THE GROUNDED MARTIN'S AXIOM

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Abstract. We introduce a variant of Martin's Axiom, called the grounded Martin's Axiom, or grMA, which asserts that the universe is a ccc forcing extension in which Martin's Axiom holds for posets in the ground model. This principle already implies several of the combinatorial consequences of MA. The new axiom is shown to be consistent with the failure of MA and a singular continuum. We prove that grMA is preserved in a strong way when adding a Cohen real and that adding a random real to a model of MA preserves grMA (even though it destroys MA itself). We also consider the analogous variant of the Proper Forcing Axiom.

The standard Solovay-Tennenbaum proof of the consistency of Martin's Axiom with a large continuum starts by choosing a suitable cardinal κ and then proceeds in an iteration of length κ by forcing with ccc posets of size less than κ, and not just those in the ground model but also those arising in the intermediate extensions. To ensure that all of the potential ccc posets are considered, some bookkeeping device is usually employed.

Consider now the following reorganization of the argument. Instead of iterating for κ many steps we build a length κ^2 iteration by first dealing with the κ many small posets in the ground model, then the small posets in that extension, and so on. The full length κ^2 iteration can be seen as a length κ iteration, all of whose iterands are themselves length κ iterations.

In view of this reformulation, we can ask what happens if we halt this construction after forcing with the first length κ iteration, when we have, in effect, ensured that Martin's Axiom holds for posets from the ground model. What combinatorial consequences of Martin's Axiom follow already from this weaker principle? We aim in this paper to answer these questions (at least partially).

1. Forcing the grounded Martin's Axiom

Definition 1. The *grounded Martin's Axiom* (grMA) asserts that V is a ccc forcing extension of some ground model W and V satisfies the conclusion of Martin's Axiom for posets $\mathbb{Q} \in W$ which are still ccc in V.

To be clear, in the definition we require that V have $\mathcal{D}$-generic filters for any $\mathcal{D} \in V$ a family of fewer than $\mathfrak{c}^V$ dense subsets of $\mathbb{Q}$. We should also note that, while the given definition is second-order, grMA is in fact first-order expressible, using the result of Reitz [7] that the ground models of the universe are uniformly definable.

If Martin's Axiom holds, we may simply take $W = V$ in the definition, which shows that the grounded Martin's Axiom is implied by Martin's Axiom. In particular, by simply performing the usual Martin's Axiom iteration, grMA $+\ \mathfrak{c} = \kappa$ can

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be forced from any model where κ is regular and satisfies $2^{<\kappa} = \kappa$. It will be shown in theorem 6, however, that the grounded Martin's Axiom is strictly weaker than Martin's Axiom. Ultimately we shall see that grMA retains some of the interesting combinatorial consequences of Martin's Axiom (corollary 11), while also being more robust with respect to mild forcing (theorems 23 and 29).

As in the case of Martin's Axiom, a key property of the grounded Martin's Axiom is that it is equivalent to its restriction to posets of small size.

Lemma 2. *The grounded Martin's Axiom is equivalent to its restriction to posets of size less than continuum, i.e. the following principle:*

> *The universe V is a ccc forcing extension of some ground model W and V satisfies the conclusion of Martin's Axiom for posets $\mathbb{Q} \in W$ of size less than $\mathfrak{c}^V$ which are still ccc in V.* $(*)$

Proof. Assume V satisfies $(*)$ and let $\mathbb{Q} \in W$ be a poset which is ccc in V and $\mathcal{D} = \{ D_\alpha ; \alpha < \kappa \} \in V$ a family of $\kappa < \mathfrak{c}^V$ many dense subsets of $\mathbb{Q}$. Let $V = W[G]$ for some W-generic $G \subseteq \mathbb{P}$ and let $\dot{D}_\alpha \in W$ be $\mathbb{P}$-names for the D_α. Choose θ large enough so that $\mathbb{P}, \mathbb{Q}$ and all of the $\dot{D}_\alpha$ are in $H^W_{\theta^+}$. We can then find an $X \in W$ of size at most κ such that $X \prec H^W_{\theta^+}$ and X contains $\mathbb{P}, \mathbb{Q}$ and the $\dot{D}_\alpha$.

Now let

$$X[G] = \{ \tau^G ; \tau \in X \text{ is a } \mathbb{P}\text{-name} \} \in W[G]$$

We can verify the Tarski-Vaught criterion to show that $X[G]$ is an elementary substructure of $H^W_{\theta^+}[G] = H^{W[G]}_{\theta^+}$. Specifically, suppose that $H^W_{\theta^+}[G] \models \exists x \colon \varphi(x, \tau^G)$ for some $\tau \in X$. Let S be the set of conditions $p \in \mathbb{P}$ which force $\exists x \colon \varphi(x, \tau)$. Since S is definable from the parameters $\mathbb{P}$ and τ we get $S \in X$. Let $A \in X$ be a antichain, maximal among those contained in S. By mixing over A we can obtain a name $\sigma \in X$ such that $p \Vdash \varphi(\sigma, \tau)$ for any $p \in A$ and it follows that $H^W_{\theta^+}[G] \models \varphi(\sigma^G, \tau^G)$, which completes the verification.

Now let $\mathbb{Q}^* = \mathbb{Q} \cap X[G]$ and $D^*_\alpha = D_\alpha \cap X[G]$. Then $\mathbb{Q}^* \prec \mathbb{Q}$ and it follows that $\mathbb{Q}^*$ is ccc (in $W[G]$) and that D^*_α is dense in $\mathbb{Q}^*$ for any $\alpha < \kappa$. Furthermore, $\mathbb{Q}^*$ has size at most κ. Finally, since $\mathbb{P}$ is ccc, the filter G is X-generic and so $\mathbb{Q}^* = \mathbb{Q} \cap X[G] = \mathbb{Q} \cap X$ is an element of W. If we now apply $(*)$ to $\mathbb{Q}^*$ and $\mathcal{D}^* = \{ D^*_\alpha ; \alpha < \kappa \}$, we find in $W[G]$ a filter $H \subseteq \mathbb{Q}^*$ intersecting every D^*_α, and thus every D_α. Thus H generates a $\mathcal{D}$-generic filter on $\mathbb{Q}$. □

The reader has likely noticed that the proof of lemma 2 is somewhat more involved than the proof of the analogous result for Martin's Axiom. The argument there hinges on the straightforward observation that elementary subposets of ccc posets are themselves ccc. While that remains true in our setting, of course, matters are made more difficult since we require that all of our posets come from a ground model that may not contain the dense sets under consideration. It is therefore not at all clear that taking appropriate elementary subposets will land us in the ground model and a slightly more elaborate argument is needed.

Let us point out a deficiency in the definition of grMA. As we have described it, the principle posits the existence of a ground model for the universe, a considerable global assumption. On the other hand, lemma 2 suggests that the operative part of the axiom is, much like Martin's Axiom, a statement about $H_\mathfrak{c}$. This discrepancy allows for some undesirable phenomena. For example, Reitz [7] shows that it is possible to perform arbitrarily closed class forcing over a given model and obtain a model of the ground axiom, the assertion that the universe is not a nontrivial set forcing extension over any ground model at all. This implies that there are models which have the same $H_{\mathfrak{c}}$ as a model of the grounded Martin's Axiom but which fail to satisfy it simply because they have no nontrivial ground models at all. To avoid this situation we can weaken the definition of grMA in a technical way.

Definition 3. The *local grounded Martin's Axiom* asserts that there are a cardinal $\kappa \geq \mathfrak{c}$ and a transitive ZFC$^-$ model $M \subseteq H_{\kappa^+}$ such that H_{κ^+} is a ccc forcing extension of M and V satisfies the conclusion of Martin's Axiom for posets $\mathbb{Q} \in M$ which are still ccc in V.

Of course, if the grounded Martin's Axiom holds, over the ground model W via the forcing notion $\mathbb{P}$, then its local version holds as well. We can simply take κ to be large enough so that $M = H_{\kappa^+}^W$ contains $\mathbb{P}$ and that $M[G] = H_{\kappa^+}^V$. One should view the local version of the axiom as capturing all of the relevant combinatorial effects of grMA (which, as we have seen, only involve $H_{\mathfrak{c}}$), while disentangling it from the structure of the universe higher up.

We now aim to give a model where the Martin's Axiom fails but the grounded version holds. The idea is to imitate the Solovay-Tennenbaum argument, but to only use ground model posets in the iteration. While it is then relatively clear that grMA will hold in the extension, a further argument is needed to see that MA itself fails. The key will be a kind of product analysis, given in the next few lemmas. We will show that the iteration of ground model posets, while not exactly a product, is close enough to a product to prevent Martin's Axiom from holding in the final extension by a result of Roitman. An extended version of this argument will also yield the consistency of grMA with a singular continuum.

Lemma 4. *Let $\mathbb{Q}_0$ and $\mathbb{R}$ be posets and τ a $\mathbb{Q}_0$-name for a poset such that $\mathbb{Q}_0 * \tau$ is ccc and $\mathbb{Q}_0 \Vdash$ "if $\check{\mathbb{R}}$ is ccc then $\tau = \check{\mathbb{R}}$". Furthermore, suppose that $\mathbb{Q}_0 * \tau \Vdash$ "$\check{\mathbb{Q}}_0$ is ccc". Then either $\mathbb{Q}_0 \Vdash$ "$\check{\mathbb{R}}$ is ccc" or $\mathbb{Q}_0 \Vdash$ "$\check{\mathbb{R}}$ is not ccc".*

Proof. Suppose the conclusion fails, so that there are conditions $q_0, q_1 \in \mathbb{Q}_0$ which force $\check{\mathbb{R}}$ to either be or not be ccc, respectively. It follows that $\mathbb{Q}_0 \restriction q_1 \times \mathbb{R}$ is not ccc. Switching the factors, there must be a condition $r \in \mathbb{R}$ forcing that $\check{\mathbb{Q}}_0 \restriction \check{q}_1$ is not ccc. Now let $G * H$ be generic for $\mathbb{Q}_0 * \tau$ with $(q_0, \check{r}) \in G * H$ (note that $(q_0, \check{r})$ is really a condition since $q_0 \Vdash \tau = \check{\mathbb{R}}$). Consider the extension $V[G * H]$. On the one hand $\mathbb{Q}_0$ must be ccc there, since this was one of the hypotheses of our statement, but on the other hand $\mathbb{Q}_0 \restriction q_1$ is not ccc there since $r \in H$ forces this. □

Lemma 5. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \tau_\alpha; \alpha < \gamma \rangle$, with $\gamma > 0$, be a finite-support ccc iteration such that for each α there is some poset $\mathbb{Q}_\alpha$ for which*

$$\mathbb{P}_\alpha \Vdash \text{"if } \check{\mathbb{Q}}_\alpha \text{ is ccc then } \tau_\alpha = \check{\mathbb{Q}}_\alpha \text{ and } \tau_\alpha \text{ is trivial otherwise"}$$

Furthermore assume that $\mathbb{P} \Vdash$ "$\check{\mathbb{Q}}_0$ is ccc". Then $\mathbb{P}$ is forcing equivalent to the product $\mathbb{Q}_0 \times \overline{\mathbb{P}}$ for some poset $\overline{\mathbb{P}}$.

