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Research Article

Controllability of Singular Linear Systems by Legendre Wavelets

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We propose a new method to design an observer and control the linear singular systems described by Legendre wavelets. The idea of the proposed approach is based on solving the generalized Sylvester equations. An example is also given to illustrate the procedure.

1. Introduction

Singular systems, also commonly called generalized or descriptor systems in the literature, appear in many practical situations including engineering systems, economic systems, network analysis, and biological systems. In fact, many systems in the real life are singular essentially. They are usually simplified or approximated by nonsingular models because there is still lacking of efficient tools to tackle problems related to such systems. The structural analysis of linear singular systems, using either algebraic or geometric approach, has attracted considerable attention from many researchers during the last three decades (see, e.g., [1–3]).

Since the introduction of Legendre wavelets method (LWM), for the resolution of variational problems, by Razzaghi and Yousefi in 2000 and 2001 [4, 5], several works applying this method were born. To mention a few, we give the resolution of differential equations [6], the study of optimal control problem with constraints [7], the resolution of linear integro-differential equations, the numerical resolution of Abel equation, and the resolution of fractional differential equations.

In this paper, we propose a new method to design an observer and control the linear singular systems described by Legendre wavelets. The method is based upon expanding various time functions in the system as their truncated Legendre wavelets. The operational matrix is introduced and utilized to reduce the solution of time singular linear system to the solution of algebraic equations. Finally, we obtain the interrelations between solution problems for the linear matrix equations of Sylvester with suitable controllability and observability conditions.

2. Properties of Legendre Wavelets

Wavelets are mathematical functions that are constructed using dilation and translation of a single function called the mother wavelet denoted by $\psi(t)$ and satisfied certain requirements.

If the dilation parameter is a and translation parameter is b, then we have the following family of wavelets:

$$\psi_{a,b}(t) = |a|^{1/2}\,\psi\left(\frac{t-b}{a}\right) \quad \text{with } a, b \in R,\ a \neq 0. \tag{1}$$

Restricting a and b to discrete values, such as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, and n, k are positive integers, we give

$$\psi_{k,n}(t) = |a|^{k/2}\,\psi\left(a_0^k t - nb_0\right), \tag{2}$$

where $\psi_{k,n}(t)$ form a basis for $L^2(R)$. If $a_0 = 2$ and $b_0 = 1$, then it is clear that the set $\{\psi_{k,n}(t)\}$ forms an orthonormal basis for $L^2(R)$.

Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments: $\hat{n} = 2n-1$, $n = 1, 2, 3, \ldots, 2^{k-1}$, k is assumed to be any positive integer, m is the order for Legendre polynomials, and t is the normalized time. They are defined on the interval $[0, 1]$ as

$$\psi_{n,m} = \begin{Bmatrix} \sqrt{m+\frac{1}{2}}2^{k/2}P_m\left(2^k t - \hat{n}\right) & \text{for } \frac{\hat{n}}{2^k} \le t \le \frac{\hat{n}+1}{2^k} \\ 0 & \text{otherwise} \end{Bmatrix} \quad (3)$$

with $m = 0, 1, \ldots, M-1$ and $n = 0, 1, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+1/2}$ is orthonormality. Here, $P_m(t)$ are the well-known Legendre polynomials of order m, which are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae [8, 9]:

$$P_0(t) = 1, \qquad P_1(t) = t,$$
$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \quad (4)$$
$$m = 1, 2, 3 \ldots.$$

A function $f(t)$ defined over $[0, 1]$ can be expanded as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m\in Z} c_{n,m}\psi_{n,m}(t)$$
$$\approx \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m}\psi_{n,m}(t) = C^T\psi(t), \quad (5)$$

where

$$c_{n,m} = \left(f(t), \psi_{n,m}(t)\right) = \int_0^1 f(t)\psi_{n,m}(t)\,dt. \quad (6)$$

C and $\psi(t)$ are $2^k(2M+1)\times 1$ vectors given by

$$C = \left[c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2,1}, \ldots, c_{2,M-1} c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}\right]^T,$$
$$\psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \ldots, \psi_{2,0}, \ldots, \psi_{2^{K-1},M-1}, \ldots, \psi_{2^{k-1},M-1}\right]^T. \quad (7)$$

The integration of the function $\psi(t)$ in (5) is given by

$$\int_0^t \psi(s)\,ds = P\psi(t), \quad (8)$$

where P is an $(2^{k-1}M)\times(2^{k-1}M)$ matrix, called the operational matrix, and is given by [10, 11]

$$P = \frac{1}{2^{k+1}} \begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & L \end{bmatrix}, \quad (9)$$

where F and L are $M\times M$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & \frac{1}{3^{1/2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{3^{1/2}}{3} & 0 & \frac{3^{1/2}}{3\times 5^{1/2}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{5^{1/2}}{5\times 3^{1/2}} & 0 & \frac{5^{1/2}}{5\times 7^{1/2}} & \ddots & 0 & 0 & 0 \\ 0 & 0 & -\frac{7^{1/2}}{7\times 5^{1/2}} & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{(2M-3)^{1/2}}{(2M-3)(2M-5)^{1/2}} & 0 & -\frac{(2M-3)^{1/2}}{(2M-3)(2M-1)^{1/2}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{(2M-1)^{1/2}}{(2M-1)(2M-3)^{1/2}} & 0 \end{bmatrix}. \quad (10)$$

The integration of the product of two Legendre wavelet function vectors is obtained as

$$\int_0^1 \psi(t)\psi(t)^T dt = 1. \quad (11)$$

3. Properties of the Sylvester Equation and Model Described

Consider the Sylvester equation

$$AX - XB = \Phi_1\Phi_2, \quad (12)$$

where $A \in C^{n\times n}$, $B \in C^{m\times m}$, $X \in C^{n\times m}$, $\Phi_1 \in C^{n\times k}$, and $\Phi_2 \in C^{k\times m}$.

