Differential cohomology in a cohesive ∞-topos

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21st century

Abstract

We formulate differential cohomology and Chern-Weil theory - the theory of connections on fiber bundles and of gauge fields - abstractly in the context of a certain class of higher toposes that we call *cohesive*. Cocycles in this differential cohomology classify higher principal bundles equipped with *cohesive structure* (topological, smooth, synthetic differential, supergeometric, etc.) and equipped with *connections*. We discuss various models of the axioms and wealth of applications revolving around fundamental notions and constructions in prequantum field theory and string theory. In particular we show that the cohesive and differential refinement of universal characteristic cocycles constitutes a higher Chern-Weil homomorphism refined from secondary caracteristic classes to morphisms of higher moduli stacks of higher gauge fields, and at the same time constitutes extended geometric prequantization – in the sense of extended/multi-tiered quantum field theory – of higher dimensional Chern-Simons-type field theories and Wess-Zumino-Witten-type field theories.

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We formulate differential cohomology and Chern-Weil theory - the theory of connections on fiber bundles and of gauge fields - abstractly in the context of a certain class of ∞-toposes that we call *cohesive*. Cocycles in this differential cohomology classify principal ∞-bundles equipped with *cohesive structure* (topological, smooth, synthetic differential, etc.) and equipped with ∞-*connections*.

We construct the cohesive ∞-topos of smooth ∞-groupoids and ∞-Lie algebroids and show that in this concrete context the general abstract theory reproduces ordinary differential cohomology (Deligne cohomology/differential characters), ordinary Chern-Weil theory, the traditional notions of smooth principal bundles with connection, abelian and nonabelian gerbes/bundle gerbes with connection, principal 2-bundles with 2-connection, connections on 3-bundles, etc. and generalizes these to higher degree and to base spaces that are orbifolds and generally smooth ∞-groupoids, such as smooth realizations of classifying spaces/moduli stacks for principal ∞-bundles and configuration spaces of gauge theories.

We exhibit a general abstract ∞-*Chern-Weil homomorphism* and observe that it generalizes the Lagrangian of Chern-Simons theory to ∞-*Chern-Simons theory*. For every invariant polynomial on an ∞-Lie algebroid it sends principal ∞-connections to *Chern-Simons circle* $(n+1)$-*bundles* (n-gerbes) with connection, whose higher parallel transport is the corresponding higher Chern-Simons Lagrangian. There is a general abstract formulation of the higher holonomy of this parallel transport and this provides the action functional of ∞-Chern-Simons theory as a morphism on its cohesive configuration ∞-groupoid. Moreover, to each of these higher Chern-Simons Lagrangian is canonically associated a differentially twisted looping, which we identify with the corresponding *higher Wess-Zumino-Witten Lagrangian*.

We show that, when in interpreted in smooth ∞-groupoids and their variants, these intrinsic constructions reproduce the ordinary Chern-Weil homomorphism, hence ordinary Chern-Simons functionals and ordinary Wess-Zumino-Witten functionals, provides their geometric prequantization in higher codimension (localized down to the point) and generalizes this to a fairly extensive list of action functionals of quantum field theories and string theories, some of them new. All of these appear in their refinement from functionals on local differential form data to global functionals defined on the full moduli ∞-stacks of field configurations/∞-connections, where they represent higher prequantum line bundles. We show that these moduli ∞-stacks naturally encode fermionic σ-model anomaly cancellation conditions, such as given by higher analogs of Spin-structures and of Spin^c-structures.

We moreover show that *higher symplectic geometry* is naturally subsumed in higher Chern-Weil theory, such that the passage from the unrefined to the refined Chern-Weil homomorphism induced from higher symplectic forms implements *geometric prequantization* of the above higher Chern-Simons and higher Wess-Zumino-Witten functionals.

We think of these results as providing a further ingredient of the recent identification of the mathematical foundations of quantum field and perturbative string theory [SaSch11]: while the cobordism theorem [LurieTQFT] identifies topological quantum field theories with a universal construction in higher category theory (representations of free symmetric monoidal (∞, n)-categories), our results indicate that the geometric structures that these arise from under geometric quantization originate in a universal construction in higher topos theory: *cohesion*.

The program discussed here was initiated around [SSS09c], following an unpublished precursor set of notes [SSSS08], presented at [Sc09], motivated in parts by the desire to put the explicit constructions of [ScWaI] [ScWaII] [ScWaIII] [BCSS07] [RoSc08] on a broad conceptual basis. The present text has grown out of and subsumes these and the series of publications [SS10, FSS10, FRS11a, FiSaScII, FiSaScIII, NSSa, NSSb]. Notes from a lecture series introducing some of the central ideas with emphasis on applications to string theory is available as [Sc12a]. (The basic idea of considering differential cohomology in the ∞-topos over smooth manifolds has then also been voiced in [Ho11][1], together with the statement that this is the context in which the seminal article [HoSi05] on differential cohomology was eventually meant to be considered.) The following text aims to provide a comprehensive theory and account of these developments. In as far as it uses paragraphs taken from the above joint publications, these paragraphs have been primarily authored by the present author.

[1]We are grateful to Alexander Kahle for pointing out this talk to us at *String-Math 2012*.

I heartily thank my coauthors for their input and work. Moreover I am grateful to Richard Williamson for an extra derived left adjoint, to David Carchedi for an extra derived right adjoint and to a talk by Peter Johnstone for making me recognize their 1-categorical shadow in Lawvere's work. I am indebted to Mike Shulman for plenty of discussion and input on higher topos theory and homotopy type theory in general. I thank Geoffrey Cruttwell for plenty of remarks on the present writeup and related discussion.

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In 1 we motivate our discussion, give an informal introduction to the main concepts involved and survey various of our constructions and applications in a more concrete, more traditional and more expository way than in the sections to follow. This may be all that some readers ever want to see, while other readers may want to skip it entirely.

In 2 we review relevant aspects of *homotopy type theory*, the theory of ∞-*categories* and ∞-*toposes*, in terms of which all of the following is formulated. This serves to introduce context and notation and to provide a list of technical lemmas which we need in the following, some of which are not, or not as explicitly, stated in existing literature.

In 3 we introduce *cohesive homotopy type theory*, a general abstract theory of differential geometry, differential cohomology and Chern-Weil theory in terms of universal constructions in ∞-topos theory. This is in the spirit of Lawvere's proposals [Lawv07] for axiomatic characterizations of those *gros toposes* that serve as contexts for abstract *geometry* in general and *differential geometry* in particular: *cohesive toposes*. We claim that the decisive role of these axioms is realized when generalizing from topos theory to ∞-topos theory and we discuss a fairly long list of geometric structures that is induced by the axioms in this case. Notably we show that every ∞-topos satisfying the immediate analog of Lawvere's axioms – every *cohesive* ∞-*topos*– comes with a good intrinsic notion of differential cohomology and Chern-Weil theory.

Then we add a further simple set of axioms to obtain a theory of what we call *differential cohesion*, a refinement of cohesion that axiomatizes the explicit ("synthetic") presence of infinitesimal objects. This is closely related to Lawvere's *other* proposal for axiomatizing toposes for differential geometry, called *synthetic differential geometry* [Lawv97], but here formulated entirely in terms of higher *closure modalities* as for cohesion itself. We find that these axioms also capture the modern synthetic-differential theory of *D-geometry* [Lurie09c]. In particular a differential cohesive ∞-topos has an intrinsic notion of (formally) *étale maps*, which makes it an axiomatic geometry in the sense of [Lurie09a] and equips it with intrinsic *manifold* theory.

In 4 we discuss models of the axioms, hence ∞-toposes of ∞-groupoids which are equipped with a geometric structure (topology, smooth structure, supergeometric structure, etc.) in a way that all the abstract differential geometry theory developed in the previous chapter can be realized. The main model of interest for our applications is the cohesive ∞-topos Smooth∞Grpd as well as its infinitesimal thickening SynthDiff∞Grpd, which we construct. Then we go step-by-step through the list of general abstract structures in cohesive ∞-toposes and unwind what these amount to in this model. We demonstrate that these subsume traditional definitions and constructions and generalize them to higher differential geometry and differential cohomology.

In 5 we discuss applications of the general theory in the context of smooth ∞-groupoids and their synthetic-differential and super-geometric refinements. We present a fairly long list of higher Spin- and Spin^c-structures, of classes of action functionals on higher moduli stacks of higher Chern-Simons type and functionals of higher Wess-Zumino-Witten type, that are all naturally induced by higher Chern-Weil theory. We exhibit a higher analog of geometric prequantization that applies to these systems and show that it captures a wealth of structures. Apart from the new constructions and results, this shows that large parts of prequantum field theory are canonically and fundamentally induced by abstract cohesion.

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1 Introduction

We give here an introduction to and exposition of central themes in *differetial geoemtry* with emphasis on its applications in *physics* in more or less traditional terms, but in a way that motivates the further formalization and investigation of these structures by *cohesive higher geometry* or *cohesive homotopy theory* which is our main theme in section 3 below.

1.1 Motivation and survey

In 1.1.1 we give a heuristic motivation from considerations in gauge theory in broad terms; then in 1.1.2 and 1.1.4 a more technical motivation proceeding from natural classes of action functionals in higher gauge theory, the problem of quantum anomaly cancellation and the inadequacy of classical Chern-Weil theory to describe this.

Finally in 1.1.5 we offer a more formal motivation from the point of view of foundations.

1.1.1 Motivation from gauge theory

The discovery of *gauge theory* is effectively the discovery of groupoids in fundamental physics. The notion of *gauge transformation* is close to synonymous to the notion *isomorphism* and more generally to *equivalence in an ∞-category*. From a modern point of view, the mathematical model for a gauge field in physics is a cocycle in (nonabelian) differential cohomology: principal bundles with connection and their higher analogs. These naturally do not form just a set, but a groupoid and generally an ∞-groupoid, whose morphisms are gauge transformations, and higher morphisms are gauge-of-gauge transformations. The development of differential cohomology has to a fair extent been motivated and influenced by its application to fundamental theoretical physics in general and gauge theory in particular.

Around 1850 Maxwell realized that the field strength of the electromagnetic field is modeled by what today we call a closed differential 2-form on spacetime. In the 1930s Dirac observed that in the presence of electrically charged quantum particles such as electrons, more precisely this 2-form is the *curvature* 2-form of a *$U(1)$-principal bundle with connection*.

In modern terms this, in turn, means equivalently that the electromagnetic field is modeled by a degree 2-cocycle in (ordinary) *differential cohomology*. This is a differential refinement of the degree-2 integral cohomology that classifies the underlying $U(1)$-principal bundles themselves via what mathematically is their *Chern class* and what physically is the topological *magnetic charge*. A coboundary in degree-2 differential cohomology is, mathematically, a smooth isomorphism of bundles with connection, hence, physically, is a *gauge transformation* between field configurations. Therefore classes in differential cohomology characterize the *gauge-invariant* information encoded in gauge field configurations, such as the electromagnetic field.

Meanwhile, in 1915, Einstein had identified also the field strength of the field of gravity as the $\mathfrak{so}(d,1)$-valued curvature 2-form of the canonical $O(d,1)$-principal bundle with connection on a $d+1$-dimensional spacetime Lorentzian manifold. This is a cocycle in differential nonabelian cohomology: in Chern-Weil theory.

In the 1950s Yang-Mills-theory identified the field strength of all the gauge fields in the standard model of particle physics as the $\mathfrak{u}(n)$-valued curvature 2-forms of $U(n)$-principal bundles with connection. This is again a cocycle in differential nonabelian cohomology.

Entities of ordinary gauge theory
Lie algebra $\mathfrak{g}$ with gauge Lie group G — connection with values in $\mathfrak{g}$ on G-principal bundle over a smooth manifold X

It is noteworthy that already in this mathematical formulation of experimentally well-confirmed fundamental physics the seed of higher differential cohomology is hidden: Dirac had not only identified the electromagnetic field as a line bundle with connection, but he also correctly identified (rephrased in modern language) its underlying cohomological Chern class with the (physically hypothetical but formally inevitable) magnetic charge located in spacetime. But in order to make sense of this, he had to resort to removing the support of the magnetic charge density from the spacetime manifold, because Maxwells equations imply that at the support of any magnetic charge the 2-form representing the field strength of the electromagnetic field is in fact not closed and hence in particular not the curvature 2-form of an ordinary connection on an ordinary bundle.

In [Free00] Diracs old argument was improved by refining the model for the electromagentic field one more step: Dan Freed notices that the charge current 3-form is itself to be regarded as a curvature, but for a connection on a circle 2-bundle with connection - also called a bundle gerbe -, which is a cocycle in degree-3 ordinary differential cohomology. Accordingly, the electromagnetic field is fundamentally not quite a line bundle, but a *twisted bundle* with connection, with the twist being the magnetic charge 3-cocycle. Freed shows that this perspective is inevitable for understanding the quantum anomaly of the action functional for electromagnetism is the presence of magnetic charge.

In summary, the experimentally verified models, to date, of fundamental physics are based on the notion of (twisted) $U(n)$-principal bundles with connection for the Yang-Mills field and $O(d,1)$-principal bundles with connection for the description of gravity, hence on nonabelian differential cohomology in degree 2 (possibly with a degree-3 twist).

In attempts to better understand the structure of these two theories and their interrelation, theoretical physicists were led to consider variations and generalizations of them that are known as *supergravity* and *string theory*. In these theories the notion of gauge field turns out to generalize: instead of just Lie algebras, Lie groups and connections with values in these, one finds structures called *Lie 2-algebras*, *Lie 2-groups* and the gauge fields themselves behave like generalized connections with values in these.

Entities of 2-gauge theory
Lie 2-algebra $\mathfrak{g}$ with gauge Lie 2-group G — connection with values in $\mathfrak{g}$ on a G-principal 2-bundle/gerbe over an orbifold X

Notably the string is charged under a field called the *Kalb-Ramond field* or *B-field* which is modeled by a $\mathbf{B}U(1)$-principal 2-bundle with connection, where $\mathbf{B}U(1)$ is the Lie 2-group delooping of the circle group: the circle Lie 2-group. Its Lie 2-algebra $\mathbf{B}\mathfrak{u}(1)$ is given by the differential crossed module $[\mathfrak{u}(1) \to 0]$ which has $\mathfrak{u}(1)$ shifted up by one in homological degree.

So far all these differential cocycles were known and understood mostly as concrete constructs, without making their abstract home in differential cohomology explicit. It is the next gauge field that made Freed and Hopkins propose [FrHo00] that the theory of differential cohomology is generally the formalism that models gauge fields in physics:

The superstring is charged also under what is called the *RR-field*, a gauge field modeled by cocycles in differential K-theory. In even degrees we may think of this as a differential cocycle whose curvature form has coefficients in the L_∞-algebra $\oplus_{n\in\mathbb{N}}\mathbf{B}^{2n}\mathfrak{u}(1)$. Here $\mathbf{B}^{2n}\mathfrak{u}(1)$ is the abelian $2n$-Lie algebra whose underlying complex is concentrated in degree $2n$ on $\mathbb{R}$. So fully generally, one finds *∞-Lie algebras*, *∞-Lie groups* and gauge fields modeled by connections with values in these.

Entities of general gauge theory
∞-Lie algebra $\mathfrak{g}$ with gauge ∞-Lie group G — connection with values in $\mathfrak{g}$ on a G-principal ∞-bundle over a smooth ∞-groupoid X

Apart from generalizing the notion of gauge Lie groups to Lie 2-groups and further, structural considerations in fundamental physics also led theoretical physicists to consider models for spacetime that are more general than the notion of a smooth manifold. In string theory spacetime is allowed to be more generally an orbifold or a generalization thereof, such as an orientifold. The natural mathematical model for these generalized spaces are Lie groupoids or, essentially equivalently, *differentiable stacks*.

It is noteworthy that the notions of generalized gauge groups and the generalized spacetime models encountered this way have a natural common context: all of these are examples of *smooth ∞-groupoids*. There is a natural mathematical concept that serves to describe contexts of such generalized spaces: a *big ∞-topos*. The notion of *differential cohomology in an ∞-topos* provides a unifying perspective on the mathematical structure encoding the generalized gauge fields and generalized spacetime models encountered in modern theoretical physics in such a general context.

1.1.2 Motivation from natural action functionals

We present here a motivation for our constructions, starting from the observation that classical Chern-Weil theory induces action functionals of Chern-Simons type, and observing that this phenomenon ought to have certain natural generalizations.

First a brief word on the general context of quantum physics.

In recent years the notion of *topological quantum field theory* (TQFT) from physics has been fully formalized and made accessible to strong mathematical tools and classifications. In its refined variant of *fully local* or *extended* n-dimensional TQFT, the fundamental concept is that of a higher category, denoted Bord_n, whose $(k \leq n)$-cells are k-dimensional smooth manifolds with boundary and corners, and whose composition operation is gluing along these boundaries. The disjoint union of manifolds equips this with a symmetric monoidal structure. Then for another symmetric monoidal n-category $n\mathrm{Vect}_{\mathrm{fd}}$, whose k-cells one thinks of as higher order linear maps between n-categorical analogs of finite dimensional (or "fully dualizable") vector spaces, an n-dimensional extended TQFT is formalized as an n-functor

$$Z : \mathrm{Bord}_n \to n\mathrm{Vect}$$

that respects this monoidal structure.

Here the higher order linear map $Z(\Sigma_{n-1})$ that is assigned to a *closed* $(n-1)$-dimensional manifold Σ_{n-1} can typically canonically be identified with a vector space, and be interpreted as the *space of states* of the physical system described by Z, for field configurations over a space of shape Σ_{n-1}. Then for Σ_n a cobordism between two such closed $(n-1)$-manifolds, $Z(\Sigma_n)$ identifies with a linear map from the space of states over the incoming to that over the outgoing boundary, and is interpreted as the ("time"-)*propagation* of states.

This idea is by now classical. A survey can for instance be found in [Ka10].

But beyond constituting a formalization of some concept motivated from physics, it is remarkable that this construction is itself entirely rooted in a universal construction in higher category theory, and would have eventually been discovered as such even in the absence of any motivation from physics. The notion of extended TQFT *derives* from higher category theory.

Namely, according to the celebrated result of [LurieTQFT], earlier hypothesized in [BaDo95], Bord_n is a *free construction* – essentially the *free symmetric monoidal n-category* generated by just the point. This means that symmetric monoidal maps $Z : \mathrm{Bord}_n \to n\mathrm{Vect}_{\mathrm{fd}}$ are equivalently encoded by n-functors from the point $Z(*) : * \to n\mathrm{Vect}_{\mathrm{fd}}$, which in turn are, of course, canonically identified simply with n-vector spaces, the *n-vector space of states* assigned by Z to the point. This adjunction is both, an intrinsic characterization of Bord_n, as well as a full *classification* of extended TQFTs: these are entirely determined by their higher space of states. All the assignments on higher dimensional Σ are obtained by forming higher order *traces* on this single higher space of states over the point.

Here we will not further dwell on extended TQFT as such, but instead use this state of affairs to motivate an investigation of a *source of examples* of *natural* TQFTs. Because the TQFTs that actually appear in fundamental physics, even when including the families of theories found in the study of theory space away from the loci of experimentally observed theories, are far from being random TQFTs allowed by the above classification.

First of all, the TQFTs that do appear are typicaly theories that arise by a process of *quantization* from a local *action functional* on a space of field configurations (recalled below). Secondly, even among all TQFTs arising by quantization from local action functionals they are special, in that they have a natural formulation in differential geometry, something that we will make precise below. The typical action functional appearing in practice is not random, but follows some natural pattern.

One may therefore ask which principle it is that selects from a universal construction in higher category theory – that of free symmetric monoidal structure – a certain subclass of "natural" geometric examples.

We will provide evidence here that this is another universal construction, but now in *higher topos theory*: *cohesion*.

Below in 3.9 (specifically in 3.9.11 and 3.9.12) we show that cohesion in an ∞-topos induces, first, a notion of *differential characteristic maps*, via a generalized *Chern-Weil theory*, and, second, from each such the corresponding spaces – in fact *moduli ∞-stacks* – of higher gauge field configurations, and, third, canonically equips these with action functionals, via a generalized higher *Chern-Simons theory*. Moreover, it induces from any such a corresponding action functional of one dimension lower, via a generalized higher *Wess-Zumino-Witten theory*. And finally the process of (geometric) quantization of these functionals on moduli stacks is itself naturally induced in a cohesive context.

1.1.2.1 Geometric quantization For completeness, we briefly recall the basic ideas of *quantization* in its formalization known as *geoemtric quantization* (which we disccuss in abstract cohesion below in 3.9.13 and in the traditional formulation in differential geometry in 4.4.20).

The input datum is, for a given manifold of the form $\Sigma = \Sigma_{n-1} \times [0,1]$ a smooth space $\mathrm{Conf}(\Sigma_n)$ of *field configurations* on Σ, equipped with a suitably smooth map, called the "action functional" of the theory,

$$S : \mathrm{Conf}(\Sigma_n) \to \mathbb{R}$$

taking values in the real numbers.

From this input one first obtains the *covariant phase space* of the system, given as the variational *critical locus* of S, schematically the subspace

$$P = \{\phi \in \mathrm{Conf}(\Sigma) \mid (dS)_\phi = 0\}$$

of field configurations on which the *variational* derivative dS of S vanishes. These field configurations are said to satisfy the *Euler-Lagrange equations of motion* of the dynamics encoded by S.

If S is a *local* action functional, in that it depends on the fields ϕ via an integral over Σ whose integran only depends on finitely many derivatives of ϕ, then this space canonically carries a *presymplectic form*, a closed 2-form $\omega \in \Omega^2_{\mathrm{cl}}(P)$.

A *symmetry* of the system is a vector field on P which is in the kernel of ω. The quotient of P by the flows of these symmetries is called the *reduced phase space*. This quotient is typically very ill-behaved if regarded in ordinary geometry, but is a natural nice space in higher geometry (modeled by *BV-BRST formalism*). The presymplectic form ω descends to a symplectic form ω_{red} on the reduced phase space.

A *geometric prequantization* of the symplectic smooth space $(P_{\mathrm{red}}, \omega_{\mathrm{red}})$ is now, if it exists, a choice of line bundle $E \to P_{\mathrm{red}}$ with connection ∇, such that $\omega = F_\nabla$ is the corresponding curvature 2-form. This becomes a *geometric quantization* proper when furthermore equipped with a choice of foliation of P_{red} by Lagrangian submanifolds (submanifolds of maximal dimension on which ω_{red} vanishes). This foliation is a choice of decomposition of phase space into "canonical coordinates and momenta" of the physical system.

Finally, the quantum space of states, $Z(\Sigma_{n-1})$, that is defined by this construction is the vector space of those sections of E that are covariantly constant along the leaves of the foliation.

The notion of fully local/extended TQFTs suggests that there ought to be an analogous fully local/extended version of geometric quantization, which produces not just the datum $Z(\Sigma_{n-1})$, but $Z(\Sigma_k)$ for all $0 \leq k \leq n$. By the above classification result it follows that the value for $k = 0$ alone will suffice to define the entire quantum theory. This should involve not just line bundles with connection, but higher analogs of these, called *circle $(n-k)$-bundles with connection* or *bundle $(n-k-1)$-gerbes with connection*.

We discuss such a *higher geometric prequantization* axiomatically in 3.9.13, and discuss examples in 4.4.20 and 5.6.

1.1.2.2 Classical Chern-Weil theory and its shortcomings Even in the space of all topological local action functionals, those that typically appear in fundamental physics are special. The archetypical example of a TQFT is 3-dimensional Chern-Simons theory (see [Fre] for a detailed review). Its action functional happens to arise from a natural construction in classical *Chern-Weil theory*. We now briefly summarize this process, which already produces a large family of natural topological action functionals on gauge equivalence classes of gauge fields. We then point out deficiencies of this classical theory, which are removed by lifting it to higher geometry.

A classical problem in topology is the classification of vector bundles over some topological space X. These are continuous maps $E \to X$ such that there is a vector space V, and an open cover $\{U_i \hookrightarrow X\}$, and such that over each patch we have fiberwise linear identifications $E|_{U_i} \simeq U_i \times V$. Examples include

- the tangent bundle TX of a smooth manifold X;
- the canonical $\mathbb{C}$-line bundle over the 2-sphere, $S^3 \times_{S^1} \mathbb{C} \to S^2$ which is associated to the Hopf fibration.

A classical tool for studying isomorphism classes of vector bundles is to assign to them simpler *characteristic classes* in the ordinary integral cohomology of the base space. For vector bundles over the complex numbers these are the *Chern classes*, which are maps

$$[c_1] : \mathrm{VectBund}_{\mathbb{C}}(X)/_\sim \to H^2(X, \mathbb{Z})$$

$$[c_2] : \mathrm{VectBund}_{\mathbb{C}}(X)/_\sim \to H^4(X, \mathbb{Z})$$

etc. natural in X. If two bundles have differing characteristic classes, they must be non-isomorphic. For instance for $\mathbb{C}$-line bundles the first Chern-class $[c_1]$ is an isomorphism, hence provides a complete invariant characterization.

In the context of *differential geometry*, where X and E are taken to be smooth manifolds and the local identifications are taken to be smooth maps, one wishes to obtain *differential* characteristic classes. To that end, one can use the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ of coefficients to obtain the map $H^{n+1}(X, \mathbb{Z}) \to H^{n+1}(X, \mathbb{R})$ from integral to real cohomology, and send any integral characteristic class $[c]$ to its real image $[c]_{\mathbb{R}}$. Due to the de Rham theorem, which identifies the real cohomology of a smooth manifold with the cohomology of its complex of differential forms,

$$H^{n+1}(X, \mathbb{R}) \simeq H^{n+1}_{\mathrm{dR}}(X)\,,$$

this means that for $[c]_{\mathbb{R}}$ one has representatives given by closed differential $(n+1)$-forms $\omega \in \Omega^{n+1}_{\mathrm{cl}}(X)$,

$$[c]_{\mathbb{R}} \sim [\omega]\,.$$

But since the passage to real cohomology may lose topological information (all torsion group elements map to zero), one wishes to keep the information both of the topological characteristic class $[c]$ as well as of its "differential refinement" ω. This is accomplished by the notion of *differential cohomology* $H^{n+1}_{\mathrm{diff}}(X)$ (see [HoSi05] for a review). These are families of cohomology groups equipped with compatible projections both to integral classes as well as to differential forms

$$\begin{array}{ccccc} & & H^{n+1}_{\mathrm{diff}}(X) & & \\ & \swarrow & & \searrow & \\ H^{n+1}(X,\mathbb{Z}) & & & & \Omega^{n+1}_{\mathrm{cl}}(X) \\ & \searrow & & \swarrow & \\ & & H^{n+1}(X,\mathbb{R}) \simeq H^{n+1}_{\mathrm{dR}}(X) & & \end{array} \qquad \begin{array}{ccccc} & & [\hat c] & & \\ & \swarrow & & \searrow & \\ [c] & & & & \omega \\ & \searrow & & \swarrow & \\ & & [c]_{\mathbb{R}} \sim [\omega] & & \end{array}\,.$$

Moreover, these differential cohomology groups come equipped with a notion of *volume holonomy*. For Σ_n an n-dimensional compact manifold, there is a canonical morphism

$$\int_\Sigma : H^{n+1}_{\mathrm{diff}}(\Sigma) \to U(1)$$

to the circle group.

