsilon, r, \{0,1\})$ for $\alpha = 1$, then

$$\|x_0 - \widehat{x}_n^M\|_\infty = \begin{cases} O(n^{-r-2\alpha}), & 0 < \alpha < 1, \\ O(n^{-r-2}(\log n)^2), & \alpha = 1. \end{cases} \tag{4.31}$$

Proof. First, by combining (4.27) and (4.29) with (4.7) the estimate (4.30) is proved since $q = \min\{\frac{r}{\alpha}, \frac{\alpha+r+1}{\alpha+\beta}\}$ and $q = \min\{\frac{r}{1-\epsilon}, \frac{\alpha+r+1}{\alpha+\beta-\epsilon}\}$ in the logarithmic case used as the graded exponent in (2.1).
To obtain the desired result of the discrete iterated version, we apply (4.10) to the following approximation

$$\|T_n'(x_0)(T(x_0) - \overline{T_n^M}(x_0)\|_\infty \leq C\overline{M}n^{-2r} + \|T_n'(x_0)(I - \pi_n^C)(T(x_0) - \overline{T_n^S}(x_0))\|_\infty. \tag{4.32}$$

The last term of (4.32) can be formulated by using Theorem 4-(i) of Graham's [12], if $x_0 \in W_1^\ell(0 < \ell \leq 2r)$ and $\kappa_s\psi_1 \in W_1^m(0 < m \leq r)$, then

$$\begin{aligned}[][T_n'(x_0)(I - \pi_n^C)(T(x_0) - \overline{T_n^S}(x_0))](s) &= \langle \kappa_s\psi_1 - u, (I - \pi_n^C)(T(x_0) - \overline{T_n^S}(x_0))\rangle_n \\ &+ \langle u, (I - \pi_n^C)((T(x_0) - \overline{T_n^S}(x_0)) - v)\rangle_n + \langle u, (I - \pi_n^C)v\rangle_n,\end{aligned} \tag{4.33}$$

for some $u \in \mathcal{S}_m^{-1}(\Pi_n)$ and some $v \in \mathcal{S}_\ell^{-1}(\Pi_n)$, with $0 < m \leq r$ and $0 < \ell \leq 2r$.
According to the proof as described in ([12], p. 362), we would like to point out that the optimal order corresponds to the first term of (4.33). Then, from estimates (2.3), (4.11), (4.27) and (4.29) one gets

$$\|\kappa_s\psi_1 - u\|_{L^1}\|(I - \pi_n^C)(T(x_0) - \overline{T_n^S}(x_0))\|_\infty \leq \begin{cases} C(1+p)n^{-r-2\alpha}, & 0 < \alpha < 1, \\ C(1+p)n^{-r-2}(\log n)^2, & \alpha = 1. \end{cases} \tag{4.34}$$

Consequently, the estimate (4.31) follows immediately by combining (2.12), (4.30), (4.32)-(4.34) with (4.8). $\square$

Corollary 4.13. *Let $\mathcal{S}_1^{-1}(\Pi_n)$ be the space of piecewise constant functions. We let $\alpha = \beta$ and let $\zeta_1 = \frac{1}{2}$ i.e., interpolation at the mid-point, then $M_1 = 0$. If $q = \frac{\alpha+r+1}{2\alpha}$ and $q > \frac{\alpha+r+1}{2\alpha}$ in the logarithmic case, then*

$$\|x_0 - \overline{x}_n^S\|_\infty = \begin{cases} O(n^{-1.5}), & \alpha = 1/2, \\ O(n^{-2}\log n), & \alpha = 1, \end{cases}$$

that is, $q = 2.5$ *and* $q > 1.5$ *for* $\alpha = 1$.
If $M_1 \neq 0$, $q = \frac{r}{\alpha}$ *and* $q > r$ *in the logarithmic case, then* $\|x_0 - \overline{x}_n^S\|_\infty = O(n^{-1})$ *for any* $0 < \alpha \leq 1$, *that is* $q = 2$ *and* $q > 1$ *for* $\alpha = 1$.
If $q = \min\{\frac{r}{\alpha}, \frac{\alpha+r+1}{2\alpha}\}$ *and* $q > \min\{r, \frac{\alpha+r+1}{2\alpha}\}$ *in the logarithmic case, then*

