Diameter of io-decomposable Riordan graphs of the Bell type*

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Abstract

Recently, in the paper [4] we suggested the two conjectures about the diameter of io-decomposable Riordan graphs of the Bell type. In this paper, we give a counterexample for the first conjecture. Then we prove that the first conjecture is true for the graphs of some particular size and propose a new conjecture. Finally, we show that the second conjecture is true for some special io-decomposable Riordan graphs.

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Key words: Riordan graph; io-decomposable Riordan graph; diameter; Catalan graph.

1 Introduction

Let $\kappa[[z]]$ be the ring of formal power series in the variable z over an integral domain κ. A Riordan matrix [13] $L = [\ell_{n,k}]_{n,k\geq 0}$ is defined by a pair of formal power series $(g, f) \in \kappa[[z]] \times \kappa[[z]]$ with $f(0) = 0$ such that $[z^n]gf^k = \ell_{n,k}$ for $k \geq 0$ where $[z^n]$ is the coefficient extraction operator. Usually, the Riordan matrix is denoted by $L = (g, f)$ and its *leading principal submatrix* of order n is denoted by $(g, f)_n$. Since $f(0) = 0$, every Riordan matrix (g, f) is an infinite lower triangular matrix. Most studies on the Riordan matrices were related to combinatorics [6, 9, 11, 14, etc.] or algebraic structures [1, 2, 3, 7, etc.].

Throughout this paper, we write $a \equiv b$ for $a \equiv b \pmod 2$.

Recently, we in [4, 5] introduced a Riordan graph by using the notion of the Riordan matrix modulo 2 as follows.

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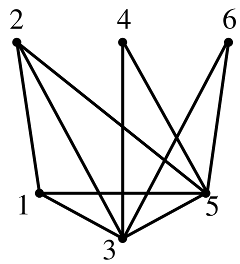

$$\mathcal{A}(CG_6) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Figure 1: The Catalan graph of order 6 and its adjacency matrix

Definition 1.1. A simple *labelled* graph G with n vertices is a *Riordan graph* of order n if the adjacency matrix $\mathcal{A}(G) = [r_{i,j}]_{1\le i,j\le n}$ of G is an $n\times n$ symmetric $(0,1)$-matrix given by

$$\mathcal{A}(G) \equiv (zg, f)_n + (zg, f)_n^T, \text{ i.e. } r_{i,j} = r_{j,i} \equiv \begin{cases} [z^{i-2}]gf^{j-1} & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}$$

for some Riordan matrix (g, f) over $\mathbb{Z}$. We denote such G by $G_n(g, f)$. In particular, the Riordan graph $G_n(g, f)$ is called *proper* if $[z^0]g = [z^1]f = 1$.

For example, consider the *Catalan graph* $CG_n = G_n(C(z), zC(z))$ where $C(z)$ is the generating function for the Catalan numbers, i.e.

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\ge 0} \frac{1}{n+1}\binom{2n}{n} z^n = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots . \quad (1)$$

When $n = 6$ we have Figure 1.

In [4], we studied the structural properties of families of Riordan graphs obtained from infinite Riordan graphs, which include a fundamental decomposition theorem and certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle. A Riordan graph $G_n(g, f)$ is called *Bell type* if $f = zg$. Moreover, we studied the following Riordan graphs of special Bell type.

Definition 1.2. *[4]* Let $G_n = G_n(g, f)$ be a proper Riordan graph with the odd and even vertex sets $V_o = \{i \in V(G_n) \mid i \equiv 1\}$ and $V_e = \{i \in V(G_n) \mid i \equiv 0\}$, respectively. The graph G_n is said to be *io-decomposable* if $\langle V_o\rangle \cong G_{\lceil n/2\rceil}(g, f)$ and $\langle V_e\rangle$ is a null graph.

A vertex in a graph G is *universal* if it is adjacent to all other vertices in G. The *distance* between two vertices u, v in a graph G is the number of edges in a shortest path between u and v. The *diameter* of G is the maximum distance between all pairs of vertices, and it is denoted by $\mathrm{diam}(G)$.

We found several properties of an io-decomposable Riordan graph of the Bell type as follows.

Lemma 1.3. *[4] Let $G_n = G_n(g, zg)$ be an io-decomposable Riordan graph of the Bell type. Then we have the following.*

(i) If $n = 2^k + 1$ for $k \geq 0$, then G_n and G_{n+1} have at least one universal vertex, namely $2^k + 1$.

(ii) G_n is a $(\lceil \log_2 n \rceil + 1)$-partite graph.

(iii) The chromatic number and the clique number of G_n are $\lceil \log_2 n \rceil + 1$.

(iv) The diameter of G_n is bounded by $\mathrm{diam}(G_n) \leq \lfloor \log_2 n \rfloor$. In particular, if $n = 2^k + 2$ or $2^{k+1} + 1$, for $k \geq 1$, then $\mathrm{diam}(G_n) = 2$.

(v) If $2^k + 1 < n < 2^{k+1}$ then $\mathrm{diam}(G_n) \leq \lfloor \log_2(n - 2^k) \rfloor + 1$.