For instance for $n = 1$, we have that $H^2(X, \mathbb{Z})$ classifies circle bundles / complex line bundles over X, $H^2_{\mathrm{diff}}(X)$ classifies such bundles *with connection* ∇, and the map $\int_\Sigma : H^2_{\mathrm{diff}}(\Sigma) \to U(1)$ is the *line holonomy* obtained from the *parallel transport* of ∇ over the 1-dimensional manifold Σ.

With such differential refinements of characteristic classes in hand, it is desireable to have them classify differential refinements of vector bundles. These are known as *vector bundles with connection.* We say a differential refinement of a characteristic class $[c]$ is a map $[\hat{c}]$ fitting into a diagram

$$\begin{array}{ccc} \mathrm{VectBund}_{\mathrm{conn}}(X)/_\sim & \xrightarrow{[\hat{c}]} & H^{n+1}_{\mathrm{diff}}(X) \\ \downarrow & & \downarrow \\ \mathrm{VectBund}(X)/_\sim & \xrightarrow{[c]} & H^{n+1}(X, \mathbb{Z}) \end{array} ,$$

where the vertical maps forget the differential refinement. Such a $[\hat{c}]$ contains information even when $[c] = 0$. Therefore one also calls $[\hat{c}]$ a *secondary characteristic class.*

All of this has a direct interpretation in terms of quantum gauge field theory.

- the elements in $\mathrm{VectBund}_{\mathrm{conn}}(X)/_\sim$ are gauge equivalence classes of *gauge fields* on X (for instance the electromagnetic field, or nuclear force fields);
- the differential class $[\hat{c}]$ defines a canonical *action functional* $S_{[c]}$ on such fields, by composition with the volume holonomy

$$\exp(iS_c(-)) : \mathrm{Conf}(\Sigma)/_\sim := \mathrm{VectBund}_{\mathrm{conn}}(\Sigma)/_\sim \stackrel{[\hat{c}]}{\to} H^{n+1}_{\mathrm{diff}}(\Sigma) \stackrel{\int_\Sigma}{\to} U(1) .$$

The action functionals that arise this way are of *Chern-Simons type.* If we write $A \in \Omega^1(\Sigma, \mathfrak{u}(n))$ for a differential form representing locally the connection on a vector bundle, then we have

- $\int_\Sigma c_1 \;:\; A \mapsto \exp(i \int_\Sigma \mathrm{tr}(A))$;
- $\int_\Sigma c_2 \;:\; A \mapsto \exp(i \int_\Sigma \mathrm{tr}(A \wedge d_{\mathrm{dR}A} + \frac{2}{3}\mathrm{tr}(A \wedge A \wedge A)))$
- etc.

Here the second expression, coming from the second Chern-class, is the standard action functional for 3-dimensional Chern-Simons theory. The first, coming from the first Chern-class, is a 1-dimensional Chern-Simons type theory. Next in the series is an action functional for a 5-dimensional Chern-Simons theory. Later we will see that by generalizing here from vector bundles to *higher bundles* of various kinds, a host of known action functionals for quantum field theories arises this way.

Despite this nice story, this traditional Chern-Weil theory has several shortcomings.

1. It is *not local*, related to the fact that it deals with cohomology classes $[c]$ instead of the cocycles c themselves. This means that there is no good obstruction theory and no information about the locality of the resulting QFTs.
2. It does not apply to *higher topological structures*, hence to *higher gauge fields* that take values in higher covers of Lie groups which are not themselves compact Lie groups anymore.
3. It is *restricted to ordinary differential geoemtry* and does not apply to variants such as supergeometry, infinitesimal geometry or derived geometry, all of which appear in examples of QFTs of interest.

1.1.2.3 Formulation in cohesive homotopy type theory We discuss now these problems in slightly more detail, together with their solution in *cohesive homotopy type theory.*

The problem with the locality is that every vector bundle is, by definition, *locally equivalent* to a trivial bundle. Also, locally on contractible patches $U \hookrightarrow X$ every integral cocycle becomes cohomologous to the trivial cocycle. Therefore the restriction of a characteristic class to local patches retains no informstion at all

$$\begin{array}{ccc} \mathrm{VectBund}(X)/_\sim & \xrightarrow{[c]} & H^{n+1}(X,\mathbb{Z}) \\ \downarrow{\scriptstyle (-)|_U} & & \downarrow{\scriptstyle (-)|_U} \\ * & \xrightarrow{\mathrm{Id}} & * \end{array} .$$

Here we may think of the singleton $*$ as the class of the trivial bundle over U. But even though on U every bundle is equivalent to the trivial bundle, this has non-trivial gauge automorphisms

$$* \xrightarrow{g} * \quad g \in C^\infty(U, G := \mathrm{GL}(V))\,.$$

These are not seen by traditional Chern-Weil theory, as they are not visible after passing to equivalence classes and to cohomology.

But by collecting this information over each U, it organizes into a *presheaf of gauge groupoids.* We shall write

$$\mathbf{B}G : U \mapsto \left\{ * \xrightarrow{g \in C^\infty(U,G)} * \right\} \quad \in \mathrm{Funct}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{Grpd})\,.$$

In order to retain all this information, we may pass to the 2-category

$$\mathbf{H} := L_W \ \mathrm{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{Grpd})$$

of such groupoid-valued functors, where we formally invert all those morphisms (natural transformations) in the class W of *stalkwise* equivalences of groupoids. This is called the *2-topos of stacks* on smooth manifolds.

For example we have

- $\mathbf{H}(U, \mathbf{B}G) \simeq \left\{ * \xrightarrow{g \in C^\infty(U,G)} * \right\}$
- $\pi_0 \mathbf{H}(X, \mathbf{B}G) \simeq \mathrm{VectBund}(X)/_\sim$

and hence the object $\mathbf{B}G \in \mathbf{H}$ constitutes a genuine smooth refinement of the classifying space for rank n-vector bundles, which sees not just their equivalence classes, but also their local smooth transformations.

The next problem of traditional Chern-Weil theory is that it cannot see beyond groupoids even in cohomology. Namely, under the standard nerve operation, groupoids embed into *simplicial sets* (described in more detail in 1.2.5.4 below)

$$N : \mathrm{Grpd} \hookrightarrow \mathrm{sSet}\,.$$

But simplicial sets model *homotopy theory.*

- There is a notion of homotopy groups π_k of simplicial sets;
- and there is a notion of *weak homotopy equivalences*, morphisms $f : X \to Y$ which induce isomorphisms on all homotopy groups.

Under the above embedding, groupoids yield only (and precisely) those simplicial sets, up to equivalence, for which only π_0 and π_1 are nontrivial. One says that these are *homotopy 1-types.* A general simplicial set presents what is called a *homotopy type* and may contain much more information.

Therefore we are lead to refine the above construction and consider the simplicial category

$$\mathbf{H} := L_W \ \mathrm{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{sSet})$$

of functors that send smooth manifolds to simplicial sets, where now we formall invert those morphisms that are stalkwise weak homotopy equivalences of simplicial sets.

This is called the ∞-*topos of* ∞-*stacks* on smooth manifolds.

For instance, there are objects $\mathbf{B}^n U(1)$ in this context which are smooth refinements of higher integral cohomology, in that

$$\pi_0 \mathbf{H}(X, \mathbf{B}^n U(1)) \simeq H^{n+1}(X, \mathbb{Z}) .$$

Finally, in this construction it is straightforward to change the geometry by changing the category of geometric test spaces. For instance we many replace smooth manifolds here by supermanifolds or by formal (synthetic) smooth manifolds. In all these cases $\mathbf{H}$ describes *homotopy types with differential geometric structure*. One of our main statements below is the following theorem.

These $\mathbf{H}$ all satisfy a simple set of axioms for "cohesive homotopy types", which were proposed for 0-types by Lawvere. In the fully homotopical context these axioms canonically induce in $\mathbf{H}$

- differential cohomology;
- higher Chern-Weil theory;
- higher Chern-Simons functionals;
- higher geometric prequantization.

This is such that it reproduces the traditional notions where they apply, and otherwise generalizes them beyond the realm of classical applicability.

1.1.2.4 Extended higher Chern-Simons theory It has become a familiar fact, known from examples as those indicated above, that there should be an n-dimensional topological quantum field theory $Z_{\mathbf{c}}$ associated to the following data:

1. a *gauge group* G: a Lie group such as $U(n)$; or more generally a higher smooth group, such as the smooth *circle* n-*group* $\mathbf{B}^{n-1}U(1)$ or the *String 2-group* or the smooth *Fivebrane 6-group* [SSS09c, FSS10];
2. a universal characteristic class $[c] \in H^{n+1}(BG, \mathbb{Z})$ and/or its image ω in real/de Rham cohomology,

where $Z_{\mathbf{c}}$ is a G-gauge theory defined naturally over all closed oriented n-dimensional smooth manifolds Σ_n, and such that whenever Σ_n happens to be the boundary of some manifold Σ_{n+1} the action fuctional on a field configuration ϕ is given by the integral of the pullback form $\hat{\phi}^*\omega$ (made precise below) over Σ_{n+1}, for some extension $\hat{\phi}$ of ϕ. These are *Chern-Simons type* gauge theories. See [Za08] for a gentle introduction to the general idea of Chern-Simons theories.

Notably for G a connected and simply connected simple Lie group, for $c \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$ any integer – the "level" – and hence for $\omega = \langle -, - \rangle$ the Killing form on the Lie algebra $\mathfrak{g}$, this quantum field theory is the original and standard Chern-Simons theory introduced in [Wi89]. See [Fre] for a comprehensive review. Familiar as this theory is, there is an interesting aspect of it that has not yet found attention, and which is an example of our constructions here.

To motivate this, it is helpful to look at the 3d Chern-Simons action functional as follows: if we write $H(\Sigma_3, \mathbf{B}G_{\mathrm{conn}})$ for the set of gauge equivalence classes of G-principal connections ∇ on Σ_3, then the (exponentiated) action functional of 3d Chern-Simons theory over Σ_3 is a function of sets

$$\exp(iS(-)) : H(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) \to U(1) .$$

Of course this function acts by picking a representative of the gauge equivalence class, given by a smooth 1-form $A \in \Omega^1(\Sigma_3, \mathfrak{g})$ and sending that to the element $\exp(2\pi i k \int_{\Sigma_3} \mathrm{CS}(A)) \in U(1)$, where $\mathrm{CS}(A) \in \Omega^3(\Sigma_3)$ is the Chern-Simons 3-form of A [ChSi74], that gives the whole theory its name. That this is well defined is the fact that for every gauge transformation $g : A \to A^g$, for $g \in C^\infty(\Sigma_3, G)$, both A as well as its gauge transform A^g, are sent to the same element of $U(1)$. A natural formal way to express this is to consider the *groupoid* $\mathbf{H}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}})$ whose objects are gauge fields A and whose morphisms are gauge transformations g as above. Then the fact that the Chern-Simons action is defined on individual gauge field configurations while being invariant under gauge transformations is equivalent the statement that it is a *functor*, hence a morphism of groupoids,

$$\exp(iS(-)) : \mathbf{H}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) \to U(1)\,,$$

where the set underlying $U(1)$ is regarded as a groupoid with only identity morphisms. Hence the fact that $\exp(iS(-))$ has to send every morphism on the left to a morphism on the right is the gauge invariance of the action.

Furthermore, the action functional has the property of being *smooth*. It takes any *smooth family* of gauge fields, over some parameter space U, to a corresponding smooth family of elements of $U(1)$ and such that these assignmens are compatible with precomposition of smooth functions $U_1 \to U_2$ between parameter spaces. The formal language that expresses this concept is that of *stacks on the site of smooth manifolds* (discussed in detail in 4.4 below): to say that for every U there is a groupoid, as above, of smooth U-families of gauge fields and smooth U-families of gauge transformations between them, in a consistent way, is to say that there is a *smooth moduli stack*, denoted $[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}]$, of gauge fields on Σ_3. Finally, the fact that the Chern-Simons action functional is not only gauge invariant but also smooth is the fact that it refines to a morphism of smooth stacks

$$\exp(i\mathbf{S}(-)) : [\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \to U(1)\,,$$

where now $U(1)$ is regarded as a smooth stack by declaring that a smooth family of elements is a smooth function with values in $U(1)$.

It is useful to think of a smooth stack simply as being a *smooth groupoid*. Lie groups and Lie groupoids are examples (and are called "differentiable stacks" when regarded as special cases of smooth stacks) but there are important smooth groupoids which are not Lie groupoids in that they have not a smooth *manifold* but a more general smooth space of objects and of morphisms. Just as Lie groups have an infinitesimal approximation given by Lie algebras, so smooth stacks/smooth groupoids have an infinitesimal approximation given by *Lie algebroids*. The smooth moduli stack $[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}]$ of gauge field configuration on Σ_3 is best known in the physics literature in the guise of its underlying Lie algebroid: this is the formal dual of the (off-shell) *BRST complex* of the G-gauge theory on Σ_3: in degree 0 this consists of the functions on the space of gauge fields on Σ_3, and in degree 1 it consists of functions on infinitesimal gauge transformations between these: the "ghost fields".

The smooth structure on the action functional is of course crucial in field theory: in particular it allows to define the *differential* $d\exp(i\mathbf{S}(-))$ of the action functional and hence its critical locus, characterized by the Euler-Lagrange equations of motion. This is the *phase space* of the theory, which is a substack

$$[\Sigma_2, \flat\mathbf{B}G] \hookrightarrow [\Sigma_2, \mathbf{B}G_{\mathrm{conn}}]$$

equipped with a presymplectic 2-form. To formalize this, write $\Omega^2_{\mathrm{cl}}(-)$ for the smooth stack of closed 2-forms (without gauge transformations), hence the rule that sends a parameter manifold U to the set $\Omega^2_{\mathrm{cl}}(U)$ of smooth closed 2-forms on U. This may be regarded as the *smooth moduli 0-stack* of closed 2-forms in that for every smooth manifold X the set of morphisms $X \to \Omega^2_{\mathrm{cl}}(-)$ is in natural bijection to the set $\Omega^2_{\mathrm{cl}}(X)$ of closed 2-forms on X. This is an instance of the *Yoneda lemma*. Similarly, a smooth 2-form on the moduli stack of field configurations is a morphism of smooth stacks of the form

$$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \to \Omega^2_{\mathrm{cl}}(-)\,.$$

Explicitly, for Chern-Simons theory this morphism sends for each smooth parameter space U a given smooth U-family of gauge fields $A \in \Omega^1(\Sigma_2 \times U, \mathfrak{g})$ to the 2-form

$$\int_{\Sigma_2} \langle d_U A \wedge d_U A \rangle \in \Omega^2_{\mathrm{cl}}(U)\,.$$

Notice that if we restrict to *genuine* families A which are functions of U but vanish on vectors tangent to U (technically these are elements in the *concretification* of the moduli stack) then this 2-form is the *fiber integral* of the Poincaré 2-form $\langle F_A \wedge F_A \rangle$ along the projection $\Sigma_2 \times U \to U$, where $F_A := dA + \frac{1}{2}[A \wedge A]$ is the curvature 2-form of A. This is the first sign of a general pattern, which we highlight in a moment.

There is more fundamental smooth moduli stack equipped with a closed 2-form: the moduli stack $\mathbf{B}U(1)_{\mathrm{conn}}$ of $U(1)$-gauge fields, hence of smooth circle bundles with connection. This is the rule that sends a smooth parameter manifold U to the groupoid $\mathbf{H}(U, \mathbf{B}U(1)_{\mathrm{conn}})$ of $U(1)$-gauge fields ∇ on U, which we have already seen above. Since the curvature 2-form $F_\nabla \in \Omega^2_{\mathrm{cl}}(U)$ of a $U(1)$-principal connection is gauge invariant, the assignment $\nabla \mapsto F_\nabla$ gives rise to a morphism of smooth stacks of the form

$$F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \to \Omega^2_{\mathrm{cl}}(-)\ .$$

In terms of this morphism the fact that every $U(1)$-gauge field ∇ on some space X has an underlying field strength 2-form ω is expressed by the existence of a commuting diagram of smooth stacks of the form

$$\begin{array}{cccc} & & \mathbf{B}U(1)_{\mathrm{conn}} & \qquad \text{gauge field / differential cocycle} \\ & {}^{\nabla}\nearrow & \downarrow{\scriptstyle F_{(-)}} & \\ X & \xrightarrow{\ \omega\ } & \Omega^2_{\mathrm{cl}}(-) & \qquad \text{field strength / curvature}\ . \end{array}$$

Conversely, if we regard the bottom morphism ω as given, and regard this closed 2-form as a (pre)symplectic form, then a *choice of lift* ∇ in this diagram is a choice of refinement of the 2-form by a circle bundle with connection, hence the choice of a *prequantum circle bundle* in the language of geometric quantization (see for instance section II in [Bry00] for a review of geometric quantization).

Applied to the case of Chern-Simons theory this means that a smooth (off-shell) prequantization of the theory is a choice of dashed morphism in a diagram of smooth stacks of the form

$$\begin{array}{ccc} & & \mathbf{B}U(1)_{\mathrm{conn}} \\ & \nearrow & \downarrow{\scriptstyle F_{(-)}} \\ [\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] & \xrightarrow[\int_{\Sigma_2} \langle F_{(-)}, F_{(-)} \rangle]{} & \Omega^2_{\mathrm{cl}}(-) \end{array}\ .$$

Similar statements apply to on-shell geometric (pre)quantization of Chern-Simons theory, which has been so successfully applied in the original article [Wi89]. In summary, this means that in the context of smooth stacks the Chern-Simons action functional and its prequantization are as in the following table:

dimension		**moduli stack description**
$k = 3$	action functional (0-bundle)	$\exp(i\mathbf{S}(-)) : [\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \to \quad U(1)$
$k = 2$	prequantum circle 1-bundle	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}U(1)_{\mathrm{conn}}$

There is a precise sense, discussed in section 4.4.16 below, in which a $U(1)$-valued function is a *circle* k*-bundle with connection* for $k = 0$. If we furthermore regard an ordinary $U(1)$-principal bundle as a *circle 1-bundle* then this table says that in dimension k Chern-Simons theory appears as a *circle* $(3 - k)$*-bundle with connection* – at least for $k = 3$ and $k = 2$.

Formulated this way, it should remind one of what is called *extended* or *multi-tiered* topological quantum field theory (formalized and classified in [LurieTQFT]) which is the full formalization of *locality* in the Schrödinger picture of quantum field theory. This says that *after quantization*, an n-dimensional topological field theory should be a rule that to a closed manifold of dimension k assigns an $(n - k)$-categorical analog of a vector space of quantum states. Since ordinary geometric quantization of Chern-Simons theory assigns to a closed Σ_2 the vector space of *polarized sections* (holomorphic sections) of the line bundle associated to the above circle 1-bundle, this suggests that there should be an *extended* or *multi-tiered* refinement of geometric (pre)quantization of Chern-Simons theory, which to a closed oriented manifold of dimension $0 \leq k \leq n$ assigns a *prequantum circle* $(n-k)$*-bundle* (bundle $(n-k-1)$-gerbe) on the moduli stack of field configurations over Σ_k, modulated by a morphism $[\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}^{(n-k)}U(1)_{\mathrm{conn}}$ to a moduli $(n-k)$-stack of circle $(n-k)$-bundles with connection.

In particular for $k = 0$ and Σ_0 connected, hence $\Sigma_0 = *$ the point, we have that the moduli stack of fields on Σ_0 is the *universal* moduli stack itself, $[*, \mathbf{B}G_{\mathrm{conn}}] \simeq \mathbf{B}G_{\mathrm{conn}}$, and so a *fully extended prequantization* of 3-dimensional G-Chern-Simons theory would have to involve a *universal characteristic* morphism

$$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$$

of smooth moduli stacks, hence a smooth circle 3-bundle with connection on the universal moduli stack of G-gauge fields. This indeed naturally exists: an explicit construction is given in [FSS10]. This morphism of smooth higher stacks is a differential refinement of a smooth refinement of the level itself: forgetting the connections and only remembering the underlying (higher) gauge bundles, we still have a morphism of smooth higher stacks

$$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)\,.$$

This expression should remind one of the continuous map of topological spaces

$$c : BG \to B^3U(1) \simeq K(\mathbb{Z}, 4)$$

from the classifying space BG to the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$, which represents the level as a class in integral cohomology $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$. Indeed, there is a canonical *derived functor* or ∞*-functor*

$$|-| : \mathbf{H} \to \mathrm{Top}$$

from smooth higher stacks to topological spaces (one of the defining properties of a cohesive ∞-topos), derived left adjoint to the operation of forming *locally constant higher stacks*, and under this map we have

$$|\mathbf{c}| \simeq c\,.$$

In this sense $\mathbf{c}$ is a *smooth refinement* of $[c] \in H^4(BG, \mathbb{Z})$ and then $\mathbf{c}_{\mathrm{conn}}$ is a further *differential refinement* of $\mathbf{c}$.

However, more is true. Not only is there an extension of the prequantization of 3d G-Chern-Simons theory to the point, but this also induces the extended prequantization in every other dimension by *tracing*: for $0 \leq k \leq n$ and Σ_k a closed and oriented smooth manifold, there is a canonical morphism of smooth higher stacks of the form

$$\exp(2\pi i \int_{\Sigma_k} (-)) : [\Sigma_k, \mathbf{B}^nU(1)_{\mathrm{conn}}] \to \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}\ ,$$

which refines the fiber integration of differential forms, that we have seen above, from curvature $(n+1)$-forms to their entire prequantum circle n-bundles (we discuss this below in section 5.7.1.1). Since, furthermore, the formation of mapping stacks $[\Sigma_k, -]$ is functorial, this means that from a morphism $\mathbf{c}_{\mathrm{conn}}$ as above we get for every Σ_k a composite morphism as such:

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) :\ [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_k, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_k, \mathbf{B}^nU(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}\ .$$

For 3d G-Chern-Simons theory and $k = n = 3$ this composite *is* the action functional of the theory (down on the set $H(\Sigma_3, \mathbf{B}G_{\mathrm{conn}})$ this is effectively the perspective on ordinary Chern-Simons theory amplified in

[CJMSW05]). Therefore, for general k we may speak of this as the *extended action functional*, with values not in $U(1)$ but in $\mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$.

This way we find that the above table, containing the Chern-Simons action functional together with its prequantum circle 1-bundle, extends to the following table that reaches all the way from dimension 3 down to dimension 0.

dim.		**prequantum $(3-k)$-bundle**	
$k=0$	differential fractional first Pontrjagin	$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$	[FSS10]
$k=1$	WZW background B-field	$[S^1, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[S^1,\mathbf{c}_{\mathrm{conn}}]} [S^1, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{S^1}(-))} \mathbf{B}^2U(1)_{\mathrm{conn}}$	
$k=2$	off-shell CS prequantum bundle	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_2,\mathbf{c}_{\mathrm{conn}}]} [\Sigma_2, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_2}(-))} \mathbf{B}U(1)_{\mathrm{conn}}$	
$k=3$	3d CS action functional	$[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_3,\mathbf{c}_{\mathrm{conn}}]} [\Sigma_3, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_3}(-))} U(1)$	[FSS10]

For each entry of this table one may compute the *total space* object of the corresponding prequantum k-bundle. This is now in general itself a higher moduli stack. In full codimension $k=0$ one finds that this is the moduli 2-stack of String(G)-2-connections described in [SSS09c, FiSaScIII]. This we discuss in section 5.7.5.1 below.

It is clear now that this is just the first example of a general class of theories which we may call *higher extended prequantum Chern-Simons theories* or just *∞-Chern-Simons theories*, for short. These are defined by a choice of

1. a smooth higher group G;
2. a smooth universal characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^nU(1)$;
3. a differential refinement $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^nU(1)_{\mathrm{conn}}$.

An example of a 7-dimensional such theory on String-2-form gauge fields is discussed in [FiSaScII], given by a differential refinement of the second fractional Pontrjagin class to a morphism of smooth moduli 7-stacks

$$\tfrac{1}{6}(\mathbf{p}_2)_{\mathrm{conn}} : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7U(1)_{\mathrm{conn}} \,.$$

We expect that these ∞-Chern-Simons theories are part of a general procedure of *extended geometric quantization* (*multi-tiered* geometric quantization) which proceeds in two steps, as indicated in the following table.

classical system	**geometric prequantization**	**quantization**
char. class c of deg. $(n+1)$ with de Rham image ω: invariant polynomial/ n-plectic form	prequantum circle n-bundle on moduli ∞-stack of fields $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^nU(1)_{\mathrm{conn}}$	extended quantum field theory $Z_{\mathbf{c}} : \Sigma_k \mapsto \left\{ \begin{array}{l} \text{polarized sections of} \\ \text{prequantum } (n-k)\text{-bundle} \\ \exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) \end{array} \right\}$

Here we are concerned with the first step, the discussion of n-dimensional Chern-Simons gauge theories (higher gauge theories) in their incarnation as prequantum circle n-bundles on their universal moduli ∞-stack of fields. A dedicated discussion of higher geometric prequantization, including the discussion of higher Heisenberg groups, higher quantomorphism groups, higher symplectomorphisms and higher Hamiltonian vector fields, and their action on higher prequantum spaces of states by higher Heisenberg operators, is given below. As shown there, plenty of interesting physical information turns out to be captured by extended prequantum n-bundles. For instance, if one regards the B-field in type II superstring backgrounds as a prequantum 2-bundle, then its extended prequantization knows all about twisted Chan-Paton bundles, the Freed-Witten anomaly cancellation condition for type II superstrings on D-branes and the associated anomaly line bundle on the string configuration space.

