Random zonotopes and valuations

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Abstract

We define a random zonotope in $\mathbb{R}^d$, by adding finitely many random segments, which are independently and identically distributed. For this random polytope, we determine, under a mild assumption on the distribution, the expectations of the intrinsic volumes, more generally, the expectations of suitable valuations. We also prove a central limit theorem for a valuation evaluated at these random zonotopes.

Keywords: random zonotope, intrinsic volume, valuation, U-statistic, central limit theorem

1 Introduction

In the stochastic geometry of $\mathbb{R}^d$, many constructions of random polytopes start with a finite number of independent, identically distributed random points. A common procedure is to take their convex hull. A first task will then be to determine the expectations of some prominent geometric functionals of the random polytope, for example, the numbers of k-dimensional faces or the intrinsic volumes. Often only asymptotic considerations, with the number of points or the dimension d tending to infinity, lead to explicit results. (Surveys on random polytopes are, in chronological order, [2], [15], [20, Sect. 8.2], [16], [14], [6], [18], [3, Sect. 1.4]). In the present note, we do not study the convex hull, but the Minkowski sum Z_p of p segments with endpoints at the origin o and at the random points. Thus, we consider random zonotopes. (Surveys on zonotopes and on zonoids, their limits in the Hausdorff metric, are [19] and [5].) In this case, the numbers of k-dimensional faces are not a problem, since under weak assumptions on the distribution of the random points they are almost surely constant. We refer to Donoho and Tanner [4, (1.6)] and the references given there, although the formulation there is slightly different (but equivalent).

In contrast, the expectations of the intrinsic volumes of the random zonotope Z_p will strongly depend on the given distribution. The intrinsic volumes are of particular interest, since they include the most common size measurements of convex bodies, namely volume, surface area, and mean width (up to normalizing constants). We show in this note that the expectations of the intrinsic volumes of Z_p can be expressed via a deterministic zonoid constructed from the distribution. In fact, the expectations of more general functionals than the intrinsic volumes can be made explicit in this way.

To be more precise, let $\mathbb{R}^d$ be the d-dimensional Euclidean vector space, with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. By $\mathcal{K}^d$ we denote the space of convex bodies (nonempty, compact, convex subsets) of $\mathbb{R}^d$, equipped with the Hausdorff metric. We recall, e.g. from [17], that the intrinsic volumes $V_0(K), \ldots, V_d(K)$ of a convex body $K \in \mathcal{K}^d$ can be defined via the Steiner formula

$$V_d(K + \rho B^d) = \sum_{j=0}^{d} \rho^{d-j} \kappa_{d-j} V_j(K).$$

Here V_d denotes the volume, $\rho \geq 0$ can be any real number, B^d is the unit ball in $\mathbb{R}^d$, and κ_d is its volume.

The intrinsic volume V_j is an element of **Val**, the real vector space of translation invariant, continuous real valuations on the space $\mathcal{K}^d$. Recall that $\varphi : \mathcal{K}^d \to \mathbb{R}$ is a valuation if $\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$ whenever $K, L, K \cup L \in \mathcal{K}^d$. (Surveys on valuations are [11], [10], [17, Chap. 6].) By Hadwiger's theorem, the intrinsic volumes form a basis of the subspace of rigid motion invariant valuations. The space **Val** is much bigger, it has infinite dimension. By a theorem of P. McMullen, $\mathbf{Val} = \bigoplus_{j=0}^{d} \mathbf{Val}_j$, where $\mathbf{Val}_j$ is the subspace of valuations that are homogeneous of degree j. For example, each function

$$K \mapsto V(\underbrace{K, \ldots, K}_{j}, L_{j+1}, \ldots, L_d),$$

where V denotes the mixed volume in $\mathbb{R}^d$ and $L_{j+1}, \ldots, L_d$ are fixed convex bodies, belongs to $\mathbf{Val}_j$. The jth intrinsic volume is obtained, up to a a normalizing factor, if $L_{j+1}, \ldots, L_d$ are unit balls. The elements of $\mathbf{Val}_j$ appear in the subsequent Theorem 1 (only for $j \geq 1$, since $\mathbf{Val}_0$ contains only constants).