A graph is called *weakly perfect* if its chromatic number equals its clique number. By (iii) of Lemma 1.3, every io-decomposable Riordan graph of the Bell type is weakly perfect. It is known [10] that almost all K_k-free graphs are $(k-1)$-partite for $k \geq 3$. By (ii) and (iii) of Lemma 1.3, every io-decomposable Riordan graph $G_n(g, zg)$ of the Bell type is $K_{\lceil \log_2 n \rceil + 2}$-free and $(\lceil \log_2 n \rceil + 1)$-partite for $n \geq 2$. Thus the io-decomposable Riordan graph of the Bell type is very interesting object in Riordan graph theory.

It is known [4] that the Pascal graph $PG_n = G_n(1/(1-z), z/(1-z))$ and the Catalan graph $CG_n = G_n(C(z), zC(z))$ are the io-decomposable Riordan graphs of the Bell type. The following two conjectures introduced in [4] show significance of the Pascal graph PG_n and the Catalan graph CG_n.

Conjecture 1. *[4]* Let G_n be an io-decomposable Riordan graph of the Bell type. Then

$$2 = \mathrm{diam}(PG_n) \leq \mathrm{diam}(G_n) \leq \mathrm{diam}(CG_n)$$

for $n \geq 4$. Moreover, PG_n is the only graph in the class of io-decomposable graphs of the Bell type whose diameter is 2 for *all* $n \geq 4$.

Conjecture 2. *[4]* We have that diam$(CG_{2^k}) = k$ and there are no io-decomposable Riordan graphs $G_{2^k} \not\cong CG_{2^k}$ of the Bell type satisfying diam$(G_{2^k}) = k$ for all $k \geq 1$.

We note that $\mathrm{diam}(PG_n) = 1$ if $n = 2, 3$ and $\mathrm{diam}(PG_n) = 2$ if $n \geq 4$ since the vertex 1 is adjacent to all other vertices, $PG_n \cong K_n$ if $n = 2, 3$ and $PG_n \not\cong K_n$ if $n \geq 4$.

In this paper, we first give a counterexample of the upper bound in Conjecture 1. Then we prove that the upper bound in Conjecture 1 is true for the graph of some particular size and we propose a new conjecture for an upper bound of the diameter of an io-decomposable Riordan graph of the Bell type. Finally, we show that Conjecture 2 is true for some special io-decomposable Riordan graphs.

2 Upper bound of Conjecture 1

It is known [12] that an infinite lower triangular matrix $L = [\ell_{i,j}]_{i,j\geq 0}$ with $\ell_{0,0} \neq 0$ is a proper Riordan matrix if and only if there is a unique sequence $(a_0, a_1, \ldots)$ with $a_0 \neq 0$ such that, for $i \geq j \geq 0$,

$$\ell_{i+1,j+1} = a_0\ell_{i,j} + a_1\ell_{i,j+1} + \cdots + a_{i-j}\ell_{i,i}.$$

The sequence $(a_i)_{i\geq 0}$ is called the *A-sequence* of the Riordan array. Also, if $L = (g, f)$ then

$$f = zA(f), \quad \text{or equivalently} \quad A = z/\bar{f} \tag{2}$$

where $A = \sum_{i\geq 0} a_i z^i$ is the generating function for the A-sequence of (g, f). In particular, if L is a binary Riordan matrix $L \equiv (g, f)$ with $f'(0) = 1$ then the sequence is called the *binary A-sequence* $(1, a_1, a_2, \ldots)$ where $a_k \in \{0, 1\}$.

We in [4] characterized the io-decomposable Riordan graph $G_n(g, zg)$ of the Bell type, see the following lemma.

Lemma 2.1. *[4] Let $G_n = G_n(g, zg)$ be a Riordan graph of the Bell type. Then G_n is io-decomposable if and only if the binary A-sequence of (g, zg) is $(1, 1, a_2, a_2, a_4, a_4, \ldots)$ where $a_{2j} \in \{0, 1\}$ for all $j \geq 1$.*

Let G_n be the io-decomposable Riordan graph of the Bell type with its binary A-sequence generating function $A(z) = \sum_{i=0}^{15} z^i$. By Lemma 2.1, the Riordan graph G_n is io-decomposable. By using the sage, we compare the diameters between CG_n and G_n up to degree $n = 100$. Then we obtain the following 13 counterexamples for the upper bound of Conjecture 1.

n	44	45	46	47	48	78	79	80	87	88	89	90	91
diam(CG_n)	3	3	3	3	3	3	3	3	3	3	3	3	3
diam(G_n)	4	4	4	4	4	4	4	4	4	4	4	4	4

If for a Riordan graph $G_n(g, f)$ with $[z^1]f = 1$, the relabelling is done by *reversing* the vertices in $[n]$, that is, by replacing a label i by $n + 1 - i$ for each $i \in [n]$, then the resulting graph will always be a Riordan graph given by the following lemma. We denote the reverse relabelling of G_n by G_n^r.

Lemma 2.2. *[4] The reverse relabelling of a Riordan graph $G_n(g, f)$ with $f'(0) = 1$ is the Riordan graph*

$$G_n^r(g, f) = G_n(g(\bar{f}) \cdot (\bar{f})' \cdot (z/\bar{f})^{n-1}, \bar{f})$$

where $\bar{f}$ is the compositional inverse of f.