Before we give the (technical) proof, let us provide some intuition for this lemma. We can define the iteration $\overline{\mathbb{P}}$ in the same way as $\mathbb{P}$ (i.e. using the same $\mathbb{Q}_\alpha$) but skipping the first step of forcing. The idea is that, by lemma 4, the posets which appear in the iteration $\mathbb{P}$ do not depend on the first stage of forcing $\mathbb{Q}_0$. We thus expect that generics $G \subseteq \mathbb{P}$ will correspond exactly to generics $H \times \overline{G} \subseteq \mathbb{Q}_0 \times \overline{\mathbb{P}}$, since the first stage G_0 of the generic G does not affect the choice of posets in the rest of the iteration.

Proof. We show the lemma by induction on γ, the length of the iteration $\mathbb{P}$. In fact, we shall work with a stronger induction hypothesis. Specifically, we shall show that for each $\alpha < \gamma$ there is a poset $\overline{\mathbb{P}}_\alpha$ such that $\mathbb{Q}_0 \times \overline{\mathbb{P}}_\alpha$ embeds densely into $\mathbb{P}_\alpha$, that the $\overline{\mathbb{P}}_\alpha$ form the initial segments of a finite-support iteration and that the dense embeddings extend one another. For the purposes of this proof we shall take all two-step iterations to be in the style of Kunen, i.e. the conditions in $\mathbb{P} * \tau$ are pairs (p, σ) such that $p \in \mathbb{P}$ and $\sigma \in \mathrm{dom}(\tau)$ and $p \Vdash \sigma \in \tau$. Furthermore we shall assume that the τ_α are full names. These assumptions make no difference for the statement of the lemma, but ensure that certain embeddings will in fact be dense.

Let us start with the base case $\gamma = 2$, when $\mathbb{P} = \mathbb{Q}_0 * \tau_1$. By lemma 4 whether or not $\check{\mathbb{Q}}_1$ is ccc is decided by every condition in $\mathbb{Q}_0$. But then, by our assumption on τ_1, if $\mathbb{Q}_0$ forces $\check{\mathbb{Q}}_1$ to be ccc then τ_1 is forced to be equal to $\check{\mathbb{Q}}_1$ and $\mathbb{Q}_0 \times \mathbb{Q}_1$ embeds densely into $\mathbb{P}$, and otherwise τ_1 is forced to be trivial and $\mathbb{Q}_0$ embeds densely into $\mathbb{P}$. Depending on which is the case, we can thus take either $\overline{\mathbb{P}}_2 = \mathbb{Q}_1$ or $\overline{\mathbb{P}}_2 = 1$.

For the induction step let us assume that the stronger induction hypothesis holds for iterations of length γ and show that it holds for iterations of length $\gamma + 1$. Let us write $\mathbb{P} = \mathbb{P}_\gamma * \tau_\gamma$. By the induction hypothesis there is a $\overline{\mathbb{P}}_\gamma$ such that $\mathbb{Q}_0 \times \overline{\mathbb{P}}_\gamma$ embeds densely into $\mathbb{P}_\gamma$.

Before we give the details, let us sketch the string of equivalences that will yield the desired conclusion. We have

$$\mathbb{P} \equiv (\mathbb{Q}_0 \times \overline{\mathbb{P}}_\gamma) * \bar{\tau}_\gamma \equiv (\overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0) * \bar{\tau}_\gamma \equiv \overline{\mathbb{P}}_\gamma * (\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma)$$

Here $\bar{\tau}_\gamma$ is the $\mathbb{Q}_0 \times \overline{\mathbb{P}}_\gamma$-name (or $\overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0$-name) resulting from pulling back the $\mathbb{P}_\gamma$-name τ_γ along the dense embedding provided by the induction hypothesis. We will specify what exactly we mean by $\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma$ later.

We can apply the base step of the induction to the iteration $\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma$ in $V^{\overline{\mathbb{P}}_\gamma}$ and obtain a $\overline{\mathbb{P}}_\gamma$-name $\dot{\mathbb{R}}$ such that $\check{\mathbb{Q}}_0 \times \dot{\mathbb{R}}$ is forced to densely embed into $\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma$. We can then continue the chain above with

$$\overline{\mathbb{P}}_\gamma * (\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma) \equiv \overline{\mathbb{P}}_\gamma * (\check{\mathbb{Q}}_0 \times \dot{\mathbb{R}}) \equiv \overline{\mathbb{P}}_\gamma * (\dot{\mathbb{R}} \times \check{\mathbb{Q}}_0) \equiv \overline{\mathbb{P}}_{\gamma+1} \times \mathbb{Q}_0$$

where $\overline{\mathbb{P}}_{\gamma+1} = \overline{\mathbb{P}}_\gamma * \dot{\mathbb{R}}$. While this is apparently enough to finish the successor step for the bare statement of the lemma, we wish to preserve the stronger induction hypothesis, and this requires a bit more work.

We first pick a specific $\overline{\mathbb{P}}_\gamma$-name for $\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma$. Let

$$\tau = \{ ((\check{q}_0, \rho), \bar{p}) ; q_0 \in \mathbb{Q}_0, \rho \in \mathrm{dom}(\bar{\tau}_\gamma), \bar{p} \in \overline{\mathbb{P}}_\gamma, (\bar{p}, q_0) \Vdash \rho \in \bar{\tau}_\gamma \}$$

Then $\overline{\mathbb{P}}_\gamma \Vdash \tau = \check{\mathbb{Q}}_0 * \tau_\gamma$. Next we pin down $\dot{\mathbb{R}}$ more. Note that $\overline{\mathbb{P}}_\gamma$ forces that $\check{\mathbb{Q}}_0$ decides whether $\check{\mathbb{Q}}_\gamma$ is ccc or not by lemma 4. Let $A = A_0 \cup A_1 \subseteq \overline{\mathbb{P}}_\gamma$ be a maximal antichain such that each $\bar{p} \in A_0$ forces $\check{\mathbb{Q}}_\gamma$ to not be ccc and each $\bar{p} \in A_1$ forces it to be ccc. Now let

$$\dot{\mathbb{R}} = \{ (\check{1}, \bar{p}) ; \bar{p} \in A_1 \} \cup \{ (\check{q}, \bar{p}) ; q \in \mathbb{Q}_\gamma, \bar{p} \in A_0 \}$$

Observe that $\dot{\mathbb{R}}$ has the properties we require of it: it is forced by $\overline{\mathbb{P}}_\gamma$ that $\dot{\mathbb{R}} = \check{\mathbb{Q}}_\gamma$ if $\check{\mathbb{Q}}_0$ forces that $\check{\mathbb{Q}}_\gamma$ is ccc, and $\dot{\mathbb{R}}$ is trivial otherwise, and that $\check{\mathbb{Q}}_0 \times \dot{\mathbb{R}}$ embeds densely into τ.

Finally, let us define $\overline{\mathbb{P}}_{\gamma+1} = \overline{\mathbb{P}}_\gamma * \dot{\mathbb{R}}$. We can now augment the equivalences given above with dense embeddings:

- The embedding $\overline{\mathbb{P}}_{\gamma+1} \times \mathbb{Q}_0 \hookrightarrow \overline{\mathbb{P}}_\gamma * (\check{\mathbb{Q}}_0 \times \dot{\mathbb{R}})$ is clear.
- To embed $\overline{\mathbb{P}}_\gamma * (\check{\mathbb{Q}}_0 \times \dot{\mathbb{R}})$ into $\overline{\mathbb{P}}_\gamma * \tau$ we can send $(\bar{p}, (\check{q}_0, \rho))$ to $(\bar{p}, (\check{q}_0, \rho'))$ where ρ' is some element of $\mathrm{dom}(\bar{\tau}_\gamma)$ for which $(\bar{p}, q_0) \Vdash \rho = \rho'$ (this is where the fullness of τ_γ is needed).
- With our specific choice of the name τ we in fact get an isomorphism between $\overline{\mathbb{P}}_\gamma * \tau$ and $(\overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0) * \bar{\tau}_\gamma$, given by sending $(\bar{p}, (\check{q}_0, \rho))$ to $((\bar{p}, q_0), \rho)$.
- The final embedding from $(\overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0) * \bar{\tau}_\gamma$ into $\mathbb{P}$ is given by the induction hypothesis.

After composing these embeddings, we notice that the first three steps essentially fixed the $\overline{\mathbb{P}}_\gamma$ part of the condition and the last step fixed the τ_γ part. It follows that the embedding $\overline{\mathbb{P}}_{\gamma+1} \times \mathbb{Q}_0 \hookrightarrow \mathbb{P}_{\gamma+1}$ we constructed extends the embedding $\overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0 \hookrightarrow \mathbb{P}_\gamma$ given by the induction hypothesis. This completes the successor step of the induction.

We now look at the limit step of the induction. The induction hypothesis gives us for each $\alpha < \gamma$ a poset $\overline{\mathbb{P}}_\alpha$ and a dense embedding $\overline{\mathbb{P}}_\alpha \times \mathbb{Q}_0 \hookrightarrow \mathbb{P}_\alpha$ and we also know that the $\overline{\mathbb{P}}_\alpha$ are the initial segments of a finite-support iteration and that the dense embeddings extend each other. If we now let $\overline{\mathbb{P}}_\gamma$ be the direct limit of the $\overline{\mathbb{P}}_\alpha$, we can easily find a dense embedding of $\overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0$ into $\mathbb{P}_\gamma$. Specifically, given a condition $(\bar{p}, q) \in \overline{\mathbb{P}}_\gamma \times \mathbb{Q}_0$, we can find an $\alpha < \gamma$ such that $\bar{p}$ is essentially a condition in $\overline{\mathbb{P}}_\alpha$, since $\overline{\mathbb{P}}_\gamma$ is the direct limit of these. Now we can map $(\bar{p}, q)$ using the stage α dense embedding, landing in $\mathbb{P}_\alpha$ and interpreting this as an element of $\mathbb{P}_\gamma$. This map is independent of the particular choice of α since all the dense embeddings extend one another and it is itself a dense embeddings since all the previous stages were. □

Theorem 6. *Let $\kappa > \omega_1$ be a cardinal of uncountable cofinality satisfying $2^{<\kappa} = \kappa$. Then there is a ccc forcing extension that satisfies* $\mathrm{grMA} + \neg\mathrm{MA} + \mathfrak{c} = \kappa$.

Proof. Fix a well-order $\lhd$ of H_κ of length κ; this can be done since $2^{<\kappa} = \kappa$. We can assume, without loss of generality, that the least element of this order is the poset $\mathrm{Add}(\omega, 1)$. We define a length κ finite-support iteration $\mathbb{P}$ recursively: at stage α we shall force with the next poset with respect to the order $\lhd$ if that is ccc at that stage and with trivial forcing otherwise. Let G be $\mathbb{P}$-generic.

