On the Existence of Tree Backbones that Realize the Chromatic Number on a Backbone Coloring

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October 17, 2018

Abstract

A proper k-coloring of a graph $G = (V, E)$ is a function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$, for every $uv \in E(G)$. The chromatic number $\chi(G)$ is the minimum k such that there exists a proper k-coloring of G. Given a spanning subgraph H of G, a q-backbone k-coloring of (G, H) is a proper k-coloring c of $V(G)$ such that $|c(u) - c(v)| \geq q$, for every edge $uv \in E(H)$. The q-backbone chromatic number $BBC_q(G, H)$ is the smallest k for which there exists a q-backbone k-coloring of (G, H). In this work, we show that every connected graph G has a spanning tree T such that $BBC_q(G, T) = \max\{\chi(G), \left\lceil \frac{\chi(G)}{2} \right\rceil + q\}$, and that this value is the best possible.

As a direct consequence, we get that every connected graph G has a spanning tree T for which $BBC_2(G, T) = \chi(G)$, if $\chi(G) \geq 4$, or $BBC_2(G, T) = \chi(G) + 1$, otherwise. Thus, by applying the Four Color Theorem, we have that every connected nonbipartite planar graph G has a spanning tree T such that $BBC_2(G, T) = 4$. This settles a question by Wang, Bu, Montassier and Raspaud (2012), and generalizes a number of previous partial results to their question.

1 Introduction

For basic notions and terminology on Graph Theory, the reader is referred to [1]. All graphs in this work are considered to be simple. Because we investigate the existence of a spanning tree with certain property, we also consider only connected graphs. However, for disconnected graphs, the statements hold by replacing “spanning tree” by “spanning forest”. A *proper k-coloring* of a graph G is a function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$, for every $uv \in E(G)$. If G admits a proper k-coloring, we say that G is *k-colorable*. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest positive integer k such that G is k-colorable. Determining the chromatic number of a graph is an

NP-hard problem on Karp's list [10] and one of the most studied problems on Graph Theory [9, 11].

Given a spanning subgraph H of G, and positive integers k and q, a *q-backbone k-coloring* of (G, H) is a proper k-coloring c of G such that $|c(u) - c(v)| \geq q$, for every $uv \in E(H)$. The *q-backbone chromatic number of* (G, H), denoted by $BBC_q(G, H)$, is the smallest integer k for which (G, H) admits a q-backbone k-coloring.

This parameter was first introduced by Broersma et al. [2] as a model for the frequency assignment problem where certain channels of communication are more demanding than others. In their seminal work, they only considered $q = 2$ and they were interested in finding out how far away from $\chi(G)$ can $BBC_2(G, H)$ be in the worst case. Concerning trees, for each positive integer k, they defined:

$$\mathcal{T}_k = \max\{BBC_2(G, T) : \chi(G) = k \text{ and } T \text{ is a spanning tree of } G\}.$$

Note that, if c is a proper $\chi(G)$-coloring of G, then by recoloring each vertex u with color $2c(u) - 1$, we obtain a proper $(2\chi(G) - 1)$-coloring of G where every color is odd. Therefore, we get $BBC_2(G, G) \leq 2\chi(G) - 1$. This gives an upper bound of $2k - 1$ for $\mathcal{T}_k$. In [2], they proved that this is actually best possible.

Theorem 1 (Broersma et al.[2])**.** *$\mathcal{T}_k = 2k - 1$, for every positive integer k.*

This means that, between all the k-colorable graphs, there is one that attains this upper bound. However, it does not give any insight on how bad can a tree backbone be for a given graph G. One could then define $\mathcal{T}_2(G)$ as the maximum $BBC_2(G, T)$, where T is a spanning tree of G. This worst case behaviour has been studied for planar graphs. If G is planar, because $\chi(G) \leq 4$ and the fact that $BBC_2(G, G) \leq 2\chi(G) - 1$, we get $\mathcal{T}_2(G) \leq 7$. Broersma et al. [3] give examples where $BBC_2(G, T) = 6$, and conjecture that $\mathcal{T}_2(G) = 6$. A partial result for their conjecture has been given in [7]. Note that this parameter can be generalized for higher values of q. In [8], Havet et al. prove that, if G is a planar graph, then $\mathcal{T}_q(G) \leq q + 6$. They also prove that this is best possible if $q \geq 4$, and conjecture that $\mathcal{T}_3(G) \leq 8$.

Now, observe that it is not clear whether G always has a spanning tree with a "good" behaviour, i.e., such that $BBC_q(G, T)$ is not much larger than $\chi(G)$. Therefore, it makes sense to define the best case behaviour of $BBC_q(G, T)$. In [12], Wang, Bu, Montassier and Raspaud asked what is the smallest value β for which the following holds: if G is a nonbipartite planar graph with girth at least β, then G has a spanning tree T such that $BBC_2(G, T) = 4$. Inspired by their question, we define the following parameter, for a given graph G and a positive integer q:

$\mathcal{B}_q(G) = \min\{BBC_q(G, T) : T \text{ is a spanning tree of } G\}$.