Now, we prove that the upper bound in Conjecture 1 is true if $n = 2^k$, $n = 2^k - 1$ or $n = 1 + 2^m + 2^k$ where $k \geq 2$ and $1 \leq m < k$.

Lemma 2.3. *Let $G_n = G_n(g, zg)$ be a proper Riordan graph and $A(z)$ be the generating function for its binary A-sequence. Then the reverse relabelling of G_n is the Riordan graph given by*

$$G_n^r = G_n((zA(z))' \cdot A^{n-2}(z), z/A(z)).$$

In particular, if G_n is an io-decomposable Riordan graph of the Bell type then the reverse relabelling of G_n is the Riordan graph given by

$$G_n^r = G_n(A'(z) \cdot A^{n-2}(z), z/A(z)).$$

Proof. Let $f = zg$ and $\bar{f}$ be the compositional inverse of f. Since (2) leads to

$$g(\bar{f}) = z/\bar{f} = A(z) \quad \text{and} \quad (\bar{f})' = \left(\frac{z}{A(z)}\right)' \equiv \frac{A(z) + zA'(z)}{A^2(z)} = (zA(z))'A^{-2}(z),$$

we obtain

$$g(\bar{f}) \cdot (\bar{f})' \cdot (z/\bar{f})^{n-1} \equiv (zA(z))' \cdot A^{n-2}(z) \quad \text{and} \quad \bar{f} = z/A(z).$$

Thus, by Lemma 2.2, we obtain the desired result. In particular, if G_n is an io-decomposable Riordan graph of the Bell type then it follows from Lemma 2.1 that $(zA(z))' \equiv A'(z)$. Hence the proof follows. □

If the base p (a prime) expansion of n and m is $n = n_0 + n_1 p + n_2 p^2 + \cdots$ and $m = m_0 + m_1 p + m_2 p^2 + \cdots$ respectively then

$$\binom{n}{m} \equiv \prod_i \binom{n_i}{m_i} \pmod{p}.$$

This is called the Lucas's theorem.

Let G_n be an io-decomposable Riordan graph of the Bell type. Since $\text{diam}(G_n) \leq k$ if $n = 2^k$ with $k \geq 0$ by (iv) of Lemma 1.3, the following theorem shows that the upper bound of Conjecture 1 is true if $n = 2^k$ for $k \geq 1$. We denote the distance between two vertices u, v in a graph G by $d_G(u, v)$.

Theorem 2.4. *For an integer $k \geq 1$, we obtain*

$$\text{diam}(CG_{2^k}) = k.$$

Proof. First we show that $CG^r_{2^k} = G_{2^k}(1, z+z^2)$ for $k \geq 1$. It follows from (1) and (2) that the generating function for A-sequence of (C, zC) is $\frac{1}{1-z}$. By Lemma 2.3, we obtain

$$A'(z) \cdot A^{2^k-2}(z) = \left(\frac{1}{1-z}\right)^2 \cdot \left(\frac{1}{1-z}\right)^{2^k-2} = \left(\frac{1}{1-z}\right)^{2^k}$$

so that by Lemma 2.2 the reverse relabelling of the Catalan graph is

$$CG^r_{2^k} = G_{2^k}((1-z)^{-2^k}, z+z^2). \tag{3}$$

Since $(1-z)^{-2^k} = \sum_{j\geq 0}\binom{2^k+j-1}{2^k-1}z^j$, by Lucas's theorem we obtain

$$\binom{2^k+j-1}{2^k-1} \equiv 1 \text{ for } j=0 \quad \text{and} \quad \binom{2^k+j-1}{2^k-1} \equiv 0 \text{ for } 1 \leq j \leq 2^k-1$$

which imply $G_{2^k}((1-z)^{-2^k}, z+z^2) = G_{2^k}(1, z+z^2)$. Thus, by (3), $CG^r_{2^k} = G_{2^k}(1, z+z^2)$. Let $m_i = \max\{j \in V(CG^r_{2^k}) \mid ij \in E(CG^r_{2^k})\}$. For each $i \in \{1, 2, \ldots, 2^{k-1}\}$, we obtain

$$m_i = \max\{j \in V(CG^r_{2^k}) \mid [z^{j-2}](z+z^2)^{i-1} \equiv 1\} = 2i. \tag{4}$$

Since $m_1 < m_2 < \cdots < m_{2^{k-1}}$, a unique shortest path from 1 to 2^k in $CG^r_{2^k}$ is $2^0 \to 2^1 \to \cdots \to 2^{k-1} \to 2^k$ so that $d_{CG^r_{2^k}}(1, 2^k) = k$. Hence, by (iv) of Lemma 1.3, we obtain the desired result. □

By Lemma 2.1, the following lemma is obtained from [4, Theorem 3.7] when $\ell = 0$.

Lemma 2.5. *Let G_n be an io-decomposable Riordan graph of the Bell type and $G = \lim_{n\to\infty} G_n$. For each $s \geq 0$, G has the following fractal properties:*

(i) $G_{2^s+1} = \langle\{1, 2, \ldots, 2^s+1\}\rangle \cong \langle\{\alpha 2^s+1, \alpha 2^s+2, \ldots, (\alpha+1)2^s+1\}\rangle$;

(ii) $G_{2^s} = \langle\{1, 2, \ldots, 2^s\}\rangle \cong \langle\{\alpha 2^s+1, \alpha 2^s+2, \ldots, (\alpha+1)2^s\}\rangle$

where $\alpha \geq 1$.

