

# THE CHROMATIC NUMBER OF RANDOM BORSUK GRAPHS

MATTHEW KAHLE AND FRANCISCO MARTINEZ-FIGUEROA

**ABSTRACT.** We study a model of random graph where vertices are  $n$  i.i.d. uniform random points on the unit sphere  $S^d$  in  $\mathbb{R}^{d+1}$ , and a pair of vertices is connected if the Euclidean distance between them is at least  $2 - \varepsilon$ . We are interested in the chromatic number of this graph as  $n$  tends to infinity.

It is not too hard to see that if  $\varepsilon > 0$  is small and fixed, then the chromatic number is  $d + 2$  with high probability. We show that this holds even if  $\varepsilon \rightarrow 0$  slowly enough. We quantify the rate at which  $\varepsilon$  can tend to zero and still have the same chromatic number. The proof depends on combining topological methods (namely the Lyusternik–Schnirelman–Borsuk theorem) with geometric probability arguments. The rate we obtain is best possible, up to a constant factor — if  $\varepsilon \rightarrow 0$  faster than this, we show that the graph is  $(d + 1)$ -colorable with high probability.

## 1. INTRODUCTION

Given  $\varepsilon > 0$  and  $d \geq 1$ , the Borsuk Graph  $\text{Bor}^d(\varepsilon)$  is the graph with vertex set corresponding to points on the  $d$ -dimensional unit sphere  $S^d \subset \mathbb{R}^{d+1}$  and edges  $\{x, y\}$  if and only if  $\|x - y\| > 2 - \varepsilon$ , that is, if the two points are  $\varepsilon$ -near to antipodal. Here distance is measured in the ambient Euclidean space  $\mathbb{R}^{d+1}$ . It is well known that when  $\varepsilon$  is sufficiently small, its chromatic number is  $d + 2$ , in fact this is equivalent to the Borsuk–Ulam theorem.

The Borsuk graph was part of Lovász’s inspiration for his proof of the Kneser conjecture [12]. Among other properties, this graph constitutes a nice example of a graph with large chromatic number and odd girth. See for example [21, 17, 5, 6]. It has also been studied because of its relation with Borsuk’s conjecture and distance graphs [19, 3, 18, 20].

We are interested in the chromatic number of random induced  $n$ -vertex subgraphs of the Borsuk graph. Our main point is that if  $\varepsilon \rightarrow 0$  slowly enough as  $n \rightarrow \infty$ , then topological lower bounds on chromatic number are tight. This contrasts with the situation studied by Kahle in [8], where topological lower bounds are not efficient for the chromatic number of Erdős–Rényi random graphs. Similar problems have also been studied for random Kneser graphs in [11] and [10].

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The rest of the paper is organized as follows. We finish this section with some definitions and notation. In section 2 we prove Theorem 1.1 when  $\varepsilon$  is fixed.

**Theorem 1.1.** Let  $d \geq 1$  and  $0 < \varepsilon < 2 - \lambda_d$  be fixed. Then a.a.s.

$$\chi(\text{Bor}^d(\varepsilon, n)) = d + 2.$$

In section 3 we prove Theorem 1.2, stating that the chromatic number is still the same when  $\varepsilon \rightarrow 0$  slowly.

**Theorem 1.2.** Let  $\varepsilon(n) = C \left( \frac{\log n}{n} \right)^{2/d}$ , where

$$C \geq \frac{64}{3} \left( \frac{3\pi^2}{4} \right)^{1/d}.$$

Then a.a.s.  $\chi(\text{Bor}^d(\varepsilon(n), n)) = d + 2$ .

Finally in section 3.2 we prove Theorem 1.3, showing that this rate is tight, up to a constant, in the sense that if  $\varepsilon \rightarrow 0$  faster, then the random Borsuk graph is  $(d + 1)$ -colorable, a.a.s.

**Theorem 1.3.** Let  $\varepsilon(n) = C(\log n/n)^{2/d}$ , where

$$C < \frac{3(4 - \lambda_d^2)}{64} \sqrt[d]{\frac{9}{4d^2}}.$$

Then a.a.s.  $\chi(\text{Bor}^d(\varepsilon(n), n)) \leq d + 1$ .

**Definition 1.4** (Random Borsuk graph). Given  $n \geq 1$ ,  $d \geq 1$ , and  $\varepsilon > 0$ , we define a *random Borsuk graph*,  $\text{Bor}^d(\varepsilon, n)$ , as follows.

- Its vertices are  $X_1, X_2, \dots, X_n$ ,  $n$  independent and identically distributed uniform random variables over the  $d$ -dimensional sphere  $S^d \subset \mathbb{R}^{d+1}$  of radius 1.
- $X_i$  and  $X_j$  for  $i \neq j$  are connected by an edge, if and only if  $\|X_i - X_j\| > 2 - \varepsilon$ , where  $\|\cdot\|$  is the Euclidean distance.