For $x \in \mathbb{R}^d$, we denote by $\overline{x}$ the closed segment with endpoints x and o.

Let X be a random vector in $\mathbb{R}^d$. Of the distribution of X, we assume that $\mathbb{E}\|X\| < \infty$. For $p \in \mathbb{N}$, let $X_1, \ldots, X_p$ be i.i.d. copies of X, and define

$$Z_p := \frac{1}{p}\left(\overline{X}_1 + \cdots + \overline{X}_p\right),$$

where the sum is Minkowski (or vector) addition.

The random set $\overline{X}$ is integrably bounded and hence has a selection expectation $\mathbb{E}\,\overline{X}$ (we refer to Molchanov [12], in particular Theorem 1.22 of Chapter 2). We denote this selection expectation by Z_X. We recall that the support function of a convex body $K \in \mathcal{K}^d$ is defined by $h(K, u) := \max\{\langle x, u \rangle : x \in K\}$ for $u \in \mathbb{R}^d$. It is known that the support function of the selection expectation Z_X is given by

$$h(\mathsf{Z}_X, u) = \mathbb{E}\, h(\overline{X}, u) = \int_{\mathbb{R}^d} h(\overline{x}, u)\, \mathbb{P}_X(\mathrm{d}x) \quad \text{for } u \in \mathbb{R}^d,$$

where $\mathbb{P}_X$ denotes the distribution of X. Thus, Z_X can be approximated by finite sums of segments and hence is a zonoid. We remark that Z_X occurs also as the 'zonoid of a measure'; see Mosler [13], in particular Theorem 2.8.

Theorem 1. *Let $j \in \{1, \ldots, d\}$, and let $p \geq j$ be an integer. If $\varphi \in \mathbf{Val}_j$, then*

$$\mathbb{E}\,\varphi(Z_p) = \frac{p!}{p^j(p-j)!}\,\varphi(\mathsf{Z}_X). \tag{1}$$

We mention that the role of the zonoid Z_X can be seen in a vague analogy to that of Matheron's [9] 'Steiner compact' (called the 'associated zonoid' in [20]) in the theory of Poisson hyperplane tessellations. The common aspect is that the distribution of a random object is governed by a measure, and this measure defines a deterministic zonoid. This zonoid can then be used to express certain expectations defined by the random object.

We prove Theorem 1 in Section 3, after explaining in the next section the ideas of Vitale on which it is based. Section 4 contains a central limit theorem for $\varphi(Z_n)$, as n tends to infinity.

2 A result of Richard A. Vitale

The following result was proved by Vitale [22].

Theorem 2. *If $X_1, \ldots, X_d$ are i.i.d. copies of X, then*

$$\mathbb{E}\, V_d(\overline{X}_1 + \cdots + \overline{X}_d) = d!V_d(\mathsf{Z}_X). \tag{2}$$

Vitale's formulation is slightly different, since he considers not the volume of the sum of d random segments (and thus of a random parallelepiped), but the absolute determinant of a $d \times d$ matrix whose columns are i.i.d. random. The formulations are clearly equivalent.

Vitale's result appears also as Theorem 2.10 in [13].

Vitale's proof is an elegant combination of two strong laws of large numbers. We recall his argument briefly, since we are going to extend it. Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d. copies of X. For each $n \in \mathbb{N}$, consider the random zonotope $Z_n := \frac{1}{n}(\overline{X}_1 + \cdots + \overline{X}_n)$. Since $\mathbb{E}\,\|X\| < \infty$, the strong law of Artstein and Vitale [1] can be applied, which yields that, with probability one,

$$Z_n \to \mathbb{E}\,\overline{X} = \mathsf{Z}_X \quad \text{as } n \to \infty, \tag{3}$$

where the convergence is in the Hausdorff metric. Since V_d is continuous on convex bodies, we get

$$\lim_{n \to \infty} V_d(Z_n) = V_d(\mathsf{Z}_X) \quad \text{a.s.} \tag{4}$$

We have (see, e.g., [17, p. 304])

$$V_d(Z_n) = \frac{1}{n^d} \sum_{1 \le i_1 < \cdots < i_d \le n} V_d(\overline{X}_{i_1} + \cdots + \overline{X}_{i_d}).$$