Notice that any poset $\mathbb{Q} \in H_\kappa$ occurs, up to isomorphism, unboundedly often in the well-order $\lhd$. Specifically, we can first find an isomorphic copy whose universe is a set of ordinals bounded in κ and then simply move this universe higher and higher up. In particular, isomorphic copies of Cohen forcing $\mathrm{Add}(\omega, 1)$ appear unboundedly often. Since these are ccc in a highly robust way (being countable), they will definitely be forced with in the iteration $\mathbb{P}$. Therefore we have at least κ many reals in the extension $V[G]$. Since the forcing is ccc of size κ and κ has uncountable cofinality, a nice-name argument shows that the continuum equals κ in the extension.

To see that MA fails in $V[G]$, notice that $\mathbb{P}$ is exactly the type of iteration considered in lemma 5. The lemma then implies that $\mathbb{P}$ is equivalent to $\overline{\mathbb{P}} \times \mathrm{Add}(\omega, 1)$ for some $\overline{\mathbb{P}}$. Therefore the extension $V[G]$ is obtained by adding a Cohen real to some intermediate model. But, as CH fails in the final extension, Roitman has shown in [8] that Martin's Axiom must also fail there.

Finally, we show that the grounded Martin's Axiom holds in $V[G]$ with V as the ground model. Before we consider the general case let us look at the easier situation when κ is regular. Thus, let $\mathbb{Q} \in V$ be a poset of size less than κ which is ccc in $V[G]$ and $\mathcal{D} \in V[G]$ a family of fewer than κ many dense subsets of $\mathbb{Q}$. We can assume without loss of generality that $\mathbb{Q} \in H_\kappa$. Code the elements of $\mathcal{D}$ into a single set $D \subseteq \kappa$ of size $\lambda < \kappa$. Since $\mathbb{P}$ is ccc, the set D has a nice $\mathbb{P}$-name $\dot{D}$ of size λ. Since the iteration $\mathbb{P}$ has finite support, the set D appears before the end of the iteration. As we have argued before, up to isomorphism, the poset $\mathbb{Q}$ appears κ many times in the well-order $\lhd$. Since $\mathbb{Q}$ is ccc in $V[G]$ and $\mathbb{P}$ is ccc, posets isomorphic to $\mathbb{Q}$ will be forced with unboundedly often in the iteration $\mathbb{P}$ and, therefore, eventually a $\mathcal{D}$-generic will be added for $\mathbb{Q}$.

If we now allow κ to be singular we run into the problem that the dense sets in $\mathcal{D}$ might not appear at any initial stage of the iteration $\mathbb{P}$. We solve this issue by using lemma 5 to factor a suitable copy of $\mathbb{Q}$ out of the iteration $\mathbb{P}$ and see it as coming after the forcing that added $\mathcal{D}$.

Let $\mathbb{Q}, \mathcal{D}, D, \lambda$ and $\dot{D}$ be as before. As mentioned, it may no longer be true that $\mathcal{D}$ appears at some initial stage of the iteration. Instead, note that, since $\dot{D}$ has size λ, there are at most λ many indices α such that some condition appearing in $\dot{D}$ has a nontrivial α-th coordinate. It follows that there is a $\delta < \kappa$ such that no condition appearing in $\dot{D}$ has a nontrivial δ-th coordinate and the poset considered at stage δ is isomorphic to $\mathbb{Q}$. Additionally, if we fix a condition $p \in G$ forcing that $\mathbb{Q}$ is ccc in $V[G]$, we can find such a δ beyond the support of p. Now argue for a moment in $V[G_\delta]$. In this intermediate extension the quotient iteration $\mathbb{P}^\delta = \mathbb{P} \restriction [\delta, \kappa)$ is of the type considered in lemma 5 and, since we chose δ beyond the support of p, we also get that $\mathbb{P}^\delta$ forces that $\check{\mathbb{Q}}$ is ccc. The lemma now implies that $\mathbb{P}^\delta$ is equivalent to $\overline{\mathbb{P}} \times \mathbb{Q}$ for some $\overline{\mathbb{P}}$. Moving back to V, we can conclude that $\mathbb{P} \restriction p$ factors as $\mathbb{P}_\delta \restriction p * (\overline{\mathbb{P}} \times \check{\mathbb{Q}}) \equiv (\mathbb{P}_\delta \restriction p * \overline{\mathbb{P}}) \times \mathbb{Q}$ and obtain the corresponding generic $(G_\delta * \overline{G}) \times H$. Furthermore, the name $\dot{D}$ is essentially a $\mathbb{P}_\delta * \overline{\mathbb{P}}$-name, since no condition in $\dot{D}$ has a nontrivial δ-th coordinate. It follows that the set D appears already in $V[G_\delta * \overline{G}]$ and that the final generic $H \in V[G_\delta * \overline{G}][H] = V[G]$ is $\mathcal{D}$-generic for $\mathbb{Q}$. We have thus shown that $(*)$ from lemma 2 holds in $V[G]$, which implies that the grounded Martin's Axiom also holds. □

We should reflect briefly on the preceding proof. If we would have been satisfied with obtaining a model with a regular continuum, the usual techniques would apply. Specifically, if κ were regular then all the dense sets in $\mathcal{D}$ would have appeared by some stage of the iteration, after which we would have forced with (a poset isomorphic to) $\mathbb{Q}$, yielding the desired $\mathcal{D}$-generic. This approach, however, fails if κ is singular, as small sets might not appear before the end of the iteration. Lemma 5 was key in resolving this issue, allowing us to factor the iteration $\mathbb{P}$ as a product and seeing the forcing $\mathbb{Q}$ as happening at the last stage, after the dense sets had already appeared. The lemma implies that the iteration factors at any stage where we considered an absolutely ccc poset (for example, we can factor out any Knaster poset from the ground model). However, somewhat fortuitously, we can also factor out any poset to which we might apply the grounded Martin's Axiom, at least below some condition. There is no surrogate for lemma 5 for the usual Solovay-Tennenbaum iteration and indeed, Martin's Axiom implies that the continuum is regular.

Corollary 7. *The grounded Martin's Axiom is consistent with the existence of a Suslin tree.*

Proof. We saw that the model of grMA constructed in the above proof was obtained by adding a Cohen real to an intermediate extension. Adding that Cohen real also adds a Suslin tree by a result of Shelah [9]. □

A further observation we can make is that the cofinality of κ plays no role in the proof of theorem 6 beyond the obvious König's inequality requirement on the value of the continuum. This allows us to obtain models of the grounded Martin's Axiom with a singular continuum and violate cardinal arithmetic properties which must hold in the presence of Martin's Axiom.

Corollary 8. *The grounded Martin's Axiom is consistent with* $2^{<\mathfrak{c}} > \mathfrak{c}$.

Proof. Starting from some model, perform the construction of theorem 6 with κ singular. In the extension the continuum equals κ. But the desired inequality is true in any model where the continuum is singular: of course $\mathfrak{c} = 2^\omega \leq 2^{\mathrm{cf}(\mathfrak{c})} \leq 2^{<\mathfrak{c}}$ is true but equalities cannot hold since the middle two cardinals have different cofinalities by König's inequality. □

On the other hand, assuming we start with a model satisfying GCH, the model of theorem 6 will satisfy the best possible alternative to $2^{<\mathfrak{c}} = \mathfrak{c}$, namely $2^{<\mathrm{cf}(\mathfrak{c})} = \mathfrak{c}$. Whether this always happens remains open.

Question 9. *Does the grounded Martin's Axiom imply that* $2^{<\mathrm{cf}(\mathfrak{c})} = \mathfrak{c}$*?*

2. The axiom's relation to other fragments of Martin's Axiom

Let us now compare some of the combinatorial consequences of the grounded Martin's Axiom with those of the usual Martin's Axiom. We first make an easy observation.

Proposition 10. *The local grounded Martin's Axiom implies* MA(countable).

Proof. Fix the cardinal $\kappa \geq \mathfrak{c}$ and the ZFC$^-$ ground model $M \subseteq H_{\kappa^+}$ witnessing local grMA. Observe that the the model M contains the poset $\mathrm{Add}(\omega, 1)$, since its elements are effectively coded by the natural numbers. This poset is therefore always a valid target for local grMA. □

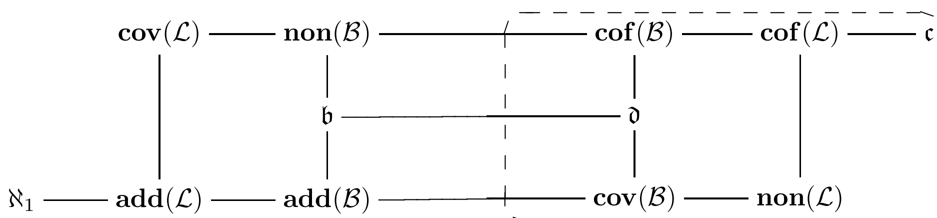
It follows from the above proposition that the (local) grounded Martin's Axiom will have some nontrivial effects on the cardinal characteristics of the continuum. In particular, we obtain the following.

Corollary 11. *The local grounded Martin's Axiom implies that the cardinals on the right side of Cichoń's diagram equal the continuum. In particular, this holds for both the covering number for category* $\mathbf{cov}(\mathcal{B})$ *and the reaping number* $\mathfrak{r}$.

Proof. All of the given equalities follow already from MA(countable); we briefly summarize the arguments from [1].

The complement of any nowhere dense subsets of the real line is dense. It follows that, given fewer than continuum many nowhere dense sets, we can apply MA(countable) to obtain a real number not contained in any of them. Therefore the real line cannot be covered by fewer than continuum many nowhere dense sets and, consequently, also not by fewer than continuum many meagre sets.

To see that the reaping number must be large, observe that, given any infinite $x \subseteq \omega$, there are densely many conditions in $\mathrm{Add}(\omega, 1)$ having arbitrarily large intersection with both x and $\omega \setminus x$. It follows that a Cohen real will split x. Starting with fewer than continuum many reals and applying MA(countable), we can therefore find a real splitting all of them, which means that the original family was not a reaping family. □



But where Martin's Axiom strictly prescribes the size of all cardinal characteristics of the continuum, the grounded Martin's Axiom allows for more leeway in some cases. Observe that, since $\kappa > \omega_1$, the iteration $\mathbb{P}$ of theorem 6 contains $\mathrm{Add}(\omega, \omega_1)$ as an iterand. Thus, by lemma 5, there is a poset $\overline{\mathbb{P}}$ such that $\mathbb{P}$ is equivalent to $\overline{\mathbb{P}} \times \mathrm{Add}(\omega, \omega_1)$.

Theorem 12. *It is consistent that the grounded Martin's Axiom holds,* CH *fails and the cardinal characteristics on the left side of Cichoń's diagram, as well as the splitting number* $\mathfrak{s}$ *are equal to* $\aleph_1$*.*

Proof. Consider a model $V[G]$ of grMA satisfying $\mathfrak{c} > \aleph_1$ which was obtained by forcing with the iteration $\mathbb{P}$ from theorem 6 over a model of GCH. We have argued that this model is obtained by adding $\aleph_1$ many Cohen reals to some intermediate extension. We again briefly summarize the standard arguments for the smallness of the indicated cardinal characteristics in such an extension (see [1] for details).