Throughout this paper we will think of random Borsuk graphs on  $S^d$  for a fixed dimension  $d$ . However we will explicitly point out the constants that depend on  $d$  in the statement of the results. We will denote the closed ball with center  $x$  and

radius  $r$  by  $B(x, r) = \{y \in \mathbb{R}^{d+1} : \|x - y\| \leq r\}$ . Similarly, we denote intersections of closed balls with the  $d$ -sphere by  $\widehat{B}(x, r)$ , and we call them spherical caps, so

$$\widehat{B}(x, r) := B(x, r) \cap S^d = \{y \in S^d : \|x - y\| \leq r\}$$

Given a Borel set  $F \subset \mathbb{R}^{d+1}$ , we denote its volume, i.e. Lebesgue measure, as  $\mathcal{V}(F)$ . Similarly, for a Borel set  $F \subset S^d$ , we denote its area on the surface of the sphere, by  $\mathcal{A}(F)$ . Also, we denote  $\omega_d = \mathcal{V}(S^d)$  and  $\alpha_d = \mathcal{A}(S^d)$ . Given a graph  $G$ , we denote its chromatic number by  $\chi(G)$ .

We say that an event happens *asymptotically almost surely (a.a.s)* if the probability approaches 1 as  $n \rightarrow \infty$ .

## 2. RANDOM BORSUK GRAPH WITH $\varepsilon$ CONSTANT

We start by proving that when  $\varepsilon > 0$  is constant and small,  $\chi(\text{Bor}^d(\varepsilon, n)) = d + 2$  a.a.s.

**Lemma 2.1.** For  $x, y \in S^d$ ,  $\|x - y\| > 2 - \varepsilon$  if and only if  $\|x + y\| < 2\sqrt{\varepsilon - \frac{\varepsilon^2}{4}}$

*Proof.* Since  $\overline{(-x)x}$  is a diameter,  $\overline{(-x)y} \perp \overline{xy}$ . Thus  $\|x + y\|^2 = 4 - \|x - y\|^2$ , so the claim follows. ■

Before getting into the analysis of the chromatic number, let us point out the fact that the odd girth of the Borsuk graph is  $> 1/\sqrt{\varepsilon}$ . While this has been observed before (see [5, 6, 21]), we include a proof for completeness.

**Lemma 2.2.** Let  $\varepsilon > 0$  and  $x_0 \in S^d$ . If  $x_0 y_1 x_1 y_2 \cdots x_n y_{n+1} = x_0$  is an odd cycle in the Borsuk graph  $\text{Bor}^d(\varepsilon)$ , then  $2n + 1 \geq 1/\sqrt{\varepsilon}$ . In other words, all odd cycles in  $\text{Bor}^d(\varepsilon)$  have length greater than  $1/\sqrt{\varepsilon}$ .

*Proof.* Since  $\|x_i - y_i\|, \|y_i - x_{i-1}\| > 2 - \varepsilon$ , for any  $i$ , by applying Lemma 2.1, we get

$$\begin{aligned} \|x_i - x_{i-1}\| &\leq \|x_i + y_i\| + \|-y_i - x_{i-1}\| \\ &= \|x_i + y_i\| + \|y_i + x_{i-1}\| \\ &\leq 4\sqrt{\varepsilon - \varepsilon^2/4} \\ &< 4\sqrt{\varepsilon}. \end{aligned}$$

Thus

$$\|x_n - x_0\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_1 - x_0\| \leq 4n\sqrt{\varepsilon}.$$

Finally,

$$2 = \|2x_0\| = \|x_0 + y_{n+1}\| \leq \|x_0 - x_n\| + \|x_n + y_{n+1}\| < 2(2n+1)\sqrt{\varepsilon}.$$

Therefore

$$\frac{1}{\sqrt{\varepsilon}} < 2n+1.$$

■

**Lemma 2.3.** For each  $d \geq 1$ , there exist a constant  $\lambda_d < 2$  such that, whenever  $0 < r < 2 - \lambda_d$ , the Borsuk graph  $\text{Bor}^d(r)$  has a proper coloring with  $d+2$  colors.

*Proof.* Let  $\Delta$  be the regular  $(d+1)$ -simplex inscribed in the unit  $d$ -sphere  $S^d$ . Consider the map  $\Phi : \partial\Delta \rightarrow S^d$  from the boundary of  $\Delta$  to  $S^d$  given by  $\Phi(x) = x/\|x\|$ . Note then that  $\Phi$  is a homeomorphism. Let  $\tau \in \partial\Delta$  be a maximal face, and let  $\lambda_d = \text{diam}(\Phi(\tau))$ . Since  $\Delta$  is regular, the value of  $\lambda_d$  does not depend on the face  $\tau$ .

Note now that  $\lambda_d < 2$ . To see this, suppose that  $\lambda_d = 2$ . Since  $\tau$  is closed, so is  $\Phi(\tau)$ , so there exist  $x, y \in \tau$  such that  $\|\Phi(x) - \Phi(y)\| = 2$ . This means  $\Phi(x)$  and  $\Phi(y)$  are antipodal, and so  $y = -\frac{\|y\|}{\|x\|}x$ . Since  $\tau$  is convex,  $0 = \frac{\|y\|}{(\|x\| + \|y\|)}x + \frac{\|x\|}{(\|x\| + \|y\|)}y$  must also be in  $\tau$ , but this is a contradiction, since  $\tau \subset \partial\Delta$ , proving the claim.

We now give a coloring for  $S^d$  as follows. We start by coloring  $\partial\Delta$ : give a different color to each of the  $d+2$  facets, and for the lower dimensional faces, assign an arbitrary color among the facets that contain them. Finally, color  $\Phi(x) \in S^d$ , with the color of  $x$ .

