Postulated colimits and left exactness of Kan-extensions*

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If $\mathbf{A}$ is a small category and $\underline{E}$ a Grothendieck topos, the Kan extension $LanF$ of a flat functor $F : \mathbf{A} \to \underline{E}$ along any functor $\mathbf{A} \to \underline{D}$ preserves whatever finite limits may exist in $\underline{D}$; this is a well known fundamental result in topos theory. We shall present a metamathematical argument to derive out of this some other left exactness results, for Kan extensions with values in a (possibly large) site $\underline{E}$.

Loosely speaking, we prove that for any specific finite limit diagram $(D_i)_{i \in I}$ in $\underline{D}$, the Kan extension $LanF$ preserves the limit diagram, provided the colimits in $\underline{E}$ used in the construction of the finitely many relevant $Lan(D_i)$'s are what we call 'postulated' colimits.

Both the notion of 'flat' $F : \mathbf{A} \to \underline{E}$, and the notion of 'postulated' colimit in $\underline{E}$ are expressed in elementary terms in terms of the site structure (the covering notion) in $\underline{E}$. If $\underline{E}$ is small with subcanonical topology, a colimit is postulated iff it is preserved by the Yoneda embedding of $\underline{E}$ into the topos $\underline{\tilde{E}}$ of sheaves on $\underline{E}$.

As a corollary, we shall conclude that if $\underline{E}$ satisfies the Giraud axioms for a Grothendieck topos, except possibly the existence of a set of generators (so $\underline{E}$ is an ∞-pretopos [2]), then any flat functor into $\underline{E}$ has left exact Kan extension (Corollary 3.3 below).

We believe that the general method presented here is well suited to give partial left exactness results[1], by for instance allowing for coarse site structure on the recipent category $\underline{E}$, so that there are relatively few flat functors into $\underline{E}$.

I am grateful to Bob Paré for bringing up the question of left Kan extensions with values in a ∞-pretopos, and for his impressive skepticism towards my original hand-waving change-of-universe arguments. I benefited much from several discussions we had on the subject.

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Also, I want to thank Francis Borceux for bringing up, at the right moment,

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[1] Added in 2003: this is corroborated by recent results of Karazeris and Velebil.

the question of whether the amalgamation theorem for groups were valid in a topos; contemplating pushouts of groups led me in the direction of postulating colimits.

1 Postulated colimits in a site

To say that a diagram

$$\begin{tikzcd} R \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

in the category of sets is a coequalizer may be expressed in elementary terms by saying that $q \circ a = q \circ b$, and that the following two assertions hold

(1.1) $$q \text{ is surjective}$$

(1.2) for any x and y in X with $q(x) = q(y)$,
there exists a finite chain $z_1, \ldots, z_m$ of elements of R
with $x = a(z_1), b(z_1) = a(z_2), \ldots, b(z_m) = y$

These assertions can be interpreted in any category where sheaf semantics is available; this means in any site, cf. [3], II.8. If they hold for a given diagram in the site, we shall say that the diagram is a *postulated* coequalizer.

We shall more generally describe what we mean by a *postulated colimit* diagram in a site. If the site is subcanonical and small, this is equivalent to saying that the diagram goes to a colimit diagram in the category of sheaves on the site. We shall only be considering subcanonical sites, with finite limits.

We introduce first a few auxiliary notions. Let $\mathbf{C}$ be a small category. By a *zig-zag* in $\mathbf{C}$, we understand a diagram $\underline{g}$ in $\mathbf{C}$ of form

$$\begin{tikzcd} & C_1 \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & & C_3 \arrow[dl, "g_2"'] \arrow[dr] & & C_n \arrow[dl] \arrow[dr, "g_n"] & \\ C_0 & & C_2 & & \ldots & & D \end{tikzcd}$$

for some (odd) integer n. The displayed $\underline{g}$ is said to be a zig-zag *from C to D*, written $d_0(\underline{g}) = C$, $d_1(\underline{g}) = D$. Let $Z_{\mathbf{C}}(C, D)$ denote the set of zig-zags in $\mathbf{C}$ from C to D. If $h : \mathbf{C} \to \mathbf{D}$ is a functor, we get a map $Z_{\mathbf{C}}(C, D) \to Z_{\mathbf{D}}(h(C), h(D))$.