Generally, all higher Chern-Simons theories that arise from extended action functionals this way enjoy a collection of very good formal properties. Effectively, they may be understood as constituting examples of a fairly extensive generalization of the *refined* Chern-Weil homomorphism with coefficients in *secondary characteristic cocycles*. Moreover, we have shown previously that the class of theories arising this way is large and contains not only several familiar theories, some of which are not traditionally recognized to be of this good form, but also contains various new QFTs that turn out to be of interest within known contexts, e.g. [FiSaScIII, FiSaScIII]. Here we further enlarge the pool of such examples.

Notably, here we are concerned with examples arising from *cup product* characteristic classes, hence of ∞-Chern-Simons theories which are decomposable or non-primitive secondary characteristic cocyles, obtained by cup-ing more elementary characteristic cocycles. The most familiar example of these is again ordinary 3-dimensional Chern-Simons theory, but now for the non-simply connected gauge group $U(1)$. In this case a gauge field configuration in $\mathbf{H}(\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}})$ is not necessarily given by a globally defined 1-form $A \in \Omega^1(\Sigma_3)$, instead it may have a non-vanishing "instanton number", the Chern-class of the underlying circle bundle. Only if that happens to vanish is the value of the action functional again given by the simple expression $\exp(2\pi i k \int_{\Sigma_3} A \wedge d_{\mathrm{dR}} A)$ as before. But in view of the above we are naturally led to ask: which circle 3-bundle (bundle 2-gerbe) with connection over Σ_3, depending naturally on the $U(1)$-gauge field, has $A \wedge d_{\mathrm{dR}} A$ as its connection 3-form in this special case, so that the correct action functional in generality is again the *volume holonomy* of this 3-bundle (see section 5.7.3 below)? The answer is that it is the *differential cup square* of the gauge field with itself. As a fully extended action functional this is a natural morphism of higher moduli stacks of the form

$$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} .$$

This morphism of higher stacks is characterized by the fact that under forgetting the differential refinement and then taking geometric realization as before, it is exhibited as a differential refinement of the ordinary cup square on Eilenberg-MacLane spaces

$$(-)^{\cup^2} : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 4)$$

and hence on ordinary integral cohomology. By the above general procedure, we obtain a well-defined action functional for $3d$ $U(1)$-Chern-Simons theory by the expression

$$\exp(2\pi i \int_{\Sigma_3} [\Sigma_3, (-)^{\cup^2_{\mathrm{conn}}}]) : [\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}] \to U(1)$$

and this is indeed the action functional of the familiar $3d$ $U(1)$-Chern-Simons theory, also on non-trivial instanton sectors, see section 5.7.5.2 below.

In terms of this general construction, there is nothing particular to the low degrees here, and we have generally a differential cup square / extended action functional for a $(4k+3)$-dimensional Chern-Simons theory

$$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}^{2k+1} U(1)_{\mathrm{conn}} \to \mathbf{B}^{4k+3} U(1)_{\mathrm{conn}}$$

for all $k \in \mathbb{N}$, which induces an ordinary action functional

$$\exp(2\pi i \int_{\Sigma_3} [\Sigma_{4k+3}, (-)^{\cup^2_{\mathrm{conn}}}]) : [\Sigma_{4k+3}, \mathbf{B}^{4k+3} U(1)_{\mathrm{conn}}] \to U(1)$$

on the moduli $(2k+1)$-stack of $U(1)$-$(2k+1)$-form gauge fields, given by the fiber integration on differential cocycles over the differential cup product of the fields. This is discussed in section 5.7.8.1 below.

Forgetting the smooth structure on $[\Sigma_{4k+3}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}]$ and passing to gauge equivalence classes of fields yields the cohomology group $H^{2k+2}_{\mathrm{conn}}(\Sigma_{4k+3})$. This is what is known as *ordinary differential cohomology* and is equivalent to the group of *Cheeger-Simons differential characters*, a review with further pointers is in [HoSi05]. That gauge equivalence classes of higher degree $U(1)$-gauge fields are to be regarded as differential characters and that the $(4k+3)$-dimensional $U(1)$-Chern-Simons action functional on these is given by the fiber integration of the cup product is discussed in detail in [FP89], also mentioned notably in [Wi96, Wi98b] and expanded on in [Free00]. Effectively this observation led to the general development of differential cohomology in [HoSi05]. Or rather, the main theorem there concerns a shifted version of the functional of $(4k+3)$-dimensional $U(1)$-Chern-Simons theory which allows to further divide it by 2. We have discussed the refinement of this to smooth moduli stacks of fields in [FiSaScIII]. These developments were largely motivated from the relation of $(4k+3)$-dimensional $U(1)$-Chern-Simons theories as the holographic duals to theories of self-dual forms in dimension $(4k+2)$ (see [BeMo06] for survey and references): a choice of conformal structure on a Σ_{4k+2} naturally induces a polarization of the prequantum 1-bundle of the $(4k+3)$-dimensional theory, and for every choice the resulting space of quantum states is naturally identified with the corresponding conformal blocks (correlators) of the $(4k+2)$-dimensional theory.

Therefore we have that regarding the differential cup square on smooth higher moduli stacks as an extended action functional yields the following table of familiar notions under extended geometric prequantization.

dim.		**prequantum** $(4k+3-d)$**-bundle**
$d=0$	differential cup square	$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \to \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}$
$\vdots$	$\vdots$	$\vdots$
$d=4k+2$	"pre-conformal blocks" of self-dual $2k$-form field	$[\Sigma_{4k+2}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{[\Sigma_{4k+2},(-)^{\cup^2_{\mathrm{conn}}}]} [\Sigma_{4k+2}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_{4k+2}}(-))} \mathbf{B}U(1)_{\mathrm{conn}}$
$d=4k+3$	CS action functional	$[\Sigma_{4k+3}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{[\Sigma_{4k+3},(-)^{\cup^2_{\mathrm{conn}}}]} [\Sigma_{4k+3}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_{4k+3}}(-))} U(1)$

This fully extended prequantization of $(4k+3)$-dimensional $U(1)$-Chern-Simons theory allows for instance to ask for and compute the total space of the prequantum circle $(4k+3)$-bundle. This is now itself a higher smooth moduli stack. For $k=0$, hence in $3d$-Chern-Simons theory it turns out to be the moduli 2-stack of *differential T-duality structures*. This we discuss in section 5.7.5.2 below.

More generally, as the name suggests, the *differential cup square* is a specialization of a general *differential cup product*. As a morphism of bare homotopy types this is the familiar cup product of Eilenberg-MacLane spaces

$$(-) \cup (-) : K(\mathbb{Z}, p+1) \times K(\mathbb{Z}, q+1) \to K(\mathbb{Z}, p+q+2)$$

for all $p, q \in \mathbb{N}$. Its smooth and then its further differential refinement is a morphism of smooth higher stacks of the form

$$(-) \cup_{\mathrm{conn}} (-) : \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^1 U(1)_{\mathrm{conn}} \to \mathbf{B}^{p+q+1}U(1)_{\mathrm{conn}} .$$

By the above discussion this now defines a higher extended gauge theory in dimension $p+q+1$ of *two different* species of higher $U(1)$-gauge fields. One example of this is the *higher electric-magnetic coupling anomaly* in higher (Euclidean) $U(1)$-Yang-Mills theory, as explained in section 2 of [Free00]. In this example one considers on an oriented smooth manifold X (here assumed to be closed, for simplicity) an *electric current* $(p+1)$-form $J_{\mathrm{el}} \in \Omega^{p+1}_{\mathrm{cl}}(X)$ and a *magnetic current* $(q+1)$-form $J_{\mathrm{mag}} \in \Omega^{q+1}_{\mathrm{cl}}(X)$, such that $p+q = \dim(X)$ is the dimension of X. A *prequantization* of these current forms in our sense of higher geometric quantization is a lift to differential cocycles

$$\begin{array}{ccc} & & \mathbf{B}^p U(1)_{\mathrm{conn}} \\ & \overset{\widehat{J}_{\mathrm{el}}}{\nearrow} & \downarrow^{F_{(-)}} \\ X & \xrightarrow{J_{\mathrm{el}}} & \Omega^{p+1}_{\mathrm{cl}}(-) \end{array} \;, \qquad \begin{array}{ccc} & & \mathbf{B}^q U(1)_{\mathrm{conn}} \\ & \overset{\widehat{J}_{\mathrm{mag}}}{\nearrow} & \downarrow^{F_{(-)}} \\ X & \xrightarrow{J_{\mathrm{mag}}} & \Omega^{q+1}_{\mathrm{cl}}(-) \end{array}$$

and here this amounts to electric and magnetic *charge quantization*, respectively: the electric charge is the universal integral cohomology class of the circle p-bundle underlying the electric charge cocycle: its *higher Dixmier-Doudy class* $[\widehat{J}_{\mathrm{el}}] \in H^{p+1}_{\mathrm{cpt}}(X, \mathbb{Z})$ (see section 5.7.3 below); and similarly for the magnetic charge. Accordingly, the higher mapping stack $[X, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}]$ is the smooth higher moduli stack of charge-quantized electric and magnetic currents on X. Recall that this assigns to a smooth test manifold U the higher groupoid whose objects are U-families of pairs of charge-quantized electric and magnetic currents, namely such currents on $X \times U$. As [Free00] explains in terms of such families of fields, the $U(1)$-principal bundle with connection that in the present formulation is the one modulated by the morphism

$$\nabla_{\mathrm{an}} := \exp(2\pi i \int_X [X, (-) \cup_{\mathrm{conn}} (-)]) : [X, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}] \to \mathbf{B} U(1)_{\mathrm{conn}}$$

is the *anomaly line bundle* of $(p-1)$-form electromagnetism on X, in the presence of electric and magnetic currents subject to charge quantization. In the language of ∞-Chern-Simons theory as above, this is equivalently the off-shell prequantum 1-bundle of the higher cup product Chern-Simons theories on pairs of $U(1)$-gauge p-form and q-form fields.

Regarded as an anomaly bundle, one calls its curvature the *local anomaly* and its *holonomy* the "global anomaly". In our contex the holonomy of ∇_{an} is (discussed again in section 5.7.3 below) the morphism

$$\mathrm{hol}(\nabla_{\mathrm{an}}) = \exp(2\pi i \int_{S^1} [S^1, \nabla_{\mathrm{an}}]) : [S^1, [X, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}} \to U(1)$$

from the loop space of the moduli stack of fields to $U(1)$. By the characteristic universal propery of higher mapping stacks, together with the "Fubini-theorem"-property of fiber integration, this is equivalently the morphism

$$\exp(2\pi i \int_{X \times S^1} [X \times S^1, (-) \cup_{\mathrm{conn}} (-)]) : [X \times S^1, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}] \to U(1)\,.$$

But from the point of view of ∞-Chern-Simons theory this is the *action functional* of the higher cup product Chern-Simons field theory induced by $\cup_{\mathrm{conn}}$. The situation is now summarized in the following table.

dim.		**prequantum** $(\dim(X) + 1 - k)$**-bundle**
$k = 0$	differential cup product	$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}^p U(1)_{\mathrm{conn}} \mathbf{B}^q U(1)_{\mathrm{conn}} \to \mathbf{B}^{d+2} U(1)_{\mathrm{conn}}$
$\vdots$	$\vdots$	$\vdots$
$k = \dim(X)$	higher E/M-charge anomaly line bundle	$\exp(2\pi i \int_X [X, (-) \cup_{\mathrm{conn}} (-)]) : [X, \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}] \longrightarrow \mathbf{B} U(1)_{\mathrm{conn}}$
$k = \dim(X) + 1$	global anomaly	$\exp(2\pi i \int_{X \times S^1} [X \times S^1, (-) \cup_{\mathrm{conn}} (-)]) : [X \times S^1, \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}] \to U(1)$

These higher electric-magnetic anomaly Chern-Simons theories are of particular interest when the higher electric/magnetic currents are themselves induced by other gauge fields. Namely if we have any two ∞-Chern-Simons theories given by extended action functionals $\mathbf{c}^1_{\mathrm{conn}} : \mathbf{B} G^1_{\mathrm{conn}} \to \mathbf{B}^p U(1)_{\mathrm{conn}}$ and $\mathbf{c}^2_{\mathrm{conn}} : \mathbf{B} G^2_{\mathrm{conn}} \to \mathbf{B}^q U(1)_{\mathrm{conn}}$, respectively, then composition of these with the differential cup product yields an extended action functional of the form

$$\mathbf{c}^1_{\mathrm{conn}} \cup_{\mathrm{conn}} \mathbf{c}^2_{\mathrm{conn}} : \; \mathbf{B}(G^1 \times G^2)_{\mathrm{conn}} \xrightarrow{(\mathbf{c}^1_{\mathrm{conn}}, \mathbf{c}^2_{\mathrm{conn}})} \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^1 U(1)_{\mathrm{conn}} \xrightarrow{(-) \cup_{\mathrm{conn}} (-)} \mathbf{B}^{p+q+1} U(1)_{\mathrm{conn}} \;,$$

which describes extended topological field theories in dimension $p+q+1$ on two species of (possibly non-abelian, possibly higher) gauge fields, or equivalently describes the higher electric/magnetic anomaly for higher electric fields induced by $\mathbf{c}^1$ and higher magnetic fields induced by $\mathbf{c}^2$.

For instance for heterotic string backgrounds $\mathbf{c}^2_{\mathrm{conn}}$ is the differential refinement of the first fractional Pontrjagin class $\frac{1}{2}p_1 \in H^4(B\mathrm{Spin}, \mathbb{Z})$ [SSS09c, FSS10] of the form

$$\mathbf{c}^2_{\mathrm{conn}} = \widehat{J}^{\mathrm{NS5}}_{\mathrm{mag}} = \tfrac{1}{2}(\mathbf{p}_1)_{\mathrm{conn}} : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}\,,$$

formalizing the *magnetic NS5-brane charge* needed to cancel the fermionic anomaly of the heterotic string by way of the Green-Schwarz mechanism. It is curious to observe, going back to the very first example of this introduction, that this $\widehat{J}^{\mathrm{NS5}}_{\mathrm{mag}}$ is at the same time the extended action functional for 3d Spin-Chern-Simons theory.

Still more generally, we may differentially cup in this way more than two factors. Examples for such *higher order cup product theories* appear in 11-dimensional supergravity. Notably plain classical 11d supergravity contains an 11-dimensional cubic Chern-Simons term whose extended action functional in our sense is

$$(-)^{\cup^3_{\mathrm{conn}}} : \mathbf{B}^3 U(1)_{\mathrm{conn}} \to \mathbf{B}^{11} U(1)_{\mathrm{conn}}\,.$$

Here for X the 11-dimensional spacetime, a field in $[X, \mathbf{B}^3 U(1)]$ is a first approximation to a model for the *supergravity C-field*. If the differential cocycle happens to be given by a globally defined 3-form C, then the induced action functional $\exp(2\pi i \int_X [X, (-)^{\cup^3_{\mathrm{conn}}}])$ sends this to element in $U(1)$ given by the familiar expression

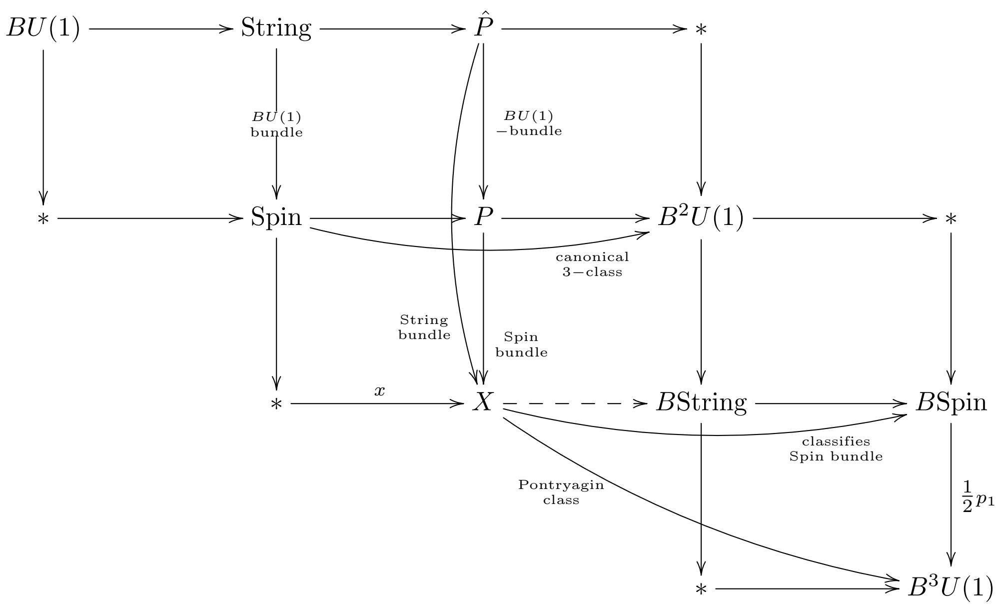
$$\exp(2\pi i \int_X [X, (-)^{\cup^3_{\mathrm{conn}}}]) : C \mapsto \exp(2\pi i \int_X C \wedge d_{\mathrm{dR}} C \wedge d_{\mathrm{dR}} C)\,.$$

More precisely this model receives quantum corrections from an 11-dimensional Green-Schwarz mechanism. In [FiSaScIII, FiSaScIII] we have discussed in detail relevant corrections to the above extended cubic cup-product action functional on the moduli stack of flux-quantized C-field configurations.

1.1.3 Motivation from long fiber sequences

It is a traditionally familiar fact that short exact sequences of (discrete) groups give rise to long sequences in cohomology with coefficients in these groups. In fact, before passing to cohomology, these long exact sequences are refined by corresponding long fiber sequences of the homotopy types obtained by the higher delooping of these groups: of the higher classifying spaces of these groups.

An example for which these long fiber sequences are of interest in the context of quantum field theory is the universal first fractional Pontryagin class $\frac{1}{2}p_1$ on the classifying space of Spin-principal bundles. The following digram displays the first steps in the long fiber sequence that it induces, together with an actual Spin-principal bundle $P \to X$ classified by a map $X \to B\mathrm{Spin}$. All squares are homotopy pullback squares of bare homotopy types.

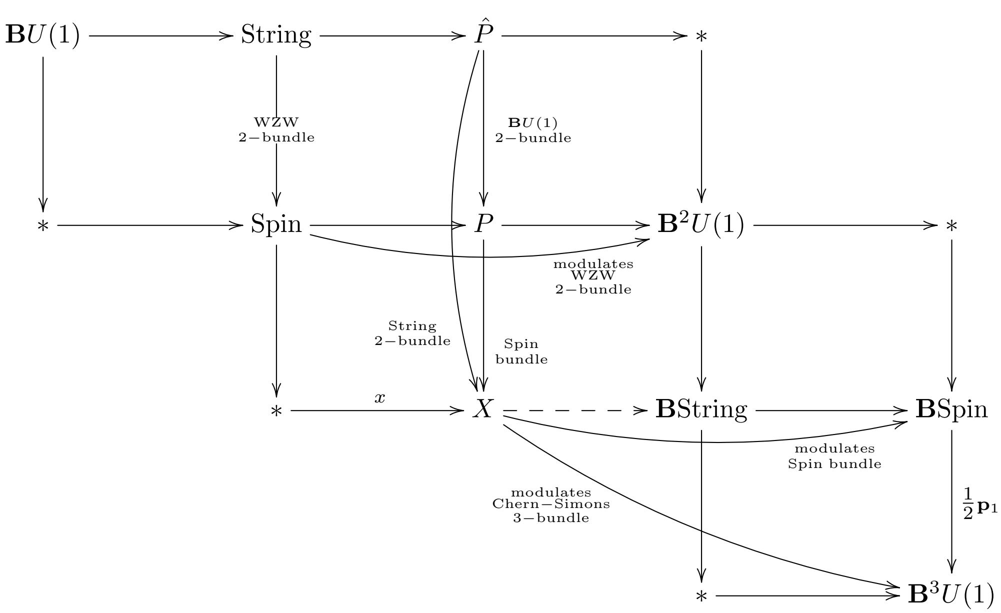


.

The topological group String which appears here as the loop space object of the homotopy fiber of $\frac{1}{2}p_1$ is the *String group*. We discuss this in detail below in 5.1. It is a $BU(1)$-extension of the Spin-group.

If X happens to be equipped with the structure of a smooth manifold, then it is natural to also equip the Spin-principal bundle $P \to X$ with the structure of a smooth bundle, and hence to lift the classifying map $X \to B\mathrm{Spin}$ to a morphism $X \to \mathbf{B}\mathrm{Spin}$ into the *smooth moduli stack* of smooth Spin-principal bundles (the morphism that not just classifies but "modulates" $P \to X$ as a smooth structure). An evident question then is: can the rest of the diagram be similarly lifted to a smooth context?

This indeed turns out to be the case, if we work in the context of *higher* smooth stacks. For instance there is a smooth moduli 3-stack $\mathbf{B}^2U(1)$ such that a morphism $\mathrm{Spin} \to \mathbf{B}^2U(1)$ not just classifies a $BU(1)$-bundle over Spin, but "modulates" a smooth *circle 2-bundle* or $U(1)$*-bundle gerbe* over Spin. One then gets the following diagram

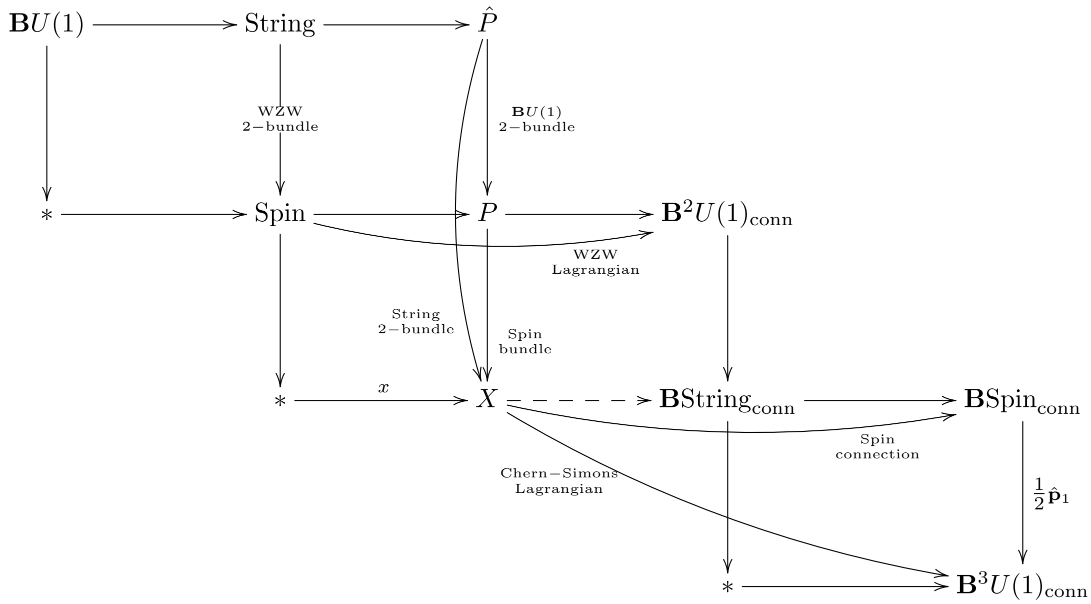


,

where now all squares are homotopy pullbacks of smooth higher stacks.

Whith this smooth geometirc structure in hand, one can then go further and ask for *differential* refinements: the smooth Spin-principal bundle $P \to X$ might be equipped with a principal connection ∇, and if so, this will be "modulated" by a morphism $X \to \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}$ into the smooth moduli stack of Spin-connections.

One of our central theorems below in 5.1 is that the universal first fractional Pontryagin class can be lifted to this situation to a *differential smooth* universal morphism of higher moduli stacks, which we write $\frac{1}{2}\hat{\mathbf{p}}_1$. Inserting this into the above diagram and then forming homotopy pullbacks as before yields further differential refinements. It turns out that these now induce the Lagrangians of 3-dimensional Spin Chern-Simons theory and of the WZW theory on Spin.



One way to understand our developments here is as a means to formalize and then analyze this setup and its variants and generalizations.

1.1.4 Motivation from quantum anomaly cancellation

One may wonder to which extent the higher gauge fields, that above in 1.1.1 we said motivate the theory of higher differential cohomology, can themselves be motivated within physics. It turns out that an important class of examples is required already by consistency of the quantum mechanics of higher dimensional fermionic ("spinning") quantum objects.

We indicate now how the full description of this *quantum anomaly cancellation* forces one to go beyond classical Chern-Weil theory to a more comprehensive theory of higher differential cohomology.

Consider a smooth manifold X. Its tangent bundle TX is a real vector bundle of rank $n = \dim X$. By the classical theorem which identifies isomorphism classes of rank-n real vector bundles with homotopy classes of *continuous* maps to the classifying space $B\mathrm{O}(n)$, for $O(n)$ the orthogonal group,

$$\mathrm{VectBund}(X)/_\sim \simeq [X, BO]\,,$$

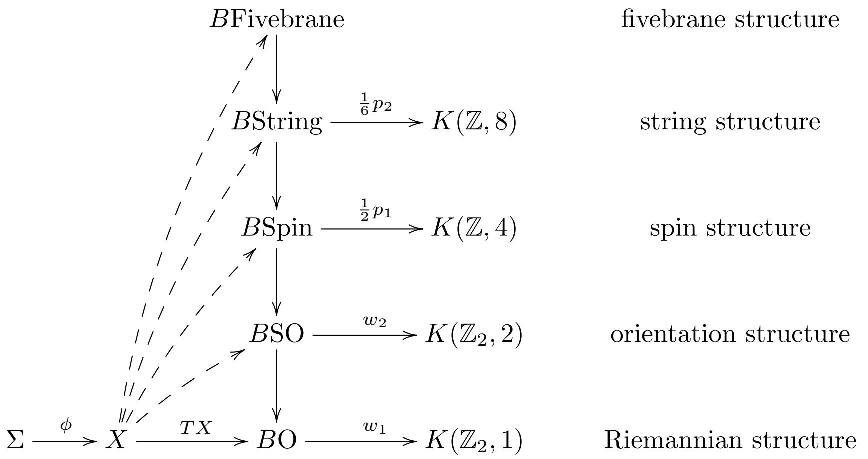
we have that TX is classified by a continuous map which we shall denote by the same symbol

$$TX : X \to B\mathrm{O}(n)\,.$$

Notice that this map takes place after passing from smooth spaces to just topological spaces. A central theme of our discussion later on are first *smooth* and then *differential* refinements of such maps.