Let X be the set of ω_1 many Cohen reals added by the final stage of forcing. We claim it is both nonmeager and splitting. Note that any real in $V[G]$ appears before all of the Cohen reals in X have appeared. It follows that every real in $V[G]$ is split by some real in X. Furthermore, if X were meager, it would be contained in a meager Borel set, whose Borel code also appears before all of the reals in X do. But this leads to contradiction, since any Cohen real will avoid any meager set coded in the ground model. □

To summarize, while the grounded Martin's Axiom implies that the right side of Cichoń's diagram is pushed up to $\mathfrak{c}$, it is consistent with the left side dropping to $\aleph_1$ (while CH fails, of course). This is the most extreme way in which the effect of the grounded Martin's Axiom on Cichoń's diagram can differ from that of Martin's Axiom. The precise relationships under grMA between the cardinal characteristics on the left warrant further exploration in the future.

We can consider further the position of the grounded Martin's Axiom within the hierarchy of the more well-known fragments of Martin's Axiom. As we have already mentioned, (local) grMA implies MA(countable). We can strengthen this slightly. Let MA(Cohen) denote Martin's Axiom restricted to posets of the form $\mathrm{Add}(\omega, \lambda)$ for some λ. It will turn out that local grMA also implies MA(Cohen).

Lemma 13. *The axiom* MA(Cohen) *is equivalent to its restriction to posets of the form* $\mathrm{Add}(\omega, \lambda)$ *for* $\lambda < \mathfrak{c}$.

Proof [1]. Let $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$ and fix a collection $\mathcal{D}$ of $\lambda < \mathfrak{c}$ many dense subsets of $\mathbb{P}$. As usual, let $\mathbb{Q}$ be an elementary substructure of $\langle \mathbb{P}, D \rangle_{D \in \mathcal{D}}$ of size λ. We shall show that $\mathbb{Q}$ is isomorphic to $\mathrm{Add}(\omega, \lambda)$. The lemma then follows easily.

To demonstrate the desired isomorphism we shall show that $\mathbb{Q}$ is determined by the single-bit conditions it contains. More precisely, $\mathbb{Q}$ contains precisely those conditions which are meets of finitely many single-bit conditions in $\mathbb{Q}$.

First note that being a single-bit conditions is definable in $\mathbb{P}$: these are precisely the coatoms of the order. Furthermore, given a coatom p, its complementary coatom $\bar{p}$ with the single bit flipped is definable from p as the unique coatom such that any condition is compatible with either p or $\bar{p}$. It follows by elementarity that the coatoms of $\mathbb{Q}$ are precisely the single-bit conditions contained in $\mathbb{Q}$ and that $\mathbb{Q}$ is closed under the operation $p \mapsto \bar{p}$. In $\mathbb{P}$ any finite collection of pairwise compatible coatoms has a meet, therefore the same holds in $\mathbb{Q}$ and the meets agree. Conversely, any given condition in $\mathbb{P}$ uniquely determines the finitely many coatoms it strengthens and therefore all the coatoms determined by conditions in $\mathbb{Q}$ are also in $\mathbb{Q}$. Taken together, this proves the claim.

It follows immediately from the claim that $\mathbb{Q}$ is isomorphic to $\mathrm{Add}(\omega, |X|)$ where X is the set of coatoms of $\mathbb{Q}$, and also that $|X| = |\mathbb{Q}|$. □

Proposition 14. *The local grounded Martin's Axiom implies* MA(Cohen).

Proof. Suppose the local grounded Martin's Axiom holds, witnessed by $\kappa \geq \mathfrak{c}$ and a ZFC^--model $M \subseteq H_{\kappa^+}$. In particular, the height of M is κ^+ and M contains all of the posets $\mathrm{Add}(\omega, \lambda)$ for $\lambda < \kappa^+$. But this means that Martin's Axiom holds for all the posets $\mathrm{Add}(\omega, \lambda)$ where $\lambda < \mathfrak{c}$ and lemma 13 now implies that MA(Cohen) holds. □

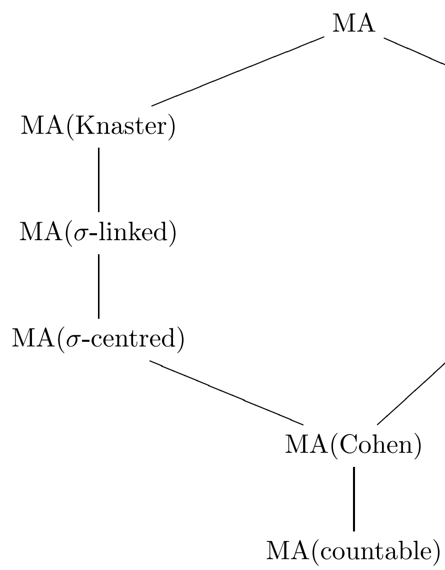
As we have seen, the local grounded Martin's Axiom implies some of the weakest fragments of Martin's Axiom. Theorem 12 tells us, however, that this behaviour stops quite quickly.

Corollary 15. *The grounded Martin's Axiom does not imply* MA(σ-centred).

Proof. By theorem 12 there is a model of the grounded Martin's Axiom where the bounding number is strictly smaller than the continuum. But this is impossible under MA(σ-centred), since applying the axiom to Hechler forcing yields for any family of fewer than continuum many reals a real dominating them all. □

As mentioned earlier, Reitz has shown that we can perform class forcing over any model in such a way that the resulting extension has the same $H_{\mathfrak{c}}$ and is also not a set-forcing extension of any ground model. Performing this construction over a model of MA(σ-centred) (or really any of the standard fragments of Martin's Axiom) shows that MA(σ-centred) does not imply grMA, for the disappointing reason that the final model is not a ccc forcing extension of anything. However, it turns out that already local grMA is independent of MA(σ-centred), and even of MA(Knaster). This places the grounded Martin's Axiom, as well as its local version, outside the usual hierarchy of fragments of Martin's Axiom.

[1]The proof of the key claim was suggested by Noah Schweber.



Theorem 16. *Assume $V = L$ and let $\kappa > \omega_1$ be a regular cardinal. Then there is a ccc forcing extension which satisfies* MA(Knaster) $+ \mathfrak{c} = \kappa$ *and in which the local grounded Martin's Axiom fails.*

Proof. Let $\mathbb{P}$ be the usual finite-support iteration forcing MA(Knaster) $+ \mathfrak{c} = \kappa$. More precisely, we consider the names for posets in H_κ using appropriate bookkeeping and append them to the iteration if they, at that stage, name a Knaster poset. Let $G \subseteq \mathbb{P}$ be generic. We claim that the local grounded Martin's Axiom fails in the extension $L[G]$.

Notice first that $\mathbb{P}$, being a finite-support iteration of Knaster posets, is Knaster. It follows that the product of $\mathbb{P}$ with any ccc poset is still ccc. In particular, forcing with $\mathbb{P}$ preserves the Suslin trees of L.

Now fix a $\lambda \geq \kappa$ and let $M \in L[G]$ be a transitive ZFC$^-$ model of height λ^+. It is straightforward to see that M builds its constructible hierarchy correctly so that, in particular, $L_{\omega_2} \subseteq M$. This implies that M has all of the Suslin trees of L. Since these trees are still Suslin in $L[G]$, partially generic filters do not exist for them and the model M does not witness local grMA in $L[G]$. As λ and M were completely arbitrary, local grMA must fail in $L[G]$. □

Let us mention that it is quite easy to perform ccc forcing over any model and have grMA fail in the extension.

Corollary 17. *Given any model V there is a ccc forcing extension $V[G]$ in which the local grounded Martin's Axiom fails.*

Proof. We may assume that CH fails in V. If $\mathbb{P}$ is the length ω_1 finite-support iteration of Hechler forcing and $G \subseteq \mathbb{P}$ is generic then it is easily seen that G is a dominating family in $V[G]$ and therefore the dominating number of $V[G]$ equals $\aleph_1$. It now follows from corollary 11 that the local grMA fails in $V[G]$. □

In the following two sections we shall explore the other side of the coin: grMA is preserved by certain kinds of ccc forcing.

3. Adding a Cohen real to a model of the grounded Martin's Axiom

An interesting question when studying fragments of Martin's Axiom is what effect adding various kinds of generic reals has on it. It was shown by Roitman [8] that $\mathrm{MA}_{\aleph_1}$ is destroyed after adding a Cohen or a random real. At the same time, it was shown that adding a Cohen real preserves a certain fragment, MA(σ-centred). In this section we follow the spirit of Roitman's arguments to show that the grounded Martin's Axiom is preserved, even with respect to the same ground model, after adding a Cohen real.

It is well known that MA + ¬CH implies that any ccc poset is Knaster (recall that a poset $\mathbb{P}$ is Knaster if any uncountable subset of $\mathbb{P}$ has in turn an uncountable subset of pairwise compatible elements). We start this section by transposing this fact to the grMA setting.

Lemma 18. *Let V satisfy the local grounded Martin's Axiom over the ground model $M \subseteq H_{\kappa^+}$ and suppose CH fails in V. Then any poset $\mathbb{P} \in M$ which is ccc in V is Knaster in V.*

Proof. Let $\mathbb{P}$ be as in the statement of the lemma and let $A = \{\, p_\alpha\,;\, \alpha < \omega_1 \,\} \in V$ be an uncountable subset of $\mathbb{P}$. We first claim that there is a $p^* \in \mathbb{P}$ such that any $q \leq p^*$ is compatible with uncountably many elements of A. For suppose not. Then there would be for any $\alpha < \omega_1$ some $q_\alpha \leq p_\alpha$ which was compatible with only countably many elements of A. We could thus choose $\beta(\alpha) < \omega_1$ in such a way that q_α would be incompatible with any p_β for $\beta(\alpha) \leq \beta$. Setting $\beta_\alpha = \beta^\alpha(0)$ (meaning the α-th iterate of β), this would mean that $\{\, q_{\beta_\alpha}\,;\, \alpha < \omega_1 \,\}$ is an uncountable antichain in $\mathbb{P}$, contradicting the fact that $\mathbb{P}$ was ccc in V.