If now $P : \mathbf{C} \to \underline{E}$ is a functor into a category with finite limits, and $\underline{g}$ is a zig-zag in $\mathbf{C}$, as displayed above, we let

$$M_P(\underline{g}) \subseteq P(C) \times P(C_1) \times \ldots \times P(C_n) \times P(D)$$

be the object

$$[\![(x, x_1, \ldots, x_n, y) \mid P(g_0)(x_1) = x \wedge P(g_1)(x_1) = x_2 \wedge \ldots$$
$$\wedge P(g_2)(x_3) = x_2 \wedge \ldots \wedge P(g_n)(x_n) = y]\!].$$

The projections from $M_P(\underline{g})$ to $P(C)$ and $P(D)$ are denoted d_0 and d_1, respectively.

By a *postulated colimit diagram* for $P : \mathbf{C} \to \underline{E}$, where $\underline{E}$ is a site, we understand a cocone on P

$$\{P(C) \xrightarrow{incl_C} L \mid C \in \mathbf{C}\} \tag{1.3}$$

which satisfies

$$\vdash \forall x \in L \bigvee_{C \in \mathbf{C}} \exists y \in P(C) : incl_C(y) = x, \tag{1.4}$$

and, for each pair C, D of objects in $\mathbf{C}$,

$$\vdash \forall x \in P(C)\ \forall y \in P(D) : incl_C(x) = incl_D(y)$$

$$\Rightarrow \bigvee_{\underline{g} \in Z_{\mathbf{C}(C,D)}} \exists \underline{x} \in M_P(\underline{g}) : d_0(\underline{x}) = x \wedge d_1(\underline{x}) = y. \tag{1.5}$$

(One may compare (1.4) and (1.5) with (1.1) and (1.2), respectively.)

It is easy to express (1.4) without sheaf semantics: it is just the assertion that (1.3) is a covering family. Also (1.5) can be expressed elementarily without sheaf semantics, but it is less economical.

Proposition 1.1 *Let $\underline{E}$ be a subcanonical site. Then every postulated colimit diagram in $\underline{E}$ is an actual colimit diagram.*

Proof. Let $P : \mathbf{C} \to \underline{E}$ be a diagram, and consider a postulated colimit for it, as in (1.3); let

$$\{P(C) \xrightarrow{f_C} B \mid C \in \mathbf{C}\} \tag{1.6}$$

be a cocone on the diagram P. Since $hom(-, B)$ is a sheaf, and the family (1.3) is a covering, we may get the desired map $L \to B$ by means of a compatible family of maps, $P(C) \to B$; we intend to prove that the family (1.6) is compatible. This means that for C, D in $\mathbf{C}$, we should prove that the square

$$\begin{tikzcd} X := P(C) \times_L P(D) \arrow[r, "y"] \arrow[d, "x"'] & P(D) \arrow[d, "f_D"] \\ P(C) \arrow[r, "f_C"'] & B \end{tikzcd} \tag{1.7}$$

commutes, where x and y denote the projections. Then x and y are elements of $P(C)$ and $P(D)$, respectively, defined at the same stage X, and we have

$$\vdash_X \ incl_C(x) = incl_D(y),$$

by commutativity of the square that defines the pull-back X. so we have also, by (1.5), that

$$\vdash_X \bigvee_{\underline{g} \in Z(C,D)} \exists \underline{x} \in M_P(\underline{g}) \ : \ d_0(\underline{x}) = x \ \wedge \ d_1(\underline{x}) = y.$$