Note this is indeed a proper coloring for  $\text{Bor}^d(r)$ , since all points in  $S^d$  of the same color lie on the image of a facet  $\Phi(\tau)$ , of diameter  $\lambda_d$ ; so if  $x$  and  $y$  have the same color,  $\|x - y\| \leq \lambda_d < 2 - r$  so they are not connected by an edge in the Borsuk graph. ■

The upper bound for the chromatic number follows immediately from Lemma 2.3. The proof we give below for the lower bound, is a direct application of the Lyusternik–Shnirelman–Borsuk Theorem [13, 4]. We state this well-known theorem without proof; for more details and a self-contained proof see, for example, Chapter 2 of Matousek’s book [15].

**Theorem** (Lyusternik–Shnirelman–Borsuk). For any cover  $U_1, \dots, U_{d+1}$  of the sphere  $S^d$  by  $d+1$  open (or closed) sets, there is at least one set containing a pair of antipodal points.

We note that Bárány gave a short proof of Kneser's conjecture using this theorem [2]. See also Greene's proof [7]. For the rest of the paper, we refer to this theorem as the LSB Theorem.

*Proof of Theorem 1.1.* By Lemma 2.3, since  $\text{Bor}^d(\varepsilon, n) \subset \text{Bor}^d(\varepsilon)$ ,  $d+2$  is an upper bound for the chromatic number of the random Borsuk graph.

Let  $F_1, F_2, \dots, F_N$  be a cover of  $S^d$  by Borel sets, such that  $\text{diam}(F_i) \leq \frac{\sqrt{\varepsilon}}{2}$  and  $\mathcal{A}(F_i) > 0$  for all  $i$ . Note we can construct such a family of sets in many ways, for instance, as we do in the next section, we can consider a  $\delta$ -net of  $S^d$  and let the sets  $F_i$  to be spherical caps centered on the  $\delta$ -net of radius  $\sqrt{\varepsilon}/4$  where  $\delta \leq \sqrt{\varepsilon}/4$ . Note here that the sets  $F_i$  and  $N$  depend only on  $\varepsilon$ , which is fixed.

Let  $c = \min_i \frac{\mathcal{A}(F_i)}{\mathcal{A}(S^d)}$  and  $G = \text{Bor}^d(\varepsilon, n)$ . The following computation shows that, a.a.s.,  $G$  contains at least one vertex in each of the sets  $F_i$ .

$$\begin{aligned} \mathbb{P} \left[ \bigwedge_{i=1}^N (V(G) \cap F_i \neq \emptyset) \right] &= 1 - \mathbb{P} \left[ \bigvee_{i=1}^N (V(G) \cap F_i = \emptyset) \right] \\ &\geq 1 - \sum_{i=1}^N \mathbb{P} [V(G) \cap F_i = \emptyset] \\ &= 1 - \sum_{i=1}^N \left( 1 - \frac{\mathcal{A}(F_i)}{\mathcal{A}(S^d)} \right)^n \geq 1 - N(1-c)^n \end{aligned}$$

since  $N$  and  $c$  are constant,  $1 - N(1-c)^n \rightarrow 1$  as  $n \rightarrow \infty$ , proving the claim.

We may assume then,  $G$  has a vertex  $y_i \in F_i$  for  $i = 1, \dots, N$ . Proceeding by way of contradiction, suppose there exists a proper coloring of  $G$  with  $d+1$  colors. For each  $j = 1, \dots, d+1$  define

$$U_j = \bigcup \widehat{B} \left( y_k, \frac{\sqrt{\varepsilon}}{2} \right)$$

where the union is taken over all the  $y_k$ 's of color  $j$ .

Since  $F_i \subset \widehat{B} \left( y_i, \frac{\sqrt{\varepsilon}}{2} \right)$ , the sets  $U_1, \dots, U_{d+1}$  are a closed cover of  $S^d$ . Thus, by the LSB Theorem, there exists an antipodal pair in one of the closed sets. Without lost

of generality, say  $x, (-x) \in U_1$ , so  $x \in \widehat{B}(y_1, \frac{\sqrt{\varepsilon}}{2})$  and  $(-x) \in \widehat{B}(y_2, \frac{\sqrt{\varepsilon}}{2})$ , with both  $y_1$  and  $y_2$  having color 1. Then,

$$\|y_1 + y_2\| \leq \|y_1 - x\| + \|x + y_2\| \leq \sqrt{\varepsilon} \leq 2\sqrt{\varepsilon - \frac{\varepsilon^2}{4}},$$

where the last inequality holds because  $\varepsilon < 2$ . Lemma 2.1 then implies  $\|y_1 - y_2\| > 2 - \varepsilon$ , so  $y_1$  and  $y_2$  are connected by an edge in  $G$ , giving the desired contradiction.  $\blacksquare$

### 3. RANDOM BORSUK GRAPH WITH $\varepsilon \rightarrow 0$

**3.1. Lower Bound.** The proof we gave for the lower bound in Theorem 1.1, suggests that we should be able to let  $\varepsilon \rightarrow 0$  and still a.a.s. get the same chromatic number. Indeed, this will be the case. We just need to control the number of sets  $N$  we use to cover the sphere and their area, in such a way that

$$\lim_{n \rightarrow \infty} 1 - N(1 - c)^n \rightarrow 1.$$

In this section we discuss how to do this using  $\delta$ -nets on  $S^d$ , and then adapt the proof of Theorem 1.1 to get Theorem 1.2.

We start with a technical lemma on spherical caps.

**Lemma 3.1.** Given  $x \in S^d$  and  $0 < r < 1$ , the following hold for the spherical cap  $B = \widehat{B}(x, r)$ .