By replacing $\mathbb{P}$ with the cone below p^* and modifying A appropriately, we may assume that in fact every element of $\mathbb{P}$ is compatible with uncountably many elements of A. We now let $D_\alpha = \bigcup_{\beta \leq \alpha} \mathbb{P} \restriction p_\beta$ for $\alpha < \omega_1$. The sets D_α are dense in $\mathbb{P}$ and by grMA + ¬CH, we can find, in V, a filter $H \subseteq \mathbb{P}$ which intersects every D_α. But then $H \cap A$ is an uncountable set of pairwise compatible elements. □

We now introduce the main technical device we will use in showing that the grounded Martin's Axiom is preserved when adding a Cohen real. In the proof we will be dealing with a two step extension $W \subseteq W[G] \subseteq W[G][c]$ where the first step is some ccc extension, the second adds a Cohen real and $W[G]$ satisfies the grounded Martin's Axiom over W. To utilize the forcing axiom in $W[G]$ in verifying it in $W[G][c]$, we need to find a way of dealing with (names for) dense sets from $W[G][c]$ in $W[G]$. The termspace forcing construction (due to Laver and possibly independently also Abraham, Baumgartner, Woodin and others) comes to mind (for more information on this construction we point the reader to [2]), however the posets arising from this construction are usually quite far from being ccc and are thus unsuitable for our context. We attempt to rectify the situation by radically thinning out the full termspace poset and keeping only the simplest conditions.

Definition 19. Given a poset $\mathbb{P}$, a $\mathbb{P}$-name τ will be called a *finite $\mathbb{P}$-mixture* if there exists a finite maximal antichain $A \subseteq \mathbb{P}$ such that for every $p \in A$ there is some x satisfying $p \Vdash_{\mathbb{P}} \tau = \check{x}$. The antichain A is called a *resolving antichain* for τ and we denote the value x of τ at p by τ^p.

Definition 20. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. The *finite-mixture termspace poset* for $\mathbb{Q}$ over $\mathbb{P}$ is

$$\mathrm{Term}_{\mathrm{fin}}(\mathbb{P},\mathbb{Q}) = \{\,\tau\,;\,\tau \text{ is a finite } \mathbb{P}\text{-mixture and } 1 \Vdash_{\mathbb{P}} \tau \in \check{\mathbb{Q}}\,\}$$

ordered by letting $\tau \leq \sigma$ iff $1 \Vdash_{\mathbb{P}} \tau \leq_{\mathbb{Q}} \sigma$.

As a side remark, let us point out that in all interesting cases the finite-mixture termspace poset is not a regular suborder of the full termspace poset and we can expect genuinely different properties. In fact, this occurs as soon as $\mathbb{P}$ and $\mathbb{Q}$ are nontrivial. To see this, suppose $\{\,p_n\,;\,n<\omega\,\}$ and $\{\,q_n\,;\,n<\omega\,\}$ are infinite antichain in $\mathbb{P}$ and $\mathbb{Q}$, respectively. By mixing we can find a $\tau \in \mathrm{Term}(\mathbb{P},\mathbb{Q})$ such that $p_n \Vdash \tau = q_n$; we claim that this τ does not have a reduction to $\mathrm{Term}_{\mathrm{fin}}(\mathbb{P},\mathbb{Q})$. Suppose $\sigma \in \mathrm{Term}_{\mathrm{fin}}(\mathbb{P},\mathbb{Q})$ were such a reduction. Then there is a condition p in its resolving antichain that is compatible with at least two conditions p_i, say p_0 and p_1. Since σ is a reduction of τ, it and all stronger conditions are compatible with τ, and this means that σ^p is compatible with q_0 and q_1. Let $q' \leq \sigma^p, q_1$ and define a strengthening $\sigma' \leq \sigma$ by setting $\sigma'^p = q'$ and keeping the rest of the mixture the same as in σ. But now σ' and τ are clearly incompatible.

In what follows, let us write $\mathbb{C} = {}^{<\omega}2$. We should mention two key issues with the finite-mixture termspace poset construction. Firstly, the construction is very sensitive to the concrete posets being used. For example, the forthcoming lemma 21 will show that $\mathrm{Term}_{\mathrm{fin}}(\mathbb{C},\mathbb{C})$ is Knaster, but it is not difficult to see that $\mathrm{Term}_{\mathrm{fin}}(\mathrm{ro}(\mathbb{C}),\mathbb{C})$ already has antichains of size continuum. Therefore we cannot freely substitute forcing equivalent posets in the construction. In fact, if $\mathbb{B}$ is a complete Boolean algebra then one easily sees that $\mathrm{Term}_{\mathrm{fin}}(\mathbb{B},\mathbb{Q})$ consists of exactly those names that have only finitely many interpretations and, if $\mathbb{Q}$ is nontrivial and $\mathbb{B}$ has no atoms, this poset will have antichains of size continuum. The second issue is that it is quite rare for a poset to have a large variety of finite maximal antichains. Some, such as ${}^{<\omega}\omega$ or various collapsing posets, have none at all except the trivial one-element maximal antichain, while others, such as ${}^{<\omega_1}2$, have a few, but they do not capture the structure of the poset very well. In all of these cases we do not expect the finite-mixture termspace poset to be of much help. Nevertheless, in the case of Cohen forcing $\mathbb{C}$ it turns out to be a useful tool.

Lemma 21. *If $\mathbb{Q}$ is a Knaster poset then* $\mathrm{Term}_{\mathrm{fin}}(\mathbb{C},\mathbb{Q})$ *is also Knaster.*

Proof. Let $T = \{\,\tau_\alpha\,;\,\alpha<\omega_1\,\}$ be an uncountable subset of $\mathrm{Term}_{\mathrm{fin}}(\mathbb{C},\mathbb{Q})$ and choose resolving antichains A_α for τ_α. By refining the A_α we may assume that each of them is a level of the tree $\mathbb{C}$ and, by thinning out T if necessary, that they are all in fact the same level A. Let us enumerate $A = \{s_0,\ldots,s_k\}$ and write τ_α^i instead of $\tau_\alpha^{s_i}$.

Since $\mathbb{Q}$ is Knaster, an uncountable subset Z_0 of ω_1 such that the set $\{\,\tau_\alpha^0\,;\,\alpha \in Z_0\,\} \subseteq \mathbb{Q}$ consists of pairwise compatible elements. Proceeding recursively, we can find an uncountable $Z \subseteq \omega_1$ such that for every $i \leq k$ the set $\{\,\tau_\alpha^i\,;\,\alpha\in Z\,\} \subseteq \mathbb{Q}$ consists of pairwise compatible elements. We can mix the lower bounds of τ_α^i and τ_β^i over the antichain A to produce a name $\sigma_{\alpha\beta} \in \mathrm{Term}_{\mathrm{fin}}(\mathbb{C},\mathbb{Q})$ such that $\sigma_{\alpha\beta}$ is a lower bound for τ_α and τ_β. Thus $\{\,\tau_\alpha\,;\,\alpha\in Z\,\}$ is an uncountable subset of T consisting of pairwise compatible elements, which proves that $\mathrm{Term}_{\mathrm{fin}}(\mathbb{C},\mathbb{Q})$ is Knaster. □

The following lemma is somewhat awkward, but it serves to give us a way of transforming a name for a dense subset of $\mathbb{Q}$ into a closely related actual dense subset of $\mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q})$. With the usual termspace forcing construction simply taking $E = \{\,\tau\,;\, \Vdash \tau \in \dot{D}\,\}$ would have sufficed, but this set is not dense in $\mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q})$, so modifications are necessary.

Lemma 22. *Let $\mathbb{Q}$ be poset and $\dot{D}$ a $\mathbb{C}$-name for a dense subset of $\mathbb{Q}$. Then for any $n<\omega$ the set*

$$\begin{aligned}E_n = \{\,\tau \in \mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q})\,;\, &\exists A \text{ a resolving antichain for } \tau\, \forall s \in A\colon\\ &n \leq |s| \wedge \exists s' \leq s\colon s' \Vdash \tau \in \dot{D}\,\}\end{aligned}$$

is a dense subset of $\mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q})$.

One can think of the set E_n as the set of those τ that have a sufficiently deep resolving antichain, none of whose elements force τ to not be in $\dot{D}$.

Proof. Let $\sigma \in \mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q})$ and let A be a resolving antichain for it. Any finite refinement of a resolving antichain is, of course, another resolving antichain, so we may assume that we already have $n \leq |s|$ for all $s \in A$. By fullness we can find a name ρ for an element of $\mathbb{Q}$ such that $1 \Vdash_{\mathbb{C}} (\rho \leq \sigma \wedge \rho \in \dot{D})$. For each $s \in A$ we can find an $s' \leq s$ such that $s' \Vdash_{\mathbb{C}} \rho = \check{q}_s$ for some $q_s \in \mathbb{Q}$. By mixing the q_s over the antichain A, we get a name $\tau \in E_n$ such that $\tau \leq \sigma$, which shows that E is dense in $\mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q})$. □

Theorem 23. *Assume the local grounded Martin's Axiom holds in V over the ground model $M \subseteq H_{\kappa^+}$ and let $V[c]$ be obtained by adding a Cohen real to V. Then $V[c]$ also satisfies the local grounded Martin's Axiom over the ground model $M \subseteq H_{\kappa^+}^{V[c]} = H_{\kappa^+}[c]$.*

Proof. By assumption there is a ccc poset $\mathbb{P} \in M$ such that $H_{\kappa^+}^V = M[G]$ for an M-generic $G \subseteq \mathbb{P}$. We may assume that CH fails in V, for otherwise it would also hold in the final extension$V[c]$, which would then satisfy the full Martin's Axiom. Consider a poset $\mathbb{Q} \in M$ which is ccc in $V[c]$. Since $\mathbb{C}$ is ccc, $\mathbb{Q}$ must also be ccc in V and by lemma 18 is in fact Knaster in V.

Let $\lambda < \mathfrak{c}^{V[c]}$ be a cardinal and let $\mathcal{D} = \{\,D_\alpha\,;\, \alpha<\lambda\,\} \in V[c]$ be a collection of dense subsets of $\mathbb{Q}$. Pick names $\dot{D}_\alpha \in V$ for these such that $1 \Vdash_{\mathbb{C}}$ "$\dot{D}_\alpha \subseteq \check{\mathbb{Q}}$ is dense".

Consider $\mathbb{R} = \mathrm{Term_{fin}}(\mathbb{C},\mathbb{Q}) \in M$. Note that $\mathbb{R}$ is computed the same in M and in V. It now follows from lemma 21 that $\mathbb{R}$ is Knaster in V (although not necessarily in M).

Let $E_{\alpha,n} \subseteq \mathbb{R}$ be the dense sets associated to the $\dot{D}_\alpha$ as in lemma 22. Write $\mathcal{E} = \{\,E_{\alpha,n}\,;\, \alpha<\lambda, n<\omega\,\}$. Applying the grounded Martin's Axiom in V, we get an $\mathcal{E}$-generic filter $H \subseteq \mathbb{R}$. We will show that the filter generated by the set $H^c = \{\,\tau^c\,;\, \tau \in H\,\}$ is $\mathcal{D}$-generic. Pick a $\dot{D}_\alpha$ and consider the set

$$\begin{aligned}B_\alpha = \{\,s' \in \mathbb{C}\,;\, &\exists \tau \in H \exists A \text{ a resolving antichain for } \tau\, \exists s \in A\colon\\ &s' \leq s \wedge s' \Vdash_{\mathbb{C}} \tau \in \dot{D}_\alpha\,\}\end{aligned}$$

We will show that B_α is dense in $\mathbb{C}$. To that end, pick a $t \in \mathbb{C}$. Since H is $\mathcal{E}$-generic, there is some $\tau \in H \cap E_{\alpha,|t|}$. Let A be a resolving antichain for τ. Since A is maximal, t must be compatible with some $s \in A$, and, since $|t| \leq |s|$, we must in fact have $s \leq t$. But then, by the definition of $E_{\alpha,|t|}$, there exists a $s' \leq s$ such that

$s' \Vdash_{\mathbb{C}} \tau \in \dot{D}$. This exactly says that $s' \in B_\alpha$ and also $s' \leq s \leq t$. Thus B_α really is dense in $\mathbb{C}$.