This means that we have a covering $\{\xi_t : X_t \to X \mid t \in T\}$, and, for each $t \in T$, we have for some $\underline{g} \in Z(C,D)$ an $\underline{x} : X_t \to M_P(\underline{g})$ with

$$\vdash_{X_t} \ d_0(\underline{x}) = x \ \wedge \ d_1(\underline{x}) = y. \tag{1.8}$$

For any given $t, \underline{g}, \underline{x}$, we have commutativity of the upper triangles in the following diagram (for simplicity, we assume that the zig-zag $\underline{g}$ has length 3)

$$\begin{tikzcd}
 & & X_t \arrow[dl, "x_1"'] \arrow[dr, "x_3"] \arrow[ddll, bend right, "x \circ \xi_t"'] \arrow[ddrr, bend left, "y \circ \xi_t"] & & \\
 & P(C_1) \arrow[dl, "P(g_0)"] \arrow[dr] \arrow[ddr, "f_{C_1}"'] & & P(C_3) \arrow[dl] \arrow[dr, "P(g_3)"'] \arrow[ddl] & \\
P(C) \arrow[drr, "f_C"'] & & P(C_2) \arrow[d] & & P(D) \arrow[dll] \\
 & & B & &
\end{tikzcd}$$

this follows from (1.8) and from the equations that defines the object $M_P(\underline{g})$.

Also, the lower triangles commute, since the f_C form a cocone. So the outer square commutes. So for each $t \in T$, (1.7) composed with $\xi_t : X_t \to X$ commutes.

Thus the family $(f_C)_{C \in \mathbf{C}}$ is compatible, and by the sheaf property of $hom(-, B)$, there exists a unique $f \in hom(L, B)$ with $f \circ incl_C = f_C$. This proves that (1.3) has the universal property required of a colimit diagram. Thus Proposition 1.1 is proved.

If we in the proof replace $hom(-, B)$ by an arbitrary sheaf $S \in \underline{\tilde{E}}$, the argument gives that the Yoneda embedding $\underline{E} \to \underline{\tilde{E}}$ takes the postulated colimit in $\underline{E}$ into a colimit in $\tilde{\underline{E}}$; this in fact characterizes the postulated colimits. But note thar $\tilde{\underline{E}}$ is an illegitgimate category, in case $\underline{E}$ is a large site.

For the case where $\mathbf{C}$ is empty, (1.5) says nothing, and (1.4) says that L is covered by the empty family. Thus, in a subcanonical site, a postulated initial object is not only initial, but strictly initial: if $f : X \to L$ is any map, X is covered by the pull-back of the empty family, which is empty, thus X itself is (postulated) initial. Let us denote such object by $\emptyset$.

For a discrete $\mathbf{C}$ in general, i.e. for a postulated coproduct, the set $Z_{\mathbf{C}}(C, D)$ of zig-zags from C to D is empty if $C \neq D$ and quite trivial if $C = D$, so that the conditions (1.4) and (1.5) may be expressed in elemetary terms by

(1.9) the inclusions into a postulated coproduct form a cover

$$\text{(for } i = j \text{ in } \mathbf{C}) : P(i) \to \coprod P(i) \text{ is monic} \tag{1.10}$$

$$\text{(for } i \neq j) : P(i) \text{ and } P(j) \text{ have intersection } \emptyset \text{ in } \coprod P(i), \tag{1.11}$$

(the coproducts in question assumed to be postulated). Except for the universality assertion, we have therefore proved

Proposition 1.2 *Postulated coproducts in a subcanonical site are disjoint and universal.*

Proof of the universality assertion. The properties (1.9) and (1.10) are clearly preserved under pull-back. And (1.10) is preserved under pull-back since postulated initial objects are strictly initial. But (1.9) –(1.11) characterize postulated coproducts.

It is not hard to see that if the site $\underline{E}$ is cocomplete, and if all coequalizers and small coproducts in $\underline{E}$ are postulated, then so are all small colimits.