We can ask that how many vertex pairs (u, v) can have the maximal distance k in CG_{2^k}. By using Lemma 2.5, the answer is given by the following theorem.

Theorem 2.6. *Let $k \geq 1$ be an integer. There exist exactly 2^{k-1} vertex pairs $(i, 2^k)$ with $i \in \{1, \ldots, 2^{k-1}\}$ such that $d_{CG_{2^k}}(i, 2^k) = k$ is the maximal distance in CG_{2^k}.*

Proof. Since $CG^r_{2^k}$ is the reverse relabelling of CG_{2^k}, this theorem is equivalent to the following:

- $d_{CG^r_{2^k}}(i,j) = k$ if $i = 1$ and $j \in \{2^{k-1}+1, \dots, 2^k\}$;
- $d_{CG^r_{2^k}}(i,j) \le k-1$ otherwise.

Since by (i) of Lemma 1.3 the vertex $2^{k-1}+1$ is adjacent to all vertices $1, \dots, 2^{k-1}$ in CG_{2^k}, the vertex 2^{k-1} is adjacent to all vertices $2^{k-1}+1, \dots, 2^k$ in $CG^r_{2^k}$. So by (4) the shortest path from 1 to j in $CG^r_{2^k}$ is

$$2^0 \to 2^1 \to \cdots \to 2^{k-1} \to j \quad \text{where} \quad j \in \{2^{k-1}+1, \dots, 2^k\}$$

and thus $d_{CG^r_{2^k}}(i,j) = k$ if $i = 1$ and $j \in \{2^{k-1}+1, \dots, 2^k\}$.

Let $V_1 = \{i \in V(CG^r_{2^k}) \mid 1 \le i \le 2^{k-1}\}$ and $V_2 = \{j \in V(CG^r_{2^k}) \mid 2^{k-1} < j \le 2^k\}$. Since $\langle V_1 \rangle \cong \langle V_2 \rangle \cong CG^r_{2^{k-1}}$ by Lemma 2.5 and $G_{2^{k-1}} \cong CG_{2^{k-1}}$, it follows from Theorem 2.4 that

$$d_{CG^r_{2^k}}(i,j) \le k-1 \ \text{ if } i,j \in V_1 \text{ or } i,j \in V_2.$$

Now it is enough to show that $d_{CG^r_{2^k}}(i,j) < k$ if $i \in V_1 \backslash \{1\}$ and $j \in V_2$ for $k \ge 2$. We prove this by induction on $k \ge 2$. Let $k = 2$. Since the adjacency matrix of CG^r_4 is given by

$$\mathcal{A}(CG^r_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

we see that $d_{CG^r_4}(2,3) = d_{CG^r_4}(2,4) = 1 < 2$. Thus it holds for $k = 2$. Let $k \ge 3$. Since $\langle V_1 \rangle \cong CG^r_{2^{k-1}}$ and the vertex 2^{k-1} is adjacent to all vertices $j \in V_2$ in $CG^r_{2^k}$, we obtain

$$\begin{aligned} d_{CG^r_{2^k}}(i,j) &\le d_{CG^r_{2^k}}(i, 2^{k-1}) + d_{CG^r_{2^k}}(2^{k-1}, j) \le d_{CG^r_{2^{k-1}}}(i, 2^{k-1}) + 1 \\ &\le k-1 \quad \text{(by induction)} \end{aligned}$$

where $i \in V_1 \backslash \{1\}$ and $j \in V_2$. Hence the proof follows. □

Example 2.7. Let us consider the Catalan graph $CG_8 = G_8(C(z), zC(z))$ of order 8. Since its reverse relabeling is $CG^r_8 = G_8(1, z+z^2)$, we obtain Figure 2 from the adjacency matrix

$$\mathcal{A}(CG^r_8) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

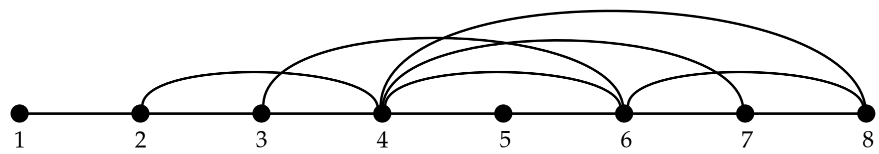

Figure 2: The graph of $CG_8^r = G_n(1, z+z^2)$

Thus we can see that the four vertex pairs $(1,5),(1,6),(1,7)$ and $(1,8)$ in CG_8^r have maximal distance 3 i.e., the four vertex pairs $(8,4),(8,3),(8,2)$ and $(8,1)$ in CG_8 have the maximal distance 3.

Let G_n be an io-decomposable Riordan graph of the Bell type. Since it follows from (iv) of Lemma 1.3 that diam$(G_n) \le k-1$ if $n = 2^k - 1$, the following corollary shows that the upper bound of Conjecture 1 is true if $n = 2^k - 1$ for $k \ge 1$.