A standard question to inquire about X is whether it is orientable. If so, a *choice* of orienation is, in terms of this classifying map, given by a lift through the canonical map $B\mathrm{SO}(n) \to B\mathrm{O}(n)$ from the classifying space of the *special* orthogonal group. Further, we may ask if X admits a Spin-*structure*. If so, a choice of Spin-structure corresponds to a further lift through the canonical map $B\mathrm{Spin}(n) \to B\mathrm{O}(n)$ from the classifying space of the Spin-group, which is the universal simply connected cover of the special orthogonal group. (Details on these basic notions are reviewed at the beginning of 5 below.)

These lifts of structure groups are just the first steps through a whole tower of higher group extensions, called the *Whitehead tower* of $B\mathrm{O}(n)$, as shown in the following picture. Here String is a *topological group* which is the universal 3-connected cover of Spin, and then Fivebrane is the universal 7-connected cover of String.



Here all subdiagrams of the form

$$\begin{array}{ccc} B\hat{G} & & \\ \downarrow & & \\ BG & \xrightarrow{c} & K(A, n) \end{array}$$

are homotopy fiber sequences. This means that $B\hat{G}$ is the homotopy fiber of the characteristic map c and $\hat{G}$ itself is the homotopy fiber of the looping Ωc of c. By the universal property of the homotopy pullback, this implies the obstruction theory for the existence of these lifts. The fist two of these are classical. For instance the orientation structure exists if the *first Stiefe-Whitney class* $[w_1(TX)] \in H^1(X, \mathbb{Z}_2)$ is trivial. Then a Spin-structure exists if moreover the *second Stiefel-Whiney class* $[w_2(TX)] \in H^2(X, \mathbb{Z}_2)$ is trivial.

Analogously, a *string structure* exists on X if moreover the *first fractional Pontryagin class* $[\frac{1}{2}p_1(TX)] \in H^4(X, \mathbb{Z})$ is trivial, and if so, a *fivebrane structure* exists if moreover the *second fractional Pontryagin class* $[\frac{1}{6}p_2(TX)] \in H^8(X, \mathbb{Z})$ is trivial.

The names of these structures indicate their role in quantum physics. Let Σ be a $d+1$-dimensional manifold and assume now that also X is smooth. Then a smooth map $\phi : \Sigma \to X$ may be thought of as modelling the trajectory of a d-dimensional object propagating through X. For instance for $d = 0$ this would be the trajectory of a point particle, for $d = 1$ it would be the worldsheet of a *string*, and for $d = 5$ the 6-dimensional worldvolume of a *5-brane*. The intrinsic "spin" of point particles and their higher dimensional analogs is described by a spinor bundle $S \to \Sigma$ equipped for each $\phi : \Sigma \to X$ with a Dirac operato $D_{\phi^* TX}$ that is twisted by the pullback of the tangent bundle of X along ϕ. The fermionic part of the *path integral* that gives the quantum dynamics of this setup computes the analog of the determinant of this Dirac operator, which is an element in a complex line called the *Pfaffian line* of $D_{\phi^* TX}$. As ϕ varies, these Pfaffian lines arrange into a line bundle on the mapping space

$$
\begin{array}{ccccc}
\{\mathrm{Pfaff}(D_{\phi^* TX})\} & & & & \\
\downarrow & & & & \\
\{\phi : \Sigma \to X\} & = & \mathrm{SmthMaps}(\Sigma, X) & \xrightarrow{\mathrm{tg}_\Sigma(c)} & K(\mathbb{Z}, 2)
\end{array} .
$$

Since the result of the fermionic part of the path integral is therefore a section of this line bundle, the resulting effective action functional can be a well defined function only if this line bundle is trivializable, hence if its Chern class vanishes. Therefore the Chern class of the Pfaffian line bundle over the bosonic configuration space is called the *global quantum anomaly* of the system. It is an obstruction to the existence of quantum dynamics of d-dimensional objects with spin on X.

Now, it turns out that this Chern class is the *transgression* $\mathrm{tg}_\Sigma(c)$ of the corresponding class c appearing in the picture of the Whitehead tower above. Therefore the vanishing of these classes implies the vanishing of the quantum anomaly.

For instance a choice of a *spin structure* on X cancels the global quantum anomaly of the quantum spinning particle. Then a choice of *string structure* cancels the global quantum anomaly of the quantum spinning string, and a choice of *fivebrane structure* cancels the global quantum anomaly of the quantum spinning 5-brane.

However, the Pfaffian line bundle turns out to be canonically equipped with more refined differential structure: it carries a *connection*. Moreover, in order to obtain a consistent quantum theory it needs to be trivialized as a bundle with connection.

For the Pfaffian line bundle with connection still to be the transgression of the corresponding obstruction class on X, evidently the entire story so far needs to be refined from cohomology to a differentially refined notion of cohomology.

Classical Chern-Weil theory achieves this, in parts, for the first few steps through the Whitehead tower (see [GHV] for a classical textbook reference and [HoSi05] for the refinement to differential cohomology that we need here). For instance, since maps $X \to B\mathrm{Spin}$ classify Spin-principal bundles on X, and since Spin is a Lie group, it is clear that the coresponding differential refinement is given by Spin-principal connections. Write $H^1(X, \mathrm{Spin})_{\mathrm{conn}}$ for the equivalence classes of these structures on X.

For every $n \in \mathbb{N}$ there is a notion of differential refinement of $H^n(X, \mathbb{Z})$ to the *differential cohomology group* $H^n(X, \mathbb{Z})_{\mathrm{conn}}$. These groups fit into square diagrams as indicated on the right of the following diagram.

$$
\begin{array}{ccccc}
H^1_{\mathrm{conn}}(X, \mathrm{Spin}) & \xrightarrow{[\frac{1}{2}\hat{\mathbf{p}}_1]} & & H^4_{\mathrm{diff}}(X, \mathbb{Z}) & \\
& & \swarrow^{\text{curvature}} & & \searrow^{\text{top. class}} \\
& & \Omega^4_{\mathrm{cl}}(X) & & H^4(X, \mathbb{Z}) \\
& & \searrow & & \swarrow \\
& & & H^4_{\mathrm{dR}}(X) \simeq H^4(X, \mathbb{R}) &
\end{array} .
$$

As shown there, an element in $H^n(X, \mathbb{Z})$ involves an underlying ordinary integral class, but also a differential n-form on X such that both structures represent the same class in real cohomology (using the de Rham isomorphism between real cohomology and de Rham cohomology). The differential form here is to be thought of as a *higher curvature form* on a higher line bundle corresponding to the given integral cohomology class.

Finally, the refined form of classical Chern-Weil theory provides differential refinements for instance of the first fractional Pontryagin class $[\frac{1}{2}p_1] \in H^4(X, \mathbb{Z})$ to a differential class $[\frac{1}{2}\hat{\mathbf{p}}_1]$ as shown in the above diagram. This is the differential refinement that under transgression produces the differential refinement of our Pfaffian line bundles.

But this classical theory has two problems.

1. Beyond the Spin-group, the topological groups String, Fivebrane etc. do not admit the structure of finite-dimensional Lie groups anymore, hence ordinary Chern-Weil theory fails to apply.

2. Even in the situation where it does apply, ordinary Chern-Weil theory only works on cohomology classes, not on cocycles. Therefore the differential refinements cannot see the homotopy fiber sequences anymore, that crucially characterized the obstruction problem of lifting through the Whitehead tower.

The source of the first problem may be thought to be the evident fact that the category Top of topological spaces does, of course, not encode smooth structure. But the problem goes deeper, even. In homotopy theory, Top is not even about topological structure. Rather, it is about homotopies and *discrete* geometric structure.

One way to make this precise is to say that there is a *Quillen equivalence* between the model category structures on topological spaces and on simplicial sets.

$$\mathrm{Top} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{sSet} \qquad \mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(\mathrm{sSet})\,.$$

Here the *singular simplicial complex functor* Sing sends a topological space to the simplicial set whose k-cells are maps from the topological k-simplex into X.

In more abstract modern language we may restate this as saying that there is an equivalence

$$\mathrm{Top} \xrightarrow[\simeq]{\Pi} \infty\mathrm{Grpd}$$

between the homotopy theory of topological spaces and that of ∞-*groupoids*, exhibited by forming the *fundamental* ∞-*groupoid* of X.

To break this down into a more basic statement, let $\mathrm{Top}_{\leq 1}$ be the subcategory of homotopy 1-types, hence of these topological spaces for which only the 0th and the first homotopy groups may be nontrivial. Then the above equivalence resticts to an equivalence

$$\mathrm{Top}_{\leq 1} \xrightarrow[\simeq]{\Pi} \mathrm{Grpd}$$

with ordinary groupoids. Restricting this even further to (pointed) connected 1-types, hence spaces for which only the first homotopy group may be non-trivial, we obtain an equivalence

$$\mathrm{Top}_{1,\mathrm{pt}} \xrightarrow[\simeq]{\pi_1} \mathrm{Grp}$$

with the category of groups. Under this equivalence a connected 1-type topological space is simply identified with its first fundamental group.

Manifestly, the groups on the right here are just bare groups with no geometric structure; or rather with *discrete* geometric structure. Therefore, since the morphism Π is an equivalence, also Top_1 is about *discrete* groups, $\mathrm{Top}_{\leq 1}$ is about *discrete* groupoids and Top is about *discrete* ∞-*groupoids*.

There is a natural solution to this problem. This solution and the differential cohomology theory that it supports is the topic of this book.

The solution is to equip discrete ∞-groupoids A with *smooth structure* by equipping them with information about what the *smooth families* of k-morphisms in it are. In other words, to assign to each smooth parameter space U an ∞-groupoid of smoothly U-parameterized families of cells in A.

If we write $\mathbf{A}$ for A equipped with smooth structure, this means that we have an assignment

$$\mathbf{A} : U \mapsto \mathbf{A}(U) =: \mathrm{Maps}(U, A)_{\mathrm{smooth}} \in \infty\mathrm{Grpd}$$

such that $\mathbf{A}(*) = A$.

Notice that here the notion of smooth maps into A is not defined before we declare $\mathbf{A}$, rather it is defined *by* declaring $\mathbf{A}$. A more detailed discussion of this idea is below in 1.2.3.1.

We can then define the homotopy theory of *smooth* ∞-groupoids by writing

$$\mathrm{Smooth}\infty\mathrm{Grpd} := L_W \mathrm{Funct}(\mathrm{SmoothMfd}^{\mathrm{op}}, \mathrm{sSet})\,.$$

Here on the right we have the category of contravariant functors on the category of smooth manifolds, such as the $\mathbf{A}$ from above. In order for this to inform this simple construction about the local nature of smoothness, we need to formally invert some of the morphisms between such functors, which is indicated by the symbol L_W on the left. The set of morphisms W that are to be inverted are those natural transformation that are *stalkwise* weak homotopy equivalences of simplicial sets.

We find that there is a canonical notion of *geoemtric realization* on smooth ∞-groupoids

$$|-| : \mathrm{Smooth}\infty\mathrm{Grpd} \stackrel{\Pi}{\to} \infty\mathrm{Grpd} \stackrel{|-|}{\to} \mathrm{Top}\,,$$

where Π is the derived left adjoint to the embedding

$$\mathrm{Disc} : \infty\mathrm{Grpd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$$

of bare ∞-groupoids as discrete smooth ∞-groupoids. We may therefore ask for *smooth refinements* of given topological spaces X, by asking for smooth ∞-groupoids $\mathbf{X}$ such that $|\mathbf{X}| \simeq X$.

A simple example is obtained from any Lie algebra $\mathfrak{g}$. Consider the functor $\exp(\mathfrak{g}) : \mathrm{SmoothMfd}^{\mathrm{op}} \to \mathrm{sSet}$ given by the assignment

$$\exp(\mathfrak{g}) : U \mapsto ([k] \mapsto \Omega^1_{\mathrm{flat,vert}} U \times \Delta^k, \mathfrak{g})\,,$$

where on the right we have the set of differential forms on the parameter space times the smooth k-simplex which are flat and vertical with respect to the projection $U \times \Delta^k \to U$.

We find that the 1-truncation of this smooth ∞-groupoid is the Lie groupoid

$$\tau_1 \exp(\mathfrak{g}) = \mathbf{B}G$$

that has a single object and whose morphisms form the simply connected Lie group G that integrates $\mathfrak{g}$. We may think of this Lie groupoid also as the *moduli stack* of smooth G-principal bundles. In particular, this is a smooth refinement of the classifying space for G-principal bundles in that

$$|\mathbf{B}G| \simeq BG\,.$$

So far this is essentially what classical Chern-Weil theory can already see. But smooth ∞-groupoids now go much further.

In the next step there is a *Lie 2-algebra* $\mathfrak{g} = \mathfrak{string}$ such that its exponentiation is

$$\tau_2 \exp(\mathfrak{string}) = \mathbf{B}\mathrm{String}$$

is a smooth 2-groupoid, which we may think of as the *moduli 2-stack of* String-*principal* which is a smooth refinement of the String-classifying space

$$|\mathbf{B}\mathrm{String}| \simeq B\mathrm{String}\,.$$

Next there is a Lie 6-algebra $\mathfrak{fivebrane}$ such that

$$\tau_6 \exp(\mathfrak{fivebrane}) = \mathbf{B}\mathrm{Fivebrane}$$

with

$$|\mathbf{B}\mathrm{Fivebrane}| \simeq B\mathrm{Fivebrane}\,.$$

Moreover, the characteristic maps that we have seen now refine first to smooth maps on these moduli stacks, for instance

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3 U(1)\,,$$

and then further to *differential* refinement of these maps

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} \,,$$

where now on the left we have the moduli stack of smooth Spin-connections, and on the right the moduli 3-stack of *circle n-bundles with connection*.

A detailed discussion of these constructions is below in 5.1.

In addition to capturing smooth and differential refinements, these constructions have the property that they work not just at the level of cohomology classes, but at the level of the full cocycle ∞-groupoids. For instance for X a smooth manifold, postcomposition with $\frac{1}{2}\hat{\mathbf{p}}$ may be regarded not only as inducing a function

$$H^1_{\mathrm{conn}}(X, \mathrm{Spin}) \to H^4_{\mathrm{conn}}(X)$$

on cohomology sets, but a morphism

$$\frac{1}{2}\hat{\mathbf{p}}(X) : \mathbf{H}^1(X, \mathrm{Spin}) \to \mathbf{H}^3(X, \mathbf{B}^3 U(1)_{\mathrm{conn}})$$

from the groupoid of smooth principal Spin-bundles with connection to the 3-groupoid of smooth circle 3-bundles with connection. Here the boldface $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ denotes the ambient ∞-topos of smooth ∞-groupoids and $\mathbf{H}(-,-)$ its hom-functor.

By this refinement to cocycle ∞-groupoids we have access to the homotopy fibers of the morphism $\frac{1}{2}\hat{\mathbf{p}}_1$. Before differential refinement the homotopy fiber

$$\mathbf{H}(X, \mathbf{B}\mathrm{String}) \longrightarrow \mathbf{H}(X, \mathbf{B}\mathrm{Spin}) \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{H}(X, B^3 U(1)) \ ,$$

is the 2-groupoid of smooth String-principal 2-bundles on X: smooth *string structures* on X. As we pass to the differential refinement, we obtain *differential string structures* on X

$$\mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \longrightarrow \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} \mathbf{H}(X, B^3 U(1)_{\mathrm{conn}}) \ .$$

A cocycle in the 2-groupoid $\mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}})$is naturally identified with a tuple consisting of

- a smooth Spin-principal bundle $P \to X$ with connection ∇;
- the Chern-Simons 2-gerbe with connection $CS(\nabla)$ induced by this;
- a choice of trivialization of this Chern-Simons 2-gerbe and its connection.

We may think of this as a refinement of secondary characteristic classes: the first Pontryagin curvature characteristic form $\langle F_\nabla \wedge F_\nabla \rangle$ itself is constrained to vanish, and so the Chern-Simons form 3-connection itself constitutes cohomological data.

More generally, we have access not only to the homotopy fiber over the 0-cocycle, but may pick one cocycle in each cohomology class to a total morphism $H^4_{\mathrm{diff}}(X) \to \mathbf{H}(X, \mathbf{B}^3 U(1)_{\mathrm{conn}})$ and consider the collection of all homotopy fibers over all connected components as the homotopy pullback

$$\begin{array}{ccc} \frac{1}{2}\hat{\mathbf{p}}_1\mathrm{Struc}_{\mathrm{tw}}(X) & \longrightarrow & H^4_{\mathrm{diff}}(X) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{H}(X, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \end{array} \quad .$$

This yields the 2-groupoid of *twisted differential string structure*. These objects, and their higher analogs given by twisted differential fivebrane structures, appear in background field structure of the heterotic string and its magnetic dual, as discussed in [SSS09c].

These are the kind of structures that ∞-Chern-Weil theory studies.

1.1.5 Motivation from higher topos theory

The history of theoretical fundamental physics is the story of a search for the suitable mathematical notions and structural concepts that naturally model the physical phenomena in question. Examples include, roughly in historical order,

1. the identification of symplectic geometry as the underlying structure of classical Hamiltonian mechanics;
2. the identification of (semi-)Riemannian differential geometry as the underlying structure of gravity;
3. the identification of group and representation theory as the underlying structure of the zoo of fundamental particles;
4. the identification of Chern-Weil theory and differential cohomology as the underlying structure of gauge theories.

All these examples exhibit the identification of the precise mathematical language that naturally captures the physics under investigation. Modern theoretical insight in theoretical fundamental physics is literally *unthinkable* without these formulations.

Therefore it is natural to ask whether one can go further. Not only have we seen above in 1.1.4 that some of these formulations leave open questions that we would want them to answer. But one is also led to wonder if this list of mathematical theories cannot be subsumed into a single more fundamental system altogether.
In a philosophical vein we should ask

Where does physics take place, conceptually?

Such philosophical-sounding questions can be given useful formalizations in terms of category theory. In this context "place" translates to *topos*, "taking place" translates to *inernalization* and whatever it is that takes places is characterized by a collection of *universal constructions* (categorical limits and colimits, categorical adjunctions).

So we translate

Physics	takes	place.
Certain universal constructions	internalize	in a suitable topos.

(For the following explanation of what precisely this means the reader only needs to know the concept of *adjoint functors*.)
The remaining question is

What characterizes a suitable topos and what are the universal constructions capturing physics.

At the bottom of it there are two aspects to physics, *kinematics* and *dynamics*. Roughly, kinematics is about the nature of *geometric spaces* appearing in physics, dynamics is about *trajectories* – paths – in these spaces. We will argue that

- the notion of a topos of geometric spaces is usefully given by what goes by the technical term *local topos*;
- the notion of a topos of spaces with trajectories is usefully given by what goes by the technical term *∞-connected topos*.

A topos that is both local and ∞-connected we call *cohesive*.

$$\begin{array}{c} \overbrace{\text{kinematic} \qquad\qquad \text{dynamics}}^{\text{physics}} \\ \\ \underbrace{\text{local topos} \qquad \infty\text{-connected topos}}_{\text{cohesive topos}} \end{array} .$$

1.1.5.1 Kinematics – local toposes. With a notion of *bare* spaces give, a notion of geometric spaces comes with a forgetful functor GeometricSpaces $\to$ BareSpaces that forgets this structure. The claim is that two extra conditions on this functor guarantee that indeed the structure it forgets is some *geometric structure*.

- There is a category C of *local models* such that every geometric space is obtained by *gluing* of local models. The operation of gluing following a blueprint is left adjoint to the inclusion of geometric spaces into blueprints for geometric spaces.
- Every bare space can canonically be equipped with the two universal cases of geometric structure, *discrete* and *indiscrete* geometric structure. (For instance a set can be equipped with discrete topology or discrete smooth structure.)

  Equipping with these structure is left and right adjoint, respectively, to forgetting geometric structure.

$$\text{BlueprintsOfGeometricSpaces} \underset{\hookleftarrow}{\xrightarrow{\text{glue local models}}} \text{GeometricSpaces} \begin{array}{c} \xleftarrow{\text{form discrete geometric structure}} \\ \xrightarrow{\text{forget geometric structure}} \\ \xleftarrow[\text{form indiscrete geometric structure}]{} \end{array} \text{BareSpaces} .$$

If we take a bare space to be a set of points, then this translates into the following formal statement.

$$\mathrm{Func}(C^{\mathrm{op}}, \mathrm{Set}) \underset{\hookleftarrow}{\xrightarrow{\text{sheafification}}} \mathrm{Sh}(C) \begin{array}{c} \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow[\mathrm{coDisc}]{} \end{array} \mathrm{Set} .$$

The category of geometric spaces embeds into the category of contravarian functors on test spaces, and this embedding has a left adjoint. It is a basic fact of topos theory that such *reflective embeddings* are precisely categories of *sheaves* on C with respect to some Grothendieck topology on C (which is defined by the reflective embedding). Therefore the first demand above says that GeometricSpaces is to be what is called a *sheaf topos*.

Another basic fact of topos theory says that this already implies the first part of the second demand, and uniquely so. There is unique pair of adjoint functors ($\mathrm{Disc} \dashv \Gamma$) as indicated. The demand of the further right adjoint embedding coDisc is what makes the sheaf topos a *local topos*.

These and the following axioms are very simple. Nevertheless, by the power of category theory, it turns out that they have rich implications. But we will we show that for them to have implications *just rich enough* to indeed formalize the kind of structures mentioned at the beginning, we want to pass to ∞-toposes instead. Then the above becomes

$$\infty\mathrm{Func}(C^{\mathrm{op}}, \infty\mathrm{Grpd}) \underset{\hookleftarrow}{\xrightarrow{\infty-\text{stackification}}} \mathrm{Sh}_\infty(C) \begin{array}{c} \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow[\mathrm{coDisc}]{} \end{array} \infty\mathrm{Grpd} .$$

1.1.5.2 Dynamics – ∞-connected toposes With a notion of *discrete ∞-groupoids* inside geometric ∞-groupoids given, we can ask for discrete ∞-bundles over any X to be characterized by the the *parallel transport* that takes their fibers into each other, as they move along paths in X. By the basic idea of *Galois theory* (see 3.8.6), this completely characterizes a notion of trajectory.

Formally this means that we require a further left adjoint $(\Pi \dashv \mathrm{Disc})$.

$$\begin{array}{ccc} \mathrm{Geometric}\infty\mathrm{Grpd}(X, \mathrm{Disc}K) & \simeq & \infty\mathrm{Grpd}(\Pi(X), K) \\ \begin{array}{l}\text{bundles of} \\ \text{discrete } \infty\text{-groupoids} \\ \text{on } X\end{array} & & \begin{array}{l}\text{parallel transport} \\ \text{of discrete } \infty\text{-groupoids} \\ \text{along trajectories} \\ \text{in } X\end{array} \end{array} .$$

This means that for any X we can think of $\Pi(X)$ as the ∞-groupoid of paths in X, of paths-between-paths in X, and so on.

In order for this to yield a consistent notion of paths in the geometric context, we want to demand that there are no non-trivial paths in the point (the terminal object), hence that

$$\Pi(*) \simeq * .$$

An ordinary topos for which Π exists and satisfies this property is called *locally connected and connected.* Hence an ∞-topos for which Π exists and satisfies this extra condition we call *∞-connected.* This terminology is good, but a bit subtle, since it refers to the meta-topology of the *collection of all geometric spaces* rather than to any that of any topological space itself. The reader is advised to regard it just as a technical term for the time being.

1.1.5.3 Physics – cohesive toposes An ∞-topos that is both local as well as ∞-connected we call *cohesive.* The idea is that the extra adjoints on it encode the information of how sets of cells in an ∞-groupoid are geometrically held together, for instance in that there are smooth paths between them. In the models of cohesive ∞-toposes that we wiil construct the local models are *open balls* with geometric structure and each such open ball can be thought of as a "cohesive blob of points".

The axioms on a cohesive topos are simple and fully formal. They involve essentially just the notion of adjoint functors.

We can ask now for universal constructions such that internalized in any cohesive ∞-topos they usefully model differental geometry, differential cohomology, action functionals for physical systems, etc. Below in 3.9 we a comprehensive discussion of an extensive list of such structures. Here we highlight one them. Differential forms.

One consequence of the axioms of cohesion is that every *connected* object in a cohesive ∞-topos $\mathbf{H}$ has am essentially unique point (whereas in general it may fail to have a point). We have an equivalence

$$\infty\mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\Omega}{\rightleftarrows}} \mathbf{H}_{*,\geq 1}$$

between group objects G in $\mathbf{H}$ and (uniquely pointed) connected objects in $\mathbf{H}$.

Define now

$$(\mathbf{\Pi} \dashv \flat) := (\mathrm{Disc}\Pi \dashv \mathrm{Disc}\Gamma) .$$

The $(\mathrm{Disc} \dashv \Gamma)$-counit gives a morphism

$$\flat\mathbf{B}G \to \mathbf{B}G .$$

We write $\flat_{\mathrm{dR}}\mathbf{B}G$ for the ∞-pullback

$$\begin{array}{ccc} \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & \flat\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array} .$$

We show in 4.4.13 that with this construction internalized in smooth ∞-groupoids, the object $\flat_{\mathrm{dR}}\mathbf{B}G$ is the coefficient object for flat $\mathfrak{g}$-valued differential forms, where $\mathfrak{g}$ is the ∞-Lie algebra of G.

Moreover, there is a canonical such form on G itself. This is obtained by forming the pasting diagram of ∞-pullbacks

$$\begin{array}{ccc} A & \longrightarrow & * \\ \downarrow{\scriptstyle \theta} & & \downarrow \\ \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & \flat\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array} .$$

We show below in 4.4.15 that this theta is canonical (Maurer-Cartan) $\mathfrak{g}$-valued form on G. Then in 4.4.16 we show that for G a shifted abelian group, this form is the *universal curvature characteristic*. Flat parallel G-valued transport that is *twisted* by this form encodes non-flat ∞-connections. Gauge fields and higher gauge fields are examples.

In 4.4.19 we show that, just as canonically, action functionals for these higher gauge fields exist in $\mathbf{H}$.

All this just from a system of adjoint ∞-functors.

1.2 The geometry of physics

The following is an introduction to fundamental differential geometric structures as they appear in a description of physics. To some extent this is classical material, but we present it from a modern perspective that should serve to motivate the formal development in section 3.

This section has an online counterpart in [ScGP] with more material and further pointers.