Here, $V_d(\overline{X}_{i_1} + \cdots + \overline{X}_{i_d}) \le \|X_{i_1}\| \cdots \|X_{i_d}\|$, hence the random variable $V_d(Z_n)$ has finite expectation. From Hoeffding's strong law of large numbers for U-statistics (see, e.g., Serfling [21, Chap. 5]) it follows that, with probability one,

$$\binom{n}{d}^{-1} \sum_{1 \le i_1 < \cdots < i_d \le n} V_d(\overline{X}_{i_1} + \cdots + \overline{X}_{i_d}) \to \mathbb{E}\, V_d(\overline{X}_1 + \cdots + \overline{X}_d) \quad \text{as } n \to \infty,$$

hence

$$\lim_{n \to \infty} V_d(Z_n) = \frac{1}{d!}\mathbb{E}\, V_d(\overline{X}_1 + \cdots + \overline{X}_d) \quad \text{a.s.} \tag{5}$$

From (4) and (5), Vitale's result (2) follows. We remark that similar arguments were used in [23].

3 An extension

Since we have given Vitale's result a geometric formulation, we can extend it in the way described in Theorem 1. The proof depends on a polynomiality property of the elements of $\mathbf{Val}_j$, which goes back to P. McMullen.

Let $j \in \{1, \ldots, d\}$ and $\varphi \in \mathbf{Val}_j$. Then there exists a continuous symmetric mapping $\phi : (\mathcal{K}^d)^j \to \mathbb{R}$ which is translation invariant and Minkowski additive in each variable, such that

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{r_1, \ldots, r_n = 0}^{j} \binom{j}{r_1 \ldots r_n} \lambda_1^{r_1} \cdots \lambda_n^{r_n} \phi(K_1[r_1], \ldots, K_n[r_n]) \tag{6}$$

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for all $K_1, \ldots, K_n \in \mathcal{K}^d$, all $\lambda_1, \ldots, \lambda_n \geq 0$ and all $n \in \mathbb{N}$. Here the multinomial coefficient is, by definition, equal to 0 if $r_1 + \cdots + r_n \neq j$. The bracket $[r_i]$ indicates that the preceding argument is repeated r_i times. We refer to [17, Thm. 6.3.6], and to the subsequent note for references and some history.

Suppose that all K_i in (6) are segments. Consider a summand on the right-hand side where some r_i is greater than 1, say $r_1 \geq 2$. This summand contains an expression $\phi(\overline{x}_1, \overline{x}_1, \overline{y}_3, \ldots, \overline{y}_j)$ with $x_1, y_3, \ldots, y_j \in \mathbb{R}^d$. For arbitrary $\lambda_1, \ldots, \lambda_j \geq 0$ we have

$$\dim\left(\lambda_1 \overline{x}_1 + \lambda_2 \overline{x}_1 + \lambda_3 \overline{y}_3 + \cdots + \lambda_j \overline{y}_j\right) < j,$$

hence $\varphi(\lambda_1 \overline{x}_1 + \lambda_2 \overline{x}_1 + \lambda_3 \overline{y}_3 + \cdots + \lambda_j \overline{y}_j) = 0$, by [17, Cor. 6.3.2] and the continuity of φ. Therefore, it follows from (6) that $\phi(\overline{x}_1, \overline{x}_1, \overline{y}_3, \ldots, \overline{y}_j) = 0$. This means that for arbitrary segments $\overline{x}_1, \ldots, \overline{x}_n$ we have

$$\phi(\overline{x}_1[r_1], \ldots, \overline{x}_n[r_n]) = 0$$

whenever $r_1 + \cdots + r_n = j$ and at least on r_i is greater than one. Therefore, if (6) is applied to segments, then on the right-hand side only the summands with $r_i \in \{0, 1\}$ can be different from zero. It follows that for segments $\overline{x}_1, \ldots, \overline{x}_n$, relation (6) can be simplified to

$$\varphi(\overline{x}_1 + \cdots + \overline{x}_n) = j! \sum_{1 \leq i_1 < \cdots < i_j \leq n} \phi(\overline{x}_{i_1}, \ldots, \overline{x}_{i_j}). \tag{7}$$