- (1) The boundary  $\partial B$ , is a  $(d - 1)$ -dimensional sphere with radius

$$r' = r\sqrt{1 - r^2/4}.$$

- (2)  $B$  is indeed a cap, i.e. there exist a  $d$ -hyperplane in  $\mathbb{R}^{d+1}$ , such that  $B$  is the portion of  $S^d$  contained in one of the semi-spaces defined by the hyperplane.
- (3) The area  $\mathcal{A}(B)$  satisfies the inequalities

$$\frac{1}{\pi} \left( \frac{\sqrt{3}}{2} \right)^{d-1} r^d \leq \frac{\mathcal{A}(\widehat{B}(x, r))}{\mathcal{A}(S^d)} \leq \frac{d}{3} r^d.$$

Spherical caps are well studied in the literature. See, for example, Lemmas 2.2. and 2.3 in [1]. We note that Lemma 2.2 in [1] is better than our Lemma 3.1 in the case that  $r$  is fixed and  $d \rightarrow \infty$ , but we are interested in the case that  $d$  is fixed.

*Proof of Lemma 3.1.* Without loss of generality, we may choose  $x = N = (0, \dots, 0, 1)$  to be the north pole by rotating the sphere. Thus  $x \in \partial B$  if and only if  $\|x\| = 1$  and  $\|x - N\| = r$ . Then

$$\begin{aligned} r^2 &= \|x - N\|^2 \\ &= x_0^2 + \dots + x_{d-1}^2 + (x_d - 1)^2 \\ &= x_0^2 + \dots + x_{d-1}^2 + x_d^2 - 2x_d + 1 \\ &= 2 - 2x_d. \end{aligned}$$

So  $x_d = 1 - \frac{r^2}{2}$ . Therefore, the hyperplane  $\{x_d = 1 - \frac{r^2}{2}\}$  determines the cap  $B$ , proving (2). To get (1), note  $1 = \|x\|^2 = x_0^2 + \dots + x_d^2$ , so we get  $x_0^2 + \dots + x_{d-1}^2 = r^2 - r^4/4 = (r')^2$ , for all  $x \in \partial B$ , proving it is indeed a  $(d-1)$ -dimensional sphere with the desired radius.

For (3), recall we get the area of  $B = \widehat{B}(x, r)$  by integrating the length  $\ell$  of the arc from  $x$  to the boundary  $\partial B$ , over all the possible unit vectors. Thus

$$\mathcal{A}(\widehat{B}(x, r)) = \int_{\hat{u} \in S^{d-1}} \ell(r')^{d-1} d\hat{u} = \ell(r')^{d-1} \int_{\hat{u} \in S^{d-1}} d\hat{u} = \ell(r')^{d-1} \alpha_{d-1}.$$

Some planar geometry gives the bounds

$$r \leq \ell = 2 \arcsin(r/2) \leq \frac{\pi}{3} r,$$

and

$$\frac{\sqrt{3}}{2} r \leq r' \leq r,$$

since  $0 < r < 1$ . This gives

$$\left(\frac{\sqrt{3}}{2}\right)^{d-1} r^d \alpha_{d-1} \leq \mathcal{A}(B) \leq \frac{\pi}{3} r^d \alpha_{d-1}$$

Recall the formula for the surface area of the unit  $d$ -dimensional sphere

$$\alpha_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.$$

By using the fact that the Gamma function is increasing on  $[2, \infty)$  and treating the first few cases separately, we have that for  $d \geq 1$ ,

$$\frac{1}{\pi} \leq \frac{\alpha_{d-1}}{\alpha_d} = \frac{\Gamma((d+1)/2)}{\sqrt{\pi} \Gamma(d/2)} \leq \frac{d}{\pi},$$

from which the desired result follows. ■

For the sake of completeness, we include the following discussion on  $\delta$ -nets. See, for example, [14, Chapter 13] for more details.

**Definition 3.2** ( $\delta$ -Nets). *Given a metric space  $X$  with metric  $d$ , a  $\delta$ -net is a subset  $\mathcal{B} \subset X$  such that for every  $x \in X$ , there exists  $y \in \mathcal{B}$  with  $d(x, y) < \delta$ .*

We can then construct a  $\delta$ -net for any compact metric space  $M$  inductively. Indeed, choose any point  $y_1 \in M$ . For each  $m \geq 2$ , if  $B_m = \cup_{i=1}^m \widehat{B}(y_i, \delta) \subsetneq M$ , choose any  $y_{m+1} \in M \setminus B_m$ . Otherwise, stop and let  $\mathcal{B} = \{y_1, y_2, \dots, y_m\}$ .

Compactness ensures that the process stops. It's clear that  $\mathcal{B}$  is a  $\delta$ -net and moreover it is also a maximal  $\delta$ -apart set. That is,  $d(y_i, y_j) > \delta$  whenever  $i \neq j$ , and we can not add any other point to  $\mathcal{B}$  without destroying this property. This implies that the balls of radius  $\delta$  and center on the points  $y_i$ 's cover  $M$ , while the open balls of radius  $\delta/2$  with center on the  $y_i$ 's are all disjoint. We now show that we can control the size of the  $\delta$ -net in the case that  $M = S^d$ .