We need to describe a notion of *postulated epi*: if $q : X \to Q$ is any map in a subcanonical site, it is easy to see that the following three conditions are equivalent; if they hold, we call q a *postulated epi*:

- i) q is a singleton covering
- ii) $\vdash \forall y \in Q\ \exists x \in X : q(x) = y$
- iii) the square

$$\begin{tikzcd} \cdot \arrow[r, "q"] \arrow[d, "q"'] & \cdot \arrow[d, "id"] \\ \cdot \arrow[r, "id"'] & \cdot \end{tikzcd}$$

is a postulated pushout.

From i), it is clear that postulated epis are stable under pull-back. Also, playing the sheaf condition for $hom(-, B)$ out against the singleton cover q proves that q is coequalizer of its kernel pair; so postulated epis are in fact stable regular epis. One may similarly see that any postulated colimit is preserved by pulling back, and that (hence) $colim(C_i) \times colim(D_j) = colim(C_i \times D_j)$, provided $colim(C_i)$ and $colim(D_j)$ are postulated (and then $colim(C_i \times D_j)$ will also be postulated).

2 Pretoposes

Let $\underline{E}$ be an ∞-pretopos. We make it into a (subcanonical) site by letting the coverings of $X \in \underline{E}$ be the jointly epi (not necessarily small) families.

Proposition 2.1 *Every small colimit diagram in an ∞-pretopos is a postulated colimit.*

Proof. In a coproduct in a pretopos, the inclusions are jointly epi, and they are monic and disjoint, by the Giraud axioms, so (1.9)–(1.11) hold, so coproducts are postulated. Consider a coequalizer diagram $R \rightrightarrows X \to Q$ in $\underline{E}$. Consider the union S in $X \times X$ of the diagonal $X \to X \times X$ and the images of $R^{(n)} \to X \times X$, where $R^{(n)}$ (described as if $\underline{E}$ were the category of sets) consists of n-tuples $z_1, \ldots, z_n \in R$ with (1.2) satisfied. This S is an equivalence relation on X , also with coequalizer Q, and hence, by another Giraud axiom, S is the kernel pair of q. But the fact that the $R^{(n)}$ and the diagonal map jointly epi to the kernel pair of q means that (1.5) (or (1.2) appropriately internalized) holds. Also (1.4) holds since q is epi, hence a singleton cover. From coproducts and coequalizers being postulated, the result follows.

Proposition 2.2 *Let $\underline{E}$ be a cocomplete finitely complete subcanonical site in which every small colimit diagram is postulated. Then $\underline{E}$ is an ∞-pretopos (and the small covering families are exactly the small jointly epi families).*

Proof. Coproducts are postulated, hence disjoint and universal, by Proposition 1.2. Let $R \rightrightarrows X$ be an equivalence relation and $q : X \to Q$ its coequalizer. Then it is a postulated coequalizer, and this implies that the map $R^{(n)} \to S$ to the kernel pair S of q, as constructed in the proof of Proposition 2.1 is postulated epi. But since $R \hookrightarrow X \times X$ is already an equivalence relation, all the maps $R^{(n)} \to X \times X$ factor through R, whence $R = S$; so R is the kernel pair of q. So equivalence relations are effective. They are also universal, because postulated epis are universal, and because the notions of postulated epi and regular epi coincide in this case, by the assumption that all coequalizers are postulated. So $\underline{E}$ is an ∞-pretopos.

For the last (parenthetical) assertion: covering families are jointly epi, in any subcanonical site. Conversely, if $\{f_i : X_i \to X \mid i \in I\}$ is a small jointly epi family, $\coprod X_i \to X$ is epi, hence postulated epi, which implies that the given family is covering.

Propositions 2.1 and 2.2 together characterize ∞-pretoposes as those "cocomplete sites where all colimits are postulated".