Let p be an integer with $j \leq p \leq n$ and consider

$$A_p := \sum_{1 \leq i_1 < \cdots < i_p \leq n} \varphi(\overline{x}_{i_1} + \cdots + \overline{x}_{i_p}).$$

(Only the case $p \geq j$ is considered, since $\varphi(\overline{x}_1 + \cdots + \overline{x}_p) = 0$ for $p < j$.) By (7) (with $\overline{x}_1 + \cdots + \overline{x}_n$ replaced by $\overline{x}_{i_1} + \cdots + \overline{x}_{i_p}$) we get

$$A_p = j! \sum_{1 \leq i_1 < \cdots < i_p \leq n} \sum_{\substack{k_1 < \cdots < k_j \\ k_1, \ldots, k_j \in \{i_1, \ldots, i_p\}}} \phi(\overline{x}_{k_1}, \ldots, \overline{x}_{k_j}).$$

A given ordered j-tuple $k_1 < \cdots < k_j$ appears in the summation as often as one can choose a set of $p - j$ distinct indices from $\{1, \ldots, n\} \setminus \{k_1, \ldots, k_j\}$, that is, $\binom{n-j}{p-j}$ times. In view of (7) it follows that $A_p = \binom{n-j}{p-j}\varphi(\overline{x}_1 + \cdots + \overline{x}_n)$ and thus

$$\varphi(\overline{x}_1 + \cdots + \overline{x}_n) = \binom{n-j}{p-j}^{-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} \varphi(\overline{x}_{i_1} + \cdots + \overline{x}_{i_p}). \tag{8}$$

Now, as in Section 2, let $X_1, X_2, \ldots$ be i.i.d. copies of X and define

$$Z_n := \frac{1}{n}(\overline{X}_1 + \cdots + \overline{X}_n).$$

Then (8) implies that

$$\varphi(Z_n) = \frac{1}{n^j} \binom{n-j}{p-j}^{-1} \binom{n}{p} U_n^{(p)}(h), \tag{9}$$

where

$$U_n^{(p)}(h) := \binom{n}{p}^{-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} h(X_{i_1}, \ldots, X_{i_p})$$

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is a U-statistic of order p with kernel function $h : (\mathbb{R}^d)^p \to \mathbb{R}$ defined by

$$h(x_1, \ldots, x_p) := \varphi(\overline{x}_1 + \cdots + \overline{x}_p)$$

(and applied to the i.i.d. sequence $X_1, X_2, \ldots$).

Now we remark that the function ϕ is in each variable Minkowski additive and continuous, hence positively homogeneous of degree 1. Therefore,

$$\phi(\overline{X}_1, \ldots, \overline{X}_j) = \|X_1\| \cdots \|X_j\| \phi(s_1, \ldots, s_j)$$

with random segments $s_1, \ldots, s_j$ having endpoints at o and at unit vectors. Since ϕ is continuous, there is a constant c, depending only on ϕ, such that $|\phi(s_1, \ldots, s_j)| \leq c$. Since $\mathbb{E}\|X\| < \infty$ and $X_1, \ldots, X_j$ are stochastically independent, we deduce that $\mathbb{E}|\phi(\overline{X}_1, \ldots, \overline{X}_j)| < \infty$. From (7) we now conclude that $\mathbb{E}|h(X_1, \ldots, X_p)| = \mathbb{E}|\varphi(\overline{X}_1 + \cdots + \overline{X}_p)| < \infty$.

