AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **34** (2006), Pages 211–228

The tough sets for the generalized Petersen graphs $G(n, 2)$

Kevin K. Ferland   Megan L. Holben

Bloomsburg University
Bloomsburg, PA 17815
U.S.A.

Abstract

The known upper bounds for the toughness of the generalized Petersen graphs $G(n, 2)$ are shown to be the exact values. Moreover, the tough sets are characterized in terms of certain key sections. Specifically, for $n \geq 9$ with $7 \nmid n$, we show that $\tau(G(n, 2)) = \frac{5\lfloor n/7 \rfloor + 3 + \delta}{4\lfloor n/7 \rfloor + 2 + \delta}$, where $\delta = 0$ if $n \equiv 1, 2,$ or $3 \pmod 7$ and $\delta = 1$ if $n \equiv 4, 5,$ or $6 \pmod 7$. These complement the known result that $\tau(G(n, 2)) = \frac{5}{4}$ if $7 \mid n$ (i.e. $\delta = 2$).

1 Basic Terminology

For each $n \geq 3$ and $0 < k < n$, $G(n, k)$ denotes the generalized Petersen graph [6] with vertex set $V = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ and edge set

$$E = \{\{u_i, u_{i+1}\} | 1 \leq i \leq n\} \cup \{\{u_i, v_i\} | 1 \leq i \leq n\} \cup \{\{v_i, v_{i+k}\} | 1 \leq i \leq n\}.$$

All subscripts are taken modulo n. The graph $G(5, 2)$ is *the* Petersen graph. The subgraph of $G(n, k)$ induced by $\{u_1, u_2, \ldots, u_n\}$ is called the outer rim, while that induced by $\{v_1, v_2, \ldots, v_n\}$ is called the inner rim. An edge of the form $\{u_i, v_i\}$ is called a spoke. The obvious dihedral symmetries of $G(n, k)$ preserve setwise the inner rim, the outer rim, and the spokes. Given a positive integer $m \leq n$, an m-section of $G(n, k)$ is a subgraph induced by a subset of V of the form $\{u_i, u_{i+1}, \ldots, u_{i+m-1}\} \cup \{v_i, v_{i+1}, \ldots, v_{i+m-1}\}$, for some i. If $n = \Sigma m_j$, then $G(n, k)$ can be built from a union of m_j-sections in the obvious way. In this paper, we focus on the case in which $k = 2$.

The toughness [1] of a connected non-complete graph $G = (V, E)$ is

$$\tau(G) = \min\{\frac{|S|}{\omega(G \setminus S)} \;:\; S \subseteq V,\ S \text{ disconnects } G\}, \tag{1.1}$$

where $\omega(G \setminus S)$ is the number of components of the graph obtained from G by removing the vertices of S. A subset $S \subseteq V$ that achieves the minimum in (1.1) is called a tough set for G.

2 Introduction

The toughness of generalized Petersen graphs $\tau(G(n,k))$ was first explored in [5], where the case in which $k = 1$ is completely settled. The case in which $k = 2$ is considered in [3], where $\frac{5}{4}$ is shown to be the critical value. More specifically, we define the function

$$t(n) = \begin{cases} \frac{5}{4} & \text{if } n \equiv 0 \pmod 7, \\ \frac{5n+16}{4n+10} & \text{if } n \equiv 1 \pmod 7, \\ \frac{5n+11}{4n+6} & \text{if } n \equiv 2 \pmod 7, \\ \frac{5n+6}{4n+2} & \text{if } n \equiv 3 \pmod 7, \\ \frac{5n+8}{4n+5} & \text{if } n \equiv 4 \pmod 7, \\ \frac{5n+3}{4n+1} & \text{if } n \equiv 5 \pmod 7, \\ \frac{5n-2}{4n-3} & \text{if } n \equiv 6 \pmod 7, \end{cases}$$

which is shown in [3] to give upper bounds for $\tau(G(n,2))$.

Theorem 2.1 ([3]). *For $n \geq 5$ and $n \neq 8$, $\frac{5}{4} \leq \tau(G(n,2)) \leq t(n)$. Also, $\tau(G(3,2)) = \frac{3}{2}$, $\tau(G(4,2)) = 1$, and $\tau(G(8,2)) = \frac{6}{5}$.*

In this paper, we complete the computation of $\tau(G(n,2))$ and show that the upper bounds displayed in Theorem 2.1 are, in fact, the exact values. Bounds on $\tau(G(n,k))$ for general k are explored in [2].

Our results are best understood through pictures. Essentially, we characterize how a tough set for $G(n,2)$ will look. Throughout this paper, sections of $G(n,2)$ are pictured with the outer rim at the top, and, to reduce excess clutter, relatively few of the inner rim edges are shown. To display a tough set S, we place a circle around each vertex in S and box off the components of $G \setminus S$. For example, what will be shown to be a tough set for $G(20,2)$ is displayed in Figure 1.

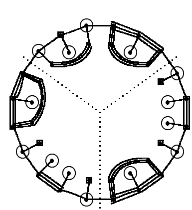

Figure 1: Tough Set for $G(20,2)$.

The important observation to make about our tough sets is that they are most naturally described in terms of certain **key m-sections**, shown in Figures 2 through 8. The full sections displayed are named M_0 through M_6, respectively, according to their lengths modulo 7. Note in Figures 3, 4, and 6 that the portion of the section up to the dotted line agrees with the shorter section used in [3] and is also called a key section here. As in [3], for each key m-section, S_m denotes the number of vertices of S it contains, and ω_m denotes the number of components of $G(n,2) \setminus S$

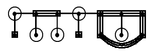

Figure 2: **Key 7-section** M_0 has $S_7 = 5$ and $\omega_7 = 4$.

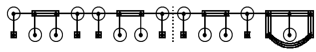

Figure 3: **Key 15-section** M_1 has $S_{15} = 13$ and $\omega_{15} = 10$. Left of the dotted line, the **key 8-section** has $S_8 = 8$ and $\omega_8 = 6$.

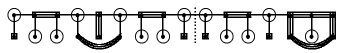

Figure 4: **Key 16-section** M_2 has $S_{16} = 13$ and $\omega_{16} = 10$. Left of the dotted line, the **key 9-section** has $S_9 = 8$ and $\omega_9 = 6$.