1.2.1 Coordinate systems

Every kind of geometry is modeled on a collection of archetypical basic spaces and geometric homomorphisms between them. In differential geometry the archetypical spaces are the abstract standard Cartesian coordinate systems, denoted $\mathbb{R}^n$, in every dimension $n \in \mathbb{N}$, and the geometric homomorphism between them are smooth functions $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, hence smooth (and possibly degenerate) coordinate transformations.

Here we discuss the central aspects of the nature of such abstract coordinate systems in themselves. At this point these are not yet coordinate systems on some other space. That is instead the topic of the next section Smooth spaces.

1.2.1.1 The continuum real (world-)line The fundamental premise of differential geometry as a model of geometry in physics is the following.

bf Premise. *The abstract worldline of any particle is modeled by the continuum real line* $\mathbb{R}$.

This comes down to the following sequence of premises.

1. There is a linear ordering of the points on a worldline: in particular if we pick points at some intervals on the worldline we may label these in an order-preserving way by integers

$$\mathbb{Z}.$$

2. These intervals may each be subdivided into n smaller intervals, for each natural number n. Hence we may label points on the worldline in an order-preserving way by the rational numbers

$$\mathbb{Q}.$$

3. This labeling is dense: every point on the worldline is the supremum of an inhabited bounded subset of such labels. This means that a worldline is the *real line*, the continuum of real numbers

$$\mathbb{R}\,.$$

The adjective"real" in "real number" is a historical shadow of the old idea that real numbers are related to observed reality, hence to physics in this way. The experimental success of this assumption shows that it is valid at least to very good approximation.

Speculations are common that in a fully exact theory of quantum gravity, currently unavailable, this assumption needs to be refined. For instance in p-adic physics one explores the hypothesis that the relevant completion of the rational numbers as above is not the reals, but p-adic numbers $\mathbb{Q}_p$ for some prime number $p \in \mathbb{N}$. Or for example in the study of QFT on non-commutative spacetime one explore the idea that at small scales the smooth continuum is to be replaced by an object in noncommutative geometry. Combining these two ideas leads to the notion of non-commutative analytic space as a potential model for space in physics. And so forth.

For the time being all this remains speculation and differential geometry based on the continuum real line remains the context of all fundamental model building in physics related to observed phenomenology. Often it is argued that these speculations are necessitated by the very nature of quantum theory applied to gravity. But, at least so far, such statements are not actually supported by the standard theory of quantization: we discuss below in Geometric quantization how not just classical physics but also quantum theory, in the best modern version available, is entirely rooted in differential geometry based on the continuum real line.

This is the motivation for studying models of physics in geometry modeled on the continuum real line. On the other hand, in all of what follows our discussion is set up such as to be maximally independent of this specific choice (this is what *topos theory* accomplishes for us). If we do desire to consider another choice of archetypical spaces for the geometry of physics we can simply "change the site", as discussed below and many of the constructions, propositions and theorems in the following will continue to hold. This is notably what we do below in Supergeometric coordinate systems when we generalize the present discussion to a flavor of differential geometry that also formalizes the notion of fermion particles: "differential supergeometry".

1.2.1.2 Cartesian spaces and smooth functions

Definition 1.2.1. A function of sets $f : \mathbb{R} \to \mathbb{R}$ is called a *smooth function* if, coinductively:

1. the derivative $\frac{df}{dx} : \mathbb{R} \to \mathbb{R}$ exists;
2. and is itself a smooth function.

Definition 1.2.2. For $n \in \mathbb{N}$, the *Cartesian space* $\mathbb{R}^n$ is the set

$$\mathbb{R}^n = \{(x^1, \cdots, x^n) | x^i \in \mathbb{R}\}$$

of n-tuples of real numbers. For $1 \leq k \leq n$ write

$$i^k : \mathbb{R} \to \mathbb{R}^n$$

for the function such that $i^k(x) = (0, \cdots, 0, x, 0, \cdots, 0)$ is the tuple whose kth entry is x and all whose other entries are $0 \in \mathbb{R}$; and write

$$|^k : \mathbb{R}^n \to \mathbb{R}$$

for the function such that $p^k(x^1, \cdots, x^n) = x^k$.

A *homomorphism* of Cartesian spaces is a smooth function

$$f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}\,,$$

hence a function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ such that all partial derivatives exist and are continuous.

Example 1.2.3. Regarding $\mathbb{R}^n$ as an $\mathbb{R}$-vector space, every linear function $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is in particular a smooth function.

Remark 1.2.4. But a homomorphism of Cartesian spaces in def. 1.2.2 is *not* required to be a linear map. We do *not* regard the Cartesian spaces here as vector spaces.

Definition 1.2.5. A smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is called a *diffeomorphism* if there exists another smooth function $\mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ such that the underlying functions of sets are inverse to each other

$$f \circ g = \mathrm{id}$$

and

$$g \circ f = \mathrm{id}\,.$$

Proposition 1.2.6. *There exists a diffeomorphism $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ precisely if $n_1 = n_2$.*

Definition 1.2.7. We will also say equivalently that

1. a Cartesian space $\mathbb{R}^n$ is an *abstract coordinate system*;
2. a smooth function $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is an *abstract coordinate transformation*;
3. the function $p^k : \mathbb{R}^n \to \mathbb{R}$ is the kth *coordinate* of the coordinate system $\mathbb{R}^n$. We will also write this function as $x^k : \mathbb{R}^n \to \mathbb{R}$.
4. for $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ a smooth function, and $1 \leq k \leq n_2$ we write
   (a) $f^k := p^k \circ f$
   (b) $(f^1, \cdots, f^n) := f$.

Remark 1.2.8. It follows with this notation that

$$\mathrm{id}_{\mathbb{R}^n} = (x^1, \cdots, x^n) : \mathbb{R}^n \to \mathbb{R}^n\,.$$

Hence an abstract coordinate transformation

$$f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$$

may equivalently be written as the tuple

$$\left(f^1\left(x^1, \cdots, x^{n_1}\right), \cdots, f^{n_2}\left(x^1, \cdots, x^{n_1}\right)\right)\,.$$

Proposition 1.2.9. *Abstract coordinate systems form a* category *– to be denoted* CartSp *– whose*

- *objects are the abstract coordinate systems $\mathbb{R}^n$ (the class of objects is the set $\mathbb{N}$ of natural numbers n);*
- *morphisms $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ are the abstract coordinate transformations = smooth functions.*

Composition of morphisms is given by composition of functions.

We have that

1. *The identity morphisms are precisely the identity functions.*
2. *The isomorphisms are precisely the diffeomorphisms.*

Definition 1.2.10. Write CartSp^{op} for the opposite category of CartSp.

This is the category with the same objects as $CartSp$, but where a morphism $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ in $CartSp^{op}$ is given by a morphism $\mathbb{R}^{n_1} \leftarrow \mathbb{R}^{n_2}$ in $CartSp$.

We will be discussing below the idea of exploring smooth spaces by laying out abstract coordinate systems in them in all possible ways. The reader should begin to think of the sets that appear in the following definition as the *set of ways* of laying out a given abstract coordinate systems in a given space.

Definition 1.2.11. A functor $X : CartSp^{op} \to Set$ (a "presheaf") is

1. for each abstract coordinate system U a set $X(U)$
2. for each coordinate transformation $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ a function $X(f) : X(\mathbb{R}^{n_1}) \to X(\mathbb{R}^{n_2})$

such that

1. identity is respected $X(id_{\mathbb{R}^n}) = id_{X(\mathbb{R}^n)}$;
2. composition is respected $X(f_2) \circ X(f_1) = X(f_2 \circ f_1)$

1.2.1.3 The fundamental theorems about smooth functions The special properties smooth functions that make them play an important role different from other classes of functions are the following.

1. existence of bump functions and partitions of unity
2. the Hadamard lemma and Borel's theorem

Or maybe better put: what makes smooth functions special is that the first of these properties holds, while the second is still retained.

1.2.2 Smooth sets

We now discuss concretely the definition of smooth sets/smooth spaces and of homomorphisms between them, together with basic examples and properties.

1.2.2.1 Plots of smooth spaces and their gluing The general kind of "smooth space" that we want to consider is something that can be *probed* by laying out coordinate systems as in def. **??** inside it, and that can be obtained by *gluing* all the possible coordinate systems in it together.

At this point we want to impose no further conditions on a "space" than this. In particular we do not assume that we know beforehand a set of points underlying X. Instead, we define smooth spaces X entirely *operationally* as something about which we can ask "Which ways are there to lay out $\mathbb{R}^n$ inside X?" and such that there is a self-consistent answer to this question. The following definitions make precise what we mean by this.

For brevity we will refer "a way to lay out a coordinate system in X" as a *plot* of X. The first set of consistency conditions on plots of a space is that they respect *coordinate transformations*. This is what the following definition formalizes.

Definition 1.2.12. A *smooth pre-space* X is

1. a collection of sets: for each Cartesian space $\mathbb{R}^n$ (hence for each natural number n) a set

   $$X(\mathbb{R}^n) \in \text{Set}$$

   – to be thought of as the *set of ways of laying out* $\mathbb{R}^n$ *inside* X;

2. for each abstract coordinate transformation, hence for each smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ a function between the corresponding sets

   $$X(f) : X(\mathbb{R}^{n_2}) \to X(\mathbb{R}^{n_1})$$

   – to be thought of as the function that sends a *plot* of X by $\mathbb{R}^{n_2}$ to the correspondingly transformed plot by $\mathbb{R}^{n_1}$ induced by laying out $\mathbb{R}^{n_1}$ inside $\mathbb{R}^{n_2}$.

such that this is compatible with coordinate transformations:

1. the identity coordinate transformation does not change the plots:

$$X(id_{\mathbb{R}^n}) = id_{X(\mathbb{R}^n)}\,,$$

2. changing plots along two consecutive coordinate transformations $f_1 \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $f_2 \colon \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$ is the same as changing them along the composite coordinate transformation $f_2 \circ f_1$:

$$X(f_1) \circ X(f_2) = X(f_2 \circ f_1)\,.$$

But there is one more consistency condition for a collection of plots to really be probes of some space: it must be true that if we glue small coordinate systems to larger ones, then the plots by the larger ones are the same as the plots by the collection of smaller ones that agree where they overlap. We first formalize this idea of "plots that agree where their coordinate systems overlap".

Definition 1.2.13. Let X be a smooth pre-space, def. 1.2.12. For $\{U_i \to \mathbb{R}^n\}_{i\in I}$ a differentially good open cover, def. **??**, let

$$\mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X) \in \mathrm{Set}$$

be the set of I-tuples of U_i-plots of X which coincide on all double intersections

$$\begin{array}{ccc} & U_i \cap U_j & \\ {}^{\iota_i}\swarrow & & \searrow^{\iota_j} \\ U_i & & U_J \\ \searrow & & \swarrow \\ & \mathbb{R}^n & \end{array}$$

(also called the *matching families* of X over the given cover):

$$\mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X) \ := \ \left\{ \ (p_i \in X(U_i))_{i\in I} \ | \ \forall_{i,j\in I} \ : \ X(\iota_i)(p_i) = X(\iota_j)(p_j) \ \right\}\,.$$

Remark 1.2.14. In def. 1.2.13 the equation

$$X(\iota_i)(p_i) = X(\iota_j)(p_j)$$

says in words:

"The plot p_i of X by the coordinate system U_i inside the bigger coordinate system $\mathbb{R}^n$ coincides with the plot p_j of X by the other coordinate system U_j inside X when both are restricted to the intersection $U_i \cap U_j$ of U_i with U_j inside $\mathbb{R}^n$."

Remark 1.2.15. For each differentially good open cover $\{U_i \to X\}_{i\in I}$ and each smooth pre-space X, def. 1.2.12, there is a canonical function

$$X(\mathbb{R}^n) \to \mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X)$$

from the set of $\mathbb{R}^n$-plots of X to the set of tuples of glued plots, which sends a plot $p \in X(\mathbb{R}^n)$ to its restriction to all the $\phi_i \colon U_i \hookrightarrow \mathbb{R}^n$:

$$p \mapsto (X(\phi_i)(p))_{i\in I}\,.$$

If X is supposed to be consistently probable by coordinate systems, then it must be true that the set of ways of laying out a coordinate system $\mathbb{R}^n$ inside it coincides with the set of ways of laying out tuples of glued coordinate systems inside it, for each good cover $\{U_i \to \mathbb{R}^n\}$ as above. Therefore:

Definition 1.2.16. A smooth pre-space X, def. 1.2.12 is a *smooth space* if for all differentially good open covers $\{U_i \to \mathbb{R}^n\}$, def. **??**, the canonical function of remark 1.2.15 from plots to glued plots is a bijection

$$X(\mathbb{R}^n) \stackrel{\simeq}{\to} \mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X)\,.$$

Remark 1.2.17. We may think of a smooth space as being a kind of space whose *local models* (in the general sense discussed at *geometry*) are Cartesian spaces:

while definition 1.2.16 explicitly says that a smooth space is something that is *consistently probeable* by such local models; by a general abstract fact that is sometimes called the *co-Yoneda lemma*, it follows in fact that smooth spaces are precisely the objects that are obtained by *gluing coordinate systems* together.

For instance we will see that two open 2-balls $\mathbb{R}^2 \simeq D^2$ along a common rim yields the smooth space version of the sphere S^2, a basic example of a smooth manifold. But before we examine such explicit constructions, we discuss here for the moment more general properties of smooth spaces.

Example 1.2.18. For $n \in \mathbb{R}^n$, there is a smooth space, def. 1.2.16, whose set of plots over the abstract coordinate systems $\mathbb{R}^k$ is the set

$$\mathrm{CartSp}(\mathbb{R}^k, \mathbb{R}^n) \in \mathrm{Set}$$

of smooth functions from $\mathbb{R}^k$ to $\mathbb{R}^n$.

Clearly this is the rule for plots that characterize $\mathbb{R}^n$ itself as a smooth space, and so we will just denote this smooth space by the same symbols "$\mathbb{R}^n$":

$$\mathbb{R}^n \colon \mathbb{R}^k \mapsto \mathrm{CartSp}(\mathbb{R}^k, \mathbb{R}^n)\,.$$

In particular the real line $\mathbb{R}$ is this way itself a smooth space.

In a moment we find a formal justification for this slight abuse of notation.

Another basic class of examples of smooth spaces are the discrete smooth spaces:

Definition 1.2.19. For $S \in \mathrm{Set}$ a set, write

$$\mathrm{Disc}S \in \mathrm{Smooth0Type}$$

for the smooth space whose set of U-plots for every $U \in \mathrm{CartSp}$ is always S.

$$\mathrm{Disc}S \colon U \mapsto S$$

and which sends every coordinate transformation $f \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ to the identity function on S.

A smooth space of this form we call a *discrete smooth space*.

More examples of smooth spaces can be built notably by intersecting images of two smooth spaces inside a bigger one. In order to say this we first need a formalization of homomorphism of smooth spaces. This we turn to now.

1.2.2.2 Homomorphisms of smooth spaces

We discuss "functions" or "maps" between smooth spaces, def. 1.2.16, which preserve the smooth space structure in a suitable sense. As with any notion of function that preserves structure, we refer to them as *homomorphisms*.

The idea of the following definition is to say that whatever a homomorphism $f : X \to Y$ between two smooth spaces is, it has to take the plots of X by $\mathbb{R}^n$ to a corresponding plot of Y, such that this respects coordinate transformations.

Definition 1.2.20. Let X and Y be two smooth spaces, def. 1.2.16. Then a homomorphism $f \colon X \to Y$ is

- for each abstract coordinate system $\mathbb{R}^n$ (hence for each $n \in \mathbb{N}$) a function $f_{\mathbb{R}^n} : X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ that sends $\mathbb{R}^n$-plots of X to $\mathbb{R}^n$-plots of Y

such that

- for each smooth function $\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we have

$$Y(\phi) \circ f_{\mathbb{R}^{n_1}} = f_{\mathbb{R}^{n_2}} \circ X(\phi)\,,$$

  hence a commuting diagram

$$\begin{array}{ccc} X(\mathbb{R}^{n_1}) & \xrightarrow{f_{\mathbb{R}^{n_1}}} & Y(\mathbb{R}^{n_1}) \\ \downarrow{\scriptstyle X(\phi)} & & \downarrow{\scriptstyle Y(\phi)} \\ X(\mathbb{R}^{n_2}) & \xrightarrow{f_{\mathbb{R}^{n_2}}} & Y(\mathbb{R}^{n_1}) \end{array} \quad .$$

For $f_1 : X \to Y$ and $f_2 : X \to Y$ two homomorphisms of smooth spaces, their composition $f_2 \circ f_1 \colon X \to Y$ is defined to be the homomorphism whose component over $\mathbb{R}^n$ is the composite of functions of the components of f_1 and f_2:

$$(f_2 \circ f_1)_{\mathbb{R}^n} := f_{2\mathbb{R}^n} \circ f_{1\mathbb{R}^n}\,.$$

Definition 1.2.21. Write Smooth0Type for the category whose objects are smooth spaces, def. 1.2.16, and whose morphisms are homomorphisms of smooth spaces, def. 1.2.20.

At this point it may seem that we have now *two different* notions for how to lay out a coordinate system in a smooth space X: on the hand, X comes by definition with a rule for what the set $X(\mathbb{R}^n)$ of its $\mathbb{R}^n$-plots is. On the other hand, we can now regard the abstract coordinate system $\mathbb{R}^n$ itself as a smooth space, by example 1.2.18, and then say that an $\mathbb{R}^n$-plot of X should be a homomorphism of smooth spaces of the form $\mathbb{R}^n \to X$.

The following proposition says that these two superficially different notions actually naturally coincide.

Proposition 1.2.22. *Let X be any smooth space, def. 1.2.16, and regard the abstract coordinate system $\mathbb{R}^n$ as a smooth space, by example 1.2.18. There is a natural bijection*

$$X(\mathbb{R}^n) \simeq Hom_{\mathrm{Smooth0Type}}(\mathbb{R}^n, X)$$

between the postulated *$\mathbb{R}^n$-plots of X and the* actual *$\mathbb{R}^n$-plots given by homomorphism of smooth spaces $\mathbb{R}^n \to X$.*

Proof. This is a special case of the *Yoneda lemma*. The reader unfamiliar with this should write out the simple proof explicitly: use the defining commuting diagrams in def. 1.2.20 to deduce that a homomorphism $f : \mathbb{R}^n \to X$ is uniquely fixed by the image of the identity element in $\mathbb{R}^n(\mathbb{R}^n) := CartSp(\mathbb{R}^n, \mathbb{R}^n)$ under the component function $f_{\mathbb{R}^n} : \mathbb{R}^n(\mathbb{R}^n) \to X(\mathbb{R}^n)$. □

Example 1.2.23. Let $\mathbb{R} \in$ Smooth0Type denote the real line, regarded as a smooth space by def. 1.2.18. Then for $X \in$ Smooth0Type any smooth space, a homomorphism of smooth spaces

$$f \colon X \to \mathbb{R}$$

is a *smooth function on* X Prop. 1.2.22 says here that when X happens to be an abstract coordinate system regarded as a smooth space by def. 1.2.18, then this general notion of smooth functions between smooth spaces reproduces the basic notion of def, 1.2.2.

Definition 1.2.24. The 0-dimensional abstract coordinate system $\mathbb{R}^0$ we also call the *point* and regarded as a smooth space we will often write it as

$$* \in \mathrm{Smooth0Type}\,.$$

For any $X \in$ Smooth0Type, we say that a homomorphism

$$x \colon * \to X$$

is a *point of* X.

Remark 1.2.25. By prop. 1.2.22 the points of a smooth space X are naturally identified with its 0-dimensional plots, hence with the "ways of laying out a 0-dimensional coordinate system" in X:

$$Hom(*, X) \simeq X(\mathbb{R}^0)\,.$$

1.2.2.3 Products and fiber products of smooth spaces

Definition 1.2.26. Let $X, Y \in Smooth0Type$ by two smooth spaces. Their *product* is the smooth space $X \times Y \in \mathrm{Smooth0Type}$ whose plots are pairs of plots of X and Y:

$$X \times Y(\mathbb{R}^n) := X(\mathbb{R}^n) \times Y(\mathbb{R}^n) \quad \in \mathrm{Set}\,.$$

The *projection on the first factor* is the homomorphism

$$p_1 \colon X \times Y \to X$$

which sends $\mathbb{R}^n$-plots of $X \times Y$ to those of X by forming the projection of the cartesian product of sets:

$$p_{1\mathbb{R}^n} \colon X(\mathbb{R}^n) \times Y(\mathbb{R}^n) \stackrel{p_1}{\to} X(\mathbb{R}^n)\,.$$

Analogously for the *projection to the second factor*

$$p_2 \colon X \times Y \to Y\,.$$

Proposition 1.2.27. *Let* $* = \mathbb{R}^0$ *be the point, regarded as a smooth space, def. 1.2.24. Then for* $X \in \mathrm{Smooth0Type}$ *any smooth space the canonical projection homomorphism*

$$X \times * \to X$$

is an isomorphism.

Definition 1.2.28. Let $f \colon X \to Z$ and $g \colon Y \to Z$ be two homomorphisms of smooth spaces, def. 1.2.20. There is then a new smooth space to be denoted

$$X \times_Z Y \in \mathrm{Smooth0Type}$$

(with f and g understood), called the *fiber product* of X and Y along f and g, and defined as follows:

the set of $\mathbb{R}^n$-plots of $X \times_Z Y$ is the set of pairs of plots of X and Y which become the same plot of Z under f and g, respectively:

$$(X \times_Z Y)(\mathbb{R}^n) = \{(p_X \in X(\mathbb{R}^n), p_Y \in Y(\mathbb{R}^n)) \mid f_{\mathbb{R}^n}(p_X) = g_{\mathbb{R}^n}(p_Y)\}\,.$$

1.2.2.4 Smooth mapping spaces and smooth moduli spaces

Definition 1.2.29. Let $\Sigma, X \in Smooth0Type$ be two smooth spaces, def. 1.2.16. Then the *smooth mapping space*

$$[\Sigma, X] \in \mathrm{Smooth0Type}$$

is the smooth space defined by saying that its set of $\mathbb{R}^n$-plots is

$$[\Sigma, X](\mathbb{R}^n) := \mathrm{Hom}(\Sigma \times \mathbb{R}^n, X)\,.$$

Here in $\Sigma \times \mathbb{R}^n$ we first regard the abstract coordinate system $\mathbb{R}^n$ as a smooth space by example 1.2.18 and then we form the product smooth space by def. 1.2.26.

Remark 1.2.30. This means in words that a $\mathbb{R}^n$-plot of the mapping space $[\Sigma, X]$ is a smooth $\mathbb{R}^n$-parameterized collection of homomorphisms $\Sigma \to X$.

Proposition 1.2.31. *There is a natural bijection*

$$\mathrm{Hom}(K, [\Sigma, X]) \simeq \mathrm{Hom}(K \times \Sigma, X)$$

for every smooth space K.

Proof. With a bit of work this is straightforward to check explicitly by unwinding the definitions. It follows however from general abstract results once we realize that $[-,-]$ is of course the *internal hom* of smooth spaces. □

Remark 1.2.32. This says in words that a smooth function from any K into the mapping space $[\Sigma, X]$ is equivalently a smooth function from $K \times \Sigma$ to X. The latter we may regard as a *K-parameterized smooth collections* of smooth functions $\Sigma \to X$. Therefore in view of the previous remark 1.2.30 this says that smooth mapping spaces have a universal property not just over abstract coordinate systems, but over all smooth spaces.

We will therefore also say that $[\Sigma, X]$ is the *smooth moduli space* of smooth functions from $\Sigma \to X$, because it is such that smooth maps $K \to [\Sigma, X]$ into it *modulate*, as we move around on K, a family of smooth functions $\Sigma \to X$, depending on K.

Proposition 1.2.33. *The set of points, def. 1.2.24, of a smooth mapping space $[\Sigma, X]$ is the bare set of homomorphism $\Sigma \to X$: there is a natural isomorphism*

$$\mathrm{Hom}(*, [\Sigma, X]) \simeq \mathrm{Hom}(\Sigma, X)\,.$$

Proof. Combine prop. 1.2.31 with prop. 1.2.27. □

Example 1.2.34. Given a smooth space $X \in Smooth0Type$, its smooth *path space* is the smooth mapping space

$$\mathbf{P}X := [\mathbb{R}^1, X]\,.$$

By prop. 1.2.33 the points of PX are indeed precisely the smooth trajectories $\mathbb{R}^1 \to X$. But PX also knows how to *smoothly vary* such smooth trajectories.

This is central for variational calculus which determines equations of motion in physics.

Remark 1.2.35. In physics, if X is a model for spacetime, then PX may notably be interpreted as the smooth space of worldlines *in* X, hence the smooth space of paths or *trajectories* of a particle in X.

Example 1.2.36. If in the above example 1.2.34 the path is constraind to be a loop in X, one obtains the *smooth loop space*

$$\mathbf{L}X := [S^1, X]\,.$$

1.2.2.5 The smooth moduli space of smooth functions In example 1.2.23 we saw that a smooth function on a general smooth space X is a homomorphism of smooth spaces, def. 1.2.20

$$f \colon X \to \mathbb{R}\,.$$

The collection of these forms the hom-set $\mathrm{Hom}_{\mathrm{Smooth0Type}}(X, \mathbb{R})$. But by the discussion in 1.2.2.4 such hom-sets are naturally refined to smooth spaces themselves.

Definition 1.2.37. For $X \in Smooth0Type$ a smooth space, we say that the *moduli space of smooth functions* on X is the smooth mapping space (def. 1.2.29), from X into the standard real line $\mathbb{R}$

$$[X, \mathbb{R}] \in \mathrm{Smooth0Type}\,.$$

We will also denote this by

$$\mathbf{C}^\infty(X) := [X, \mathbb{R}]\,,$$

since in the special case that X is a Cartesian space this is the smooth refinement of the set $C^\infty(X)$ of smooth functions, def. **??**, on X.

Remark 1.2.38. We call this a *moduli space* because by prop. 1.2.31 above and in the sense of remark 1.2.32 it is such that smooth functions into it *modulate* smooth functions $X \to \mathbb{R}$.

By prop. 1.2.33 a point $* \to [X, \mathbb{R}^1]$ of the moduli space is equivalently a smooth function $X \to \mathbb{R}^1$.