Corollary 2.8. *For an integer $k \ge 1$, we obtain*

$$\mathrm{diam}(CG_{2^k-1}) = k-1.$$

Proof. By Theorem 2.6, we obtain diam$(CG_{2^k-1}) \le k-1$. It follows from Lemma 2.3 that one can show $CG^r_{2^k-1} = G_n(1+z, z+z^2)$. By using the similar proof in Theorem 2.4, we can show that $2^1-1 \to 2^2-1 \to \cdots \to 2^k-1$ is the shortest path from 1 to 2^k-1 in $CG^r_{2^k-1}$, i.e. $d_{CG^r_{2^k-1}}(1, 2^k-1) = k-1$. Since it follows from Theorem 2.6 that diam$(CG_{2^k-1}) \le k-1$, we obtain diam$(CG_{2^k-1}) = k-1$. Hence the proof follows. $\square$

The following lemma is useful to obtain Theorem 2.10 and Conjecture 3.

Lemma 2.9. *Let $n = 1 + 2^m + \sum_{j=0}^{s} 2^{k+j}$ be an integer with $k > m \ge 1$. If G_n be the io-decomposable Riordan graph of the Bell type, then we obtain*

$$\mathrm{diam}(G_n) \le \begin{cases} s+2 & \text{if } m = 1; \\ s+3 & \text{otherwise.} \end{cases}$$

Proof. We prove this by induction on $s \ge 0$. Let $s = 0$, i.e. $n = 1 + 2^m + 2^k$. If $m = 1$ then it follows from (v) of Lemma 1.3 that $\mathrm{diam}(G_{2^k+3}) = 2$. For $k > m \ge 2$, let $V_1 = \{i \in V(G_n) \mid 1 \le i \le 2^k+1\}$ and $V_2 = \{j \in V(G_n) \mid 2^k+1 \le j \le n\}$. Since $\langle V_1\rangle \cong G_{2^k+1}$ and $\langle V_2\rangle \cong G_{2^m+1}$ by Lemma 2.5, it follows from (iv) of Lemma 1.3 that $\mathrm{diam}(\langle V_1\rangle) = \mathrm{diam}(\langle V_2\rangle) = 2$. Let $i \in V_1 \backslash \{2^k+1\}$ and $j \in V_2 \backslash \{2^k+1\}$. Now it is enough to show that $d(i,j) \le 3$. Since the vertices 2^k+1 and 2^k+2^m+1 are the universal vertices in $\langle V_1\rangle$ and $\langle V_2\rangle$ respectively, we obtain

$$\begin{aligned} d_{G_n}(i,j) &\le d_{G_n}(i, 2^k+1) + d_{G_n}(2^k+1, 2^k+2^m+1) + d_{G_n}(2^k+2^m+1, j) \\ &\le d_{\langle V_1\rangle}(i, 2^k+1) + d_{\langle V_2\rangle}(2^k+1, 2^k+2^m+1) + d_{\langle V_2\rangle}(2^k+2^m+1, j) \\ &\le 3. \end{aligned}$$

Thus the theorem holds for $s = 0$.

Let $s \geq 1$, i.e. $n = 1 + 2^m + \sum_{j=0}^{s} 2^{k+j}$. For $k > m \geq 1$, let $W_1 = \{i \in V(G_n) \mid 1 \leq i \leq 2^{k+s} + 1\}$ and $W_2 = \{j \in V(G_n) \mid 2^{k+s} + 1 \leq j \leq n\}$. Since by Lemma 2.5 we obtain $\langle W_1 \rangle \cong G_{2^{k+s}+1}$ and $\langle W_2 \rangle \cong G_{n-2^{k+s}}$, by (iv) of Lemma 1.3 we obtain $\mathrm{diam}(\langle W_1 \rangle) = 2$ and by induction we obtain $\mathrm{diam}(\langle W_2 \rangle) \leq s+1$ if $m = 1$ or $\mathrm{diam}(\langle W_2 \rangle) \leq s+2$ if $k > m > 1$. Let $i \in W_1 \backslash \{2^{k+s}+1\}$ and $j \in W_2 \backslash \{2^{k+s}+1\}$. Now it is enough to show that $d_{G_n}(i,j) \leq s+2$ if $m = 1$ or $d_{G_n}(i,j) \leq s+3$ if $k > m > 1$. Since the vertices $2^{k+1}+1$ are the universal vertices in $\langle W_1 \rangle$, we obtain

$$d_{G_n}(i,j) \leq d_{G_n}(i, 2^{k+s}+1) + d_{G_n}(2^{k+s}+1, j) \leq 1 + d_{G_{n-2^s}}(2^{k+s}+1, j).$$

Hence, by induction, we obtain the desired result. □

From Lemma 2.9, the following theorem shows that the upper bound of Conjecture 1 is true if $n = 1 + 2^m + 2^k$ for $k > m \geq 1$.