Therefore, the strong law for U-statistics (see, e.g., Serfling [21, p. 190] or Lee [8, p. 122]) can be applied. It yields that, with probability one,

$$\lim_{n \to \infty} U_n^{(p)}(h) = \mathbb{E}\, h(X_1, \ldots, X_p) = \mathbb{E}\, \varphi(\overline{X}_1 + \cdots + \overline{X}_p) = p^j\, \mathbb{E}\, \varphi(Z_p).$$

Therefore (9), together with

$$\lim_{n \to \infty} \frac{1}{n^j} \binom{n-j}{p-j}^{-1} \binom{n}{p} = \frac{(p-j)!}{p!},$$

gives

$$\lim_{n \to \infty} \varphi(Z_n) = \frac{(p-j)!}{p!} p^j\, \mathbb{E}\, \varphi(Z_p) \quad \text{a.s.}$$

By (3) and the continuity of φ, we have

$$\lim_{n \to \infty} \varphi(Z_n) = \varphi(\mathsf{Z}_X) \quad \text{a.s.} \tag{10}$$

Both results together yield the value for the expectation $\mathbb{E}\, \varphi(Z_p)$ as stated in Theorem 1.

4 A central limit theorem

Besides the strong law of large numbers expressed by (10), we may also state a central limit theorem for $\varphi(Z_n)$. For this, we need to compute a variance in our present case. We consider the kernel function h for $p = j$, that is, $h(x_1, \ldots, x_j) = \varphi(\overline{x}_1 + \cdots + \overline{x}_j)$. We assume now that $\mathbb{E}\|X\|^2 < \infty$. Then we can conclude as in Section 2, using (7), that

$$\mathbb{E}\, h^2(X_1, \ldots, X_j) = \mathbb{E}\,(j! \phi(\overline{X}_1, \ldots, \overline{X}_j))^2 = (j!)^2 \mathbb{E}\left(\|X_1\|^2 \cdots \|X_j\|^2 \phi(s_1, \ldots, s_j)^2\right)$$

with segments $s_1, \ldots, s_j$ of unit length, and hence that $\mathbb{E}\, h^2(X_1, \ldots, X_j) < \infty$.

Adopting the notation of [21], we write

$$\theta := \mathbb{E}\, h(X_1, \ldots, X_j) = \mathbb{E}\, \varphi(\overline{X}_1 + \cdots + \overline{X}_j) = j^j\, \mathbb{E}\, \varphi(Z_j) = j! \varphi(\mathsf{Z}_X)$$

and, for $x_1 \in \mathbb{R}^d$,

$$h_1(x_1) := \mathbb{E}\, h(x_1, X_2, \ldots, X_j) = \mathbb{E}\, \varphi(\overline{x}_1 + \overline{X}_2 + \cdots + \overline{X}_j),$$

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further $\widetilde{h}_1 := h_1 - \theta$ and

$$\zeta_1 := \mathbb{E}\,\widetilde{h}_1^2(X_1).$$

Now we define a random zonotope by

$$Z_n(x_1) := \overline{x}_1 + \frac{1}{n}(\overline{X}_2 + \cdots + \overline{X}_n) \overset{d}{=} \overline{x}_1 + \frac{n-1}{n} Z_{n-1}$$

for $x_1 \in \mathbb{R}^d$. Then by (3), with probability one,

$$Z_n(x_1) \to \overline{x}_1 + \mathbb{E}\,\overline{X} = \overline{x}_1 + \mathsf{Z}_X \quad \text{as } n \to \infty$$

and hence

$$\lim_{n \to \infty} \varphi(Z_n(x_1)) = \varphi(\overline{x}_1 + \mathsf{Z}_X) \quad \text{a.s.} \tag{11}$$

We write (8) for $p = j$ in the form

$$\varphi(\overline{x}_1 + \cdots + \overline{x}_n)$$
$$= \sum_{2 \le i_2 < \cdots < i_j \le n} \varphi(\overline{x}_1 + \overline{x}_{i_2} + \cdots + \overline{x}_{i_j}) + \sum_{2 \le i_1 < \cdots < i_j \le n} \varphi(\overline{x}_{i_1} + \cdots + \overline{x}_{i_j}).$$