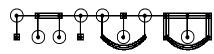

Figure 5: **Key 10-section** M_3 has $S_{10} = 8$ and $\omega_{10} = 6$.

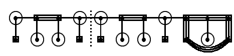

Figure 6: **Key 11-section** M_4 has $S_{11} = 9$ and $\omega_{11} = 7$. Left of the dotted line, the **key 4-section** has $S_4 = 4$ and $\omega_4 = 3$.

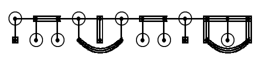

Figure 7: **Key 12-section** M_5 has $S_{12} = 9$ and $\omega_{12} = 7$.

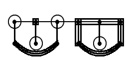

Figure 8: **Key 6-section** M_6 has $S_6 = 4$ and $\omega_6 = 3$.

it contains. A critical feature of these key m-sections is that, when they are linked together to form $G(n,2)$, the components local to one key section do not join up with the components local to another. Consequently, if $n = \sum m_j$ and $G(n,2)$ is built as a union of key m_j-sections, then $|S| = \sum S_{m_j}$ and $\omega(G(n,2) \setminus S) = \sum \omega_{m_j}$. The dotted lines in Figure 1 split $G(20,2)$ into 3 sections. All but one are the 7-section M_0. The remaining one, at the top of Figure 1, is M_6 and is special to the congruence class of 20 modulo 7.

The following is our main theorem.

Theorem 2.2. *For* $n \geq 9$, $\tau(G(n,2)) = t(n)$.

Observe that the values in Theorem 2.2 can be seen to come from building $G(n,2)$ in a natural way based upon the congruence class of n modulo 7. Take $r \in \{6, 7, 15, 16, 10, 11, 12\}$ with $n \equiv r \pmod 7$, and link $M_{n \bmod 7}$ with $\frac{n-r}{7}$ copies of M_0. We write $M_{n \bmod 7} M_0^{\frac{n-r}{7}}$. This forms both $G(n,2)$ and what will be shown to be a tough set S. By adding the appropriate values S_m and ω_m, it is easy to see that the values listed for $t(n)$ are obtained. Thus, it suffices to show that this method does indeed yield a tough set.

The proof of Theorem 2.2 is given in Section 5 after several preliminary results are established. With the additional machinery we develop, we can also give a characterization of all of the tough sets for $G(n, 2)$. That subject is addressed after the proof.

3 Local Structure of Tough Sets

Before characterizing the tough sets for $G(n, 2)$, we need a detailed understanding of their local structure. We thus make frequent use of the following easily proven rules governing this structure.

Lemma 3.1 ([3, 4]). *Let S be a tough set for a graph G.*

(a) ***Separation Rule****. If $v \in S$, then v is adjacent to at least two components of $G \setminus S$.*

(b) ***Cutpoint Rule****. If $\tau(G) > 1$, then no component of $G \setminus S$ has a cutpoint.*

(c) ***Trade Rule****. If a vertex v in S is adjacent to exactly two components of $G \setminus S$ and one of the components consists of more than one vertex but contains exactly one neighbor u of v, then $S \cup \{u\} \setminus \{v\}$ must also be a a tough set for G.*

Some special cases of Lemma 3.1 warrant special mention. The first follows from the separation rule, and the second results by combining the cutpoint rule with the trade rule.

Corollary 3.2. *Let S be a tough set for a graph G.*

(a) **3*-in-a-row Rule****. A vertex of degree 3 together with two of its neighbors cannot all be in S.*

(b) **2*-in-a-row Rule****. If $\tau(G) > 1$, then a vertex in S of degree 3 cannot have one neighbor in S and another neighbor whose removal leaves a cutpoint in its component of $G \setminus S$.*

Given a tough set S for $G(n, 2)$, we say that a component of $G(n, 2) \setminus S$ is **tame** if it is contained within an m-section from some $m < n$. Otherwise, it is said to be **wild**. Our principal result in this section is that, for $n \geq 13$, all components are tame and there are only three isomorphism types, namely, the complete graphs K_1 and K_2 and the 5-cycle C_5.

Lemma 3.3. *Let S be a tough set for $G(n, 2)$. If a component of $G(n, 2) \setminus S$ is tame, then it is one of K_1, K_2, or C_5.*

Proof. Denote $G(n,2)$ by G. Let T be a tame component of $G \setminus S$, and let M be the smallest section of G containing T.

First, we claim that either T is a K_1 component, T is a K_2 component, or the intersections of T with the 3-sections at the ends of M each form C_5. By reindexing if necessary, we may assume that M is induced by the vertices $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$.
Case 1: $u_1 \in T$ and $v_1 \in S$. So $u_0 \in S$. If $u_2 \in S$, then T is K_1. So suppose $u_2 \in T$. The cutpoint rule at u_2 now forces $v_2, u_3 \in S$. Thus, T is K_2.
Case 2: $u_1 \in S$ and $v_1 \in T$. So $v_{-1} \in S$. If $v_3 \in S$, then T is K_1. So suppose $v_3 \in T$. The cutpoint rule at v_3 now forces $u_3, v_5 \in S$. Thus, T is K_2.
Case 3: $u_1, v_1 \in T$. So $u_0, v_{-1} \in S$. Figure 9 shows the beginning structure of T in M. If either u_2 or v_3 is in S, then the cutpoint rule forces T to be K_2. So assume

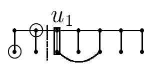

Figure 9: Tame component.

that $u_2, v_3 \in T$. If $u_3 \in S$, then the cutpoint rule forces $v_2 \in T$, the definition of M forces $v_0 \in S$, and the 2-in-a-row rule is violated at v_0. Hence, $u_3 \in T$. If $v_2 \notin S$, then $v_0 \in S$ and the 2-in-a-row rule is violated at u_0. So $v_2 \in S$. We thus see that, in the 3-section at the left end of M, the intersection with T forms C_5. By symmetry, the same must be true at the right end.