Theorem 2.10. *Let k and m be integers with $k > m \geq 1$. Then*

$$\mathrm{diam}(CG_{1+2^m+2^k}) = \begin{cases} 2 & \text{if } m = 1; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Since by Lemma 2.9 we obtain $\mathrm{diam}(CG_{2^k+3}) = 2$, it is enough to show that $\mathrm{diam}(CG_{1+2^m+2^k}) = 3$ for $k > m > 1$. Now let k and m be integers with $k > m > 1$. By Lemma 2.2, the reverse relabelling of the Catalan graph $CG_{2^k+2^m+1}$ is

$$CG^r_{1+2^m+2^k} = G_{1+2^m+2^k}((1-z)^{-1-2^m-2^k}, z + z^2). \tag{5}$$

Let $\mathcal{A}(CG_{1+2^m+2^k}) = [c_{i,j}]$ and $\mathcal{A}(CG^r_{1+2^m+2^k}) = [r_{i,j}]$. By (5), we obtain

$$c_{2^m+2^k,j} = r_{2^m+2^k+2-j,2} \equiv \begin{cases} 1 & \text{if } j = 2^k + 2^m + 1; \\ 0 & \text{if } j = 2^k + 2^m; \\ [z^{2^k+2^m-j}]z(1-z)^{-2^k-2^m} & \text{otherwise.} \end{cases} \tag{6}$$

Since

$$[z^{2^k+2^m-j}]z(1-z)^{-2^k-2^m} = \binom{2^{k+1}+2^{m+1}-j-2}{2^k+2^m-1},$$

by Lucas's theorem we obtain for $j = 1, \ldots, 2^k + 2^m - 1$

$$\begin{aligned} c_{2^k+2^m,j} &\equiv \binom{2^{k+1}+2^{m+1}-j-2}{2^k+2^m-1} \\ &\equiv \begin{cases} 1 & \text{if } j \in \{2^{m+1}+t2^m-1 \mid t = 0, \ldots, 2^{k-m}-1\}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

By (6) and (7), the set $N(2^k + 2^m)$ of neighbors of the vertex $2^k + 2^m$ in $CG_{2^k+2^m+1}$ is

$$N(2^k + 2^m) = \{2^{m+1} + t2^m - 1 \mid t = 0, \ldots, 2^{k-m} - 1\} \cup \{2^k + 2^m + 1\}.$$

It is known [8] that $[z^n]C(z) \equiv 1$ if and only if $n = 2^k - 1$ for $k \geq 1$. It implies

$$c_{i,1} = \begin{cases} 1 & \text{if } j \in \{2^s + 1 \mid s = 0, 1, \ldots, k\}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus the set $N(1)$ of neighbors of the vertex 1 in $CG_{2^k+2^m+1}$ is

$$N(1) = \{2^s + 1 \mid s = 0, \ldots, k\}.$$

Since

$$2^k + 2^m \notin N(1), \quad 1 \notin N(2^k + 2^m) \ \text{ and } \ N(1) \cap N(2^k + 2^m) = \emptyset,$$

the distance between vertices 1 and $2^k + 2^m$ in $CG_{1+2^m+2^k}$ is at least 3 so that by Lemma 2.9 we obtain $\text{diam}(CG_{1+2^m+2^k}) = 3$. Hence the proof follows. □

We end this section with the following conjecture.

Conjecture 3. *Let $n = 1 + 2^m + \sum_{j=0}^{s} 2^{k+j}$ be an integer with $k > m \geq 1$ and $s \geq 1$. Then*

$$\text{diam}(CG_n) = \begin{cases} s + 2 & \text{if } m = 1; \\ s + 3 & \text{otherwise.} \end{cases}$$

Remark 2.11. *If Conjecture 3 is true, then by Lemma 2.9 the upper bound of Conjecture 1 is true if $n = 1 + 2^m + \sum_{j=0}^{s} 2^{k+j}$ for $k > m \geq 1$ and $s \geq 1$. By using the sage, we have checked that Conjecture 3 is true for $n \leq 2^8$.*

3 Conjecture 2

In this section, we show that Conjecture 2 is true for some special io-decomposable Riordan graphs of the Bell type.

Lemma 3.1. *Let $G_n = G_n(g, zg)$ be an io-decomposable Riordan graph. If there exists $k \geq 2$ such that diam$(G_{2^k}) = s$ then diam$(G_{2^{k+m}}) \leq s + m$ for all $m \geq 1$.*

Proof. Let $V_1 = \{i \in V(G_n) \mid 1 \leq i \leq 2^{k+m-1} + 1\}$ and $V_2 = \{j \in V(G_n) \mid 2^{k+m-1} + 1 \leq j \leq 2^{k+m}\}$ be the vertex subsets of $V(G_{2^{k+m}})$. Since $\langle V_1 \rangle \cong G_{2^{k+m-1}+1}$ has a universal vertex $2^{k+m-1} + 1$ and by Lemma 2.5 we obtain $\langle V_2 \rangle \cong G_{2^{k+m-1}}$, we obtain

$$\text{diam}(G_{2^{k+m}}) \leq \text{diam}(\langle V_2 \rangle) + 1 = \text{diam}(G_{2^{k+m-1}}) + 1. \tag{8}$$

Let $\mathrm{diam}(G_{2^k}) = s$. Applying for $m = 1$ in (8), we obtain $\mathrm{diam}(G_{2^{k+1}}) \leq s + 1$. Applying again for $m = 2$ in (8), we obtain $\mathrm{diam}(G_{2^{k+2}}) \leq s + 2$. By repeating this process, we obtain the desired result. $\square$