By genericity we can find an $s' \in B_\alpha \cap c$. If $\tau \in H$ is the corresponding name, the definition of B_α now implies that $\tau^c \in D_\alpha \cap H^c$. Thus H^c really does generate a $\mathcal{D}$-generic filter. □

The proof is easily adapted to show that, starting from the full grounded Martin's Axiom in V over a ground model W, we obtain the full grounded Martin's Axiom in $V[c]$ over the same ground model W.

4. Adding a random real to a model of the grounded Martin's Axiom

Our next goal is to prove a preservation theorem for adding a random real. The machinery of the proof in the Cohen case will be slightly modified to take advantage of the measure theoretic structure in this context.

Recall that a measure algebra is a pair $(\mathbb{B}, m)$ where $\mathbb{B}$ is a complete Boolean algebra and $m\colon \mathbb{B} \to [0,1]$ is a countably additive map such that $m(b) = 1$ iff $b = 1$.

Definition 24. Let $(\mathbb{B}, m)$ be a measure algebra and $0 < \varepsilon < 1$. A $\mathbb{B}$-name τ will be called an *ε-deficient finite $\mathbb{B}$-mixture* if there is a finite antichain $A \subseteq \mathbb{B}$ such that $m(\sup A) > 1 - \varepsilon$ and for every $w \in A$ there exists some x such that $w \Vdash_{\mathbb{B}} \tau = \check{x}$. The antichain A is called a *resolving antichain* and we denote the value x of τ at w by τ^w.

Definition 25. Let $(\mathbb{B}, m)$ be a measure algebra, $\mathbb{Q}$ a poset and $0 < \varepsilon < 1$. The *ε-deficient finite mixture termspace poset* for $\mathbb{Q}$ over $(\mathbb{B}, m)$ is

$$\mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}, \mathbb{Q}) = \{\, \tau\,;\, \tau \text{ is an } \varepsilon\text{-deficient finite } \mathbb{B}\text{-mixture and } 1 \Vdash_{\mathbb{B}} \tau \in \check{\mathbb{Q}} \,\}$$

ordered by letting $\tau \leq \sigma$ iff there are resolving antichains A_τ and A_σ such that A_τ refines A_σ and $\sup A_\tau \Vdash_{\mathbb{B}} \tau \leq \sigma$.

The following lemma is the analogue of lemma 22 for ε-deficient finite mixtures.

Lemma 26. *Let $(\mathbb{B}, m)$ be a measure algebra, $\mathbb{Q}$ a poset and $0 < \varepsilon < 1$. If $\dot{D}$ is a $\mathbb{B}$-name for a dense subset of $\mathbb{Q}$ then*

$$E = \{\, \tau \in \mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}, \mathbb{Q})\,;\, \exists A \text{ a resolving antichain for } \tau\colon \sup A \Vdash_{\mathbb{B}} \tau \in \dot{D} \,\}$$

is dense in $\mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}, \mathbb{Q})$.

Proof. Let $\sigma \in \mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}, \mathbb{Q})$ and pick a resolving antichain $A = \{w_0, \ldots, w_n\}$ for it. Let $\delta = m(\sup A) - (1 - \varepsilon)$. By fullness there are $\mathbb{B}$-names ρ_i for elements of $\mathbb{Q}$ such that $w_i \Vdash \rho_i \leq \sigma \wedge \rho \in \dot{D}$. There are maximal antichains A_i below w_i such that each element of A_i decides the value of ρ_i. We now choose finite subsets $A_i' \subseteq A_i$ such that $m(\sup A_i) - m(\sup A_i') < \frac{\delta}{n}$. Write $A' = \bigcup_i A_i'$. We then have $m(\sup A') > 1 - \varepsilon$. By mixing we can find a $\mathbb{B}$-name τ for an element of $\mathbb{Q}$ which is forced by each element of A' to be equal to the appropriate ρ_i. Thus A' is a resolving antichain for τ and we have ensured that τ is in E and $\sigma \leq \tau$. □

In what follows we let $\mathbb{B}_{\mathrm{null}}$ be the random Boolean algebra with the induced Lebesgue measure μ. The next lemma is the analogue of lemma 21.

Lemma 27. *Let $\mathbb{Q}$ be a Knaster poset and $0 < \varepsilon < 1$. Then $\mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}_{\mathrm{null}}, \mathbb{Q})$ is Knaster as well.*

Proof. Let $\{\, \tau_\alpha\,;\, \alpha < \omega_1 \,\}$ be an uncountable subset of $\mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}_{\mathrm{null}}, \mathbb{Q})$. Choose resolving antichains A_α for the τ_α. We may assume that there is a fixed δ such that $1-\varepsilon < \delta < \mu(\sup A_\alpha)$ for all α. We may also assume that all of the A_α have the same size n and enumerate them as $A_\alpha = \{w^0_\alpha, \ldots, w^{n-1}_\alpha\}$; we shall write τ^i_α instead of $\tau_\alpha^{w^i_\alpha}$. By inner regularity of the measure we may assume further that the elements of each A_α are compact. Using this and the outer regularity of the measure we can find open neighbourhoods $w^i_\alpha \subseteq U^i_\alpha$ such that U^i_α and U^j_α are disjoint for all α and distinct i and j and additionally satisfy

$$\mu(U^i_\alpha \setminus w^i_\alpha) < \frac{\delta - (1-\varepsilon)}{n}$$

Fix a countable basis for the topology. Since the w^i_α are compact, we may take the U^i_α to be finite unions of basic opens. Since there are only countably many such finite unions, we can assume that there are fixed U^i such that $U^i_\alpha = U^i$ for all α.

We now obtain

$$\begin{aligned} \mu(w^i_\alpha \cap w^i_\beta) &= \mu(w^i_\alpha) - \mu(w^i_\alpha \cap (U^i \setminus w^i_\beta)) \\ &\geq \mu(w^i_\alpha) - \mu(U^i \setminus w^i_\beta) > \mu(w^i_\alpha) - \frac{\delta-(1-\varepsilon)}{n} \end{aligned}$$

In particular, this gives that $\sum_i \mu(w^i_\alpha \cap w^i_\beta) > 1-\varepsilon$.

Since $\mathbb{Q}$ is Knaster we may assume that the elements of $\{\, \tau^i_\alpha\,;\, \alpha < \omega_1 \,\}$ are pairwise compatible and that this holds for any i. Pick lower bounds $q^i_{\alpha\beta}$ for the τ^i_α and τ^i_β. By mixing we can construct $\mathbb{B}_{\mathrm{null}}$-names $\sigma_{\alpha\beta}$ for elements of $\mathbb{Q}$ such that $w^i_\alpha \cap w^i_\beta \Vdash \sigma_{\alpha\beta} = q^i_{\alpha\beta}$ for all i. By construction the $\sigma_{\alpha\beta}$ are ε-deficient finite $\mathbb{B}_{\mathrm{null}}$-mixtures and are lower bounds for τ_α and τ_β. □

While the concept of ε-deficient finite mixtures makes sense for any measure algebra, finding a good analogue of the preceding proposition for algebras of uncountable weight has proven difficult.

Lemma 28. *Let $(\mathbb{B}, m)$ be a measure algebra and $0 < \varepsilon < 1$. Suppose $\mathcal{A}$ is a family of finite antichains in $\mathbb{B}$, downward directed under refinement, such that $m(\sup A) > 1-\varepsilon$ for any $A \in \mathcal{A}$. If we let $d_{\mathcal{A}} = \inf\{\sup A\,;\, A \in \mathcal{A}\}$ then $m(d_{\mathcal{A}}) \geq 1-\varepsilon$.*

Proof. By passing to complements it suffices to prove the following statement: if I is an upward directed subset of $\mathbb{B}$ all of whose elements have measure less than ε then $\sup I$ has measure at most ε.

Using the fact that $\mathbb{B}$ is complete, we can refine I to an antichain Z that satisfies $\sup I = \sup Z$. Since $\mathbb{B}$ is ccc, Z must be countable. Applying the upward directedness of I and the countable additivity of the measure, we can conclude that $m(\sup Z) \leq \varepsilon$. □

We are finally ready to state and prove the preservation theorem we have been building towards.

Theorem 29. *Assume Martin's Axiom holds in V and let $V[r]$ be obtained by adding a random real to V. Then $V[r]$ satisfies the grounded Martin's Axiom over the ground model V.*

Proof. If CH holds in V then it holds in $V[r]$ as well, implying that $V[r]$ satisfies the full Martin's Axiom. We may therefore assume without loss of generality that V satisfies MA + ¬CH.

Assume toward a contradiction that $V[r]$ does not satisfy the grounded Martin's Axiom over V. Then there exist a poset $\mathbb{Q} \in V$ which is ccc in $V[r]$, a cardinal $\kappa < \mathfrak{c}$ and a collection $\mathcal{D} = \{ D_\alpha ; \alpha < \kappa \} \in V[r]$ of dense subsets of $\mathbb{Q}$ such that $V[r]$ has no $\mathcal{D}$-generic filters on $\mathbb{Q}$. There must be a condition $b_0 \in \mathbb{B}_{\text{null}}$ forcing this. Let $\varepsilon < \mu(b_0)$.

Since $\mathbb{B}_{\text{null}}$ is ccc, $\mathbb{Q}$ must be ccc in V and, since MA + ¬CH holds there, is also Knaster there. Thus $\text{Term}^{\varepsilon}_{\text{fin}}(\mathbb{B}_{\text{null}}, \mathbb{Q}) \in V$ is Knaster by lemma 27. We now choose names $\dot{D}_\alpha$ for the dense sets D_α such that $\mathbb{B}_{\text{null}}$ forces that the $\dot{D}_\alpha$ are dense subsets of $\mathbb{Q}$. Then, by lemma 26, the E_α are dense in $\text{Term}^{\varepsilon}_{\text{fin}}(\mathbb{B}_{\text{null}}, \mathbb{Q})$, where E_α is defined from $\dot{D}_\alpha$ as in that lemma. We can thus obtain, using Martin's Axiom in V, a filter H on $\text{Term}^{\varepsilon}_{\text{fin}}(\mathbb{B}_{\text{null}}, \mathbb{Q})$ which meets all of the E_α.