**Lemma 3.3.** For every  $d \geq 1$  and  $0 < \delta < 1$  there exists a  $\delta$ -net  $\mathcal{B} \subset S^d$ , such that (1) for every two points  $y_i, y_j \in \mathcal{B}$ ,  $\|y_i - y_j\| > \delta$ , and (2) its cardinality  $\mathcal{N} = |\mathcal{B}|$  satisfies:

$$\frac{3}{d\delta^d} \leq \mathcal{N} \leq \frac{2(3^d)(d+1)}{\delta^d}.$$

In the literature, it seems more common to find an upper bound such as

$$\mathcal{N} \leq (4/\delta)^{d+1},$$

which is a better bound when  $\delta$  is constant and  $d \rightarrow \infty$  [14, Lemma 13.1.1]. However, the bound we give is more useful for us since we are dealing with  $d$  constant and  $\delta \rightarrow 0$ .

*Proof of Lemma 3.3.* By letting  $\mathcal{B}$  be the  $\delta$ -net defined above, we already have a  $\delta$ -net for the sphere  $S^d$  that is also a  $\delta$ -apart maximal set. To prove the inequalities on its cardinality we give a volume and an area arguments.



Since the points  $y_i \in \mathcal{B}$  are  $\delta$  apart, the open balls  $\text{int}\left(B\left(y_i, \frac{\delta}{2}\right)\right)$  are disjoint. So

$$\mathcal{V}\left(\bigcup_{i=1}^N B\left(y_i, \frac{\delta}{2}\right)\right) = \mathcal{N} \omega_d \left(\frac{\delta}{2}\right)^{d+1},$$

where  $\omega_d$  denotes the volume of the  $d+1$ -dimensional unit ball in  $\mathbb{R}^{d+1}$ . Moreover, all such balls are contained in the set  $B\left(0, 1 + \frac{\delta}{2}\right) \setminus B\left(0, 1 - \frac{\delta}{2}\right)$ . Thus

$$\begin{aligned} \mathcal{V}\left(\bigcup_{i=1}^N B\left(y_i, \frac{\delta}{2}\right)\right) &\leq \mathcal{V}\left(B\left(0, 1 + \frac{\delta}{2}\right)\right) - \mathcal{V}\left(B\left(0, 1 - \frac{\delta}{2}\right)\right) \\ &= \omega_d \left( \left(1 + \frac{\delta}{2}\right)^{d+1} - \left(1 - \frac{\delta}{2}\right)^{d+1} \right) \\ &= \omega_d \delta \sum_{r=0}^d \binom{d}{r} \left(1 + \frac{\delta}{2}\right)^{d-r} \left(1 - \frac{\delta}{2}\right)^r \\ &\leq \omega_d \delta \sum_{r=0}^d \binom{d}{r} \left(1 + \frac{\delta}{2}\right)^d \\ &= \omega_d \delta (d+1) \left(1 + \frac{\delta}{2}\right)^d \\ &\leq \omega_d \delta (d+1) \left(\frac{3}{2}\right)^d, \end{aligned}$$

and the upper bound for  $\mathcal{N}$  follows.

For the lower bound, we consider the area of the spherical caps. Since all points in  $S^d$  are within distance  $\delta$  of the points  $y_i \in \mathcal{B}$  we must have

$$\mathcal{A}(S^d) = \mathcal{A}\left(\bigcup_{i=1}^{\mathcal{N}} \widehat{B}(y_i, \delta)\right) \leq \sum_{i=1}^{\mathcal{N}} \mathcal{A}(\widehat{B}(y_i, \delta)) = \mathcal{N} \mathcal{A}(\widehat{B}(y_1, \delta))$$

Therefore, Lemma 3.1 yields  $\mathcal{N} \geq \frac{\mathcal{A}(S^d)}{\mathcal{A}(\widehat{B}(y_0, \delta))} \geq \frac{3}{d\delta^d}$ . ■

We now proceed to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let  $G = \text{Bor}^d(\varepsilon(n), n)$ . Since  $\varepsilon \rightarrow 0$ , eventually  $\varepsilon < 2 - \lambda_d$ , so by Lemma 2.3,  $\chi(G) \leq d+2$ . We now proceed to prove the lower bound by a modification of the proof of theorem 1.1.

Let  $\delta = \sqrt{\varepsilon}/4$ . Let  $\mathcal{B}$  be the  $\delta$ -net given by Lemma 3.3. Say  $\mathcal{B} = \{y_1, y_2, \dots, y_N\}$ , where

$$N \leq \frac{2(3^d)(d+1)}{\delta^d} = \frac{A_d}{\varepsilon^{d/2}},$$

and  $A_d$  is a constant which only depends on  $d$ . For each  $i = 1, \dots, N$ , define  $F_i = \widehat{B}(y_i, \delta)$ . Note then the  $F_i$ 's cover the sphere and  $\text{diam}(F_i) \leq 2\delta = \frac{\sqrt{\varepsilon}}{2}$ .

Applying Lemma 3.1, we have

$$\frac{\mathcal{A}(F_i)}{\mathcal{A}(S^d)} \geq \frac{1}{4\pi} \left( \frac{\sqrt{3}}{8} \right)^{d-1} \varepsilon^{d/2} = B_d \varepsilon^{d/2} =: c,$$

where  $B_d$  is constant.