Second, having established our claim, we need only consider the case in which $m > 3$ and obtain a contradiction. In this case, $v_4 \notin S$, since otherwise the 2-in-a-row rule would be violated at v_2. So, in fact, it must be that $m \geq 6$. Moreover, observe that T must be the only component in M. This follows from our earlier claim, since any other component in M would have to be a K_1 component or a K_2 component, and it is straightforward to verify that any such component in M would force T to have a cutpoint, violating the cutpoint rule.

Take $6 \leq r \leq 12$ so that $m \equiv r \pmod 7$. Form a new m-section M' by starting with the key r-section, taken from Figures 2 through 8, and appending to the right $\frac{m-r}{7}$ copies of the key 7-section M_0. Replacing M by M', we obtain a new disconnecting set S' for G. From the observation that the inner vertices second from each end of M and M' agree in S and S', it follows that

$$|S'| \leq |S| + (S_r - 2) + 5\left(\frac{m-r}{7}\right)$$

and

$$\omega(G \setminus S') = \omega(G \setminus S) + (\omega_r - 1) + 4\left(\frac{m-r}{7}\right).$$

Since S is a tough set,

$$\frac{|S|}{\omega(G \setminus S)} \leq \frac{|S'|}{\omega(G \setminus S')} \leq \frac{|S| + (S_r - 2) + 5\left(\frac{m-r}{7}\right)}{\omega(G \setminus S) + (\omega_r - 1) + 4\left(\frac{m-r}{7}\right)}.$$

From the inequality

$$|S|\left[\omega(G\setminus S)+(\omega_r-1)+4\left(\tfrac{m-r}{7}\right)\right]\leq\left[|S|+(S_r-2)+5\left(\tfrac{m-r}{7}\right)\right]\omega(G\setminus S),$$

it follows that

$$\frac{|S|}{\omega(G\setminus S)}\leq\frac{(S_r-2)+5\left(\frac{m-r}{7}\right)}{(\omega_r-1)+4\left(\frac{m-r}{7}\right)}. \tag{3.1}$$

Since, in each case, $\frac{S_r-2}{\omega_r-1}<\frac{5}{4}$, the right-hand side of (3.1) is less than $\frac{5}{4}$. This contradicts the lower bound given in Theorem 2.1. □

Lemma 3.4. *Let S be a tough set for $G(n,2)$. Then, $G(n,2)\setminus S$ has at most one wild component.*

Proof. The result is trivial for $n\leq 4$, so assume $n\geq 5$. Each wild component must cross over a dotted line of the form shown in Figure 10. Note that there are three

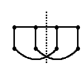

Figure 10: Cutline for wild components.

edges that cross this line, but at most two of them can be from distinct components of $G(n,2)\setminus S$. Hence, if there is more than one wild component of $G(n,2)\setminus S$, then there must be just two wild components and these must be the only two components. Since $|S|\geq 3$, this forces $\tau(G(n,2))\geq\frac{3}{2}$, which is a contradiction. □

We have seen in the proof of Lemma 3.3 the need to study the local structure of a tough set S, as in the argument surrounding Figure 9. In subsequent arguments, we need to consider whether or not certain vertices are in S or are in some component of $G(n,2)\setminus S$. However, at various points of these arguments, we may be unaware of where a known portion of a component ends. If a vertex is known not to be in S, but the size of its component is unknown, then that vertex may be marked with the symbol $\times$. For example, this is used in our arguments surrounding Figure 11.

Proposition 3.5. *Let $n\geq 5$ with $n\notin\{6,8,12\}$, and let S be a tough set for $G(n,2)$. Each component of $G(n,2)\setminus S$ is one of K_1, K_2, or C_5.*

Proof. Appendix A in [3] shows that this result holds for $5\leq n\leq 15$. So assume $n\geq 16$, and let $G=G(n,2)$. Since $\tau(G)<\frac{4}{3}$, it follows that $\omega(G\setminus S)\geq 4$. Suppose to the contrary that $G\setminus S$ has a wild component. By Lemma 3.4, there must be exactly one wild component W. By Lemma 3.3, each of the remaining components is K_1, K_2, or C_5. That the tame components can be neither C_5 nor a non-spoke K_2 can be easily seen by trying to depict how W crosses through the section containing such a tame component. We first show that a tame component also cannot be a spoke K_2.

Suppose to the contrary that there is a spoke K_2 component. Say $u_0, v_0, \notin S$ and $u_{-1}, u_1, v_{-2}, v_2\in S$. By the 3-in-a-row rule, $u_{-2}, u_2\notin S$. To accommodate W, we need $v_{-3}, v_{-1}, v_1, v_3\in W$. The cutpoint rule for W now forces $u_{-3}, u_3\in S$, and hence

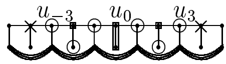

Figure 11: Wild component avoids K_2.

we must have $v_{-5}, v_5 \in W$. Since having $u_4 \in S$ would violate the 2-in-a-row rule at u_3, we have $u_4 \notin S$. Similarly we have $u_{-4} \notin S$, and therefore we have the structure shown in Figure 11. If $u_4 \in W$, then the set $S' = S \cup \{u_0, v_{-5}, v_{-1}\} \setminus \{u_{-1}, u_1, u_3\}$ satisfies $\frac{|S'|}{\omega(G \setminus S')} = \frac{|S|}{\omega(G \setminus S)}$ and is thus also a tough set for G. However, since $n \geq 16$, it follows that $G \setminus S'$ has a tame component that is neither K_1, K_2, nor C_5. This violates Lemma 3.3. By symmetry, $u_{-4}, u_4 \notin W$. So $u_{-5}, u_5 \in S$. Therefore, $v_{-7}, v_7 \in W$, and we have the structure shown in Figure 12. We consider two situations.

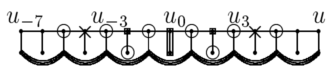

Figure 12: Wild component continues to avoid K_2.

Situation 1: $v_4 \notin S$. By the cutpoint rule (at v_4), $v_6 \in S$. So u_4 and v_4 form a spoke K_2 component. By the arguments used above for the K_2 component formed by u_0 and v_0, we now get $u_7, u_9 \in S$ and $u_6, u_8, v_9, v_{11} \notin S$ with $u_6, u_8 \notin W$. See Figure 13.