$$\|x_0 - \overline{x}_n^M\|_\infty = \begin{cases} O(n^{-1.5}), & \alpha = 1/2, \\ O(n^{-2}\log n), & \alpha = 1, \end{cases}$$

that is $q = 2$ *and* $q > 1$ *for* $\alpha = 1$. *In addition,*

$$\|x_0 - \widehat{x}_n^M\|_\infty = \begin{cases} O(n^{-2}), & \alpha = 1/2, \\ O(n^{-3}(\log n)^2), & \alpha = 1. \end{cases}$$

5. Numerical results

This section includes two numerical examples that demonstrate the theoretical estimates derived in the preceding sections. Let $\mathbb{X}_n$ be the space of piecewise constant functions ($r = 1$) were used as approximating subspaces. In this framework, a *Newton–Raphson* method was used to solve different nonlinear systems. It should be noted that all the necessary integrals were computed with a highly accurate *Gauss*-type quadrature rule [18]. Furthermore, the numerical algorithms were implemented using `WOLFRAM MATHEMATICA`.

Example 1. Consider the following *Hammerstien* equation

$$x(s) - \int_0^1 \frac{1}{\sqrt{|s-t|}} \left[\frac{1}{1+x(t)}\right] dt = f(s), \quad 0 \leq s \leq 1, \tag{5.1}$$

where f is selected so that the exact solution is $x(s) = \sqrt{s}$, which is non smooth. In the following, we verify and confirm the bounds described in Corollary 1 and 2, since the solution is of *Type*$(\frac{1}{2}, r, \{0,1\})$. The convergence rate is affected by the value of parameter q which is used to establish a graded mesh.

From Corollary 1, if $q = 2$, then the expected orders of convergence for the discrete *Galerkin*-type and modified *Galerkin*-type solutions are 1.5, whereas for the discrete iterated modified *Galerkin*-type solution it is 2.

From Corollary 2, if $q = 2.5$, then the expected order of convergence for the discrete collocation-type is 1.5. If $q = 2$, then the expected order of convergence for the discrete collocation-type is 1, and 1.5 for the discrete modified collocation-type, whereas for the discrete iterated modified collocation-type solution it is 2.

Example 2. We solve the following integral equation

$$x(s) - \int_0^1 \log(|s-t|)x^2(t)dt = f(s), \quad 0 \leq s \leq 1, \tag{5.2}$$

where f is selected so that the exact solution is $x(s) = s\log(s)$, which is of *Type*$(1, k, \{0, 1\})$.

From Corollary 1, if $q > 1$, then the expected orders of convergence for the discrete *Galerkin*-type and modified *Galerkin*-type solutions are 2, whereas for the iterated modified *Galerkin*-type solution it is 3.

According to Corollary 2, if $q > 1.5$, then the expected order of convergence for the discrete collocation-type is 2. If $q > 1$, then the expected order of convergence for the discrete collocation-type is 1, and 2 for the discrete modified collocation-type, whereas for the discrete iterated modified collocation-type solution it is 3. The results of these estimates are confirmed by the numerical computations.

From the integral equations (5.1) and (5.2), we compute the maximum errors and orders of convergence of the approximation solution obtained by the discrete modified projection-type method and its iterated version and we compare them with those obtained by the projection-type method in Table 1-4, respectively.