Pick a resolving antichain A_τ for each $\tau \in H$ and consider $\mathcal{A} = \{ A_\tau ; \tau \in H \}$. This family satisfies the hypotheses of lemma 28, whence we can conclude that $\mu(d_{\mathcal{A}}) \geq 1 - \varepsilon$, where $d_{\mathcal{A}}$ is defined as in that lemma. Interpreting H as a $\mathbb{B}_{\text{null}}$-name for a subset of $\mathbb{Q}$, we now observe that

$$d_{\mathcal{A}} \Vdash_{\mathbb{B}_{\text{null}}} \text{“}H \text{ generates a } \dot{\mathcal{D}}\text{-generic filter on } \mathbb{Q}\text{”}$$

Now, crucially, since we have chosen $\varepsilon < \mu(b_0)$, the conditions b_0 and d_H must be compatible in $\mathbb{B}_{\text{null}}$. But this is a contradiction, since they force opposing statements. Therefore $V[r]$ really does satisfy the grounded Martin's Axiom over V. □

Corollary 30. *The grounded Martin's Axiom is consistent with there being no Suslin trees.*

Proof. If Martin's Axiom holds in V and r is random over V then $V[r]$ satisfies the grounded Martin's Axiom by the above theorem and also has no Suslin trees by a theorem of Laver [6]. □

Unfortunately, the employed techniques do not seem to yield the full preservation result as in theorem 23. If V satisfied merely the grounded Martin's Axiom over a ground model W we would have to argue that the poset $\text{Term}^{\varepsilon}_{\text{fin}}(\mathbb{B}_{\text{null}}, \mathbb{Q})$ as computed in V was actually an element of W, so that we could apply grMA to it. But we cannot expect this to be true if passing from W to V added reals; not only will the termspace posets be computed differently in W and in V, even the random Boolean algebras of these two models will be different. Still, these considerations lead us to the following improvement to the theorem above.

Theorem 31. *Assume the grounded Martin's Axiom holds in V over the ground model W via a forcing which is countably distributive (or, equivalently, does not add reals), and let $V[r]$ be obtained by adding a random real to V. Then $V[r]$ also satisfies the grounded Martin's Axiom over the ground model W.*

Proof. By assumption there is a ccc countably distributive poset $\mathbb{P} \in W$ such that $V = W[G]$ for some W-generic $G \subseteq \mathbb{P}$. Since W and V thus have the same reals, they must also have the same Borel sets. Furthermore, since the measure is inner regular, a Borel set having positive measure is witnessed by a positive measure compact (i.e. closed) subset, which means that W and V agree on which Borel sets are null. It follows that the random Boolean algebras as computed in W and in V are the same.

Now let $0 < \varepsilon < 1$ and let $\mathbb{Q} \in W$ be a poset which is ccc in V. We claim that V and W compute the poset $\mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}_{\mathrm{null}}, \mathbb{Q})$ the same. Clearly any ε-deficient finite mixture in W is also such in V, so we really only need to see that V has no new such elements. But $\mathbb{B}_{\mathrm{null}}$ is ccc, which means that elements of $\mathbb{Q}$ have countable nice names and these could not have been added by G. So V and W in fact agree on the whole termspace poset $\mathrm{Term}(\mathbb{B}_{\mathrm{null}}, \mathbb{Q})$, and therefore also on the ε-deficient finite mixtures.

The rest of the proof proceeds as in theorem 29. The key step there, where we apply Martin's Axiom to the poset $\mathrm{Term}^{\varepsilon}_{\mathrm{fin}}(\mathbb{B}_{\mathrm{null}}, \mathbb{Q})$, goes through, since we have shown that this poset is in W and we may therefore apply grMA to it. □

Just as in theorem 23 we may replace the grounded Martin's Axiom in the above theorem with its local version.

It is not immediately obvious that the hypothesis of the above theorem is ever satisfied in a nontrivial way, that is, whether grMA can ever hold via a *nontrivial* countably distributive extension. The following theorem, due to Larson, shows that this does happen and gives yet another construction of a model of grMA. For the purposes of this theorem we shall call a Suslin tree T *homogeneous* if for any two nodes $p, q \in T$ of the same height, the cones below them are isomorphic. Note that homogeneous Suslin trees may be constructed from $\diamondsuit$.

Theorem 32 (Larson [5]). *Let $\kappa > \omega_1$ be a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let T be a homogeneous Suslin tree. Then there is a ccc poset $\mathbb{P}$ such that, given a V-generic $G \subseteq \mathbb{P}$, the tree T remains Suslin in $V[G]$ and, if b is a generic branch through T, the extension $V[G][b]$ satisfies $\mathfrak{c} = \kappa$ and the grounded Martin's Axiom over the ground model $V[G]$.*

Proof. The idea is to attempt to force $\mathrm{MA} + \mathfrak{c} = \kappa$, but only using posets that preserve the Suslin tree T. More precisely, fix a well-order $\lhd$ of H_κ of length κ and define $\mathbb{P}$ as the length κ finite support iteration which forces at stage α with the next $\mathbb{P}_\alpha$-name for a poset $\dot{\mathbb{Q}}_\alpha$ such that $\mathbb{P}_\alpha$ forces that $T \times \dot{\mathbb{Q}}_\alpha$ is ccc.

Let $G \subseteq \mathbb{P}$ be V-generic. It is easy to see by induction that $\mathbb{P}_\alpha \times T$ is ccc for all $\alpha \leq \kappa$; the successor case is clear from the definition of the iteration $\mathbb{P}$ and the limit case follows by a Δ-system argument. We can thus conclude that T remains a Suslin tree in $V[G]$. Furthermore, standard arguments show that there are exactly κ many reals in $V[G]$ and that this extension satisfies Martin's Axiom for small posets which preserve T, i.e. those $\mathbb{Q}$ such that $\mathbb{Q} \in H_\kappa^{V[G]}$ and $\mathbb{Q} \times T$ is ccc.

Finally, let us see that adding a branch b through T over $V[G]$ yields a model of the grounded Martin's Axiom over $V[G]$. Thus let $\mathbb{Q} \in V[G]$ be a poset which is ccc in $V[G][b]$ and has size less than κ there. There is a condition in T forcing that $\mathbb{Q}$ is ccc, so by our homogeneity assumption T forces this, meaning that $\mathbb{Q} \times T$ is ccc in $V[G]$. The key point now is that, since T is countably distributive, all of the maximal antichains (and open dense subsets) of $\mathbb{Q}$ in $V[G][b]$ are already in $V[G]$. Furthermore, any collection $\mathcal{D}$ of less than κ many of these in $V[G][b]$ can be covered by some $\widetilde{\mathcal{D}}$ in $V[G]$ of the same size. Our observation from the previous paragraph then yields a $\widetilde{\mathcal{D}}$-generic filter for $\mathbb{Q}$ in $V[G]$ and therefore $V[G][b]$ satisfies grMA over $V[G]$ by lemma 2. □

Starting from a Suslin tree with a stronger homogeneity property, Larson also shows that there are no Suslin trees in the extension $V[G][b]$ above. This gives an alternative proof of corollary 30.

From the argument of theorem 32 we can actually extract another preservation result for grMA.

Theorem 33. *Assume the grounded Martin's Axiom holds in V over the ground model W and let $T \in V$ be a Suslin tree. If $b \subseteq T$ is a generic branch then $V[b]$ also satisfies the grounded Martin's Axiom over the ground model W.*

Proof. The point is that, just as in the proof of theorem 32, forcing with T does not add any new maximal antichains to posets from W that remain ccc in $V[b]$ and any collection of these antichains in $V[b]$ can be covered by a collection of the same size in V. □

Starting from a Suslin tree with a stronger homogeneity property, Larson also shows that there are no Suslin trees at all in the extension $V[G][b]$ above. This shows that grMA + ¬MA + "there are no Suslin trees" is consistent (although this is also true in our model from theorem 29 by a result of Laver [6]). On the other hand, our models from theorems 6 and 23 show the consistency of grMA + "there is a Suslin tree" by a result of Shelah [9].

If grMA holds over a ground model that reals have been added to, it seems harder to say anything about preservation after adding a further random real. Nevertheless, we fully expect the answers to the following questions to be positive.

Question 34. *Does adding a random real to a model of* grMA *preserve* grMA*? Does it preserve it with the same witnessing ground model?*

Generalizations of theorems 23 and 29 to larger numbers of reals added seem the natural next step in the exploration of the preservation phenomena of the grounded Martin's Axiom. Such preservation results would also help in determining the compatibility of grMA with various configurations of the cardinal characteristics on the left side of Cichoń's diagram. The constructions $\mathrm{Term}_{\mathrm{fin}}$ and $\mathrm{Term}_{\mathrm{fin}}^{\varepsilon}$ seem promising, but obtaining a good chain condition in any case at all, except those shown, has proven difficult.

5. The grounded Proper Forcing Axiom

We can, of course, also consider grounded versions of other forcing axioms. We define one and note that similar definitions can be made for grSPFA, grMM and so on.

Definition 35. The *grounded Proper Forcing Axiom* (grPFA) asserts that V is a forcing extension of some ground model W by a proper poset and V satisfies the conclusion of the Proper Forcing Axiom for posets $\mathbb{Q} \in W$ which are still proper in V.

Theorem 36. *Let κ be supercompact. Then there is a proper forcing extension that satisfies* $\mathrm{grPFA} + \mathfrak{c} = \kappa = \omega_2$ *and in which* PFA*, and even* MA*, fails.*

Proof. Start with a Laver function ℓ for κ and build a countable-support forcing iteration $\mathbb{P}$ of length κ which forces at stage α with $\dot{\mathbb{Q}}$, some full name for the poset $\mathbb{Q} = \ell(\alpha)$ if it is proper at that stage and with trivial forcing otherwise. Note that

$\mathbb{P}$ is proper. Now let $V[G][H]$ be a forcing extension by $\mathbb{R} = \mathrm{Add}(\omega, 1) \times \mathbb{P}$. We claim that $V[G][H]$ is the required model.

Since the Laver function ℓ will quite often output the poset $\mathrm{Add}(\omega, 1)$ and this will always be proper, the iteration $\mathbb{P}$ will add reals unboundedly often. Furthermore, since $\mathbb{R}$ is κ-cc, we will obtain $\mathfrak{c} = \kappa$ in $V[G][H]$.

Next we wish to see that $\kappa = \omega_2^{V[G][H]}$. For this it suffices to see that any $\omega_1 < \lambda < \kappa$ is collapsed at some point during the iteration. Recall the well-known fact that any countable-support iteration of nontrivial posets adds a Cohen subset of ω_1 at stages of cofinality ω_1 and therefore collapses the continuum to ω_1 at those stages. Now fix some $\omega_1 < \lambda < \kappa$. Since $\mathbb{P}$ ultimately adds κ many reals and is κ-cc, there is some stage α of the iteration such that $\mathbb{P}_\alpha$ has already added λ many reals and therefore $\mathfrak{c}^{V[H_\alpha]} \geq \lambda$. Since $\mathbb{P}_\alpha$ is proper, the rest of the iteration $\mathbb{P} \restriction [\alpha, \kappa)$ is a countable-support iteration in $V[H_\alpha]$ and the fact mentioned above implies that λ is collapsed to ω_1 by this tail of the iteration.