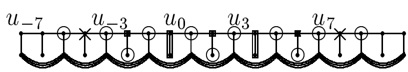

Figure 13: Situation 1 structure.

Situation 2: $v_4 \in S$. By the 3-in-a-row rule we have $v_6 \notin S$. Since having $u_6 \in S$ would violate the 2-in-a-row rule at u_5, we have $u_6 \notin S$. Since having $u_7 \notin S$ would violate the 2-in-a-row rule at v_4, we have $u_7 \in S$. By the cutpoint rule, $v_8 \in S$. So u_6 and v_6 form a spoke K_2 component and again applying the arguments used above for the K_2 component formed by u_0 and v_0, we get $u_9, u_{11} \in S$, and

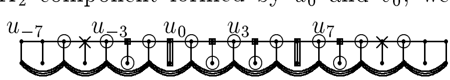

Figure 14: Situation 2 structure.

$u_8, v_9, u_{10}, v_{11}, v_{13} \notin S$. See Figure 14.

Observe that the right-most 8-sections from Figures 12 through 14 are the same. Hence, we can continue our arguments to the right, considering both situations at each stage, and certain properties are common to all cases. Namely, for j odd, we have $u_j \in S$ and $v_j \in W$, and, for j even, we have $u_j \notin S$ and $v_j \notin W$. Since this pattern persists, it must be that n is even, $\omega(G \setminus S) = \frac{n}{2} + 1$, and $|S| \geq \frac{3n}{4}$. Since $n \geq 16$,

$$\frac{|S|}{\omega(G \setminus S)} \geq \frac{3n}{2n+4} \geq \frac{4}{3},$$

contradicting the assumption that S is a tough set. This contradiction shows that there must be no spoke K_2 components. That is, all of the components of $G \setminus S$, besides W, are K_1 components.

Next, we claim that no vertex in S can be adjacent to two K_1 components from different rims. If $u_0 \in S$ is adjacent to K_1 components at v_0 and u_1, then $v_1, v_2 \in S$, and there is no way to accommodate W. So suppose to the contrary that $v_0 \in S$ is adjacent to K_1 components at u_0 and v_2. So $u_{-1}, u_1, u_2, v_4 \in S$. To accommodate W, we must have $v_{-3}, v_{-1}, v_1, v_3 \in W$. That the 2-in-a-row rule is now violated at u_1 gives a contradiction and establishes our claim. Thus, each vertex in S must be adjacent either to one K_1 component or to two K_1 components from the same rim.

Suppose there is a K_1 component on the outer rim, say u_0. Thus we have the structure shown in Figure 15 in the case that $j = 0$. Note that v_0 is adjacent

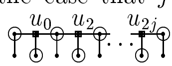

Figure 15: Outer rim chain.

to no other K_1 components and u_1 (and u_{-1}) can only be adjacent to another K_1 component on the outer rim. If u_1 is adjacent to another K_1 component, then we have the structure shown in Figure 15 in the case that $j = 1$. Of course, u_3 might be adjacent to another K_1 component on the outer rim, and so on. The j in Figure 15 simply reflects the number of times we go on.

Suppose there is a K_1 component on the inner rim, say v_0. Thus we have the structure shown in Figure 16 in the case that $j = 0$. Note that u_0 is adjacent

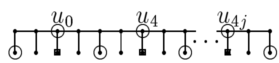

Figure 16: Inner rim chain.

to no other K_1 components and v_2 (and v_{-2}) can only be adjacent to another K_1 component on the inner rim. If v_2 is adjacent to another K_1 component, then we have the structure shown in Figure 16 in the case that $j = 1$. Of course, v_6 might be adjacent to another K_1 component on the inner rim, and so on. The j in Figure 16 simply reflects the number of times we go on.

Regard the K_1 components and vertices of S in sections of the types shown in Figures 15 and 16 as full chains if their endmost vertices in S are adjacent to exactly one K_1 component. For each type, the index j is some reflection of the length of the full chain. Additionally, a full chain might wrap around the entire graph $G(n, 2)$. Since each K_1 component must be adjacent to some vertex in S, each K_1 component is in some full chain. Since no vertex in S can be adjacent to K_1 components from different rims, all of the full chains are disjoint. Notice, within each full chain, that the number of K_1 components is at most half of the number of vertices in S. Summing over all full chains gives $|S| \geq 2(\omega(G \setminus S) - 1)$. So,

$$\frac{|S|}{\omega(G \setminus S)} \geq 2 - \frac{2}{\omega(G \setminus S)} \geq \frac{3}{2},$$

which contradicts the assumption that S is a tough set. □

As a consequence of Proposition 3.5, we can show that K_1 components must occur.

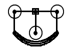

Figure 17: **Section** A has $S_A = 3$ and $\omega_A = 2$.

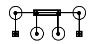

Figure 18: **Section** B has $S_B = 4$ and $\omega_B = 3$.

Corollary 3.6. *Let $n \geq 6$ with $n \neq 10$, and let S be a tough set for $G(n, 2)$. There must be a K_1 component in $G(n, 2) \setminus S$.*

Proof. In light of Appendix A of [3], it suffices to assume that $n \geq 16$. Suppose there are no K_1 components. By Proposition 3.5, each component must be K_2 or C_5. Thus each component is adjacent to at least 4 vertices in S. Since each vertex in S is adjacent to at most 3 components, it follows that $4\omega(G(n, 2) \setminus S) \leq 3|S|$. This contradicts the fact that $\tau(G(n, 2)) < \frac{4}{3}$. □

4 Forced Sections

Definition 4.1. We say that a tough set S is **trade equivalent** to a tough set S' if S' can be obtained from S by a sequence of applications of the trade rule. Moreover, if a sequence of trades is restricted to vertices in some section M, then we say that the resulting section M' with the same vertex set is trade equivalent to M and write $M \approx M'$.