3 Left exactness of Kan-extensions

We consider a situation

$$\begin{tikzcd} \mathbf{A} \arrow[r, "H"] \arrow[d, "F"'] & \underline{D} \arrow[dl, "Lan_H F"] \\ \underline{E} & \end{tikzcd} \tag{3.1}$$

where $\mathbf{A}$ is a small category, $\underline{E}$ is a cocomplete category, and $\underline{D}$ is arbitrary; the left Kan extension functor $Lan_H F$ is described by

$$(Lan_H F)(D) = colim((H \downarrow D) \xrightarrow{\delta} \mathbf{A} \xrightarrow{F} \underline{E}) \tag{3.2}$$

(where $\delta(H(A) \to D) = A$). To study which finite limits $Lan_H F$ preserves is immediately reducible to the special case where $\underline{D} = \underline{Set}^{\mathbf{A}^{op}} =: \hat{\mathbf{A}}$, the category of presheaves on $\mathbf{A}$. For, given $H : \mathbf{A} \to \underline{D}$, as in (3.1), we have the 'singular' functor $S_H : \underline{D} \to \hat{\mathbf{A}}$, given by

$$S_H(D) = hom_{\underline{D}}(H(-), D).$$

It preserves whatever finite limits may exist in $\underline{D}$. Let also $y : \mathbf{A} \to \hat{\mathbf{A}}$ denote the Yoneda embedding. From the Yoneda lemma follows that, for any $D \in \underline{D}$

$$H \downarrow D = y \downarrow S_H(D),$$

and from this, and the general colimit formula for Kan extensions, one concludes easily that

$$Lan_H F = (Lan_y F) \circ S_H.$$

So it suffices to study left exactness properties of $Lan_y F$; so consider the situation (where $\mathbf{A}$ is small and $\underline{E}$ is cocomplete):

$$\begin{tikzcd} \mathbf{A} \arrow[r, "y"] \arrow[d, "F"'] & \underline{Set}^{\mathbf{A}^{op}} = \hat{\mathbf{A}} \arrow[dl, "Lan_y F"] \\ \underline{E} & \end{tikzcd} \tag{3.3}$$

(it is actually commutative up to isomorphism, and $Lan_y F$ has a right adjoint, namely the singular functor S_F).

We consider the language $\mathcal{L}_{\mathbf{A}}$ for "functors on $\mathbf{A}$": it has one sort $F(A)$ for each object A of $\mathbf{A}$, and one unary operation $F(\alpha)$, from $F(A)$ to $F(A')$, for each arrow $\alpha : A \to A'$ in $\mathbf{A}$. Any functor $F : \mathbf{A} \to \underline{E}$ defines a structure for this language, in an evident way.

We can now present our pivotal, but somewhat technical, result:

Theorem 3.1 *For each finite cone D in $\hat{\mathbf{A}}$, there exists a set $\Lambda(D)$ of geometric sentences in the language $\mathcal{L}_{\mathbf{A}}$, such that for any functor $F : \mathbf{A} \to \underline{E}$ into a (cocomplete, finitely complete, subcanonical) site $\underline{E}$, if the colimits used for constructing $(Lan_y F)(D_i)$ (for those $D_i \in \hat{\mathbf{A}}$ that occur in D) are postulated, then*

$$F \text{ satisfies } \Lambda(D)$$

iff

$$Lan_y F \text{ takes } D \text{ into a limit diagram in } \underline{E}.$$

Proof. We shall only consider in detail the case where D is an equalizer shaped diagram. So let $H \to K \rightrightarrows L$ be such in $\hat{\mathbf{A}}$. We shall consider the categories of elements of H, K, and L, i.e. $\mathbf{H} = (y \downarrow H)$, etc., so that we have a diagram of categories (all small except for $\underline{E}$):

$$\begin{tikzcd}
\mathbf{H} \arrow[r, "\eta"] \arrow[dr, "h"'] & \mathbf{K} \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] \arrow[d, "k"] & \mathbf{L} \arrow[dl, "l"] \\
& \mathbf{A} \arrow[d, "F"] & \\
& \underline{E} &
\end{tikzcd}$$