1.2.2.6 Outlook Later we define/see the following:

- A *smooth manifold* is a smooth space that is *locally equivalent* to a coordinate system;
- A *diffeological space* is a smooth space such that every coordinate labels a point in the space. In other words, a diffeological space is a smooth space that has an underlying set $X_s \in Set$ of points such that the set of $\mathbb{R}^n$-plots is a subset of the set of all functions:

$$X(\mathbb{R}^n) \hookrightarrow \mathrm{Functions}(\mathbb{R}^n, S_s)\,.$$

We discuss below a long sequence of faithful inclusions

$\{$coordinate systems $\} \hookrightarrow \{$smooth manifolds$\} \hookrightarrow \{$diffeological spaces$\} \hookrightarrow \{$smooth spaces$\} \hookrightarrow \{$smooth groupoids$\} \hookrightarrow \cdots$

1.2.3 Smooth homotopy types

Here we give an introduction to and a survey of the general theory of cohesive differential geometry that is developed in detail below in 3 below.

The framework of all our constructions is *topos theory* [John03] or rather, more generally, ∞-*topos theory* [LuHTT]. In 1.2.3.1 and 1.2.3.2 below we recall and survey basic notions with an eye towards our central example of an ∞-topos: that of smooth ∞-groupoids. In these sections the reader is assumed to be familiar with basic notions of category theory (such as adjoint functors) and basic notions of homotopy theory (such as weak homotopy equivalences). A brief introduction to relevant basic concepts (such as Kan complexes and homotopy pullbacks) is given in section 1.2.3, which can be read independently of the discussion here.

Then in 1.2.3.3 and 1.2.3.4 we describe, similarly in a leisurely manner, the intrinsic notions of cohomology and geometric homotopy in an ∞-topos. Several aspects of the discussion are fairly well-known, we put them in the general perspective of (cohesive) ∞-topos theory and then go beyond.

Finally in 1.2.11.2 we indicate how the combination of the intrinsic cohomology and geometric homotopy in a locally ∞-connected ∞-topos yields an intrinsic notion of differential cohomology in an ∞-topos.

- 1.2.3.1 – Toposes;
- 1.2.3.2 – ∞-Toposes;
- 1.2.3.3 – Cohomology;
- 1.2.3.4 – Homotopy;
- 1.2.11.2 – Differential cohomology.

Each of these topics surveyed here are discussed in technical detail below in 3.

1.2.3.1 Toposes There are several different perspectives on the notion of *topos*. One is that a topos is a category that looks like a category of spaces that sit by local homeomorphisms over a given base space: all spaces that are locally modeled on a given base space.

The archetypical class of examples are sheaf toposes over a topological space X denoted $\mathrm{Sh}(X)$. These are equivalently categories of étale spaces over X: topological spaces Y that are equipped with a local homeomorphisms $Y \to X$. When $X = *$ is the point, this is just the category Set of all sets: spaces that are modeled on the point. This is the archetypical topos itself.

What makes the notion of toposes powerful is the following fact: even though the general topos contains objects that are considerably different from and possibly considerably richer than plain sets and even richer than étale spaces over a topological space, the general abstract category theoretic properties of every topos are essentially the same as those of Set. For instance in every topos all small limits and colimits exist and it is cartesian closed (even locally). This means that a large number of constructions in Set have immediate analogs internal to every topos, and the analogs of the statements about these constructions that are true in Set are true in every topos.

This may be thought of as saying that toposes are *very nice categories of spaces* in that whatever construction on spaces one thinks of – for instance formation of quotients or of intersections or of mapping spaces – the resulting space with the expected general abstract properties will exist in the topos. In this sense toposes are *convenient categories for geometry* - as in: *convenient category of topological spaces*, but even more convenient than that.

On the other hand, we can de-emphasize the role of the objects of the topos and instead treat the topos itself as a "generalized space" (and in particular, a categorified space). We then consider the sheaf topos $\mathrm{Sh}(X)$ as a representative of X itself, while toposes not of this form are "honestly generalized" spaces. This point of view is supported by the fact that the assignment $X \mapsto \mathrm{Sh}(X)$ is a full embedding of (sufficiently nice) topological spaces into toposes, and that many topological properties of a space X can be detected at the level of $\mathrm{Sh}(X)$.

Here we are mainly concerned with toposes that are far from being akin to sheaves over a topological space, and instead behave like abstract *fat points with geometric structure*. This implies that the objects of these toposes are in turn generalized spaces modeled locally on this geometric structure. Such toposes are called *gros toposes* or *big toposes*. There is a formalization of the properties of a topos that make it behave like a big topos of generalized spaces inside of which there is geometry: this is the notion of *cohesive toposes*.

1.2.3.1.1 Sheaves More concretely, the idea of sheaf toposes formalizes the idea that any notion of space is typically modeled on a given collection of simple test spaces. For instance differential geometry is the geometry that is modeled Cartesian spaces $\mathbb{R}^n$, or rather on the category $C = \mathrm{CartSp}$ of Cartesian spaces and smooth functions between them.

A presheaf on such C is a functor $X : C^{\mathrm{op}} \to \mathrm{Set}$ from the opposite category of C to the category of sets. We think of this as a rule that assigns to each test space $U \in C$ the set $X(U) :=: \mathrm{Maps}(U, X)$ of structure-preserving maps from the test space U into the would-be space X - the *probes* of X by the test space U. This assignment defines the generalized space X modeled on C. Every category of presheaves over a small category is an example of a topos. But these presheaf toposes, while encoding the *geometry* of generalized spaces by means of probes by test spaces in C fail to correctly encode the *topology* of these spaces. This is captured by restricting to *sheaves* among all presheaves.

Each test space $V \in C$ itself specifies presheaf, by forming the hom-sets $\mathrm{Maps}(U, V) := \mathrm{Hom}_C(U, V)$ in C. This is called the *Yoneda embedding* of test spaces into the collection of all generalized spaces modeled on them. Presheaves of this form are the *representable presheaves*. A bit more general than these are the *locally representable presheaves*: for instance on $C = \mathrm{CartSp}$ this are the smooth manifolds $X \in \mathrm{SmoothMfd}$, whose presheaf-rule is $\mathrm{Maps}(U, X) := \mathrm{Hom}_{\mathrm{SmoothMfd}}(U, X)$. By definition, a manifold is locally isomorphic to a Cartesian space, hence is locally representable as a presheaf on CartSp.

These examples of presheaves on C are special in that they are in fact *sheaves*: the value of X on a test space U is entirely determined by the restrictions to each U_i in a *cover* $\{U_i \to U\}_{i \in I}$ of the test space U by other test spaces U_i. We think of the subcategory of sheaves $\mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C)$ as consisting of those special presheaves that are those rules of probe-assignments which respect a certain notion of ways in which test spaces $U, V \in C$ may glue together to a bigger test space.

One may axiomatize this by declaring that the collections of all covers under consideration forms what is called a *Grothendieck topology* on C that makes C a *site*. But of more intrinsic relevance is the equivalent fact that categories of sheaves are precisely the subtoposes of presheaves toposes

$$\mathrm{Sh}(C) \underset{\hookrightarrow}{\overset{L}{\leftarrow}} \mathrm{PSh}(C) = [C^{\mathrm{op}}, \mathrm{Set}] \ ,$$

meaning that the embedding $\mathrm{Sh}(X) \hookrightarrow \mathrm{PSh}(X)$ has a left adjoint functor L that preserves finite limits. This may be taken to be the *definition* of Grothendieck toposes. The left adjoint is called the *sheafification functor*. It is determined by and determines a Grothendieck topology on C.

For the choice $C = \mathrm{CartSp}$ such is naturally given by the good open cover coverage, which says that a bunch of maps $\{U_i \to U\}$ in C exhibit the test object U as being glued together from the test objects $\{U_i\}$ if these form a good open cover of U. With this notion of coverage every smooth manifold is a sheaf on CartSp.

But there are important genenralized spaces modeled on CartSp that are not smooth manifolds: topological spaces for which one can consistently define which maps from Cartesian spaces into them count as smooth in a way that makes this assignment a sheaf on CartSp, but which are not necessarily locally isomorphic to a Cartsian space: these are called *diffeological spaces*. A central example of a space that is naturally a diffeological space but not a finite dimensional manifold is a mapping space $[\Sigma, X]$ of smooth functions between smooth manifolds Σ and X: since the idea is that for U any Cartesian space the smooth U-parameterized families of points in $[\Sigma, X]$ are smooth U-parameterized families of smooth maps $\Sigma \to X$, we can take the plot-assigning rule to be

$$[\Sigma, X] : U \mapsto \mathrm{Hom}_{\mathrm{SmoothMfd}}(\Sigma \times U, X) \, .$$

It is useful to relate all these phenomena in the topos $\mathrm{Sh}(C)$ to their image in the archetypical topos Set. This is simply the category of sets, which however we should think of here as the category $\mathrm{Set} \simeq \mathrm{Sh}(*)$ of sheaves on the category $*$ which contains only a single object and no nontrivial morphism: objects in here are generalized spaces *modeled on the point*. All we know about them is how to map the point into them, and as such they are just the sets of all possible such maps from the point.

Every category of sheaves $\mathrm{Sh}(C)$ comes canonically with an essentially unique topos morphism to the topos of sets, given by a pair of adjoint functors

$$\mathrm{Sh}(C) \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \mathrm{Sh}(*) \simeq \mathrm{Set} \ .$$

Here Γ is called the *global sections functor*. If C has a terminal object $*$, then it is given by evaluation on that object: the functor Γ sends a plot-assigning rule $X : C^{\mathrm{op}} \to \mathrm{Set}$ to the set of plots by the point $\Gamma(X) = X(*)$. For instance in $C = \mathrm{CartSp}$ the terminal object exists and is the ordinary point $* = \mathbb{R}^0$. If $X \in \mathrm{Sh}(C)$ is a smooth manifold or diffeological space as above, then $\Gamma(X) \in \mathrm{Set}$ is simply its underlying set of points. So the functor Γ can be thought of as forgetting the *cohesive structure* that is given by the fact that our generalized spaces are modeled on C. It remembers only the underlying point-set.

Conversely, its left adjoint functor Disc takes a set S to the sheafification $\mathrm{Disc}(S) = L\mathrm{Const}(S)$ of the constant presheaf $\mathrm{Const} : U \mapsto S$, which asserts that the set of its plots by any test space is always the same set S. This is the plot-rule for the *discrete space* modeled on C given by the set S: a plot has to be a constant map of the test space U to one of the elements $s \in S$. For the case $C = \mathrm{CartSp}$ this interpretation is literally true in the familiar sense: the generalized smooth space $\mathrm{Disc}(S)$ is the discrete smooth manifold or discrete diffeological space with point set S.

1.2.3.1.2 Concrete and non-concrete sheaves The examples for generalized spaces X modeled on C that we considered so far all had the property that the collection of plots $U \to X$ into them was a subset of the set of maps of sets from U to their underlying set $\Gamma(X)$ of points. These are called *concrete sheaves*. Not every sheaf is concrete. The concrete sheaves form a subcategory inside the full topos which is itself almost, but not quite a topos: it is the *quasitopos* of concrete objects.

$$\mathrm{Conc}(C) \hookrightarrow \mathrm{Sh}(C) \; .$$

Non-concrete sheaves over C may be exotic as compared to smooth manifolds, but they are still usefully regarded as generalized spaces modeled on C. For instance for $n \in \mathbb{N}$ there is the sheaf $\kappa(n, \mathbb{R})$ given by saying that plots by $U \in \mathrm{CartSp}$ are identified with closed differential n-forms on U:

$$\kappa(n, \mathbb{R}) : U \mapsto \Omega^n_{\mathrm{cl}}(U) \, .$$

This sheaf describes a very non-classical space, which for $n \geq 1$ has only a single point, $\Gamma(\kappa(n, \mathbb{R})) = *$, only a single curve, a single surface, etc., up to a single $(n-1)$-dimensional probe, but then it has a large number of n-dimensional probes. Despite the fact that this sheaf is very far in nature from the test spaces that it is modeled on, it plays a crucial and very natural role: it is in a sense a model for an Eilenberg-MacLane space $K(n, \mathbb{R})$. We shall see in 4.4.14 that these sheaves are part of an incarnation of the ∞-Lie-algebra $b^n\mathbb{R}$ and the sense in which it models an Eilenberg-MacLane space is that of Sullivan models in rational homotopy theory. In any case, we want to allow ourselves to regard non-concrete objects such as $\kappa(n, \mathbb{R})$ on the same footing as diffeological spaces and smooth manifolds.

1.2.3.2 ∞-Toposes While therefore a general object in the sheaf topos $\mathrm{Sh}(C)$ may exhibit a considerable generalization of the objects $U \in C$ that it is modeled on, for many natural applications this is still not quite general enough: if for instance X is a *smooth orbifold* (see for instance [MoPr97]), then there is not just a set, but a *groupoid* of ways of probing it by a Cartesian test space U: if a probe $\gamma : U \to X$ is connected by an orbifold transformation to another probe $\gamma' : U \to X$, then this constitutes a morphism in the groupoid $X(U)$ of probes of X by U.

Even more generally, there may be a genuine ∞-*groupoid* of probes of the generalized space X by the test space U: a set of probes with morphisms between different probes, 2-morphisms between these 1-morphisms, and so on.

Such structures are described in ∞-*category theory*: where a category has a set of morphisms between any two objects, an ∞-category has an ∞-grouopoid of morphisms, whose compositions are defined up to higher coherent homotopy. The theory of ∞-categories is effectively the combination of category theory and homotopy theory. The main fact about it, emphasized originally by André Joyal and then further developed in [LuHTT], is that it behaves formally entirely analously to category theory: there are notions of ∞-functors, ∞-limits, adjoint ∞-functors etc., that satisfy all the familiar relations from category theory.

1.2.3.2.1 ∞-Groupoids We first look at bare ∞-groupoids and then discuss how to equip these with smooth structure.

An ∞-groupoid is first of all supposed to be a structure that has k-*morphisms* for all $k \in \mathbb{N}$, which for $k \geq 1$ go between $(k-1)$-morphisms. A useful tool for organizing such collections of morphisms is the notion of a *simplicial set*. This is a functor on the opposite category of the simplex category Δ, whose objects are the abstract cellular k-simplices, denoted $[k]$ or $\Delta[k]$ for all $k \in \mathbb{N}$, and whose morphisms $\Delta[k_1] \to \Delta[k_2]$ are all ways of mapping these into each other. So we think of such a simplicial set given by a functor

$$K : \Delta^{\mathrm{op}} \to \mathrm{Set}$$

as specifying

- a set $[0] \mapsto K_0$ of *objects*;
- a set $[1] \mapsto K_1$ of *morphisms*;
- a set $[2] \mapsto K_2$ of *2-morphisms*;

- a set $[3] \mapsto K_3$ of *3-morphisms*;

and generally

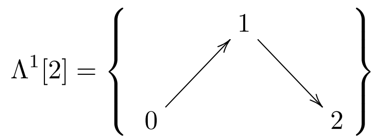
- a set $[k] \mapsto K_k$ of *k-morphisms*.

as well as specifying

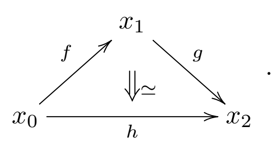
- functions $([n] \hookrightarrow [n+1]) \mapsto (K_{n+1} \to K_n)$ that send $n+1$-morphisms to their boundary n-morphisms;
- functionss $([n+1] \to [n]) \mapsto (K_n \to K_{n+1})$ that send n-morphisms to identity $(n+1)$-morphisms on them.

The fact that K is supposed to be a functor enforces that these assignments of sets and functions satisfy conditions that make consistent our interpretation of them as sets of k-morphisms and source and target maps between these. These are called the *simplicial identities*. But apart from this source-target matching, a generic simplicial set does not yet encode a notion of *composition* of these morphisms.

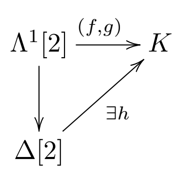
For instance for $\Lambda^1[2]$ the simplicial set consisting of two attached 1-cells



and for $(f,g) : \Lambda^1[2] \to K$ an image of this situation in K, hence a pair $x_0 \stackrel{f}{\to} x_1 \stackrel{g}{\to} x_2$ of two *composable* 1-morphisms in K, we want to demand that there exists a third 1-morphisms in K that may be thought of as the *composition* $x_0 \stackrel{h}{\to} x_2$ of f and g. But since we are working in higher category theory, we want to identify this composite only up to a 2-morphism equivalence

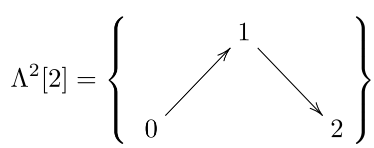


From the picture it is clear that this is equivalent to demanding that for $\Lambda^1[2] \hookrightarrow \Delta[2]$ the obvious inclusion of the two abstract composable 1-morphisms into the 2-simplex we have a diagram of morphisms of simplicial sets



A simplicial set where for all such (f,g) a corresponding such h exists may be thought of as a collection of higher morphisms that is equipped with a notion of composition of adjacent 1-morphisms.

For the purpose of describing groupoidal composition, we now want that this composition operation has all inverses. For that purpose, notice that for



the simplicial set consisting of two 1-morphisms that touch at their end, hence for

$$(g,h) : \Lambda^2[2] \to K$$

two such 1-morphisms in K, then if g had an inverse g^{-1} we could use the above composition operation to compose that with h and thereby find a morphism f connecting the sources of h and g. This being the case is evidently equivalent to the existence of diagrams of morphisms of simplicial sets of the form

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{(g,h)} & K \\ \downarrow & \nearrow_{\exists f} & \\ \Delta[2] & & \end{array} .$$

Demanding that all such diagrams exist is therefore demanding that we have on 1-morphisms a composition operation with inverses in K.

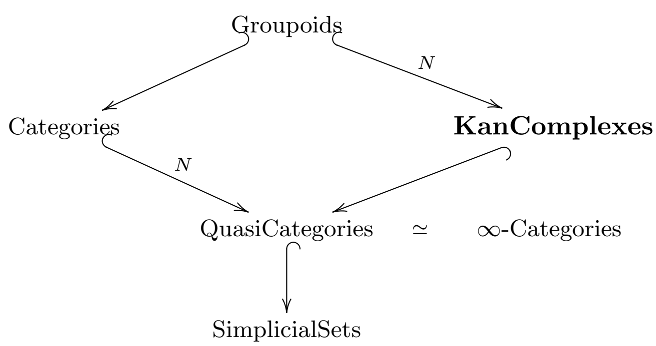
In order for this to qualify as an ∞-groupoid, this composition operation needs to satisfy an associativity law up to 2-morphisms, which means that we can find the relevant tetrahedra in K. These in turn need to be connected by *pentagonators* and ever so on. It is a nontrivial but true and powerful fact, that all these coherence conditions are captured by generalizing the above conditions to all dimensions in the evident way:

let $\Lambda^i[n] \hookrightarrow \Delta[n]$ be the simplicial set – called the ith n-horn – that consists of all cells of the n-simplex $\Delta[n]$ except the interior n-morphism and the ith $(n-1)$-morphism.

Then a simplicial set is called a *Kan complex*, if for all images $f : \Lambda^i[n] \to K$ of such horns in K, the missing two cells can be found in K – in that we can always find a *horn filler* σ in the diagram

$$\begin{array}{ccc} \Lambda^i[n] & \xrightarrow{f} & K \\ \downarrow & \nearrow_{\exists \sigma} & \\ \Delta[n] & & \end{array} .$$

The basic example is the *nerve* $N(C) \in \mathrm{sSet}$ of an ordinary groupoid C, which is the simplicial set with $N(C)_k$ being the set of sequences of k composable morphisms in C. The nerve operation is a full and faithful functor from 1-groupoids into Kan complexes and hence may be thought of as embedding 1-groupoids in the context of general ∞-groupoids.



But we need a bit more than just bare ∞-groupoids. In generalization to Lie groupoids, we need *smooth ∞-groupoids*. A useful way to encode that an ∞-groupoid has extra structure modeled on geometric test objects that themselves form a category C is to remember the rule which for each test space U in C produces the ∞-groupoid of U-parameterized families of k-morphisms in K. For instance for a smooth ∞-groupoid we could test with each Cartesian space $U = \mathbb{R}^n$ and find the ∞-groupoids $K(U)$ of smooth n-parameter families of k-morphisms in K.

This data of U-families arranges itself into a presheaf with values in Kan complexes

$$K : C^{\mathrm{op}} \to \mathrm{KanCplx} \hookrightarrow \mathrm{sSet}\,,$$

hence with values in simplicial sets. This is equivalently a simplicial presheaf of sets. The functor category $[C^{\mathrm{op}}, \mathrm{sSet}]$ on the opposite category of the category of test objects C serves as a model for the ∞-category of ∞-groupoids with C-structure.

While there are no higher morphisms in this functor 1-category that could for instance witness that two ∞-groupoids are not isomorphic, but still equivalent, it turns out that all one needs in order to reconstruct *all* these higher morphisms (up to equivalence!) is just the information of which morphisms of simplicial presheaves would become invertible if we were keeping track of higher morphism. These would-be invertible morphisms are called *weak equivalences* and denoted $K_1 \stackrel{\simeq}{\to} K_2$.

For common choices of C there is a well-understood way to define the weak equivalences $W \subset \mathrm{Mor}[C^{\mathrm{op}}, \mathrm{sSet}]$, and equipped with this information the category of simplicial presheaves becomes a *category with weak equivalences*. There is a well-developed but somewhat intricate theory of how exactly this 1-cagtegorical data models the full higher category of structured groupoids that we are after, but for our purposes here we essentially only need to work inside the category of *fibrant* objects of a model structure on presheaves, which in practice amounts to the fact that we use the following three basic constructions:

1. **∞-anafunctors** –
2. ∞-anafunctor A morphisms $X \to Y$ between ∞-groupoids with C-structure is not just a morphism $X \to Y$ in $[C^{\mathrm{op}}, \mathrm{sSet}]$, but is a span of such ordinary morphisms

$$\begin{array}{ccc} \hat{X} & \longrightarrow & Y \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array} ,$$

   where the left leg is a weak equivalence. This is sometimes called an ∞-*anafunctor* from X to Y.
3. **homotopy pullback** – For $A \to B \stackrel{p}{\leftarrow} C$ a diagram, the ∞-pullback of it is the ordinary pullback in $[C^{\mathrm{op}}, \mathrm{sSet}]$ of a replacement diagram $A \to B \stackrel{\hat{p}}{\leftarrow} \hat{C}$, where $\hat{p}$ is a *good replacement* of p in the sense of the following factorization lemma.
4.

Proposition 1.2.39 (factorization lemma)**.** *For $p : C \to B$ a morphism in $[C^{\mathrm{op}}, \mathrm{sSet}]$, a* good replacement *$\hat{p} : \hat{C} \to B$ is given by the composite vertical morphism in the ordinary pullback diagram*

$$\begin{array}{ccc} \hat{C} & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle p} \\ B^{\Delta[1]} & \longrightarrow & B \\ \downarrow & & \\ B & & \end{array} ,$$

where $B^{\Delta[1]}$ is the path object of B: the presheaf that is over each $U \in C$ the simplicial path space $B(U)^{\Delta[1]}$.

1.2.3.2.2 ∞-Sheaves / ∞-stacks In particular, there is a notion of ∞-presheaves on a category (or ∞-category) C: ∞-functors

$$X : C^{\mathrm{op}} \to \infty\mathrm{Grpd}$$

to the ∞-category $\infty\mathrm{Grpd}$ of ∞-groupoids – there is an ∞-Yoneda embedding, and so on. Accordingly, ∞-topos theory proceeds in its basic notions along the same lines as we sketched above for topos theory:

an ∞-topos of ∞-sheaves is defined to be a reflective sub-∞-category

$$\mathrm{Sh}_{(\infty,1)}(C) \underset{\hookrightarrow}{\overset{L}{\leftarrow}} \mathrm{PSh}_{(\infty,1)}(C)$$

of an ∞-category of ∞-presheaves. As before, such is essentially determined by and determines a Grothendieck topology or coverage on C. Since a 2-sheaf with values in groupoids is usually called a *stack*, an ∞-sheaf is often also called an ∞*-stack*.

In the spirit of the above discussion, the objects of the ∞-topos of ∞-sheaves on $C = \mathrm{CartSp}$ we shall think of as *smooth* ∞*-groupoids*. This is our main running example. We shall write $\mathrm{Smooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ for the ∞-topos of smooth ∞-groupoids.

Let

- $C :=$ SmthMfd be the category of all smooth manifolds (or some other site, here assumed to have enough points);
- $\mathrm{gSh}(C)$ be the category of groupoid-valued sheaves over C,
  for instance $X = \{\, X \rightrightarrows X \,\}, \mathbf{B}G = \{\, G \rightrightarrows * \,\} \in \mathrm{gSh}(C)$;
- $\mathrm{Ho}_{\mathrm{gSh}(C)}$ the *homotopy category* obtained by universally turning the *stalkwise groupoid-equivalences* into isomorphisms.

Fact: $\qquad H^1(X, G) \simeq \mathrm{Ho}_{\mathrm{gSh}(C)}(X, \mathbf{B}G)$.

- $\mathrm{sSet}(C)_{\mathrm{lfib}} \hookrightarrow \mathrm{Sh}(C, \mathrm{sSet})$ be the stalkwise Kan simplicial sheaves;
- $L_W \mathrm{sSh}(C)_{\mathrm{lfib}}$ the *simplicial localization* obtained by universally turning *stalkwise homotopy equivalences* into *homotopy equivalences.*

Definition/Theorem. This is the ∞-category theory analog of the sheaf topos over C, the ∞*-stack* ∞*-topos*: $\qquad \mathbf{H} := \mathrm{Sh}_\infty(C) \simeq L_W \mathrm{sSh}(C)_{\mathrm{lfib}}$.
Example. $\mathrm{Smooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{SmthMfd})$ is the ∞-topos of *smooth* ∞*-groupoids.*
Proposition. Every object in $\mathrm{Smooth}\infty\mathrm{Grpd}$ is presented by a simplicial manifold, but not necessarily by a *locally Kan* simplicial manifold (see below).

But a crucial point of developing our theory in the language of ∞-toposes is that all constructions work in great generality. By simply passing to another site C, all constructions apply to the theory of generalized spaces modeled on the test objects in C. Indeed, to really capture all aspects of ∞-Lie theory, we should and will adjoin to our running example $C = \mathrm{CartSp}$ that of the slightly larger site $C = \mathrm{CartSp}_{\mathrm{synthdiff}}$ of infinitesimally thickened Cartesian spaces. Ordinary sheaves on this site are the generalized spaces considered in *synthetic differential geometry*: these are smooth spaces such as smooth loci that may have infinitesimal extension. For instance the first order jet $D \subset \mathbb{R}$ of the origin in the real line exists as an infinitesimal space in $\mathrm{Sh}(\mathrm{CartSp}_{\mathrm{synthdiff}})$. Accordingly, ∞-groupoids modeled on $\mathrm{CartSp}_{\mathrm{synthdiff}}$ are smooth ∞-groupoids that may have k-morphisms of infinitesimal extension. We will see that a smooth ∞-groupoid all whose morphisms has infinitesimal extension is a Lie algebra or Lie algebroid or generally an ∞-Lie algebroid.