Since Proposition 3.5 tells us that the complement of a tough set contains only three kinds of components, the potential changes in a tough set resulting from trades are rather limited. In particular, C_5 components are never affected by the trade rule. The main result of this section is that each tough set for $G(n, 2)$ is trade equivalent to one built from sections of the types A, B, B', B'', and C displayed and defined in Figures 17 through 21. In each case, the displayed components are assumed to end in the section as shown. That is, there are some vertices outside of the visible section assumed to be in S. We also define

$$N_3 = B''C,$$

and shall see that N_3 needs to be considered as an alternative to M_3M_0. Essentially, our main result in this paper is that, for $n \geq 13$, each tough set is trade equivalent to one built from the sections $M_0, \ldots, M_6, N_3$. This is seen to be true for $n = 13, 14, 15$ in Appendix A of [3]. We have additionally verified by computer that this is true for $n = 16, 17, 18, 19$. Thus, throughout this section, we assume that $n \geq 20$.

To simplify some of our proofs, we define an additional section type b. It is defined and displayed in Figure 22 and is assumed to contain a K_1 component as shown. Note that b serves as the beginning of B, B', and B''. The reflection of b is called d (of course). Note that the section type a in Figure 23 serves as the left end of A and b.

In this section, we use our knowledge of the local structure of a tough set to march around the graph and determine larger chunks of that structure.

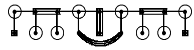

Figure 19: **Section** B' has $S_{B'} = 8$ and $\omega_{B'} = 6$.

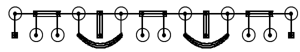

Figure 20: **Section** B'' has $S_{B''} = 12$ and $\omega_{B''} = 9$.

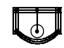

Figure 21: **Section** C has $S_C = 1$ and $\omega_C = 1$.

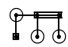

Figure 22: **Section** b has $S_b = 3$ and $\omega_b = 2$.

Figure 23: **Section** a has $S_a = 2$. Its components may extend beyond a.

Lemma 4.2. *A tough set for $G(n, 2)$ cannot contain sections of the types shown in Figure 24.*

Figure 24: Sections forbidden by Lemma 4.2.

Proof. Suppose a tough set S does contain a section of one of the types shown in Figure 24. We may assume that $i = 0$. For part (a), let $S' = S \setminus \{u_{-1}, u_1\}$. For part (b), let $S' = S \cup \{u_0\} \setminus \{u_{-1}, u_1, v_0\}$. In both cases, we have

$$\frac{|S'|}{\omega(G \setminus S')} = \frac{|S| - 2}{\omega(G \setminus S) - 1} < \frac{|S|}{\omega(G \setminus S)},$$

which contradicts the assumption that S is a tough set. □

Lemma 4.3. *Let S be a tough set for $G(n, 2)$. If S contains a section of the type shown on the left of Figure 25, then that section must be the left-end subsection of that shown on the right.*

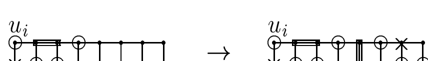

Figure 25: Structure forced by Lemma 4.3.

Proof. Suppose S contains a section of the type shown on the left of Figure 25. We may assume that $i = 1$. Proposition 3.5 forces $u_6, v_8 \in S$, and the 3-in-a-row rule gives $u_5 \notin S$.

Suppose toward a contradiction that $v_5 \in S$. The 3-in-a-row rule gives $v_7 \notin S$. By the trade rule with u_3 and v_3 and Lemma 4.2(a), we must have $u_7 \notin S$. By Proposition 3.5 and the 2-in-a-row rule (at v_5), we must have $u_8 \in S$. However, we can now trade u_3 with v_3, and then trade u_7 with u_8, to arrive at a contradiction with Lemma 4.2(a).

We conclude that $v_5 \notin S$. Proposition 3.5 then gives $v_7 \in S$, and the 3-in-a-row rule forces $u_7 \notin S$. □

4.1 Existence of A, B, B', or B''

In this subsection we strengthen Corollary 3.6. Recall that throughout Section 4 we are retaining the assumption that $n \geq 20$.

Lemma 4.4. *Every tough set for $G(n,2)$ is trade equivalent to one containing a section of type A, b, or d.*

Proof. Let S be a tough set for $G(n,2)$. By Corollary 3.6, $G(n,2) \setminus S$ must have a K_1 component.

Suppose a K_1 component is on the outer rim. We may assume $u_0 \notin S$ and $u_{-1}, u_1, v_0 \in S$. By the 3-in-a-row rule, we cannot have both v_{-1} and $v_1 \in S$. If $v_{-1}, v_1 \notin S$, then we have a section of type A. So, assume $v_{-1} \notin S$ and $v_1 \in S$. Then $u_2, v_3 \notin S$ by the 3-in-a-row rule, and $u_3 \in S$ by Proposition 3.5. This gives the structure shown in Figure 26. If $v_2 \notin S$, then $v_4 \in S$ by Proposition 3.5, and

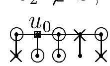

Figure 26: An outer rim K_1 not part of an A section.

trading u_1 and u_2 gives d. So, assume $v_2 \in S$. Then $v_4 \notin S$ by the 3-in-a-row rule. By Lemma 4.2(b), we must have $u_4 \notin S$. However, we may now trade v_2 and v_4, and then u_1 and u_2 to obtain d.

Suppose a K_1 component is on the inner rim. We may assume $v_0 \notin S$ and $u_0, v_{-2}, v_2 \in S$. It follows from Proposition 3.5 and the 3-in-a-row rule that we cannot have both $v_{-1}, v_1 \notin S$. So we may assume $v_1 \in S$. The 3-in-a-row rule then forces $u_1 \notin S$, and we have the structure shown in Figure 27. If $u_2 \in S$, then we

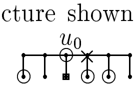

Figure 27: An inner rim K_1.

have an outer rim K_1 as handled above. If $u_2 \notin S$, then we have b. □

Lemma 4.5. *Let S be a tough set for $G(n,2)$ containing a section of type b. Then, there is a trade equivalence with the section to its right so that b becomes the left-end subsection of one of B, B', or B''. The symmetric result holds for d.*

Proof. We may assume $u_1, v_2, v_3 \in S, u_2, u_3, v_1 \notin S$. Proposition 3.5 gives $u_4 \in S$, and the 3-in-a-row rule gives $v_4 \notin S$. If $v_6 \in S$, then we have B. So assume

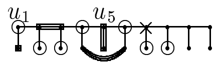

Figure 28: Section b starting more than B.