(Then $(Lan_y F)(H) = colim(F \circ h)$, and similarly for K and L.) For each pair of objects v and w of $\mathbf{H}$, we write down the following sentence in the language $\mathcal{L}_{\mathbf{A}}$ (recalling the M_F and Z's of §1):

(3.4)

$$\forall x \in F(h(v))\, \forall y \in F(h(w)) \;:\; \bigvee_{\underline{g} \in Z_{\mathbf{K}(\eta(v),\eta(w))}} \exists \underline{x} \in M_F(k(\underline{g})) : d_0\underline{x} = x \wedge d_1\underline{x} = y$$
$$\Rightarrow \bigvee_{\underline{g}' \in Z_{\mathbf{H}(v,w)}} \exists \underline{x}' \in M_F(h(\underline{g}')) : d_0\underline{x}' = x \wedge d_1\underline{x}' = y.$$

This is to be read informally: if (x, v) and (y, w) represent elements in $colim(F \circ h)$ which become equivalent in $colim(F \circ k)$ (namely in virtue of $\underline{g}, \underline{x}$) then they are already equivalent in $colim(F \circ h)$ (namely in virtue of $\underline{g}', \underline{x}'$). The intended meaning of the sentences of this form is: "$(Lan_y F)(H \to K)$ is monic". – Furthermore, for each u in $\mathbf{K}$, we write down the sentence

$$
\begin{aligned}
&\forall x \in F(k(u)) \ : \bigvee_{\underline{g} \in Z_{\mathbf{L}}(\phi u, \psi u)} \exists \underline{x} \in M_F(l(\underline{g})) \ : d_0(\underline{x}) = d_1(\underline{x}) = x \\
(3.5) \qquad & \qquad \Rightarrow \bigvee_{v \in \mathbf{H}} \bigvee_{\underline{g}' \in Z_{\mathbf{K}}(u, \eta(v))} \exists \underline{x} \in M_F(k(\underline{g}')) \ : d_0(\underline{x}) = x.
\end{aligned}
$$

This is to be read informally: if (x, u) represents an element in $colim(F \circ k)$ which is equalized (in virtue og $\underline{g}, \underline{x}$) by the maps induced by ϕ and ψ, then (x, u) is equivalent, (in virtue of $\underline{g}', \underline{x}'$) to an element (namely $(d_1(\underline{x}), \eta(v))$ which is in the image of $colim(F \circ h) \to colim(F \circ k)$. The collection of sentences (3.4) and (3.5) is to be our $\Lambda(D)$, for the case where D is the diagram $H \to K \rightrightarrows L$.

Consider now the diagram in $\underline{E}$

$$
(3.6) \qquad colim(F \circ h) \to colim(F \circ k) \rightrightarrows colim(F \circ l),
$$

where the maps are induced by η, ϕ and ψ, respectively. Assuming the three colimits are postulated, and that $\Lambda(D)$ holds for F, we should prove that (3.6) is an equalizer. The two composites are equal, since they are so in $H \to K \rightrightarrows L$. The existence and uniqueness aspects of proving (3.6) to be an equalizer is taken care of by (3.5) and (3.4), respectively; we shall only do uniqueness, i.e. proving that the left hand arrow in (3.6) is monic:

Let $x, y : X \to colim(F \circ h)$ be a pair of elements equalized by the left hand map in (3.6). Since the inclusions $incl_u : F(h(u)) \to colim(F \circ h)$ form a covering, by (1.4), it is easy to see that X may be covered by a cover $\{\xi_t : X_t \to X \mid t \in T\}$, such that on each part X_t of this covering, x and y factor through inclusion maps $incl_v$ and $incl_w$, respectively, $(v, w \in \mathbf{H})$. To prove x and y equal, it suffices to see that they are equal on each X_t. For fixed $t \in T$, we change notation, and denote X_t by X; in other words, we now assume that the given pair of elements is of the form

$$
(3.7) \qquad X \xrightarrow{x} F(A) \xrightarrow{incl_v} colim(F \circ h), \quad X \xrightarrow{y} F(A') \xrightarrow{incl_w} colim(F \circ h),
$$

where $A = h(v)$, $A' = h(w)$. The assumption that this pair is equalized in $colim(F \circ k)$ then means that

$$
X \xrightarrow{x} F(A) \xrightarrow{incl_{\eta(v)}} colim(F \circ k) = X \xrightarrow{y} F(A') \xrightarrow{incl_{\eta(w)}} colim(F \circ k).
$$