In Tables 1 and 2, we observe that a satisfactory precision is obtained even when the polynomials are of low degree. As expected the performance of the discrete projection-type and the discrete modified projection-type methods are similar. Note that, this remark remains valid in the case of discrete collocation-type method except when we choose the interpolation at the mid-points. By observing the results, we can note that in order to achieve an error of order 10^{-4}, the discrete modified collocation-type method requires a system of size 64 to be solved. In contrast, to achieve a similar order of accuracy in the discrete modified *Galerkin*-type method, a system of size 128 needs to be solved. However, when computing $\|x_0 - \widehat{x}_{16}^M\|_\infty$ by a discrete modified projection-type method, which is obtained by solving a system of size 16 in the modified collocation-type and 32 in the modified *Galerkin*-type method, we get an error of the order of 10^{-4}.

As a result, the discrete projection-type and the modified projection-type methods have almost the same order of convergence, the iterated discrete modified projection-type method converges less than both of them in terms of the error and the order of convergence. It should be mentioned that the discrete modified collocation-type method has benefits theoretically and computationally over the discrete modified *Galerkin*-type method, which require solving an extremely large nonlinear system that is computationally very expensive. There are similar observations to be made in the Tables 3 and 4.

To ensure that all relevant information is included, we present in Figures 1 and 2 the errors in absolute value obtained by different methods when employing example (5.1). These methods include the discrete projection-type (shown in yellow), the discrete modified projection-type (shown in blue), and the discrete iterated modified method (shown in green) with different values of n.

n	$\|x_0 - \overline{x}_n^S\|_\infty$	order	$\|x_0 - \overline{x}_n^M\|_\infty$	order	$\|x_0 - \widehat{x}_n^M\|_\infty$	order
2	1.16×10^{-1}		1.14×10^{-1}		4.72×10^{-2}	
4	4.18×10^{-2}	1.46	3.87×10^{-2}	1.55	9.77×10^{-3}	2.27
8	1.56×10^{-2}	1.42	1.33×10^{-2}	1.53	2.12×10^{-3}	2.20
16	5.30×10^{-3}	1.56	4.15×10^{-3}	1.68	4.85×10^{-4}	2.12
32	1.66×10^{-3}	1.66	1.46×10^{-3}	1.50	1.14×10^{-4}	2.07
64	5.60×10^{-4}	1.57	5.13×10^{-4}	1.50	2.77×10^{-5}	2.05

TABLE 1. Discrete *Galerkin*-type and modified *Galerkin*-type methods for algebraic singularity.

n	$\|x_0 - \overline{x}_n^S\|_\infty$	order	$\|x_0 - \overline{x}_n^S\|_\infty$	order	$\|x_0 - \overline{x}_n^M\|_\infty$	order	$\|x_0 - \widehat{x}_n^M\|_\infty$	order
2	1.49×10^{-1}		1.06×10^{-1}		1.19×10^{-1}		5.55×10^{-2}	
4	5.37×10^{-2}	1.47	6.09×10^{-2}	0.80	4.55×10^{-2}	1.39	1.11×10^{-3}	2.31
8	1.98×10^{-2}	1.43	2.65×10^{-2}	1.20	1.15×10^{-2}	1.54	2.39×10^{-3}	2.21
16	6.66×10^{-3}	1.57	1.19×10^{-2}	1.15	5.45×10^{-3}	1.51	5.90×10^{-4}	2.12
32	2.03×10^{-3}	1.71	5.48×10^{-3}	1.12	1.91×10^{-3}	1.51	1.51×10^{-4}	1.96
64	6.28×10^{-4}	1.69	2.52×10^{-3}	1.12	6.82×10^{-4}	1.48	3.48×10^{-5}	2.12

TABLE 2. Discrete collocation-type and modified collocation-type methods for algebraic singularity.

n	$\|x_0 - \overline{x}_n^S\|_\infty$	order	$\|x_0 - \overline{x}_n^M\|_\infty$	order	$\|x_0 - \widehat{x}_n^M\|_\infty$	order
2	6.97×10^{-2}		5.31×10^{-2}		3.65×10^{-2}	
4	3.18×10^{-2}	1.13	1.65×10^{-2}	1.68	7.97×10^{-3}	2.19
8	1.10×10^{-2}	1.52	3.04×10^{-3}	2.43	9.82×10^{-4}	3.02
16	3.08×10^{-3}	1.83	5.34×10^{-4}	2.51	1.07×10^{-4}	3.19
32	8.01×10^{-4}	1.94	1.18×10^{-4}	2.17	1.20×10^{-5}	3.15
64	2.02×10^{-4}	1.98	2.71×10^{-5}	2.12	1.40×10^{-6}	3.09