$v_6 \notin S$. Lemma 4.3 then forces the structure shown in Figure 28 with $u_6, v_7, v_8 \in S$, $u_5, u_7, v_5 \notin S$.

Case 1: $u_8 \in S$. So $u_9, v_{10} \notin S$ by the 3-in-a-row rule, and $u_{10} \in S$ by Proposition 3.5. If $v_9 \in S$, then $S' = S \cup \{v_6\} \setminus \{u_6, v_7, v_8\}$ is a disconnecting set with

$$\frac{|S'|}{\omega(G \setminus S')} = \frac{|S| - 2}{\omega(G \setminus S) - 1} < \frac{|S|}{\omega(G \setminus S)},$$

a contradiction. So we must have $v_9 \notin S$. Proposition 3.5 then gives $v_{11} \in S$, and trading u_8 and u_9 gives B'.

Case 2: $u_8 \notin S$. So $u_9 \in S$ by Proposition 3.5, and $v_9 \notin S$ by the 3-in-a-row rule. If $v_{11} \in S$, then we have B'. So assume $v_{11} \notin S$. Lemma 4.3 then forces the structure in Figure 29 with $u_{11}, v_{12}, v_{13} \in S, u_{10}, u_{12}, v_{10} \notin S$.

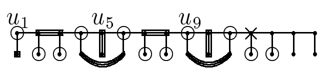

Figure 29: Section b starting more than B'.

Subcase 2a: $u_{13} \in S$. So $u_{14}, v_{15} \notin S$ by the 3-in-a-row rule, and $u_{15} \in S$ by Proposition 3.5. If $v_{14} \in S$, then, similar to above, the disconnecting set $S' = S \cup \{v_{11}\} \setminus \{u_{11}, v_{12}, v_{13}\}$ contradicts the assumption that S is a tough set. Hence we must have $v_{14} \notin S$. Proposition 3.5 then gives $v_{16} \in S$, and trading u_{13} and u_{14} gives B''.

Subcase 2b: $u_{13} \notin S$. So $u_{14} \in S$ by Proposition 3.5, and $v_{14} \notin S$ by the 3-in-a-row rule. If $v_{16} \in S$, then we have B''. So assume $v_{16} \notin S$. Lemma 4.3 forces the structure displayed in Figure 30 with $u_{16} \in S$, $u_{15}, v_{15} \notin S$. However, $S' = S \cup \{u_5, u_8, u_{12}, v_6, v_{10}, v_{11}, v_{14}\} \setminus \{u_6, u_{11}, u_{14}, v_8, v_{12}, v_{13}\}$ is then a disconnecting set with

$$\frac{|S'|}{\omega(G \setminus S')} = \frac{|S| + 1}{\omega(G \setminus S) + 1} < \frac{|S|}{\omega(G \setminus S)},$$

which shows that this subcase cannot occur. □

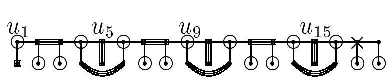

Figure 30: Section b starting more than B''.

Corollary 4.6. *Every tough set for $G(n, 2)$ is trade equivalent to one containing a section of type A, B, B', or B''.*

4.2 Neighbors of Sections

Proposition 3.5 tells us that C is a possible section in a tough set for $G(n,2)$. Corollary 4.6 establishes the importance of A, B, B', and B''. We now explore the possible combinations of these important section types, still under the assumption that $n \geq 20$.

Lemma 4.7. *The sections dB'', AB', AB'', BB'', $B'B''$, $B''B''$, $B'B'$, AA, ABA, ABB, ABB', BAB, $BB'B$, $BBBA$, $BBBB$, $BBBB'$, $B'BB$, $B'BB'$, and their mirror images cannot occur in a tough set for $G(n,2)$.*

Proof. Let S be a tough set containing one of these sections.

If S contains AB' or AB'', then replacing AB' with BBB or replacing AB'' with BBB', respectively, yields a disconnecting set S' with

$$\frac{|S'|}{\omega(G \setminus S')} = \frac{|S|+1}{\omega(G \setminus S)+1} < \frac{|S|}{\omega(G \setminus S)}.$$

If S contains dB'' or $B'B'$, then replacing B'' with $CBCB$ or replacing $B'B'$ with $BCBCB$, respectively, yields a disconnecting set S' with

$$\frac{|S'|}{\omega(G \setminus S')} = \frac{|S|-2}{\omega(G \setminus S)-1} < \frac{|S|}{\omega(G \setminus S)}.$$

Note that dB'' is the right end of each of BB'', $B'B''$, and $B''B''$.

If S contains AA, then trading u_4 and v_4 (regarding the left end of AA as u_1 and v_1) contradicts the mirror image of Lemma 4.3.

If S contains ABa, then the fact that $ABa \approx BAa$ contradicts Lemma 4.2(a). Note that ABa is the left end of each of ABA, ABB, and ABB'. Also, Lemma 4.2(a) directly forbids BAB.

If S contains $BBBa$ or $B'Ba$, then replacing BBB with $B'C$ or replacing $B'B$ with $BCAC$ yields a disconnecting set S' with

$$\frac{|S'|}{\omega(G \setminus S')} = \frac{|S|-3}{\omega(G \setminus S)-2} < \frac{|S|}{\omega(G \setminus S)}.$$

Note that $BBBa$ is the left end of each of $BBBA$, $BBBB$, and $BBBB'$, and $B'Ba$ is the left end of both $B'BB$ and $B'BB'$. Also, $BB'B \approx B'BB$. □

Lemma 4.8. *Let S be a tough set for $G(n,2)$ containing a section of type d. The section to its right either is C or is trade equivalent to A, B, or B'.*

Proof. Assume that the left end of d is at vertices u_1 and v_1. That is, $u_3, v_1, v_2, v_5 \in S$ and $u_1, u_2, v_3 \notin S$. By the 3-in-a-row rule, we cannot have both $u_4, v_4 \in S$.