By the assumption that $colim(F \circ k)$ is postulated, it follows from (1.5) that X can be covered by $\{X_t \to X \mid t \in T\}$, such that for each X_t, there is a zig-zag $\underline{g}$ in $\mathbf{K}$ from $\eta(v)$ to $\eta(w)$, such that

$$
\vdash_{X_t} \exists \underline{x} \in M_F(k(\underline{g})) \ : \ d_0(\underline{x}) = x \wedge d_1(\underline{x}) = y.
$$

(Observe $M_F(k(\underline{g})) = M_{F\circ k}(\underline{g})$.) For this X_t, the assumption (3.4) therefore gives

$$\vdash_{X_t} \bigvee_{\underline{g}' \in Z_{\mathbf{H}(v,w)}} \exists \underline{x}' \in M_F(h(\underline{g}')) \ : \ d_0(\underline{x}') = x \wedge d_1(\underline{x}') = y$$

which implies $\vdash_{X_t} incl_v(x) = incl_w(y)$. Since the X_t cover X, we conclude $\vdash_X incl_v(x) = incl_w(y)$, so the two elements in the pair (3.7) are equal.

The converse implication is proved much the same way.

For the case of binary products $H \times K$ in $\hat{\mathbf{A}}$, sentences in $\mathcal{L}_{\mathbf{A}}$ that intend to express that the comparison map $(Lan_y F)(H \times K) \to (Lan_y F)(H) \times (Lan_y F)(K)$ is monic, are easy to write down, much like the sentences (3.4). To express its surjectivity, one writes down, for each $u' \in H(A')$, $u'' \in K(A'')$ the sentence

(3.8)

$$\forall x' \in F(A') \, \forall x'' \in F(A'') \bigvee_{A \in \mathbf{A}} \bigvee_{\substack{u_1 \in H(A) \\ u_2 \in K(A)}} \bigvee_{\substack{\underline{g}' \in Z_{\mathbf{H}}(u_1, u') \\ \underline{g}'' \in Z_{\mathbf{K}}(u_2, u'')}}$$

$$\exists \underline{x}' \in M_F(h(\underline{g})) \, \exists \underline{x}'' \in M_F(k(\underline{g}'')) \ : \ d_1 \underline{x}' = x' \wedge d_1 \underline{x}'' = x'' \wedge d_0 \underline{x}' = d_0 \underline{x}''.$$

To prove that validity of these sentences does indeed force the comparison map to be (postulated) epi, provided the colimits for $(LanyF)(H)$ and $(Lan_y F)(K)$ [and $(Lan_y F)(H \times K)$] are postulated, one uses that $limC_i \times limD_j \cong lim(C_i \times D_j)$, for any postulaated colimits. – We omit the rest of the details.

If in particular one takes $H = yA'$, $K = yA''$, and x' and x'' the generic elements ($x' = id_{A'}$, $x'' = id_{A''}$), the category $\mathbf{H}$ is $\mathbf{A}/A'$ which has a terminal object, and similarly for $\mathbf{K}$, which means that the zig-zags may as well be chosen to be of length one, so that (3.8) in this case is equivalent to the simpler