TABLE 3. Discrete *Galerkin*-type and modified *Galerkin*-type methods for logarithmic singularity.

n	$\|x_0 - \overline{x}_n^S\|_\infty$	order	$\|x_0 - \overline{x}_n^S\|_\infty$	order	$\|x_0 - \overline{x}_n^M\|_\infty$	order	$\|x_0 - \widehat{x}_n^M\|_\infty$	order
2	5.88×10^{-2}		4.57×10^{-2}		4.23×10^{-2}		7.43×10^{-3}	
4	3.54×10^{-2}	0.73	5.35×10^{-2}	0.22	1.17×10^{-2}	1.84	1.16×10^{-3}	2.67
8	7.39×10^{-3}	2.25	2.64×10^{-2}	1.01	2.81×10^{-3}	2.06	2.63×10^{-4}	2.13
16	1.56×10^{-3}	2.24	1.17×10^{-2}	1.17	5.58×10^{-4}	2.33	4.30×10^{-5}	2.61
32	3.72×10^{-4}	2.07	5.74×10^{-3}	1.02	1.53×10^{-4}	1.83	1.44×10^{-5}	1.57
64	8.99×10^{-5}	2.05	2.85×10^{-3}	1.00	3.76×10^{-5}	2.03	1.68×10^{-6}	3.10

TABLE 4. Discrete collocation-type and modified collocation-type methods for logarithmic singularity.

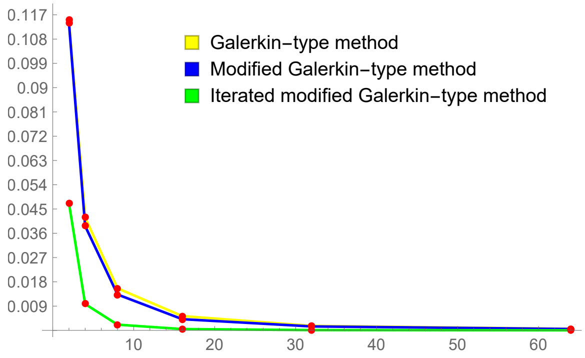

FIGURE 1. Discrete *Galerkin*-type and modified *Galerkin*-type methods for algebraic singularity.

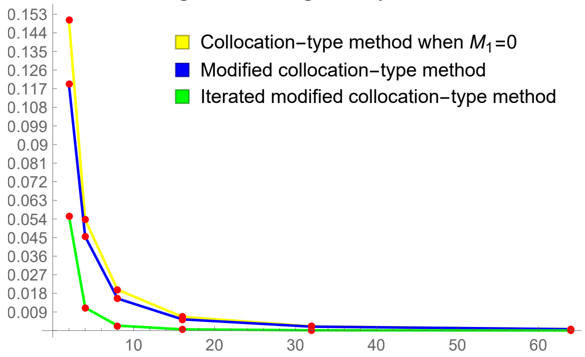

FIGURE 2. Discrete collocation-type and modified collocation-type methods for algebraic singularity.

6. Conclusion

The primary objective of this research paper is to examine a modified projection-type method in discrete version for solving *Hammerstein* integral equations. The integral operator in question has a singularity that is either algebraic or logarithmic in nature. The paper presents theoretical calculations for both the error bound and the convergence rate of the method. Furthermore, numerical examples are provided to demonstrate the practical effectiveness of the proposed approach and to validate the theoretical error estimates. The results in this paper have the potential to be extended to derivative-dependent *Hammerstein* integral equations, although that topic would require another research paper to address.

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