Let $\mathcal{B}(g, f)$ denote a binary Riordan matrix, i.e. $\mathcal{B}(g, f) \equiv (g, f)$. We note that a Riordan matrix $[b_{i,j}]_{i,j\geq 0}$ is of the Bell type given by $\mathcal{B}(g, zg)$ with $g(0) = 1$ if and only if, for $i \geq j \geq 0$,

$$\begin{aligned} b_{i+1,0} &\equiv a_1 b_{i,0} + a_2 b_{i,1} + \cdots + a_{i+1} b_{i,i}, \\ b_{i+1,j+1} &\equiv b_{i,j} + a_1 b_{i,j+1} + \cdots + a_{i-j} b_{i,i} \end{aligned} \tag{9}$$

where $(1, a_1, \ldots)$ is the binary A-sequence of $\mathcal{B}(g, zg)$. Let $G_n = G_n(g, zg)$ and $\mathcal{A}(G_n) = [r_{i,j}]_{1\leq i,j\leq n}$ where $g(0) = 1$. Since $r_{i,j} = b_{i-2,j-1}$ for $i > j \geq 1$, by (9) we need the finite term $(1, a_1, \ldots, a_{n-2})$ of the binary A-sequence to determine $\mathcal{A}(G_n)$.

Theorem 3.2. *Let $G_{2^k} = G_{2^k}(g, zg)$ be an io-decomposable Riordan graph. If the binary A-sequence of (g, zg) is of the following form*

$$(\underbrace{1, 1, \ldots, 1}_{2^m - 2 \text{ copies}}, 0, 0, a_{2^m}, a_{2^m}, a_{2^m+2}, a_{2^m+2}, \ldots), \quad a_j \in \{0, 1\}, \ m \geq 4 \tag{10}$$

then for $k \geq 4$ we obtain

$$\mathit{diam}(G_{2^k}) < \mathit{diam}(CG_{2^k}) = k.$$

Proof. First we show that $\mathrm{diam}(G_{2^m}) = m - 1$. Since the induced subgraph H of $\{1, 2 \ldots, 2^m - 1\}$ in G_{2^m} is $H = CG_{2^m-1}$ and $CG^r_{2^m-1} = G_{2^m-1}(1 + z, z + z^2)$, the $(2^m - 1)$th row of $\mathcal{A}(G_{2^m}) = [r_{i,j}]$ is given by

$$(0, \ldots, 0, 1, 1, 0, 1) = (r_{2^m-1,i})_{i=1}^{2^m}. \tag{11}$$

By (9), (10) and (11), the 2^mth row in $\mathcal{A}(G_{2^m}) = [r_{i,j}]$ is given by

$$(1, 0, \ldots, 0, 1, 0) = (r_{2^m,i})_{i=1}^{2^m}$$

which means the only two vertices 1 and $2^m - 1$ are adjacent to the vertex 2^m in G_{2^m}. Let $V_1 = \{1, \ldots, 2^{m-1} + 1\}$ and $V_2 = \{2^{m-1} + 1, \ldots, 2^m - 1\}$ be the vertex subsets of $V(G_{2^m})$. Since $\langle V_1 \rangle$ has the universal vertex $2^{m-1} + 1$ and $\langle V_2 \rangle \cong CG_{2^{m-1}-1}$, if $v_1 \in V_1$ and $v_2 \in V_2$ then we respectively obtain $d_{G_{2^m}}(v_1, 2^m) \leq 3$ and

$$d_{G_{2^m}}(v_2, 2^m) \leq \mathrm{diam}(CG_{2^{m-1}-1}) + 1 \leq 2^m - 1 \quad \text{(by Corollary 2.8)}$$

which implies $\mathrm{diam}(G_{2^m}) = m - 1$. Hence, by Lemma 3.1, we obtain the desired result. $\square$

A-seq. of G_8	diam(G_8)	A-seq. of G_8	diam(G_8)
$(1,1,0,0,0,0,0)$	2	$(1,1,1,1,0,0,1)$	2
$(1,1,1,1,0,0,0)$	2	$(1,1,0,0,1,1,1)$	2
$(1,1,0,0,1,1,0)$	2	$(1,1,1,1,1,1,0)$	3
$(1,1,0,0,0,0,1)$	2	$(1,1,1,1,1,1,1)$	3
$(1,1,1,1,1,1,0)$	2		

Table 1 Diameters of io-decomposable Riordan graphs of the Bell type with degree 8