Case 1: $u_4, v_4 \notin S$. If $u_5 \notin S$, then Proposition 3.5 gives $u_6, v_6 \notin S$, and we have dC. So assume $u_5 \in S$. Proposition 3.5 gives $v_6 \in S$, the 3-in-a-row rule gives $u_6 \notin S$, and we have the structure shown in Figure 31. We can now trade u_4 and u_5 to get db, and Lemma 4.5 tells us that we can obtain one of dB, dB', or dB''. However, by Lemma 4.7, dB'' is impossible.

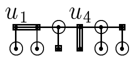

Figure 31: d meets $u_4, v_4 \notin S$.

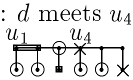

Figure 32: d meets $u_4 \notin S, v_4 \in S$.

Case 2: $u_4 \notin S$ and $v_4 \in S$. The 3-in-a-row rule gives $v_6 \notin S$ and the structure shown in Figure 32. If $u_5 \notin S$, then Proposition 3.5 gives $u_6 \in S$, and we can trade u_4 and v_4 to obtain dA. So assume $u_5 \in S$. The 3-in-a-row rule now gives $u_6 \notin S$. Trading first u_5 and u_6, and then u_4 and v_4, gives dA.

Case 3: $u_4 \in S$ and $v_4 \notin S$. The 3-in-a-row rule gives $u_5 \notin S$, and we have the structure shown in Figure 33. If $u_6 \notin S$, then Proposition 3.5 gives $v_6 \in S$, and, as

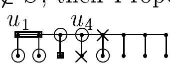

Figure 33: d meets $u_4 \in S, v_4 \notin S$.

above, we can obtain dB or dB'. So, assume $u_6 \in S$. If $v_6 \notin S$, then we have dA. So assume $v_6 \in S$. Now, $u_7, v_8 \notin S$ by the 3-in-a-row rule. Further, $v_7 \notin S$ by the reflection of Lemma 4.2(b), and $v_9 \in S$ by Proposition 3.5. Trading u_6 and u_7 gives dB. □

Proposition 4.9. *Let S be a tough set for $G(n, 2)$.*

(a) If S contains a section of type A, then the section next to it either is C or is trade equivalent to B.

(b) If S contains a section of type B, then the section next to it either is C or is trade equivalent to A, B, or B'.

(c) If S contains a section of type B', then the section next to it either is C or is trade equivalent to B.

(d) If S contains a section of type B'', then the section next to it is C.

(e) If S contains a section of type C, then the section next to it is trade equivalent to A, B, B', or B''.

Proof. (a) Assume that the left end of A is at u_1 and v_1. That is, $u_1, u_3, v_2, v_5 \in S$ and $u_2, v_1, v_3 \notin S$. By the 3-in-a-row rule, we cannot have both $u_4, v_4 \in S$.

Case 1: $u_4, v_4 \notin S$. If $u_5 \notin S$, then Proposition 3.5 gives $u_6, v_6 \notin S$, and we have AC. So assume $u_5 \in S$. Proposition 3.5 now gives $v_6 \in S$, the 3-in-a-row rule gives $u_6 \notin S$, and we have the structure shown in Figure 34. Trading u_4 and u_5 gives Ab.

Figure 34: A meets $u_4, v_4 \notin S$.

Lemma 4.5 now tells us that we can obtain one of AB, AB', or AB''. However, by Lemma 4.7, AB' and AB'' are impossible.

Case 2: $u_4 \notin S$ and $v_4 \in S$. The 3-in-a-row rule gives $v_6 \notin S$, and we have the structure shown in Figure 35. If $u_5 \notin S$, then Proposition 3.5 gives $u_6 \in S$,

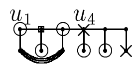

Figure 35: A meets $u_4 \notin S$ and $v_4 \in S$.

and the reflection of Lemma 4.3 contradicts the fact that $v_2 \in S$. So, we must have $u_5 \in S$. Further, $u_6 \notin S$ by the 3-in-a-row rule. We can now trade u_5 and u_6, and the reflection of Lemma 4.3 again contradicts $v_2 \in S$.

Case 3: $u_4 \in S$ and $v_4 \notin S$. So $u_5 \notin S$ by the 3-in-a-row rule. Since we could trade u_3 and v_3, the 3-in-a-row rule forces $v_7 \notin S$, and we have the structure shown in Figure 36. If $u_6 \notin S$, then Proposition 3.5 gives $v_6 \in S$, and, similar to case 1, we

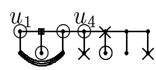

Figure 36: A meets $u_4 \in S$ and $v_4 \notin S$.

obtain AB. So assume $u_6 \in S$. Since Lemma 4.7 forbids AA, we must have $v_6 \in S$. The 3-in-a-row rule gives $u_7 \notin S$. Now, trade u_6 and u_7. As before, we obtain AB.

(b), (c), and (d) Since d is the right end of B, B', and B'', Lemma 4.8 tells us that the section next to them either is C or is trade equivalent to A, B, or B'. However, Lemma 4.7 says that BB'', $B'A$, $B'B'$, $B'B''$, $B''A$, $B''B$, $B''B'$, and $B''B''$ are impossible.

(e) Assume that the left end of C is at u_1 and v_1. That is, $u_1, u_2, u_3, v_1, v_3 \notin S$ and $v_2, u_4, v_5 \in S$. So $u_5, v_4 \notin S$ by the 3-in-a-row rule, and we have Ca. If $u_6 \notin S$, then Proposition 3.5 gives $v_6 \in S$, we have Cb, and, by Lemma 4.5, we can obtain one of CB, CB', or CB''. So assume $u_6 \in S$. If $v_6 \notin S$, then we have CA. So assume $v_6 \in S$. By the 3-in-a-row rule, $u_7 \notin S$. By the 2-in-a-row rule (at v_5), $v_7 \notin S$. This gives the structure shown in Figure 37, and trading u_6 and u_7 gives Cb, as handled above. □

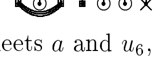

Figure 37: C meets a and $u_6, v_6 \in S$, $u_7, v_7 \notin S$.

Starting from the sections listed in Corollary 4.6, we explore what can follow to their right until a section of type C is encountered.