Our main result is the following:

Theorem 2. *For every graph G and positive integer q,*

$$\mathcal{B}_q(G) = \max\{\chi(G), \left\lceil \frac{\chi(G)}{2} \right\rceil + q\}.$$

This gives us the following value for bipartite graphs:

Corollary 1. *If G is bipartite, then $\mathcal{B}_q(G) = q + 1$.*

Considering $q \geq 2$, observe that if G has at least one edge and T is a spanning tree of G, then $BBC_2(G,T) \geq 3$, and that $BBC_2(G,T) = 3$ if, and only if, G is bipartite. Also, observe that, when G is a nonbipartite planar graph, we get that $\max\{\chi(G), \lceil \chi(G)/2 \rceil + 2\}$ is always equal to 4. Therefore, the answer to Wang et al's question is $\beta = 3$, i.e., having high girth is not a necessary condition for having the desired spanning tree.

Corollary 2. *If G is a nonbipartite planar graph and $q \geq 2$, then $\mathcal{B}_q(G) = q+2$. In particular, G always has a spanning tree T for which $BBC_2(G,T) = 4$.*

We mention that this generalizes results in a number of papers: [4, 5, 13, 6, 12]. We also mention that, in [12], Wang et al. wrongly state that β is at least 4 due to the existence of a nonbipartite planar graph G and a spanning tree T of G such that $BBC_2(G,T) = 6$. However, they fail to notice that, in order for β to be at least 4, this should hold for every spanning tree of G.

2 Proof of Theorem 2

Roughly, the idea of the proof is to show that any graph G has a *nice* proper k-coloring, where $k = \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$. By nice we mean that the subgraph of G induced by the edges whose endpoint colors differ by at least q form a *connected spanning* subgraph of G. Then, we select among these edges a spanning tree to form its backbone. Before presenting the main result, let us recall some definitions, and present some new ones.

Consider a proper k-coloring c of a graph G. For $i \in \{1, \ldots, k\}$, the *color class i of c* is the subset $c_i = \{u \in V(G) : c(u) = i\}$. Observe that if H is a component of $G[c_i \cup c_j]$, a.k.a. Kempe's chain, then the k-coloring c' obtained from c by switching colors i and j in $V(H)$ is also a proper k-coloring of G. We denote the set of edges $\{uv \in E(G) : u \in V(H) \text{ and } v \in V(G) \setminus V(H)\}$ by $[H, \overline{H}]$. Given an integer q, and $i \in \{1, \ldots, k\}$, we denote by $[i]_q$ the set $\{j \in \{1, \ldots, k\} : |i - j| < q\}$. The *$q$-subgraph of c*, denoted by $G_{c,q}$, is the subgraph $(V(G), E_{c,q})$, where $E_{c,q} = \{uv \in E(G) : |c(u) - c(v)| \geq q\}$. Alternatively, one can see that $uv \in E_{c,q}$ if and only if $c(u) \notin [c(v)]_q$ if and only if $c(v) \notin [c(u)]_q$. Our upper bound is obtained as a corollary of the following theorem:

Theorem 3. *If G is a connected graph and $k \geq \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$, then there exists a proper k-coloring c of G such that $G_{c,q}$ is connected.*

Proof. Consider $k = \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$ and let c be a proper k-coloring of G that uses the following $\chi(G)$ colors: $\{1, \ldots, x, x + k' + 1, \ldots, k\}$, where $x = \lceil \chi(G)/2 \rceil$ and $k' = k - \chi(G)$. Let H be a component of $G_{c,q}$ with maximum number of vertices. Suppose, without loss of generality, that c maximizes the size of H. We claim that such a coloring c satisfies that $G_{c,q}$ is connected, which means that H is a spanning subgraph of G.

By contradiction, suppose that $V(H) \subset V(G)$, i.e., H does not contain every vertex of G. Since G is connected, there must be an edge $uv \in [H, \overline{H}]$. By the definition of $G_{c,q}$, we know that $[c(u)]_q \cap [c(v)]_q \neq \emptyset$.

First, suppose that there exists $j \in \{1, \ldots, k\} \setminus ([c(u)]_q \cup [c(v)]_q)$, and let H' be the component of $G[c_j \cup c_{c(v)}]$ containing v. We claim that $V(H') \cap V(H) = \emptyset$. Suppose otherwise and let $v' \in V(H') \cap V(H)$ be closest to v in H'; also, let $w \in N_{H'}(v') \setminus V(H)$ (it exists by the choice of v'). By the definition of H', we know that $\{c(v'), c(w)\} = \{j, c(v)\}$. This contradicts the construction of H since $wv' \notin E_{c,q}$ and $j \notin [c(v)]_q$. Now, let c' be obtained from c by switching colors j and $c(v)$ in H'. Because $V(H') \cap V(H) = \emptyset$, nothing changes in H; additionally, $c'(v) \notin [c'(u)]_q$, which means that $uv \in E_{c',q}$ and that there is a component in $G_{c',q}$ that strictly contains H, a contradiction to the choice of c.