This gives

$$\varphi(Z_n(x_1))$$
$$= \sum_{2 \le i_2 < \cdots < i_j \le n} \varphi\left(\overline{x}_1 + \frac{1}{n}\overline{X}_{i_2} + \cdots + \frac{1}{n}\overline{X}_{i_j}\right) + \sum_{2 \le i_1 < \cdots < i_j \le n} \varphi\left(\frac{1}{n}\overline{X}_{i_1} + \cdots + \frac{1}{n}\overline{X}_{i_j}\right)$$
$$= \frac{1}{n^{j-1}} \sum_{2 \le i_2 < \cdots < i_j \le n} \varphi\left(\overline{x}_1 + \overline{X}_{i_2} + \cdots + \overline{X}_{i_j}\right) + \frac{1}{n^j} \sum_{2 \le i_1 < \cdots < i_j \le n} \varphi\left(\overline{X}_{i_1} + \cdots + \overline{X}_{i_j}\right).$$

Here we have used that φ is homogeneous of degree j and that (7), together with the Minkowski-linearity of ϕ in each argument, implies that for segments $S_1, \ldots, S_j$ we have $\varphi(nS_1 + S_2 + \cdots + S_j) = n\varphi(S_1 + \cdots + S_j)$.

The result can be written as

$$\varphi(Z_n(x_1)) = \frac{1}{n^{j-1}} \binom{n-1}{j-1} U_{n-1}^{(j-1)}(g_{x_1}) + \frac{1}{n^j} \binom{n-1}{j} U_{n-1}^{(j)}(h)$$

with $g_{x_1}(x_2, \ldots, x_j) = \varphi(\overline{x}_1 + \overline{x}_2 + \cdots + \overline{x}_j)$. Therefore, the strong law for U-statistics gives that, with probability one,

$$\lim_{n \to \infty} \varphi(Z_n(x_1)) = \frac{1}{(j-1)!} \mathbb{E}\,\varphi(\overline{x}_1 + \overline{X}_2 + \cdots + \overline{X}_j) + \frac{1}{j!} \mathbb{E}\,\varphi(\overline{X}_1 + \cdots + \overline{X}_j)].$$

Together with (11) this yields

$$h_1(x_1) = (j-1)!\,[\varphi(\overline{x}_1 + \mathsf{Z}_X) - \varphi(\mathsf{Z}_X)].$$

The central limit theorem requires that $\zeta_1 > 0$. To achieve this, we need assumptions on the distribution of the random vector X and on the valuation φ.

Lemma 1. *Let $j \in \{1, \ldots, d\}$ and $\varphi \in \mathbf{Val}_j$. Suppose that the support of the distribution $\mathbb{P}_X$ of X contains the origin and is not contained in a $(j-1)$-dimensional linear subspace. Suppose further that $\varphi(K) \neq 0$ for convex bodies K with $\dim K \geq j$ (as it is satisfied for the jth intrinsic volume). Then $\zeta_1 > 0$.*

Proof. Since the distribution $\mathbb{P}_X$ of X is not concentrated on a $(j-1)$-dimensional linear subspace, the zonoid Z_X has dimension at least j. Assume that $\zeta_1 = 0$. Since h_1 is continuous, we then have $(h_1(x_1) - \theta)^2 = 0$ for all x_1 in the support of $\mathbb{P}_X$ and hence

$$\varphi(\overline{x}_1 + \mathsf{Z}_X) = (j+1)\varphi(\mathsf{Z}_X)$$

for these x_1. Since o is in the support of $\mathbb{P}_X$, this yields $\varphi(\mathsf{Z}_X) = 0$, a contradiction. $\qquad\square$

We can now formulate a central limit theorem.

Theorem 3. *Suppose that $\mathbb{E}\|X\|^2 < \infty$. Under the assumptions of Lemma 1,*

$$\sqrt{n}\left(\varphi(Z_n) - \varphi(\mathsf{Z}_X)\right) \xrightarrow{d} \mathcal{N}(0, (j!j)^2\zeta_1).$$

Proof. We have seen above that $\mathbb{E}\|X\|^2 < \infty$ implies $\mathbb{E}\,h^2(X_1, \ldots, X_j) < \infty$. The assumptions of Lemma 1 imply that $\zeta_1 > 0$. Therefore, Hoeffding's central limit theorem for U-statistics (see [21, p. 192], [8, p. 76], [7, p. 128]) says that

$$\sqrt{n}\left(U_n^{(j)}(h) - \theta\right) \xrightarrow{d} \mathcal{N}(0, j^2\zeta_1).$$

Since

$$\varphi(Z_n) = \frac{1}{n^j}\binom{n}{j}U_n^{(j)}(h), \qquad \theta = j!\varphi(\mathsf{Z}_X),$$

Slutsky's theorem (e.g., [21, Sect. 1.5.4]) gives the assertion. $\qquad\square$

Finally, we want to show how in a simple case the expectation and variance appearing in Theorem 3 can be computed explicitly. We assume that X has a standard normal distribution and $\varphi = V_j$, the jth intrinsic volume, for some $j \in \{1, \ldots, d\}$.