Finally, all that remains to prove is that  $1 - N(1-c)^n \rightarrow 1$  as  $n \rightarrow \infty$ , even when  $N$  and  $c$  depend on  $n$ . This is as follows

$$\begin{aligned} N(1-c)^n &\leq \frac{A_d}{\varepsilon^{d/2}} (1-c)^n \\ &= \frac{A_d}{\varepsilon^{d/2}} (1 - B_d \varepsilon^{d/2})^n \\ &= \frac{A_d n}{C^{d/2} \log n} \left( 1 - \frac{B_d C^{d/2} \log n}{n} \right)^n \\ &\leq \frac{A_d n}{C^{d/2} \log n} \exp(-B_d C^{d/2} \log n) \\ &= \frac{A_d}{C^{d/2} \log n} n^{1-B_d C^{d/2}} \end{aligned}$$

The last expression goes to zero as  $n \rightarrow \infty$ , since

$$C \geq \frac{64}{3} \sqrt[d]{\frac{3\pi^2}{4}},$$

so  $B_d C^{d/2} \geq 1$  and this completes the proof.  $\blacksquare$

**Corollary.** If

$$\frac{64}{3} \sqrt[d]{\frac{3\pi^2}{4}} \left( \frac{\log n}{n} \right)^{2/d} \leq \varepsilon(n) \leq 2 - \lambda_d$$

for all sufficiently large  $n$ , then  $\chi(\text{Bor}^d(\varepsilon(n), n)) = d + 2$  a.a.s.

*Proof.* The chromatic number is monotone with respect to  $\varepsilon$ , so this follows directly.  $\blacksquare$

**3.2. Upper Bound.** Theorem 1.2 and its Corollary shows that if  $\varepsilon \rightarrow 0$  sufficiently slowly then the chromatic number of the random Borsuk graph is a.a.s.  $d+2$ . In this section we show that the rate obtained is tight, up to a constant factor. That is, we show an upper bound for  $\varepsilon$  for which the random Borsuk graph is  $(d+1)$ -colorable.

We start our analysis by constructing a proper coloring of  $\text{Bor}^d(\varepsilon) \setminus \widehat{B}(x, \delta)$  with exactly  $d+1$  colors, for a suitable  $\delta$  that depends on  $\varepsilon$ . Lemma 3.1 establishes that the boundary of a spherical cap on  $S^d$  is a  $S^{d-1}$  with radius  $\delta'$ , and Lemma 2.3 allows to color it with  $d+1$  colors. We will provide the technical details to translate this coloring into a proper coloring of the desired graph.

For the following analysis consider the spherical cap  $A = \widehat{B}(N, r)$ , where  $N$  is the north pole. For each  $x \in S^d \setminus \{N, -N\}$ , let  $\gamma_x : [0, \pi] \rightarrow S^d$  be the great semi-circle going from  $N$  to  $-N$  and passing through  $x$ . Define  $f : S^d \rightarrow \partial A$  by letting  $f(x)$  be the intersection of  $\gamma_x$  with  $\partial A$ . Note this is a well defined function, since if  $x = (x_0, \dots, x_d)$ , we can parametrize

$$\gamma_x(t) = \left( \frac{\sin t}{\sqrt{1-x_d^2}}x_0, \dots, \frac{\sin t}{\sqrt{1-x_d^2}}x_{d-1}, \cos t \right)$$

so its last coordinate takes all values in  $[-1, 1]$  exactly once for  $0 \leq t \leq \pi$ , and from Lemma 3.1 we know  $\partial A$  consists of all points with last coordinate  $a := 1 - \frac{r^2}{2}$ .

The following lemmas construct the desired coloring.

**Lemma 3.4.** Let  $x, y \in S^d \setminus \{N, -N\}$  such that  $\|x - y\| \leq \delta$ . Define  $y' = (y'_0, \dots, y'_d)$  to be the point in the geodesic  $\gamma_y$ , such that  $y'_d = x_d$ . Then  $\|x - y'\| \leq 2\delta$ .

*Proof.* Without loss of generality, we may assume  $y = (0, 0, \dots, 0, \sqrt{1-y_d^2}, y_d)$ , since we can get this by a rotation of the sphere that leaves the last coordinate fixed. This rotation fixes the north and south poles, so it also transforms the geodesic through  $y$  into another geodesic through  $y$ . Thus, the formula for the geodesic simplifies to

$$\gamma_y = (0, \dots, 0, \sin t, \cos t), \text{ for } 0 \leq t \leq \pi.$$

So,  $y' = (0, \dots, 0, \sqrt{1-x_d^2}, x_d)$ . Then

$$\begin{aligned} \|x - y'\|^2 &= x_0^2 + \dots + x_{d-2}^2 + \left(x_{d-1} - \sqrt{1-y_d^2}\right)^2 + (x_d - y_d)^2 \\ &= (1 - x_d^2) + (1 - y_d^2) - 2x_{d-1}\sqrt{1-y_d^2} + (x_d - y_d)^2 \end{aligned}$$

and

$$\begin{aligned}\|y - y'\|^2 &= \left(\sqrt{1 - y_d^2} - \sqrt{1 - x_d^2}\right)^2 + (x_d - y_d)^2 \\ &= (1 - x_d^2) + (1 - y_d^2) - 2\sqrt{1 - x_d^2}\sqrt{1 - y_d^2} + (x_d - y_d)^2\end{aligned}$$