A-seq. of G_{16}	diam(G_{16})	A-seq. of G_{16}	diam(G_{16})
$(1,1,1,1,1,1,0,0,0,0,0,0,0,0,0)$	3	$(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0)$	3
$(1,1,1,1,1,1,0,0,1,1,0,0,0,0,0)$	3	$(1,1,1,1,1,1,1,1,1,1,0,0,0,0,0)$	3
$(1,1,1,1,1,1,0,0,0,0,1,1,0,0,0)$	3	$(1,1,1,1,1,1,1,1,0,0,1,1,0,0,0)$	3
$(1,1,1,1,1,1,0,0,0,0,0,0,1,1,0)$	3	$(1,1,1,1,1,1,1,1,0,0,0,0,1,1,0)$	3
$(1,1,1,1,1,1,0,0,0,0,0,0,0,0,1)$	3	$(1,1,1,1,1,1,1,1,0,0,0,0,0,0,1)$	3
$(1,1,1,1,1,1,0,0,1,1,1,1,0,0,0)$	3	$(1,1,1,1,1,1,1,1,1,1,1,1,0,0,0)$	3
$(1,1,1,1,1,1,0,0,1,1,0,0,1,1,0)$	3	$(1,1,1,1,1,1,1,1,1,1,0,0,1,1,0)$	3
$(1,1,1,1,1,1,0,0,1,1,0,0,0,0,1)$	3	$(1,1,1,1,1,1,1,1,1,1,0,0,0,0,1)$	3
$(1,1,1,1,1,1,0,0,0,0,1,1,1,1,0)$	3	$(1,1,1,1,1,1,1,1,0,0,1,1,1,1,0)$	3
$(1,1,1,1,1,1,0,0,0,0,1,1,0,0,1)$	3	$(1,1,1,1,1,1,1,1,0,0,1,1,0,0,1)$	3
$(1,1,1,1,1,1,0,0,1,1,1,1,1,1,0)$	3	$(1,1,1,1,1,1,1,1,0,0,0,0,1,1,1)$	3
$(1,1,1,1,1,1,0,0,1,1,0,0,1,1,1)$	3	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)$	3
$(1,1,1,1,1,1,0,0,1,1,1,1,0,0,1)$	3	$(1,1,1,1,1,1,1,1,1,1,1,1,0,0,1)$	3
$(1,1,1,1,1,1,0,0,1,1,1,1,1,1,0)$	3	$(1,1,1,1,1,1,1,1,1,1,0,0,1,1,1)$	3
$(1,1,1,1,1,1,0,0,1,1,1,1,1,1,1)$	3	$(1,1,1,1,1,1,1,1,0,0,1,1,1,1,1)$	3
		$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	4

Table 2 Diameters of io-decomposable Riordan graphs of the Bell type with degree 16 such that the first 6 entries of its A-sequence are all 1s

By Lemma 3.1, using the results in Table 1 and 2 we obtain the following theorem.

Theorem 3.3. *For $k \geq 4$, let $G_{2^k} = G_{2^k}(g, zg)$ be an io-decomposable Riordan graph and $G_{2^k} \ncong CG_{2^k}$. If the first 16 entries in the binary A-sequence of (g, zg) are not all 1s then*

$$\mathit{diam}(G_{2^k}) < \mathit{diam}(CG_{2^k}) = k.$$

References

[1] G.-S. Cheon, I.-C. Huang, S. Kim, *Multivariate Riordan groups and their representations*, Linear Algebra Appl. 514 (2017), 198–207.

[2] G.-S. Cheon, S.-T. Jin, *The group of multi-dimensional Riordan arrays*, Linear Algebra Appl. 524 (2017), 263–277.

[3] G.-S. Cheon, S.-T. Jin, H. Kim, L. W. Shapiro, *Riordan group involutions and the Δ-sequence*, Discrete Appl. Math. 157 (2009), 1696–1701

[4] G.-S. Cheon, J.-H. Jung, S. Kitaev, S. A. Mojallal, *Riordan graphs I: Structural properties*, arXiv:1710.04604 [math.CO].

[5] G.-S. Cheon, J.-H. Jung, S. Kitaev, S. A. Mojallal, *Riordan graphs II: Spectral properties*, arXiv:1801.07021 [math.CO].

[6] G.-S. Cheon, H. Kim, L. W. Shapiro, *Combinatorics of Riordan arrays with identical A and Z sequences*, Discrete Math. 312 (2012), 2040–2049.

[7] G.-S. Cheon, A. Luzón, M. A. Morón, L. F. Prieto-Martinez, M. Song, *Finite and infinite dimensional Lie group structures on Riordan groups*, Advances in Mathematics 319 (2017), 522–566.

[8] E. Deutsch, B. E. Sagan, *Congruences for Catalan and Motzkin numbers and related sequences*, J. Number Theory 117 (2006), 191–215.

[9] H. Kim, R. P. Stanley, *A refined enumeration of hex trees and related polynomials*, Eur. J. Combin. 54 (2016), 207–219.

[10] Ph.G. Kolaitis, H.J. Promel, B.L. Rothschild, *$K_{\ell+1}$-free graphs: asymptotic structure and a 0-1 law*, Trans. Amer. Math. Soc. 303 (1987), 637–671.

[11] D. Merlini, R. Sprugnoli, *Algebraic aspects of some Riordan arrays related to binary words avoiding a pattern*, Theoretical Computer Science 412(27) (2011), 2988–3001.

[12] D. Merlini, D.G. Rogers, R. Sprugnoli, M.C. Verri, *On some alternative characterizations of Riordan arrays*, Canad. J. Math. 49 (1997), 301–320.

[13] L.W. Shapiro, S. Getu, W.-J. Woan, L. Woodson, *The Riordan group*, Discrete Appl. Math. 34 (1991), 229–239.

[14] R. Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math. 132 (1994), 267–290.