Note that $\mathbb{R}$ is proper, since $\mathbb{P} * \mathrm{Add}(\omega, 1)$ is proper and has a dense subset isomorphic to $\mathbb{R}$. To verify that grPFA holds in $V[G][H]$ let $\mathbb{Q} \in V$ be a poset that is proper in $V[G][H]$ and let $\mathcal{D} = \{ D_\alpha \,;\, \alpha < \omega_1 \} \in V[G][H]$ be a family of dense subsets of $\mathbb{Q}$. In V we can fix (for some large enough θ) a θ-supercompactness embedding $j \colon V \to M$ such that $j(\ell)(\kappa) = \mathbb{Q}$. Since the Cohen real forcing is small, the embedding j lifts to a θ-supercompactness embedding $j \colon V[G] \to M[G]$. We can factor $j(\mathbb{P})$ in $M[G]$ as $j(\mathbb{P}) = \mathbb{P} * \mathbb{Q} * \mathbb{P}_{\mathrm{tail}}$. Let $h * H_{\mathrm{tail}} \subseteq \mathbb{Q} * \mathbb{P}_{\mathrm{tail}}$ be $V[G][H]$-generic. As usual, we can now lift the embedding j in $V[G][H * h * H_{\mathrm{tail}}]$ to $j \colon V[G][H] \to \overline{M} = M[G][H * h * H_{\mathrm{tail}}]$. Note that the closure of this embedding implies that $j[h] \in \overline{M}$. But $j[h]$ is a $j(\mathcal{D})$-generic filter on $j(\mathbb{Q})$ in $\overline{M}$ and so, by elementarity, there is a $\mathcal{D}$-generic filter on $\mathbb{Q}$ in $V[G][H]$.

Finally, since we can see $V[G][H]$ as obtained by adding a Cohen real to an intermediate extension and since CH fails there, PFA and even Martin's Axiom must fail there by Roitman's [8]. □

With regard to the above proof, we should mention that one usually argues that κ becomes ω_2 after an iteration similar to ours because at many stages the poset forced with was explicitly a collapse poset $\mathrm{Coll}(\omega_1, \lambda)$. In our case, however, the situation is different. It turns out that a significant number of proper posets from V (the collapse posets among them) cease to be proper as soon as we add the initial Cohen real. Therefore the possibility of choosing $\mathrm{Coll}(\omega_1, \lambda)$ never arises in the construction of the iteration $\mathbb{P}$ and a different argument is needed. We recount a proof of this fact below. The argument is essentially due to Shelah, as communicated by Goldstern in [4].

Theorem 37 (Shelah)**.** *Let $\mathbb{P}$ be a ccc poset and let $\mathbb{Q}$ be a countably distributive poset which collapses ω_2. Let $G \subseteq \mathbb{P}$ be V-generic. If $V[G]$ has a new real then $\mathbb{Q}$ is not proper in $V[G]$.*

Proof. Fix at the beginning a $\mathbb{Q}$-name $\dot{f}$, forced to be a bijection between ω_1 and ω_2^V. Let θ be a sufficiently large regular cardinal. By claim XV.2.12 of [10] we can label the nodes $s \in {}^{<\omega}2$ with countable models $M_s \prec H_\theta$ such that:

- the M_s are increasing along each branch of the tree ${}^{<\omega}2$;
- $\mathbb{P}, \mathbb{Q}, \dot{f} \in M_\emptyset$;
- there is an ordinal δ such that $M_s \cap \omega_1 = \delta$ for all s;

- for any s there are ordinals $\alpha_s < \beta_s < \omega_2$ such that $\alpha_s \in M_{s^\frown 0}$ and $\alpha_s \notin M_{s^\frown 1}$ and $\beta_s \in M_{s^\frown 1}$.

Now consider, in $V[G]$, the tree of models $M_s[G]$. By the argument given in the proof of lemma 2, the models $M_s[G]$ are elementary in $H_\theta^{V[G]}$ and, since $\mathbb{P}$ is ccc, we still have $M_s[G] \cap \omega_1 = \delta$. Let $M = M_r[G]$ be the branch model determined by the new real $r \in V[G]$. We shall show that there are no M-generic conditions in $\mathbb{Q}$.

Suppose that q were such a generic condition. We claim that q forces that $\dot{f} \restriction \delta$ maps onto $M \cap \omega_2^V$ (note that $\dot{f}$ still names a bijection $\omega_1 \to \omega_2^V$ over $V[G]$). First, suppose that q does not force that $\dot{f}[\delta] \subseteq M$. Then we can find $q' \leq q$ and an $\alpha < \delta$ such that $q' \Vdash \dot{f}(\alpha) \notin M$. But if $q' \in H \subseteq \mathbb{Q}$ is generic then $M[H]$ is an elementary substructure of $H_\theta^{V[G][H]}$ and, of course, $f, \alpha \in M[H]$, leading to a contradiction.

Conversely, suppose that q does not force that $M \cap \omega_2^V \subseteq \dot{f}[\delta]$. We can again find $q' \leq q$ and an $\alpha \in M \cap \omega_2$ such that $q' \Vdash \alpha \notin \dot{f}[\delta]$. Let $q' \in H \subseteq \mathbb{Q}$ be generic. As before, $M[H]$ is an elementary substructure of $H_\theta^{V[G][H]}$ and $f, \alpha \in M[H]$. Since $f\colon \omega_1 \to \omega_2^V$ is a bijection, we must have $f^{-1}(\alpha) \in M[H]$. But by construction $f^{-1}(\alpha)$ is an ordinal greater than δ while simultaneously $M[H] \cap \omega_1 = \delta$ by the M-genericity of q, giving a contradiction.

Fixing our putative generic condition q, we can use the countable distributivity of $\mathbb{Q}$ in V to see that $\dot{f} \restriction \delta$, and consequently $M \cap \omega_2$, exist already in V. But we can extract r from $M \cap \omega_2$.

Notice that, given a model $M_{s^\frown 1}$ in our original tree, no elementary extension $M_{s^\frown 1} \prec X$ satisfying $X \cap \omega_1 = \delta$ can contain α_s. This is because $M_{s^\frown 1}$ contains a bijection $g\colon \omega_1 \to \beta_s$ and, by elementarity, g must restrict to a bijection between δ and $M_{s^\frown 1} \cap \beta_s$. But seen from the viewpoint of X, that same function g must restrict to a bijection between δ and $X \cap \beta_s$ and so $X \cap \beta_s = M_{s^\frown 1} \cap \beta_s$.

Using this fact, we can now extract r from $M \cap \omega_2$ in V. Specifically, we can decide at each stage whether the branch determined by r went left or right depending on whether $\alpha_s \in M \cap \omega_2$ or not. We conclude that r appears already in V, contradicting our original assumption. Therefore there is no generic condition q as above and $\mathbb{Q}$ is not proper in $V[G]$. □

Ultimately, one hopes that by grounding the forcing axiom we lower its consistency strength while still being able to carry out at least some of the usual arguments and obtain at least some of the standard consequences. However, theorem 37 severely limits the kind of arguments we can carry out under grPFA. Many arguments involving PFA use, among other things, collapsing posets such as $\mathrm{Coll}(\omega_1, 2^\omega)$. In contrast, if the poset witnessing grPFA in a model is any kind of iteration that at some stage added, say, a Cohen real, the theorem prevents us from applying the forcing axiom to any of these collapsing posets. It is thus unclear exactly how much strength of PFA can be recovered from grPFA. In particular, while grPFA implies that CH fails, the following key question remains open:

Question 38. *Does* grPFA *imply that the continuum equals ω_2?*

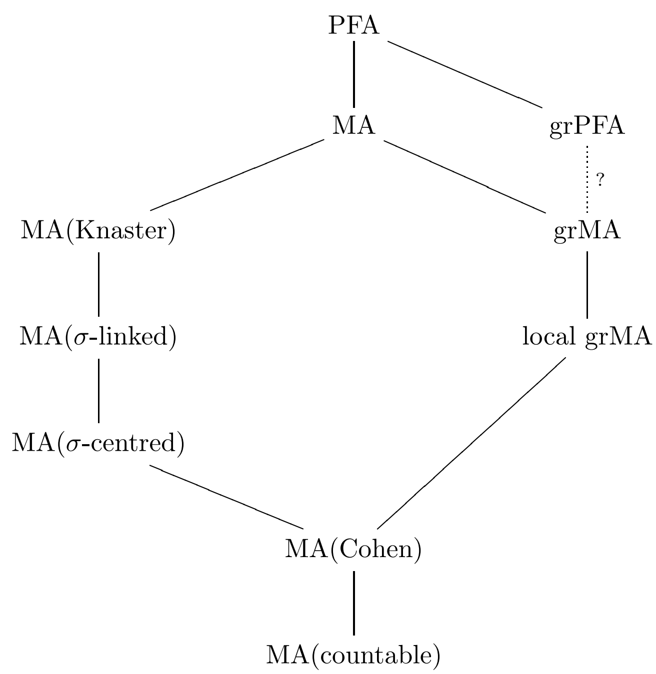
Regarding the relation of grPFA to other forcing axioms, a lot remains unknown. Theorem 36 shows that grPFA does not imply MA. Beyond this a few more things can be said.

Proposition 39. *Martin's Axiom does not imply* grPFA.

Proof. Starting over some model, force with the Solovay-Tennenbaum iteration to produce a model $V[G]$ satisfying MA and $\mathfrak{c} = \omega_3$. Now perform Reitz's ground axiom forcing (cf. [7]) above ω_3 to produce a model $V[G][H]$ still satisfying MA and $\mathfrak{c} = \omega_3$ but which is not a set-forcing extension of any model (note that H is added by class-sized forcing). Therefore the only way $V[G][H]$ could satisfy grPFA is if it actually satisfied PFA in full. But that cannot be the case since PFA implies that the continuum equals ω_2. □

Proposition 40. *The grounded Proper Forcing Axiom does not imply* MA(σ-centred) *(and not even* $\mathrm{MA}_{\aleph_1}$(σ-centred)*).*

Proof. We could have replaced the forcing $\mathrm{Add}(\omega, 1) \times \mathbb{P}$ in the proof of theorem 36 with $\mathrm{Add}(\omega, \omega_1) \times \mathbb{P}$ without issue. As in the proof of theorem 12 we get a model whose bounding number equals $\aleph_1$, but this contradicts $\mathrm{MA}_{\aleph_1}$(σ-centred) as in the proof of corollary 15. □



While it is easy to see that grPFA implies $\mathrm{MA}_{\aleph_1}$(Cohen), whether or not it even implies MA(countable) is unclear (a large part of the problem being that we do not have an answer to question 38). But an even more pressing question concerns the relationship between grPFA and grMA:

Question 41. *Does* grPFA *imply* grMA*?*

Even if the answer to question 38 turns out to be positive, we conjecture that the answer to this last question is negative. We expect that it is possible to use the methods of [7] or, more generally, [3] in combination with the forcing construction of theorem 36 to produce a model of grPFA which has no ccc ground models and in which MA (and consequently also grMA) fails.

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