Proposition 4.10. *Let S be a tough set for $G(n,2)$.*

(a) If S contains a section of type A, then there is a trade equivalence with the section to its right so that A becomes the left-end subsection of $AC = M_6$ or $ABC \approx M_3$.

(b) If S contains a section of type B, then there is a trade equivalence with the section to its right so that B becomes the left-end subsection of $BC = M_0$, $BAC = M_3$, $BBC = M_4$, $BBBC = M_1$, or $BB'C \approx M_2$.

(c) If S contains a section of type B', then there is a trade equivalence with the section to its right so that B' becomes the left-end subsection of $B'C = M_5$ or $B'BC = M_2$.

(d) If S contains a section of type B'', then B'' is the left-end subsection of $B''C = N_3$.

Proof. By Proposition 4.9, we need only consider certain sections.

(a) We consider AC, ABC, ABA, ABB, and ABB'. However, Lemma 4.7 excludes ABA, ABB, and ABB'.

(b) We consider BC, BAC, BAB, BBC, BBA, $BBBC$, $BBBA$, $BBBB$, $BBBB'$, BBB', $BB'C$, and $BB'B$. However, Lemma 4.7 excludes BAB, BBA, $BBBA$, $BBBB$, $BBBB'$, BBB', and $BB'B$.

(c) We consider $B'C$, $B'BC$, $B'BA$, $B'BB$, and $B'BB'$. However, Lemma 4.7 excludes $B'BA$, $B'BB$, and $B'BB'$.

Part (d) simply recasts Proposition 4.9(d). □

Corollary 4.11. *Every tough set for $G(n, 2)$ contains a section of type C.*

5 Global Structure of Tough Sets

Proof of Theorem 2.2. By our computer verification, it suffices to assume $n \geq 20$. Let S be a tough set for $G(n, 2)$. By Corollary 4.11, it has a section of type C. By Proposition 4.9(e), the section to the right of this C is trade equivalent to A, B, B', or B''. By Proposition 4.10, the section to the right of our initial C must in fact be trade equivalent to one of $M_0, \ldots, M_6, N_3$. Since each of these ends in C, we can continue this process and conclude that S must be trade equivalent to a tough set built exclusively of key sections from the list $M_0, \ldots, M_6, N_3$. We further claim that we can assume that at most one of the key sections is not M_0. Of course, which section is used besides M_0 depends upon the congruence class of n modulo 7.

Suppose that two or more sections are used from $M_1, \ldots, M_6, N_3$. Let m_1 and m_2 be their lengths. Since, permuting the key sections does not change the size of S or $\omega(G(n, 2) \setminus S)$, we may assume that the m_1- and m_2-section are next to each other forming an $(m_1 + m_2)$-section M. Since $m_1, m_2 \geq 6$, we have $m_1 + m_2 \geq 12$. Consequently, there is some $r \in \{6, 7, 15, 16, 10, 11, 12\}$ such that $r \leq m_1 + m_2$ and $m_1 + m_2 \equiv r \pmod 7$. Replace M by a new $(m_1 + m_2)$-section M' built by attaching an r-section from $M_1, \ldots, M_6$ to $\frac{m_1+m_2-r}{7}$ copies of M_0. It is straightforward to check in each case that

$$\frac{S_r + 5(\frac{m_1+m_2-r}{7})}{\omega_r + 4(\frac{m_1+m_2-r}{7})} \leq \frac{S_{m_1} + S_{m_2}}{\omega_{m_1} + \omega_{m_2}}. \tag{5.1}$$

That is, M' is no worse a choice for an $(m_1 + m_2)$-section than M. By repeating this process, as necessary, we can replace the tough set S by one that uses at most one section from $M_1, \dots, M_6$. Now,

$$\frac{|S|}{\omega(G(n,2) \setminus S)}$$

is easily computed and seen to have the asserted value. □

In verifying (5.1), there are four cases in which equality occurs. In each of those cases, a certain pair of sections from $M_0, \dots, M_6$ can be replaced by some other combination of sections from that list without changing the size of the tough set. Consequently, we say that the pair-section and its replacement are size equivalent sections. We denote this weaker notion of size equivalence between sections and tough sets by the symbol $\sim$. Our proof of Theorem 2.2 establishes that any two tough sets for $G(n,2)$ are size equivalent. Moreover, with the additional consideration of the relationship between N_3 and M_3, there are five basic size equivalences

$$\begin{aligned} M_1 M_0 &\sim M_4^2 \\ M_2 M_0 &\sim M_4 M_5 \\ M_3 M_0 &\sim M_4 M_6 \\ M_3 M_0^2 &\sim M_5^2 \\ M_3 M_0 &\sim N_3. \end{aligned}$$

The last two are the most intriguing since they do not preserve the number of C_5 components. It is straightforward to verify that there are no further combinations of sections from $M_1, \dots, M_6, N_3$ resulting in size equivalences. After we strip out the dihedral symmetries, the trade equivalences, and the freedom to permute the key sections, the five relations listed above generate all size equivalences. Note that, modulo dihedral symmetries, tough sets are unique when $n \equiv 5, 6$, or $0 \pmod 7$. There are two when $n \equiv 4 \pmod 7$, but they are trade equivalent.

References

[1] V. Chvátal, *Tough graphs and Hamiltonian circuits*, Discrete Math. **5** (1973), 215–228.

[2] K. K. Ferland, *Toughness of generalized Petersen graphs*, Ars Combinatoria **66**, 49–63.

[3] K. K. Ferland, *On the toughness of some generalized Petersen graphs*, Ars Combinatoria **36** (1993), 65–88.

[4] K. K. Ferland and L. L. Doty, *Maximally tough cubic graphs*, Congressus Numerantium **161** (2003), 75–96.

[5] B. Piazza, R. Ringeisen, and S. Stueckle, *On the vulnerability of cycle permutation graphs*, Ars Combinatoria **29** (1990), 289–296.

[6] M. E. Watkins, *A theorem on Tait colorings with an application to the generalized petersen graphs*, Journal of Combinatorial Theory **6** (1969), 152–164.

(Received 20 June 2004; revised 17 Mar 2005)