While ∞-category theory provides a good abstract definition and theory of ∞-groupoids modeled on test objects in a category C in terms of the ∞-category of ∞-sheaves on C, for concrete manipulations it is often useful to have a presentation of the ∞-categories in question in terms of generators and relations in ordinary category theory. Such a generators-and-relations-presentation is provided by the notion of a *model category* structure. Specifically, the ∞-toposes of ∞-presheaves that we are concerned with are presented in this way by a model structure on simplicial presheaves, i.e. on the functor category $[C^{\mathrm{op}}, \mathrm{sSet}]$ from C to the category sSet of simplicial sets. In terms of this model, the corresponding ∞-category of ∞-sheaves is given by another model structure on $[C^{\mathrm{op}}, \mathrm{sSet}]$, called the *left Bousfield localization* at the set of covers in C.

These models for ∞-stack ∞-toposes have been proposed, known and studied since the 1970s and are therefore quite well understood. The full description and proof of their abstract role in ∞-category theory was established in [LuHTT].

As before for toposes, there is an archetypical ∞-topos, which is $\infty\mathrm{Grpd} = \mathrm{Sh}_{(\infty,1)}(*)$ itself: the collection of generalized ∞-groupoids that are modeled on the point. All we know about these generalized spaces is how to map a point into them and what the homotopies and higher homotopies of such maps are, but no further extra structure. So these are bare ∞-groupoids without extra structure. Also as before, every ∞-topos comes with an essentially unique geometric morphism to this archetypical ∞-topos given by a pair of adjoint ∞-functors

$$\mathrm{Sh}_{(\infty,1)}(C) \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \infty\mathrm{Grpd} \ .$$

Again, if C happens to have a terminal object $*$, then Γ is the operation that evaluates an ∞-sheaf on the point: it produces the bare ∞-groupoid underlying an ∞-groupoid modeled on C. For instance for $C = \mathrm{CartSp}$ a smooth ∞-groupoid $X \in \mathrm{Sh}_{(\infty,1)}(C)$ is sent by Γ to to the underlying ∞-groupoid that forgets the smooth structure on X.

Moreover, still in direct analogy to the 1-categorical case above, the left adjoint Disc is the ∞-functor that sends a bare ∞-groupoid S to the ∞-stackification $\mathrm{Disc}S = L\mathrm{Const}S$ of the constant ∞-presheaf on S. This models the discretely structured ∞-groupoid on S. For instance for $C = \mathrm{CartSp}$ we have that $\mathrm{Disc}S$ is a smooth ∞-groupoid with discrete smooth structure: all smooth families of points in it are actually constant.

1.2.3.2.3 Structured ∞-Groups It is clear that we may speak of *group objects* in any topos, (or generally in any category with finite products): objects G equipped with a multiplication $G \times G \to G$ and a neutral element $* \to G$ such that the multiplication is unital, associative and has inverses for each element. In a sheaf topos, such a G is equivalently a *sheaf of groups.* For instance every Lie group canonically becomes a group object in Sh(CartSp).

As we pass to an ∞-topos the situation is essentially the same, only that the associativity condition is replaced by *associativity up to coherent homotopy* (also called: up to *strong homotopy*), and similarly for the unitalness and the existence of inverses. One way to formalize this is to say that a group object in an ∞-topos $\mathbf{H}$ is an A_∞-algebra object G such that its 0-truncation $\tau_0 G$ is a group object in the underlying 1-topos. (This is discussed in [Lurie11].)

For instance in the ∞-topos over CartSp a Lie group still naturally is a group object, but also a *Lie 2-group* or *differentiable group stack* is. Moreover, every sheaf of *simplicial groups* presents a group object in the ∞-topos, and we will see that all group objects

A group object in $\infty\mathrm{Grpd} \simeq \mathrm{Top}$ we will for emphasis call an *∞-group.* In this vein a group object in an ∞-topos over a non-trivial site is a *structured ∞-group* (for instance a topological ∞-group or a smooth ∞-group).

A classical source of ∞-groups are *loop spaces*, where the group multiplication is given by concatenation of based loops in a given space, the homotopy-coherent associativity is given by reparameterizations of concatenations of loops, and inverses are given by reversing the parameterization of a loop. A classical result of Milnor says, in this language, that every ∞-group arises as a loop space this way. This statement generalizes from discrete ∞-groups (group objects in $\infty\mathrm{Grpd} \simeq \mathrm{Top}$) to structured ∞-groups.

Theorem. (Milnor-Lurie) There is an equivalence

$$\{ \text{ groups in } \mathbf{H} \ \} \underset{\text{delooping } \mathbf{B}}{\overset{\text{looping } \Omega}{\underset{\simeq}{\rightleftarrows}}} \left\{ \begin{array}{c} \text{pointed connected} \\ \text{objects in } \mathbf{H} \end{array} \right\}$$

This equivalence is a most convenient tool. In the following we will almost exclusively handle ∞-groups G in terms of their pointed connected delooping objects $\mathbf{B}G$. We discuss this in more detail below in 3.6.8. This is all the more useful as the objects $\mathbf{B}G$ happen to be the *moduli ∞-stacks* of *G-principal ∞-bundles.* We come to this in 1.2.5.5.

1.2.3.3 Cohomology Where the archetypical topos is the category Set, the archetypical ∞-topos is the ∞-category ∞Grpd of ∞-groupoids. This, in turn, is equivalent by a classical result (see 4.1) to Top, the category of topological spaces, regarded as an ∞-category by taking the 2-morphisms to be homotopies between continuous maps, 3-morphisms to be homotopies of homotopy, and so forth:

$$\infty\mathrm{Grpd} \simeq \mathrm{Top}\,.$$

In Top it is familiar – from the notion of *classifying spaces* and from the *Brown representability theorem* (the reader in need of a review of such matter might try [May]) – that the cohomology of a topological space X may be identified as the set of homotopy classes of continuous maps from X to some coefficient space A

$$H(X,A) := \pi_0 \mathrm{Top}(X,A)\,.$$

For instance for $A = K(n,\mathbb{Z}) \simeq B^n\mathbb{Z}$ the topological space called the nth *Eilenberg-MacLane space* of the additive group of integers, we have that

$$H(X,A) := \pi_0 \mathrm{Top}(X, B^n\mathbb{Z}) \simeq H^n(X,\mathbb{Z})$$

is the ordinary integral (singular) cohomology of X. Also *nonabelian cohomology* is famously exhibited this way: for G a (possibly nonabelian) topological group and $A = BG$ its classifying space (we discuss this construction and its generalization in detail in 4.3.4.1) we have that

$$H(X,A) := \pi_0 \mathrm{Top}(X, BG) \simeq H^1(X,G)$$

is the degree-1 nonabelian cohomology of X with coeffients in G, which classifies *G-principal bundles* over X (more on that in a moment).

Since this only involves forming ∞-categorical hom-spaces and since this is an entirely categorical operation, it makes sense to *define* for X, A any two objects in an arbitrary ∞-topos $\mathbf{H}$ the intrinsic cohomology of X with coefficients in A to be

$$H(X,A) := \pi_0 \mathbf{H}(X,A)\,,$$

where $\mathbf{H}(X,A)$ denotes the ∞-groupoid of morphism from X to A in $\mathbf{H}$. This general identification of cohomology with hom-spaces in ∞-toposes is central to our developments here. We indicate now two classes of justification for this definition.

1. Essentially every notion of cohomology already considered in the literature is an example of this definition. Moreover, those that are not are often improved on by fixing them to become an example.

2. The use of a good notion of G-cohomology on X should be that it *classifies* “G-structures over X” and exhibits the *obstruction theory* for extensions or lifts of such structures. We find that it is precisely the context of an ambient ∞-topos (precisely: the ∞-Giraud axioms that characterize an ∞-topos) that makes such a classification and obstruction theory work.

1.2.3.3.1 Equivariant structured nonabelian twisted generalized cohomology We discuss a list of examples of ∞-toposes $\mathbf{H}$ together with notions of cohomology whose cocycles are given by morphisms $c \in \mathbf{H}(X,A)$ between a domain object X and coefficient object A in this ∞-topos. Some of these examples are evident and classical, modulo our emphasis on the ∞-topos theoretic perspective, others are original. Even those cases that are classical receive new information from the ∞-topos theoetic perspective.

Details are below in the relevant parts of 4 and 5.

In view of the unification that we discuss, some of the traditional names for notions of cohomology are a bit suboptimal. For instance the term *generalized cohomology* for theories satisfying the Eilenberg-Steenrod axioms does not well reflect that it is a generalization of ordinary cohomology of topological spaces (only) which is, in a quite precise sense, *orthogonal* to the generalizations given by passage to sheaf cohomology or to nonabelian cohomology, all of which are subsumed by cohomology in an ∞-topos. In order to usefully distinguish the crucial aspects here we will use the following terminology

- We speak of *structured cohomology* to indicate that a given notion is realized in an ∞-topos other than the archetypical $\infty\mathrm{Grpd} \simeq \mathrm{Top}$ (which representes "discrete structre" in the precise sense discussed in 4.1). Hence traditional sheaf cohomology is "structured" in this sense, while ordinary cohomology and Eilenberg-Steenrod cohomology is "unstructured".

- We speak of *nonabelian cohomology* when coefficient objects are not *required* to be abelian (groups) or stable (spectra), but may generally be deloopings $A := \mathbf{B}G$ of arbitrary (structred) ∞-groups G.

  More properly this might be called *not-necessarily abelian cohomology*, but following common practice (as in "noncommutative geometry") we stick with the slightly imprecise but shorter term. One point that we will dwell on (see the discussion of examples in 5.4) is that the traditional notion of *twisted cohomology* (already twisted abelian cohomology) is naturally a special case of nonabelian cohomology.

Notice that the "generalized" in "generalized cohomology" of Eilenberg-Steenrod type refers to allowing coefficient objects which are abelian ∞-groups more general than Eilenberg-MacLane objects. Hence this is in particular subsumed in *nonabelian cohomology.*

In this terminology, the notion of cohomology in ∞-toposes that we are concerned with here is *structured nonabelian/twisted generalized cohomology.*

Finally, not only is it natural to allow the coefficient objects A to be general objects in a general ∞-topos, but also there is no reason to restrict the nature of the domain objects X. For instance traditional sheaf cohomology always takes X, in our language, to be the *terminal object* $X = *$ of the ambient ∞-topos. This is also called the *(-2)-truncated object* (see 3.6.2 below) of the ∞-topos, being the unique member of the lowest class in a hierarchy of *n-truncated objects* for $(-2) \leq n \leq \infty$. As we increase n here, we find that the domain object is generalized to

- $n = -1$: subspaces of X;
- $n = 0$: étale spaces over X;
- $n = 1$: orbifolds / orbispaces / groupoids over X;
- $n \geq 2$: higher orbifolds / orbispaces / groupoids

One finds then that cohomology of an n-truncated object for $n \geq 1$ reproduces the traditional notion of *equivariant cohomology.* In particular this subsumes *group cohomology*: ordinary group cohomology in the unstructured case (in $\mathbf{H} = \infty\mathrm{Grpd}$) and generally structured group cohomology such as *Lie group cohomology.*

Therefore, strictly speaking, we are here concerned with *equivariant structured nonabelian/twisted generalized cohomology.* All this is neatly encapsulated by just the fundamental notion of hom-spaces in ∞-toposes.

Cochain cohomology

The origin and maybe the most elementary notion of cohomology is that appearing in *homological algebra*: given a *cochain complex* of abelian groups

$$V^\bullet = \left[\cdots \xleftarrow{d^2} V_2 \xleftarrow{d^1} V_0 \xleftarrow{d^0} V_0 \right],$$

its cohomology group in degree n is defined to be the quotient group

$$H^n(V) := \ker(d^n)/\mathrm{im}(d^{n-1}).$$

To see how this is a special case of cohomology in an ∞-topos, consider a fixed abelian group A and suppose that this cochain complex is the A-dual of a *chain* complex

$$V_\bullet = \left[\cdots \longrightarrow V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \right],$$

in that $V^\bullet = \mathrm{Hom}_{\mathrm{Ab}}(V_\bullet, A)$. For instance if $A = \mathbb{Z}$ and V_n is the free abelian group on the set of n-simplices in some topological space, then V^n is the group of *singular n-cochains* on X.

Write then $A[n]$ (or $A[-n]$, if preferred) for the chain complex concentrated in degree n on A. In terms of this

1. morphisms of chain complexes $c : V_\bullet \to A[n]$ are in natural bijection with *closed* elements in V^n, hence with $\ker(d^n)$;

2. chain homotopies $\eta : c_1 \to c_2$ between two such chain morphisms are in natural bijection with elements in $\mathrm{im}(d^{n-1})$.

This way the cohomology group $H^n(V^\bullet)$ is naturally identified with the *homotopy classes* of maps $V_\bullet \to A[n]$.

Consider then again an example as that of singular cochains as above, where $V_\bullet$ is degreewise a free abelian group in a simplicial set X. Then this cohomology is the group of connected components of a hom-space in an ∞-topos. To see this, one observes that the category of chain complexes $\mathrm{Ch}_\bullet$ is but a convenient presentation for the category of ∞-groupoids that are equipped with *strict abelian group structure* in their incarnation as Kan complexes: simplicial abelian groups. This equivalence $\mathrm{Ch}_\bullet \simeq \mathrm{sAb}$ is known as the *Dold-Kan correspondence*, to be discussed in more detail in 2.2.6. We write $\Xi(V_\bullet)$ for the Kan complex corresponding to a chain complex under this equivalence. Moreover, for chain complexes of the form $A[n]$ we write

$$\mathbf{B}^n A := \Xi(A[n])\,.$$

With this notation, the ∞-groupoid of chain maps $V_\bullet \to A[n]$ is equivalently that of ∞-functors $X \to \mathbf{B}^n A$ and hence the cochain cohomology of $V^\bullet$ is

$$H^n(V^\bullet) \simeq \pi_0 \mathbf{H}(X, \mathbf{B}^n A)\,.$$

Lie group cohomology

There are some definitions in the literature of cohomology theories that are not special cases of this general concept, but in these cases it seems that the failure is with the traditional definition, not with the above notion. We will be interested in particular in the group cohomology of Lie groups. Originally this was defined using a naive direct generalization of the formula for bare group cohomology as

$$H^n_{\mathrm{naive}}(G, A) = \{\text{smooth maps } G^{\times n} \to A\}/\sim\,.$$

But this definition was eventually found to be too coarse: there are structures that ought to be cocycles on Lie groups but do not show up in this definition. Graeme Segal therefore proposed a refined definition that was later rediscovered by Jean-Luc Brylinski, called *differentiable Lie group cohomology* $H^n_{\mathrm{diffbl}}(G, A)$. This refines the naive Lie group cohomology in that there is a natural morphism $H^n_{\mathrm{naive}}(G, A) \to H^n_{\mathrm{diffbl}}(G, A)$.

But in the ∞-topos of smooth ∞-groupoids $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{CartSp})$ we have the natural intrinsic definition of Lie group cohomology as

$$H^n_{\mathrm{Smooth}}(G, A) := \pi_0 \mathbf{H}(\mathbf{B}G, \mathbf{B}^n A)$$

and one finds that this naturally includes the Segal/Brylinski definition

$$H^n_{\mathrm{naive}}(G, A) \to H^n_{\mathrm{diffrbl}}(G, A) \to H^n_{\mathrm{Smooth}}(G, A) := \pi_0 \mathbf{H}(\mathbf{B}G, \mathbf{B}^n A)\,.$$

and at least for A a discrete group, or the group of real numbers or a quotient of these such as $U(1) = \mathbb{R}/\mathbb{Z}$, the notions coincide

$$H^n_{\mathrm{diffrbl}}(G, A) \simeq H^n_{\mathrm{Smooth}}(G, A)\,.$$

Details on this discussion about refined Lie group cohomology are below in 4.4.6.2.

For instance one of the crucial aspects of the notion of cohomology is that a cohomology class on X *classifies* certain structures over X.

It is a classical fact that if G is a (discrete) group and BG its delooping in Top, then the structure classified by a cocycle $g : X \to BG$ is the G-principal bundle over X obtained as the 1-categorical pullback $P \to X$

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & BG \end{array}$$

of the universal G-principal bundle $EG \to BG$. But one finds that this pullback construction is just a 1-categorical *model* for what intrinsically is something simpler: this is just the *homotopy pullback* in Top of the point

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & \swarrow_{\simeq} & \downarrow \\ X & \xrightarrow[g]{} & BG \end{array}$$

This form of the construction of the G-principal bundle classified by a cocycle makes sense in any ∞-topos $\mathbf{H}$:

we shall say that for $G \in \mathbf{H}$ a group object in $\mathbf{H}$ and $\mathbf{B}G$ its delooping and for $g : X \to \mathbf{B}G$ a cocycle (any morphism in $\mathbf{H}$) that the *G-principal ∞-bundle* classified by g is the ∞-pullback/homotopy pullback

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & \swarrow_{\simeq} & \downarrow \\ X & \xrightarrow[g]{} & BG \end{array}$$

in $\mathbf{H}$. (Beware that usually we will notationally suppress the homotopy filling this square diagram.)

Let G be a Lie group and X a smooth manifold, both regarded naturally as objects in the ∞-topos of smooth ∞-groupoids. Let $g : X \to \mathbf{B}G$ be a morphism in $\mathbf{H}$. One finds that in terms of the presentation of Smooth∞Grpd by the model structure on simplicial presheaves this is a Čech 1-cocycle on X with values in G. The corresponding ∞-pullback P is (up to equivalence or course) the smooth G-principal bundle classified in the usual sense by this cocycle.

The analogous proposition holds for G a Lie 2-group and P a G-principal 2-bundle.

Generally, we can give a natural definition of G-principal ∞-bundle in any ∞-topos $\mathbf{H}$ over any ∞-group object $G \in \mathbf{H}$. One finds that it is the Giraud axioms that characterize ∞-toposes that ensure that these are equivalently classified as the ∞-bullbacks of morphisms $g : X \to \mathbf{B}G$. Therefore the intrinsic cohomology

$$H(X, G) := \pi_0 \mathbf{H}(X, BG)$$

in $\mathbf{H}$ classifies G-principal ∞-bundles over X. Notice that X here may itself be any object in $\mathbf{H}$.

1.2.3.4 Homotopy Every ∞-sheaf ∞-topos $\mathbf{H}$ canonically comes equipped with a geometric morphism given by pair of adjoint ∞-functors

$$(L\mathrm{Const} \dashv \Gamma) : \mathbf{H} \underset{\Gamma}{\overset{L\mathrm{Const}}{\rightleftarrows}} \infty\mathrm{Grpd}$$

relating it to the archeytpical ∞-topos of ∞-groupoids. Here Γ produces the global sections of an ∞-sheaf and LConst produces the constant ∞-sheaf on a given ∞-groupoid.

In the cases that we are interested in here $\mathbf{H}$ is a big topos of ∞-groupoids equipped with cohesive structure, notably equipped with smooth structure in our motivating example. In this case Γ has the interpretation of sending a cohesive ∞-groupoid $X \in \mathbf{H}$ to its underlying ∞-groupoid, after forgetting the cohesive structure, and LConst has the interpretation of forming ∞-groupoids equipped with discrete cohesive structure. We shall write Disc $:= L$Const to indicate this.

But in these cases of cohesive ∞-toposes there are actually more adjoints to these two functors, and this will be essentially the general abstract definition of cohesiveness. In particular there is a further left adjoint

$$\Pi : \mathbf{H} \to \infty\mathrm{Grpd}$$

to Disc: the *fundamental ∞-groupoid functor on a locally ∞-connected ∞-topos.* Following the standard terminology of *locally connected toposes* in ordinary topos theory we shall say that $\mathbf{H}$ with such a property is a *locally ∞-connected ∞-topos.* This terminology reflects the fact that if X is a locally contractible topological space then $\mathbf{H} = \mathrm{Sh}_\infty(X)$ is a locally contractible ∞-topos. A classical result of Artin-Mazur implies, that in this case the value of Π on $X \in \mathrm{Sh}_\infty(X)$ is, up to equivalence, the *fundamental ∞-groupoid of X*:

$$\Pi : (X \in \mathrm{Sh}_\infty(X)) \mapsto (\mathrm{Sing}X \in \infty\mathrm{Grpd}),$$

which is the ∞-groupoid whose

- objects are the points of X;
- morphisms are the (continuous) paths in X;
- 2-morphisms are the continuous homotopies between such paths;
- k-morphisms are the higher order homotopies between $(k-1)$-dimensional paths.

This is the object that encodes all the homotopy groups of X in a canonical fashion, without choice of fixed base point.

Also the big ∞-topos Smooth∞Grpd $= \mathrm{Sh}_\infty(\mathrm{CartSp})$ turns out to be locally ∞-connected

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma) : \ \mathrm{Smooth}\infty\mathrm{Grpd} \underset{\Gamma}{\overset{\Pi}{\underset{\xleftarrow{\mathrm{Disc}}}{\rightarrow}}} \infty\mathrm{Grpd}$$

as a reflection of the fact that every Cartesian space $\mathbb{R}^n \in \mathrm{CartSp}$ is contractible as a topological space. We find that for X any paracompact smooth manifold, regarded as an object of Smooth∞Grpd, again $\Pi(X) \in$ Smooth∞Grpd is the corresponding fundamental ∞-groupoid. More in detail, under the *homotopy-hypothesis*-equivalence $(|-| \dashv \mathrm{Sing}) : \ \mathrm{Top} \underset{\mathrm{Sing}}{\overset{|-|}{\underset{\to}{\overset{\leftarrow}{\simeq}}}} \infty\mathrm{Grpd}$ we have that the composite

$$|\Pi(-)| : \mathbf{H} \overset{\Pi}{\to} \infty\mathrm{Grpd} \overset{|-|}{\to} \mathrm{Top}$$

sends a smooth manifold X to its homotopy type: the underlying topological space of X, up to weak homotopy equivalence.

Analogously, for a general object $X \in \mathbf{H}$ we may think of $|\Pi(X)|$ as the generalized geometric realization in Top. For instance we find that if $X \in$ Smooth∞Grpd is presented by a simplicial paracompact manifold, then $|\Pi(X)|$ is the ordinary geometric realization of the underlying simplicial topological space of X. This means in particular that for $X \in$ Smooth∞Grpd a Lie groupoid, $\Pi(X)$ computes its *homotopy groups of a Lie groupoid* as traditionally defined.

The ordinary homotopy groups of $\Pi(X)$ or equivalently of $|\Pi(X)|$ we call the *geometric homotopy groups* of $X \in \mathbf{H}$, because these are based on a notion of homotopy induced by an intrisic notion of geometric paths in objects in X. This is to be contrasted with the *categorical homotopy groups* of X. These are the homotopy groups of the underlying ∞-groupoid $\Gamma(X)$ of X. For instance for X a smooth manifold we have that

$$\pi_n(\Gamma(X)) \simeq \begin{cases} X \in \mathrm{Set} & |n = 0 \\ 0 & |n > 0 \end{cases}$$

but

$$\pi_n(\Pi(X)) \simeq \pi_n(X \in \mathrm{Top})\,.$$

This allows us to give a precise sense to what it means to have a *cohesive refinement* (continuous refinement, smooth refinement, etc.) of an object in Top. Notably we are interested in smooth refinements of classifying spaces $BG \in \mathrm{Top}$ for topological groups G by deloopings $\mathbf{B}G \in \mathrm{Smooth}\infty\mathrm{Grpd}$ of ∞-Lie groups G and we may interpret this as saying that

$$\Pi(\mathbf{B}G) \simeq BG$$

in $\mathrm{Top} \simeq \mathrm{Smooth}\infty\mathrm{Grpd}$.

1.2.4 Groups

(...)

1.2.5 Principal bundles

The following is an exposition of the notion of *principal bundles* in higher but low degree.

We assume here that the reader has a working knowledge of groupoids and at least a rough idea of 2-groupoids. For introductions see for instance [BrHiSi11] [Por]

Below in 1.2.5.4 a discussion of the formalization of ∞-groupoids in terms of Kan complexes is given and is used to present a systematic way to understand these constructions in all degrees.

1.2.5.1 Principal 1-bundles Let G be a Lie group and X a smooth manifold (all our smooth manifolds are assumed to be finite dimensional and paracompact). We give a discussion of smooth G-principal bundles on X in a manner that paves the way to a straightforward generalization to a description of principal ∞-bundles. From X and G are naturally induced certain Lie groupoids.

From the group G we canonically obtain a groupoid that we write BG and call the *delooping groupoid* of G. Formally this groupoid is

$$BG = (\, G \rightrightarrows * \,)$$

with composition induced from the product in G. A useful depiction of this groupoid is

$$BG = \left\{ \begin{array}{ccc} & * & \\ {}^{g_1}\nearrow & \| & \searrow^{g_2} \\ * & \xrightarrow{g_2 \cdot g_1} & * \end{array} \right\},$$

where the $g_i \in G$ are elements in the group, and the bottom morphism is labeled by forming the product in the group. (The order of the factors here is a convention whose choice, once and for all, does not matter up to equivalence.)

But we get a bit more, even. Since G is a Lie group, there is smooth structure on BG that makes it a Lie groupoid, an internal groupoid in the category SmoothMfd of smooth manifolds: its collection of objects (trivially) and of morphisms each form a smooth manifold, and all structure maps (source, target, identity, composition) are smooth functions. We shall write

$$\mathbf{B}G \in \mathrm{LieGrpd}$$

for BG regarded as equipped with this smooth structure. Here and in the following the boldface is to indicate that we have an object equipped with a bit more structure - here: smooth structure - than present on the object denoted by the same symbols, but without the boldface. Eventually we will make this precise by having the boldface symbols denote objects in the ∞-topos $\mathrm{Smooth}\infty\mathrm{Grpd}$ which are taken by a suitable functor to objects in $\infty\mathrm{Grpd}$ denoted by the corresponding non-boldface symbols.