Now, suppose that

$$[c(u)]_q \cup [c(v)]_q = \{1, \ldots, k\}, \text{ for all } uv \in [H, \overline{H}]. \tag{1}$$

Recall that c uses the colors that are in the set $\{1, \ldots, x, x + k' + 1, \ldots, k\}$, where $k = \max\{\chi(G), \lceil \chi(G)/2 \rceil + q\}$, $x = \lceil \chi(G)/2 \rceil$ and $k' = k - \chi(G)$. We want to prove that $1 \notin [i]_q$, for every $i \in \{x + k' + 1, \ldots, k\}$, and that $k \notin [i]_q$, for every $i \in \{1, \ldots, x\}$. We analyse the cases below.

- $q \geq \lfloor \chi(G)/2 \rfloor$: in this case, $k = x + q$. If $i \in \{1, \ldots, x\}$, then $k - i \geq k - x = x + q - x = q$. In case, $i \in \{x + k' + 1, \ldots, k\}$, then $i - 1 \geq x + k' + 1 - 1 = x + k - \chi(G) = x + x + q - \chi(G) \geq q$.
- $q < \lfloor \chi(G)/2 \rfloor$: observe that $k = \chi(G)$ and $k' = 0$. If $i \in \{1, \ldots, x\}$, then $k - i \geq k - x = \chi(G) - x = \lfloor \chi(G)/2 \rfloor > q$. Similarly, if $i \in \{x + k' + 1, \ldots, k\}$, then $i - 1 \geq x + k' + 1 - 1 = x > q$.

Now, consider any edge $uv \in [H, \overline{H}]$. Suppose that $c(u) \leq x$, in which case $k \notin [c(u)]_q$; if this is not the case, we get $1 \notin [c(u)]_q$ and the argument is analogous. By Equation 1, we get $k \in [c(v)]_q$, and therefore $c(v) \geq x + k' + 1$. Let H' be the component of $G[c_k \cup c_{c(v)}]$ containing v. We claim that $V(H') \cap V(H) = \emptyset$. Suppose otherwise, and let $v' \in V(H') \cap V(H)$ be the closest to v in H' and let $w \in N_{H'}(v') \setminus V(H)$. By the choice of H', we know that $\{c(v'), c(w)\} = \{c(v), k\}$, in which case $1 \notin [c(v')]_q \cup [c(w)]_q$, contradicting Equation 1. Finally, the theorem follows by the same argument used on the previous case. □

It remains to prove that this is also a lower bound. Our proof actually holds for any spanning backbone that does not contain isolated vertices.

Lemma 1. *If G is a graph and H is a spanning subgraph of G such that $\delta(H) \geq 1$, then, for every positive integer q the following holds:*

$$BBC_q(G,H) \geq \max\{\chi(G), \left\lceil \frac{\chi(G)}{2} \right\rceil + q\}.$$

Proof. Let H be any spanning subgraph of G with $\delta(H) \geq 1$, and let $k = BBC_q(G,H)$. Furthermore, let c be a q-backbone k-coloring of G. Since any q-backbone coloring of (G,H) is also a proper coloring of G, we have that $k \geq \chi(G)$. Now, if either $q \leq \lfloor \chi(G)/2 \rfloor$, or $q \geq \lceil \chi(G)/2 \rceil$ and $k \geq 2q$, we are done. So, suppose $q \geq \lceil \chi(G)/2 \rceil$ and $k < 2q$, and let $k' = 2q - k$. We claim that $[i]_q = \{1, \dots, k\}$, for every $i \in \{q - k' + 1, \dots, q\}$. Because $d_H(u) \geq 1$, we know that none of these k' colors can be used on u, for every $u \in V(G)$, and the following holds:

$$k - k' = k - 2q + k \geq \chi(G).$$

This inequality implies that:

$$k \geq \left\lceil \frac{\chi(G)}{2} \right\rceil + q.$$

It remains to prove our claim. So, let i be any color in $\{q - k' + 1, \dots, q\}$. It suffices to show that $\{1, k\} \subseteq [i]_q$. Clearly, $1 \in [i]_q$, since $i \leq q$. Also, since $k = 2q - k'$ and $i \geq q - k' + 1$, we get $k - i \leq 2q - k' - q + k' - 1 = q - 1$. Thus, $k \in [i]_q$ and the lemma follows. □

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