(3.9)
$$\forall x' \in F(A') \, \forall x'' \in F(A'') \bigvee_{A \in \mathbf{A}} \bigvee_{\substack{u_1 : A \to A' \\ u_2 A \to A''}}$$
$$\exists x \in F(A) \ : \ F(u_1)(x) = x' \wedge F(u_2)(x) = x'',$$

which is one of the two sentences that defines the notion of *flat* functor on $\mathbf{A}$. The other two groups of sentences that defines this notion could be similarly motivated; they are

(3.10)
$$\bigvee_{a \in \mathbf{A}} \exists x \in F(A)$$

and, for each pair $a, b : A' \rightrightarrows A''$ of parallel arrows in $\mathbf{A}$, the sentence

(3.11)

$$\forall x \in F(A') : \ F(a)(x) = F(b)(x) \Rightarrow \bigvee_{A} \bigvee_{\substack{c : A \to A' \\ \text{with } ac = bc}} \exists y \in F(A) : F(c)(y) = x.$$

The sentencues (3.9), (3.10), and (3.11) are geometric sentences in the language $\mathcal{L}_{\mathbf{A}}$. Together with the equations needed to ensure that F commutes with composition and identities, they define the geometric theory $\mathbf{F}lat_{\mathbf{A}}$ of flat functors on $\mathbf{A}$. Thus it makes sense to talk about flat functors from $\mathbf{A}$ into any site $\underline{E}$. Like any geometric theory, $\mathbf{F}lat_{\mathbf{A}}$ has a classifying topos with a generic model. A fundamental result in topos theory is the so-called "Diaconescu Theorem" (cf. [1] 4.3) which asserts that the classifying topos for $\mathbf{F}lat_{\mathbf{A}}$ is $\hat{\mathbf{A}}$, with $y : \mathbf{A} \to \hat{\mathbf{A}}$ as the generic flat model. Diaconescu proved it in the context of elementary toposes. In the context of Grothendieck toposes, the theorem is virtually identical to the assertion that, for any Grothendieck topos $\underline{E}$, F flat $\Rightarrow Lan_y F$ is left exact , which may be essentially found in [4], Exposé 1.

Note that $Lan_y(y) =$ identity functor on $\hat{\mathbf{A}}$.

From the fact that all colimits in $\hat{\mathbf{A}}$ are postulated (by Proposition 2.1), and from the implication '$\Leftarrow$' in Theorem 3.1, we therefore immediately conclude that the generic flat functor y satisfies $\Lambda(D)$, for any finite limit diagram D in $\hat{\mathbf{A}}$. (This fact could of course also be easily seen by direct inspection.)

From the Completeness Theorem for Geometric Logic, we therefore conclude

$$\mathbf{F}lat_{\mathbf{A}} \vdash \Lambda(D) \tag{3.12}$$

for any finite limit diagram D in $\hat{\mathbf{A}}$.

From the soundness of geometric logic for interpretations in subcanonical sites $\underline{E}$, we conclude from (3.12) that if $\mathbf{A} \to \underline{E}$ is any flat functor into such site, then F satisfies the sentences $\Lambda(D)$. This does not imply that $Lan_y F$ preserves the limit diagram in D, but it does, by Theorem 3.1, if the appropriate colimits in $\underline{E}$ are postulated. We thus have

Corollary 3.2 *Let $\underline{E}$ be a (cocomplete, finitely complete, subcanonical) site, and let $F : \mathbf{A} \to \underline{E}$ be a flat functor. Let D be a finite limit diagram in $\hat{\mathbf{A}}$. If the colimits used for constructing $(Lan_y F)(D_i)$ (for those $D_i \in \hat{\mathbf{A}}$ that occur in D) are postulated, then $Lan_y F$ preserves the limit diagram D.*

Corollary 3.3 *Let $\underline{E}$ be an ∞-pretopos, and let $F : \mathbf{A} \to \underline{E}$ be flat. Then $Lan_y F : \hat{\mathbf{A}} \to \underline{E}$ is left exact.*

Proof. All small colimits in $\underline{E}$ are postulated, by Proposition 2.1. Now apply Corollary 3.2.

References

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