Since $h(\overline{x}, u) + h(-\overline{x}, u) = |\langle x, u \rangle|$ and the distribution of X is invariant under reflection in the origin, we get

$$h(\mathsf{Z}_X, u) = \int_{\mathbb{R}^d} h(\overline{x}, u)\,\mathbb{P}_X(\mathrm{d}x) = \frac{1}{2\sqrt{2\pi}^d}\int_{\mathbb{R}^d}|\langle x, u\rangle|e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x.$$

Using polar coordinates and denoting the spherical Lebesgue measure on the unit sphere S^{d-1} by σ_{d-1}, we obtain

$$
\begin{aligned}
h(\mathsf{Z}_X, u) &= \frac{1}{2\sqrt{2\pi}^d}\int_{S^{d-1}}\int_0^\infty |\langle rv, u\rangle|e^{-\frac{1}{2}r^2}r^{d-1}\,\mathrm{d}r\,\sigma_{d-1}(\mathrm{d}v) \\
&= \frac{1}{2\sqrt{2\pi}^d}\int_{S^{d-1}}|\langle v, u\rangle|\,\sigma_{d-1}(\mathrm{d}v)\int_0^\infty e^{-\frac{1}{2}r^2}r^d\,\mathrm{d}r \\
&= \frac{1}{2\sqrt{2\pi}^d}\cdot\frac{\kappa_{d-1}}{2}\cdot 2^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right).
\end{aligned}
$$

Thus, Z_X is a ball with center o and radius

$$\mathsf{R} = \frac{1}{4\sqrt{2\pi}}.$$

It follows from the Steiner formula for the volume (e.g., [17, (4.1)]) that

$$V_j(\mathsf{Z}_X) = \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}} \mathsf{R}^j.$$

To compute the function h_1, given by

$$h_1(x) = (j-1)! \left[V_j(\overline{x} + \mathsf{Z}_X) - V_j(\mathsf{Z}_X) \right]$$

for $x \in \mathbb{R}^d$, we note that $\mathsf{Z}_X = \mathsf{R}B^d$ and

$$V_j(K + \mathsf{R}B^d) = \sum_{r=0}^{j} \binom{d-r}{j-r} \frac{\kappa_{d-r}}{\kappa_{d-j}} V_r(K) \mathsf{R}^{j-r}$$

for convex bodies K, as can be deduced from the Steiner formula for the volume. If $K = \overline{x}$, we have $V_0(K) = 1$, $V_1(K) = \|x\|$ and $V_r(K) = 0$ for $r \geq 2$. This gives

$$h_1(x) = \frac{(d-1)!}{(d-j)!} \frac{\kappa_{d-1}}{\kappa_{d-j}} \mathsf{R}^{j-1} \|x\|$$

and

$$\widetilde{h}_1(x) = h_1(x) - j! V_j(\mathsf{Z}_X) = a(\|x\| - b)$$

with

$$a = \frac{(d-1)!}{(d-j)!} \frac{\kappa_{d-1}}{\kappa_{d-j}} \mathsf{R}^{j-1}, \qquad b = \frac{d\kappa_d}{\kappa_{d-1}} \mathsf{R}.$$

Thus,

$$\zeta_1 = a^2 \left[\mathbb{E} \|X\|^2 - 2b \, \mathbb{E} \|X\| + b^2 \right]$$

with

$$\mathbb{E} \|X\|^2 = d, \qquad \mathbb{E} \|X\| = \frac{d}{\sqrt{2\pi}} \frac{\kappa_d}{\kappa_{d-1}},$$

which altogether yields an explicit expression for ζ_1.

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