Since  $x_{d-1} \leq |x_{d-1}| \leq \sqrt{x_0^2 + \dots + x_{d-1}^2} = \sqrt{1 - x_d^2}$ , we get  $\|y - y'\| \leq \|x - y\|$ , and so

$$\|x - y'\| \leq \|x - y\| + \|y - y'\| \leq 2\|x - y\| \leq 2\delta.$$

■

**Lemma 3.5.** Let  $x, y \in S^d \setminus \{N, -N\}$  such that  $x \notin A \cup (-A)$  and  $\|x - y\| \leq \delta$ . Then

$$\|f(x) - f(y)\| \leq 2\delta.$$

*Proof.* Let  $y' \in S^d$  such that its last coordinate is  $y'_d = x_d$ . From the parametrization for  $\gamma_x$ , we see  $f(x) = \gamma_x(t_1)$ , where  $\cos t_1 = a$  and  $\sin t_1 = \sqrt{1 - a^2} = \delta'$ , the radius of  $\delta A$ , hence

$$f(x) = \left( \frac{\delta'}{\sqrt{1 - x_d^2}} x_0, \dots, \frac{\delta'}{\sqrt{1 - x_d^2}} x_{d-1}, a \right).$$

A similar expression holds for  $f(y')$ , with  $y'_d = x_d$ , so we get

$$\begin{aligned}\|f(x) - f(y')\| &= \sqrt{\sum_{i=0}^{d-1} \frac{\delta'^2}{1 - x_d^2} (x_i - y'_i)^2} \\ &= \frac{\delta'}{\sqrt{1 - x_d^2}} \sqrt{\sum_{i=0}^{d-1} (x_i - y'_i)^2} \\ &\leq \frac{\delta'}{\sqrt{1 - x_d^2}} \|x - y'\|\end{aligned}$$

Moreover, since  $x \notin A \cup (-A)$ ,  $|x_d| < a$ , so  $\frac{\delta'}{\sqrt{1 - x_d^2}} < 1$ , so  $\|f(x) - f(y')\| \leq \|x - y'\|$ . Finally, if we let  $y'$  be the one defined in Lemma 3.4,  $f(y') = f(y)$ , and therefore  $\|f(x) - f(y)\| = \|f(x) - f(y')\| \leq \|x - y'\| \leq 2\delta$ .

■

**Lemma 3.6.** Let  $0 < \varepsilon < 1$ , such that

$$r = \frac{8\sqrt{\varepsilon}}{\sqrt{3(4 - \lambda_{d-1}^2)}} < 1,$$

$x \in S^d$ , and  $A = \widehat{B}(x, r)$ . Let  $H$  be the induced subgraph of  $\text{Bor}^d(\varepsilon)$  by the vertex set  $S^d \setminus A$ . Then  $\chi(H) \leq d + 1$ .

*Proof.* Without loss of generality let  $x = N$  the north pole, so  $A = \widehat{B}(N, r)$ . Lemma 3.1 says  $\partial A$  is a  $S^{d-1}$  sphere of radius  $r' = r\sqrt{1 - \frac{r^2}{4}} \geq \frac{\sqrt{3}}{2}r$ . Thus adapting Lemma 2.3, we can color it in such a way that every two points with the same color are at a distance of at most  $\lambda_{d-1}r'$ . We then color  $H$  by giving each point  $y \in S^d \setminus A \setminus \{-N\}$  the color of  $f(y)$ , and giving the south pole  $-N$  any color. We proceed to prove this is a proper coloring of  $H$ .

From Lemma 2.1, the neighbors of the south pole lie in  $\widehat{B}(N, \sqrt{\varepsilon - \varepsilon^2/4}) \subset A$ , so  $-N$  is isolated in  $H$ . Let  $y, z \in S^d \setminus A \setminus \{-N\}$  such that  $\|y - z\| > 2 - \varepsilon$ . Lemma 2.1 implies  $\|y + z\| < \delta := 2\sqrt{\varepsilon - \frac{\varepsilon^2}{4}}$ . If we had  $(-y), (-z) \in A$ , that would mean  $y, z \in -A$ , but then  $\|y - z\| \leq r \leq 2 - \varepsilon$  for small  $\varepsilon$ . So we may assume  $(-y) \notin A$ , and since  $y \notin A$ ,  $(-y) \notin -A$ . Thus  $(-y) \notin A \cup (-A)$  and  $\| -y - z \| \leq \delta$ , thus Lemma 3.5 implies

$$\|f(-y) - f(z)\| \leq 2\delta = 4\sqrt{\varepsilon - \frac{\varepsilon^2}{4}} < 4\sqrt{\varepsilon} = \sqrt{4 - \lambda_{d-1}^2} \frac{\sqrt{3}}{2}r \leq \sqrt{4 - \lambda_{d-1}^2}r'$$

From the definition of  $f$ , it is clear that  $f(-y) = -f(y)$ , thus Lemma 2.1 implies  $\|f(y) - f(z)\| > \lambda_{d-1}r'$ , and so  $f(y)$  and  $f(z)$  have different colors, meaning  $y$  and  $z$  have different colors as well. Therefore  $\chi(H) \leq d + 1$ . ■

As an immediate application, if a random Borsuk graph leaves some spherical cap in  $S^d$  of radius bigger than  $r$  with no vertices, then it can be colored with  $d + 1$  colors. We will show that this is indeed the case when  $\varepsilon \rightarrow 0$  at the said rate. We now include some theorems about Poisson Point Processes and Poisson distributions. For their proofs and a complete discussion refer to [16] or [9].

**Theorem 3.7 (Poissonization).** Let  $X_1, X_2, \dots$ , be uniform random variables on  $S^d$ . Let  $M \sim \text{Pois}(\lambda)$  and let  $\eta$  be the random counting measure associated to the point process  $P_\lambda = \{X_1, X_2, \dots, X_M\}$ . Then  $P_\lambda$  is a Poisson Point Process and for a Borel  $A \subset S^d$ ,  $\eta(A) \sim \text{Pois}\left(\lambda \frac{\mathcal{A}(A)}{\mathcal{A}(S^d)}\right)$ .