For some recent developments in the theory of rational matrix functions and in the theory of linear systems and control leading to equations, one can refer to [11–15].

Definition 1. Identify the least positive integer k for which a solution X, Φ_1, Φ_2, with(A, Φ_1) is controllable and (Φ_2, B) is observable

$$k(A,B) = \max_{\lambda\in C}\{\dim\mathrm{Ker}(\lambda I - A) + \dim\mathrm{Ker}(\lambda I - B)\}. \quad (13)$$

Lemma 2. *Let $A \in C^{n\times n}$ be given. There exists $\Phi_1 \in C^{n\times k}$ such that (A, Φ_1) is controllable, if and only if $k \geq \max_{\lambda\in C} \dim\mathrm{Ker}(\lambda I - A)$.*

Theorem 3. *Given $A \in C^{n\times n}$, $B \in C^{m\times m}$. The minimal integer K for which there exist $X \in C^{n\times m}$, $\Phi_1 \in C^{n\times r}$, and $\Phi_2 \in C^{r\times m}$ satisfying (12) such that (A, Φ_1) is controllable and (Φ_2, B) is observable, which is equal to $k(A, B)$.*

A singular linear system can be described as follows:

$$K\dot{X}(t) = EX(t) + FU(t), \quad (14)$$

$$X(0) = X_0, \quad (15)$$

where $X \in R^n$, $U(t) \in R^m$, $K \in R^{n\times n}$, $E \in R^{n\times n}$, and $F \in R^{n\mp\times m}$.

4. Solution of Singular Linear System

In this section, the Legendre wavelet method is used for solving the singular linear system by approximated functions. Assume that $U(t)$ is square integral in the interval $[0, 1)$. In the matrix forms,

$$U(t) = G\psi(t), \quad (16)$$

where G is a $p \times m$ matrix. G can be obtained by the method described in Section 2.

Avoiding impulse functions, we straight away expand $\dot{X}(t)$ instead of $X(t)$ itself into Legendre wavelet given by $\dot{X}(t) = S\psi(t)$. So we get

$$X(t) = \int_0^t \dot{X}(\tau)\,d\tau = S\int_0^t \psi(\tau)\,d\tau + X_0 = SP\psi(t) + X_0. \quad (17)$$

By substituting (16) and (17) into (14), we obtain

$$KS\psi(t) = E[SP\psi(t) + X_0] + FG\psi(t). \quad (18)$$

We have

$$[KS - ESP]\psi(t) = \{Q + FG\}\psi(t), \quad (19)$$

where $Q\psi(t) = EX_0$.

In this work (19) can be written as

$$KS - ESP = W, \quad (20)$$

where

$$W = \{Q + BG\}. \quad (21)$$

Since the unknown matrix in (20), it can be solved using Kronecker product as

$$S = [K \otimes I - E \otimes P^T]^{-1} W^T, \quad (22)$$

where

$$S^T = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{m-1} \end{bmatrix}, \qquad W^T = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{m-1} \end{bmatrix}. \quad (23)$$

So $X(t) = S\psi(t)$.

5. Some Theorems

For $KS - ESP = W$, if $E^{-1} \in R^{n\times n}$, then $E^{-1}KS - SP = E^{-1}W$; let $\widetilde{A} = E^{-1}K$, $\widetilde{B} = P$, $\widetilde{\Phi}_1 = E^{-1}$, and $\widetilde{\Phi}_2 = W$; we obtain $\widetilde{A}S - S\widetilde{B} = \widetilde{\Phi}_1\widetilde{\Phi}_2$.

Accordingly, we propose some results.

Lemma 4. *Let $\widetilde{A} \in R^{n\times n}$ be given. There exists $\widetilde{\Phi}_1 \in R^{n\times k}$ such that $(\widetilde{A}, \widetilde{\Phi}_1)$ is controllable, if and only if $k \geq \max_{\lambda\in C} \dim\mathrm{Ker}(\lambda I - \widetilde{A})$.*

Theorem 5. *Given $\widetilde{A} \in R^{n\times n}$, $\widetilde{B} \in R^{m\times m}$. The minimal integer K for which there exist $S \in R^{n\times m}$, $\widetilde{\Phi}_1 \in R^{n\times r}$, and $\widetilde{\Phi}_2 \in R^{r\times m}$ satisfying (24) such that $(\widetilde{A}, \widetilde{\Phi}_1)$ is controllable and $(\widetilde{\Phi}_2, \widetilde{B})$ is observable, which is equal to $k(\widetilde{A}, \widetilde{B})$.*

6. Example

For Legendre Wavelets, let $M = 3$, $k = 2$; let us consider the first order linear singular system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u. \quad (24)$$

Set

$$x_0 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}, \qquad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (25)$$

We solve (24) using the algorithm described in Section 5 for the case corresponds to Theorem 5, so we can get (24) that is controllable and observable.

7. Conclusions

By the analysis of the above, we find that the method proposed in the paper is efficient in tackling the singular linear systems. Furthermore, the method is also applied for solving the interrelations between solution problems for singular linear systems and the linear matrix equations of Sylvester with suitable controllability and observability conditions. The design example is good enough to illustrate our idea.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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