**Lemma 3.8.** For  $n \geq 0$ ,  $\mathbb{P}[\text{Pois}(2n) < n] \leq e^{-0.306n}$ .

We are now ready to prove the Theorem 1.3.

*Proof of Theorem 1.3.* Let  $X_1, X_2, \dots$ , be uniform random variables on  $S^d$ . Let  $M \sim \text{Pois}(2n)$ . Let  $\eta$  be the random counting measure of the Poisson Point Process  $\{X_1, \dots, X_M\}$ . Similarly, let  $\eta_1^n$  be the counting measure of the Random points  $\{X_1, \dots, X_n\}$ .

Let

$$\delta = \frac{16\sqrt{\varepsilon}}{\sqrt{3(4 - \lambda_{d-1}^2)}} = A_d\sqrt{\varepsilon},$$

where  $A_d$  is a constant which only depends on  $d$ . Let  $\mathcal{B} = \{y_1, \dots, y_N\}$  be the  $\delta$ -net given by Lemma 3.3, so

$$N \geq \frac{3}{d\delta^d} = \frac{B_d}{\varepsilon^{d/2}},$$

and  $B_d$  is constant. Let  $F_i = \widehat{B}(y_i, \delta/2)$  for  $i = 1, \dots, N$  be spherical caps centered at the  $\delta$ -net. Thus, as in the proof of 3.3, the  $F_i$ 's are disjoint. Lemma 3.1 gives

$$\frac{\mathcal{A}(F_i)}{\mathcal{A}(S^d)} \leq \frac{d}{3} \left(\frac{\delta}{2}\right)^d = D_d \varepsilon^{d/2},$$

where  $D_d$  is constant.

Note that these spherical caps have the same radius required by Lemma 3.6, so if we prove that a.a.s. one of these  $F_i$ 's doesn't contain any vertices of the random Borsuk graph, then it must be contained in  $S^d \setminus F_i$ , and the Lemma 3.6 gives a proper  $(d+1)$  coloring. This is what we do.

Note that

$$\mathbb{P} \left[ \min_{1 \leq i \leq N} \eta(F_i) = 0 \right] \leq \mathbb{P} \left[ \min_{1 \leq i \leq N} \eta_1^n(F_i) = 0 \right] + \mathbb{P}[M < n]. \quad (1)$$

We have

$$\begin{aligned}
\mathbb{P}\left[\min_{1 \leq i \leq N} \eta(F_i) = 0\right] &= 1 - \mathbb{P}\left[\bigwedge_{i=1}^N \eta(F_i) > 0\right] = 1 - \prod_{i=1}^N \mathbb{P}[\eta(F_i) > 0] \\
&= 1 - \left(1 - \mathbb{P}\left[\text{Pois}\left(2n \frac{\mathcal{A}(F_1)}{\alpha_d}\right) = 0\right]\right)^N \\
&\geq 1 - \exp\left(-\exp\left(-2n \frac{\mathcal{A}(F_1)}{\alpha_d}\right) N\right) \\
&\geq 1 - \exp\left(-\exp(-2n D_d \varepsilon^{d/2}) \frac{B_d}{\varepsilon^{d/2}}\right) \\
&= 1 - \exp\left(-\frac{B_d}{C^{d/2} \log n} n^{1-2D_d C^{d/2}}\right).
\end{aligned}$$

This last expression tends to 1 as  $n \rightarrow \infty$ , since  $C$  is such that  $1 - 2D_d C^{d/2} > 0$ .

Lemma 3.8 assures that

$$\mathbb{P}[M < n] = \mathbb{P}[\text{Pois}(2n) < n] \rightarrow 0$$

as  $n \rightarrow \infty$ , and therefore (1) gives  $\mathbb{P}\left[\min_{1 \leq i \leq N} \eta_i^n(F_i) = 0\right] \rightarrow 1$ , as desired. ■

#### 4. FURTHER QUESTIONS

- (1) It might be possible to find sharper constants in Theorems 1.2 and 1.3. For  $d = 1$ , it is certainly possible. The following can be achieved with similar methods to the ones used throughout this paper, so we include the statement without proof.

**Theorem 4.1.** Let  $\varepsilon = C(\log n/n)^2$ .

- (a) If  $C \geq 9\pi^2/4$ , then a.a.s.  $\chi(\text{Bor}^1(\varepsilon, n)) = 3$ .
- (b) If  $C < \pi^2/4$ , then a.a.s.  $\chi(\text{Bor}^1(\varepsilon, n)) \leq 2$ .

- (2) We wonder whether there exist functions  $\varepsilon = \varepsilon(n)$  such that the chromatic number of the random Borsuk graph  $\text{Bor}^d(\varepsilon, n)$  a.a.s. equals  $i$ , for  $1 \leq i \leq d + 1$ .

- (3) We only studied here the case that  $d$  is fixed and  $\varepsilon$  is either fixed or tends to zero at some rate. It also seems interesting to let  $d \rightarrow \infty$  at some rate, or to let  $d$  be fixed and  $\varepsilon \rightarrow 2$ . See, for example, Raigorodskii's work on coloring high-dimensional spheres [19].

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OHIO STATE UNIVERSITY

*E-mail address:* kahle.70@osu.edu

OHIO STATE UNIVERSITY

*E-mail address:* martinezfigueroa.2@osu.edu