Also the smooth manifold X may be regarded as a Lie groupoid - a groupoid with only identity morphisms. Its depiction is simply

$$X = \{ \; x \xrightarrow{\mathrm{Id}} x \; \}$$

for all $x \in X$ But there are other groupoids associated with X: let $\{U_i \to X\}_{i \in I}$ be an open cover of X. To this is canonically associated the Čech-groupoid $C(\{U_i\})$. Formally we may write this groupoid as

$$C(\{U_i\}) = \left\{ \; \coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i \; \right\} .$$

A useful depiction of this groupoid is

$$C(\{U_i\}) = \left\{ \begin{array}{ccccc} & & (x,j) & & \\ & \nearrow & & \searrow & \\ (x,i) & & \xrightarrow{\quad \| \quad} & & (x,k) \end{array} \right\} ,$$

This indicates that the objects of this groupoid are pairs (x,i) consisting of a point $x \in X$ and a patch $U_i \subset X$ that contains x, and a morphism is a triple (x,i,j) consisting of a point and two patches, that both contain the point, in that $x \in U_i \cap U_j$. The triangle in the above depiction symbolizes the evident way in which these morphisms compose. All this inherits a smooth structure from the fact that the U_i are smooth manifolds and the inclusions $U_i \hookrightarrow X$ are smooth functions. Hence also $C(\{U_i\})$ becomes a Lie groupoid.

There is a canonical projection functor

$$C(\{U_i\}) \to X : (x,i) \mapsto x .$$

This functor is an internal functor in SmoothMfd and moreover it is evidently essentially surjective and full and faithful. However, while essential surjectivity and full-and-faithfulness implies that the underlying bare functor has a homotopy-inverse, that homotopy-inverse never has itself smooth component maps, unless X itself is a Cartesian space and the chosen cover is trivial.

We do however want to think of $C(\{U_i\})$ as being equivalent to X even as a Lie groupoid. One says that a smooth functor whose underlying bare functor is an equivalence of groupoids is a *weak equivalence* of Lie groupoids, which we write as $C(\{U_i\}) \stackrel{\simeq}{\to} X$. Moreover, we shall think of $C(\{U_i\})$ as a *good* equivalent replacement of X if it comes from a cover that is in fact a *good open cover* in that all its non-empty finite intersections $U_{i_0,\cdots,i_n} := U_{i_0} \cap \cdots \cap U_{i_n}$ are diffeomorphic to the Cartesian space $\mathbb{R}^{\dim X}$.

We shall discuss later in which precise sense this condition makes $C(\{U_i\})$ *good* in the sense that smooth functors out of $C(\{U_i\})$ model the correct notion of morphism out of X in the context of smooth groupoids (namely it will mean that $C(\{U_i\})$ is cofibrant in a suitable model category structure on the category of Lie groupoids). The formalization of this statement is what ∞-topos theory is all about, to which we will come. For the moment we shall be content with accepting this as an ad hoc statement.

Observe that a functor

$$g : C(\{U_i\}) \to \mathbf{B}G$$

is given in components precisely by a collection of smooth functions

$$\{g_{ij} : U_{ij} \to G\}_{i,j \in I}$$

such that on each $U_i \cap U_j \cap U_k$ the equality $g_{jk} g_{ij} = g_{ik}$ of functions holds.

It is well known that such collections of functions characterize G-principal bundles on X. While this is a classical fact, we shall now describe a way to derive it that is true to the Lie-groupoid-context and that will make clear how smooth principal ∞-bundles work.

First observe that in total we have discussed so far spans of smooth functors of the form

$$\begin{array}{ccc} C(\{U_i\}) & \xrightarrow{g} & \mathbf{B}G \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array} .$$

Such spans of functors, whose left leg is a weak equivalence, are sometimes known, essentially equivalently, as *Morita morphisms*, as *generalized morphisms* of Lie groupoids, as *Hilsum-Skandalis morphisms*, or as *groupoid bibundles* or as *anafunctors*. We are to think of these as concrete models for more intrinsically defined direct morphisms $X \to \mathbf{B}G$ in the ∞-topos of smooth ∞-groupoids.

Now consider yet another Lie groupoid canonically associated with G: we shall write $\mathbf{E}G$ for the groupoid – the *smooth universal G-bundle* – whose formal description is

$$\mathbf{E}G = \left(G \times G \underset{p_1}{\overset{(-)\cdot(-)}{\rightrightarrows}} G \right)$$

with the evident composition operation. The depiction of this groupoid is

$$\left\{ \begin{array}{ccccc} & & g_2 & & \\ & {\scriptstyle g_2 g_1^{-1}} \nearrow & \| & \searrow {\scriptstyle g_3 g_2^{-1}} & \\ g_1 & & \xrightarrow[g_3 g_1^{-1}]{} & & g_3 \end{array} \right\} ,$$

This again inherits an evident smooth structure from the smooth structure of G and hence becomes a Lie groupoid.

There is an evident forgetful functor

$$\mathbf{E}G \to \mathbf{B}G$$

which sends

$$(g_1 \to g_2) \mapsto (\bullet \overset{g_2 g_1^{-1}}{\to} \bullet) .$$

Consider then the pullback diagram

$$\begin{array}{ccc} \tilde{P} & \longrightarrow & \mathbf{E}G \\ \downarrow & & \downarrow \\ C(\{U_i\}) & \xrightarrow{g} & \mathbf{B}G \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array}$$

in the category Grpd(SmoothMfd). The object $\tilde{P}$ is the Lie groupoid whose depiction is

$$\tilde{P} = \left\{ \; (x, i, g_1) \longrightarrow (x, j, g_2 = g_{ij}(x) g_1) \; \right\} ;$$

where there is a unique morphism as indicated, whenever the group labels match as indicated. Due to this uniqueness, this Lie groupoid is weakly equivalent to one that comes just from a manifold P (it is 0-truncated)

$$\tilde{P} \overset{\simeq}{\to} P .$$

This P is traditionally written as

$$P = \left(\coprod_i U_i \times G \right) / \sim \,,$$

where the equivalence relation is precisely that exhibited by the morphisms in $\tilde{P}$. This is the traditional way to construct a G-principal bundle from cocycle functions $\{g_{ij}\}$. We may think of $\tilde{P}$ as *being* P. It is a particular representative of P in the ∞-topos of Lie groupoids.

While it is easy to see in components that the P obtained this way does indeed have a principal G-action on it, for later generalizations it is crucial that we can also recover this in a general abstract way. For notice that there is a canonical action

$$(\mathbf{E}G) \times G \to \mathbf{E}G \,,$$

given by the group action on the space of objects. Then consider the pasting diagram of pullbacks

$$\begin{array}{ccc} \tilde{P} \times G & \longrightarrow & \mathbf{E}G \times G \\ \downarrow & & \downarrow \\ \tilde{P} & \longrightarrow & \mathbf{E}G \\ \downarrow & & \downarrow \\ C(U) & \stackrel{g}{\longrightarrow} & \mathbf{B}G \\ \downarrow{\simeq} & & \\ X & & \end{array} \,.$$

Here the morphism $\tilde{P} \times G \to \tilde{P}$ exhibits the principal G-action of G on $\tilde{P}$.

In summary we find the following

Observation 1.2.40. For $\{U_i \to X\}$ a good open cover, there is an equivalence of categories

$$\mathrm{SmoothFunc}(C(\{U_i\}), \mathbf{B}G) \simeq G\mathrm{Bund}(X)$$

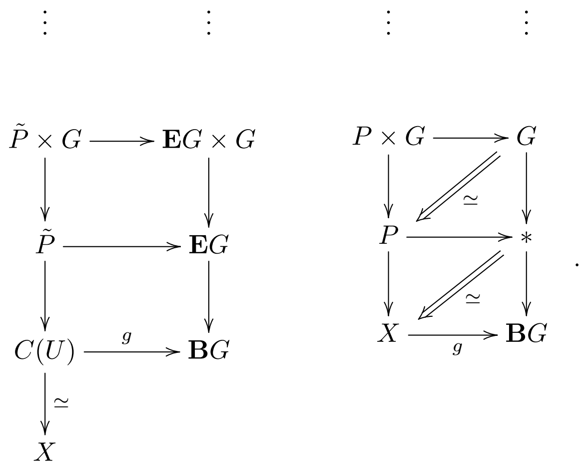
between the functor category of smooth functors and smooth natural transformations, and the groupoid of smooth G-principal bundles on X.

It is no coincidence that this statement looks akin to the maybe more familiar statement which says that equivalence classes of G-principal bundles are classified by homotopy-classes of morphisms of topological spaces

$$\pi_0 \mathrm{Top}(X, BG) \simeq \pi_0 G\mathrm{Bund}(X) \,,$$

where $BG \in \mathrm{Top}$ is the topological classifying space of G. What we are seeing here is a first indication of how cohomology of bare ∞-groupoids is lifted inside a richer ∞-topos to cohomology of ∞-groupoids with extra structure.

In fact, all of the statements that we considered so far becomes conceptually simpler in the ∞-topos. We had already remarked that the anafunctor span $X \stackrel{\simeq}{\leftarrow} C(\{U_i\}) \stackrel{g}{\to} \mathbf{B}G$ is really a model for what is simply a direct morphism $X \to \mathbf{B}G$ in the ∞-topos. But more is true: that pullback of $\mathbf{E}G$ which we considered is just a model for the homotopy pullback of just the *point*



in the model category $\qquad$ in the ∞-topos

The traditional statement which identifies the classifying topological space BG as the quotient of the contractible EG by the free G-action

$$BG \simeq EG/G$$

becomes afte the refinement to smooth groupoids the statement that $\mathbf{B}G$ is the *homotopy quotient* of G acting on the point:

$$\mathbf{B}G \simeq *//G\,.$$

Generally:

Definition 1.2.41. For V a smooth manifold equipped with a smooth action by G (not necessarily free), the *action groupoid* $V//G$ is the Lie groupoid whose space of objects is V, and whose morphisms are group elements that connect two points (which may coincide) in V.

$$V//G = \left\{ v_1 \xrightarrow{g} v_2 \;| v_2 = g(v_1) \right\}\,.$$

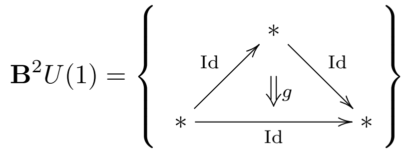
Such an action groupoid is canonically equipped with a morphism to $\mathbf{B}G \simeq *//G$ obtained by sending all objects to the single object and acting as the identity on morphisms. Below in 3.6.13 we discuss that the sequence

$$V \to V//G \to \mathbf{B}G$$

entirely encodes the action of G on V. Also we will see in 5.4.2 that the morphism $V//G \to \mathbf{B}G$ is the smooth refinement of the V-bundle which is *associated to the universal G-bundle* via the given action. If V is a vector space acted on linearly, then this is an associated vector bundle. Its pullbacks along anafunctors $X \to \mathbf{B}G$ yield all V-vector bundles on X.

1.2.5.2 Principal 2-bundles and twisted 1-bundles The discussion above of G-principal bundles was all based on the Lie groupoids $\mathbf{B}G$ and $\mathbf{E}G$ that are canonically induced by a Lie group G. We now discuss the case where G is generalized to a *Lie 2-group*. The above discussion will go through essentially verbatim, only that we pick up 2-morphisms everywhere. This is the first step towards higher Chern-Weil theory. The resulting generalization of the notion of principal bundle is that of *principal 2-bundle*. For historical reasons these are known in the literature often as *gerbes* or as *bundle gerbes*, even though strictly speaking there are some conceptual differences.

Write $U(1) = \mathbb{R}/\mathbb{Z}$ for the circle group. We have already seen above the groupoid $\mathbf{B}U(1)$ obtained from this. But since $U(1)$ is an abelian group this groupoid has the special property that it still has itself the structure of a group object. This makes it what is called a *2-group*. Accordingly, we may form its delooping once more to arrive at a Lie 2-groupoid $\mathbf{B}^2U(1)$. Its depiction is



for $g \in U(1)$. Both horizontal composition as well as vertical composition of the 2-morphisms is given by the product in $U(1)$.

Let again X be a smooth manifold with good open cover $\{U_i \to X\}$. The corresponding Čech groupoid we may also think of as a Lie 2-groupoid,

$$C(U) = \left(\coprod_{i,j,k} U_i \cap U_j \cap U_k \rightrightarrows \coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i \right).$$

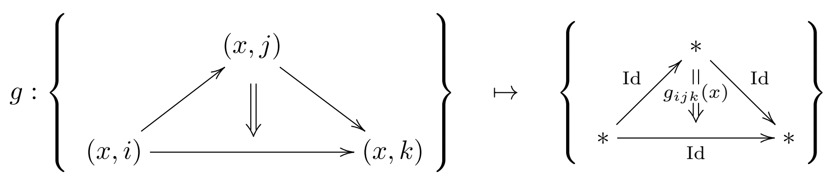
What we see here are the first stages of the full *Čech nerve* of the cover. Eventually we will be looking at this object in its entirety, since for all degrees this is always a *good* replacement of the manifold X, as long as $\{U_i \to X\}$ is a good open cover. So we look now at 2-anafunctors given by spans

$$\begin{array}{ccc} C(\{U_i\}) & \xrightarrow{g} & \mathbf{B}^2U(1) \\ \downarrow \simeq & & \\ X & & \end{array}$$

of internal 2-functors. These will model direct morphisms $X \to \mathbf{B}^2U(1)$ in the ∞-topos. It is straightforward to read off the following

Observation 1.2.42. A smooth 2-functor $g : C(\{U_i\}) \to \mathbf{B}^2U(1)$ is given by the data of a 2-cocycle in the Čech cohomology of X with coefficients in $U(1)$.

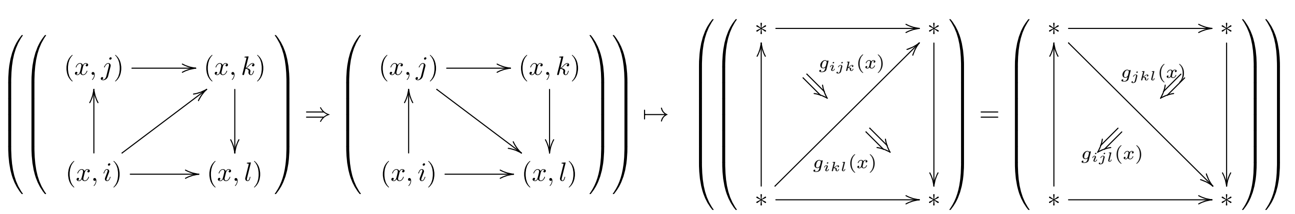
Because on 2-morphisms it specifies an assignment



that is given by a collection of smooth functions

$$(g_{ijk} : U_i \cap U_j \cap U_k \to U(1)).$$

On 3-morphisms it gives a constraint on these functions, since there are only identity 3-morphisms in $\mathbf{B}^2U(1)$:

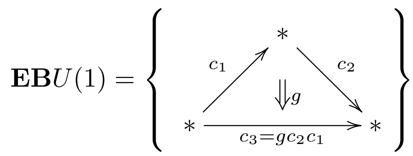


This relation

$$g_{ijk} \cdot g_{ikl} = g_{ijl} \cdot g_{jkl}$$

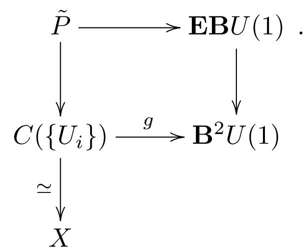
defines degree-2 cocycles in *Čech cohomology* with coefficients in $U(1)$.

In order to find the circle principal 2-bundle classified by such a cocycle by a pullback operation as before, we need to construct the 2-functor $\mathbf{E}\mathbf{B}U(1) \to \mathbf{B}^2U(1)$ that exhibits the universal principal 2-bundle over $U(1)$. The right choice for $\mathbf{E}\mathbf{B}U(1)$ – which we justify systematically in 1.2.5.4 – is indicated by

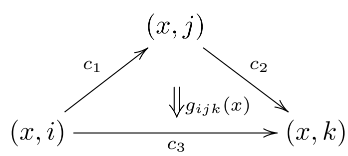


for $c_1, c_2, c_3, g \in U(1)$, where all possible composition operations are given by forming the product of these labels in $U(1)$. The projection $\mathbf{E}\mathbf{B}U(1) \to \mathbf{B}^2U(1)$ is the obvious one that simply forgets the labels c_i of the 1-morphisms and just remembers the labels g of the 2-morphisms.

Definition 1.2.43. With $g : C(\{U_i\}) \to \mathbf{B}^2U(1)$ a Čech cocycle as above, the $U(1)$-*principal 2-bundle* or *circle 2-bundle* that it defines is the pullback



Unwinding what this means, we see that $\tilde{P}$ is the 2-groupoid whose objects are that of $C(\{U_i\})$, whose morphisms are finite sequences of morphisms in $C(\{U_i\})$, each equipped with a label $c \in U(1)$, and whose 2-morphisms are generated from those that look like



subject to the condition that

$$c_1 \cdot c_2 = c_3 \cdot g_{ijk}(x)$$

in $U(1)$. As before for principal 1-bundles P, where we saw that the analogous pullback 1-groupoid $\tilde{P}$ was equivalent to the 0-groupoid P, here we see that this 2-groupoid is equivalent to the 1-groupoid

$$P = \left(C(U)_1 \times U(1) \rightrightarrows C(U) \right)$$

with composition law

$$((x,i) \xrightarrow{c_1} (x,j) \xrightarrow{c_2} (x,k)) = ((x,i) \xrightarrow{(c_1 \cdot c_2 \cdot g_{ijk}(x))} (x,k)) \,.$$

This is a groupoid central extension

$$\mathbf{B}U(1) \to P \to C(\{U_i\}) \simeq X \,.$$

Centrally extended groupoids of this kind are known in the literature as *bundle gerbes* (over the surjective submersion $Y = \coprod_i U_i \to X$). They may equivalently be thought of as given by a line bundle

$$\begin{array}{ccc} L & & \\ \downarrow & & \\ (C(U)_1 = \coprod_{i,j} U_i \cap U_j) & \longrightarrow & (C(U)_0 = \coprod_i U_i) \\ & & \downarrow \\ & & X \end{array}$$

over the space $C(U)_1$ of morphisms, and a line bundle morphism

$$\mu_g : \pi_1^* L \otimes \pi_2^* L \to \pi_1^* L$$

that satisfies an evident associativity law, equivalent to the cocycle codition on g. In summary we find that:

Observation 1.2.44. Bundle gerbes are presentations of Lie groupoids that are total spaces of $\mathbf{B}U(1)$-principal 2-bundles, def. 1.2.43.

Notice that, even though there is a close relation, the notion of *bundle gerbe* is different from the original notion of $U(1)$*-gerbe*. This point we discuss in more detail below in 1.2.56 and more abstractly in 4.3.10.

This discussion of *circle 2-bundles* has a generalization to 2-bundles that are principal over more general *2-groups*.

Definition 1.2.45.
1. A smooth *crossed module* of Lie groups is a pair of homomorphisms $\partial : G_1 \to G_0$ and $\rho : G_0 \to \mathrm{Aut}(G_1)$ of Lie groups, such that for all $g \in G_0$ and $h, h_1, h_2 \in G_1$ we have $\rho(\partial h_1)(h_2) = h_1 h_2 h_1^{-1}$ and $\partial\rho(g)(h) = g\partial(h)g^{-1}$.
2. For $(G_1 \to G_0)$ a smooth crossed module, the corresponding *strict Lie 2-group* is the smooth groupoid $G_0 \times G_1 \rightrightarrows G_0$, whose source map is given by projection on G_0, whose target map is given by applying ∂ to the second factor and then multiplying with the first in G_0, and whose composition is given by multiplying in G_1.

   This groupoid has a strict monoidal structure with strict inverses given by equipping $G_0 \times G_1$ with the semidirect product group structure $G_0 \ltimes G_1$ induced by the action ρ of G_0 on G_1.
3. The corresponding one-object strict smooth 2-groupoid we write $\mathbf{B}(G_1 \to G_0)$. As a simplicial object (under the Duskin nerve of 2-categories) this is of the form

$$\mathbf{B}(G_1 \to G_0) = \mathrm{cosk}_3 \left(G_0^{\times 3} \times G_1^{\times 3} \, \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \, G_0^{\times 2} \times G_1 \rightrightarrows G_0 \longrightarrow * \right) .$$

The infinitesimal analog of a crossed module of groups is a *differential crossed module*.

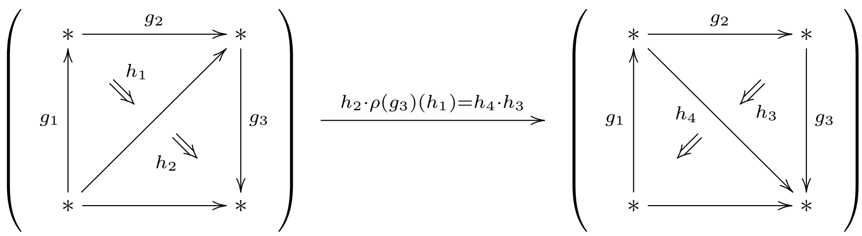
Definition 1.2.46. A *differential crossed module* is a chain complex of vector space of length 2 $V_1 \to V_0$ equipped with the structure of a dg-Lie algebra.

Example 1.2.47. For $G_1 \to G_0$ a smooth crossed module of Lie groups, differentiation of all structure maps yields a corresponding differential crossed module $\mathfrak{g}_1 \to \mathfrak{g}_0$.

Observation 1.2.48. For $G := [G_1 \xrightarrow{\delta} G_0]$ a crossed module, the 2-groupoid delooping a 2-group coming from a crossed module is of the form

$$\mathbf{B}G = \left\{ \begin{array}{ccc} & * & \\ {}^{g_1}\nearrow & \Downarrow k & \searrow^{g_2} \\ * & \xrightarrow[\delta(k) g_2 \cdot g_1]{} & * \end{array} \; \middle| \; g_1, g_2 \in G_0, k \in G_1 \right\} ,$$

where the 3-morphisms – the composition identities – are



Remark 1.2.49. All ingredients here are functorial, so that the above statements hold for presheaves over sites, hence in particular for cohesive 2-groups such as smooth 2-groups. Below in corollarly 3.6.136 it is shown that every cohesive 2-group has a presentation by a crossed module this way.

Notice that there are different equivalent conventions possible for how to present $\mathbf{B}G$ in terms of the correspondiung crossed module, given by the choices of order in the group products. Here we are following convention "LB" in [RoSc08].

Example 1.2.50 (shift of abelian Lie group)**.** For K an abelian Lie group then $\mathbf{B}K$ is the delooping 2-group coming from the crossed module $[K \to 1]$ and $\mathbf{BB}K$ is the 2-group coming from the complex $[K \to 1 \to 1]$.

Example 1.2.51 (automorphism 2-group)**.** For H any Lie group with automorphism Lie group $\mathrm{Aut}(H)$, the morphism $H \stackrel{\mathrm{Ad}}{\to} \mathrm{Aut}(H)$ that sends group elements to inner automorphisms, together with $\rho = \mathrm{id}$, is a crossed module. We write $\mathrm{AUT}(H) := (H \to \mathrm{Aut}(H))$ and speak of the *automorphism 2-group* of H.

Example 1.2.52. The inclusion of any normal subgroup $N \hookrightarrow G$ with conjugation action of G on N is a crossed module, with the canonical induced conjugation action of G on N.

Example 1.2.53 (string 2-group)**.** For G a compact, simple and simply connected Lie group, write PG for the smooth group of based paths in G and $\hat{\Omega} G$ for the universal central extension of the smooth group of based loops. Then the evident morphism $(\hat{\Omega} G \to PG)$ equipped with a lift of the adjoint action of paths on loops is a crossed module [BCSS07]. The corresponding strict 2-group is (a presentation of what is) called the *string 2-group* extension of G. The string 2-group we discuss in detail in 5.1.10.

It follows immediately that

Observation 1.2.54. For $G = (G_1 \to G_0)$ a 2-group coming from a crossed module, a cocycle

$$X \stackrel{\simeq}{\leftarrow} C(U_i) \stackrel{g}{\to} \mathbf{B}G$$

is given by data

$$\{h_{ij} \in C^\infty(U_{ij}, G_0), g_{ijk} \in C^\infty(U_{ijk}, G_1)\}$$

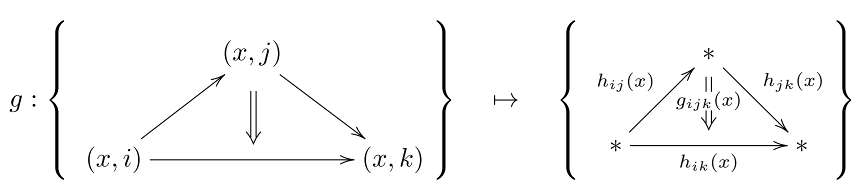
such that on each U_{ijk} we have

$$h_{ik} = \delta(h_{ijk}) h_{jk} h_{ij}$$

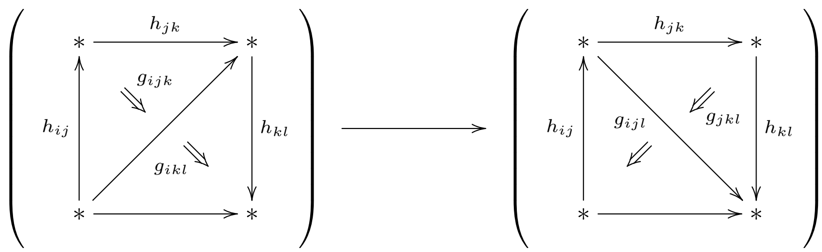
and on each U_{ijkl} we have

$$g_{ikl} \cdot \rho(h_{jk})(g_{ijk}) = g_{ijk} \cdot g_{jkl} \, .$$

Because under the above correspondence between crossed modules and 2-groups, this is the data that encodes assignments



that satisfy



For the case of the crossed module $(U(1) \to 1)$ this recovers the cocycles for circle 2-bundles from observation 1.2.42.

Apart from the notion of *bundle gerbe*, there is also the original notion of *gerbe*. The terminology is somewhat unfortunate, since neither of these concepts is, in general, a special case of the other. But they are of course closely related. We consider here the simple cocycle-characterization of gerbes and the relation of these to cocycles for 2-bundles.

Definition 1.2.55 (G-gerbe)**.** Let G be a smooth group. Then a cocycle for a smooth G*-gerbe* over a manifold X is a cocycle for a $\mathrm{AUT}(G)$-principal 2-bundle, where $\mathrm{AUT}(G)$ is the automorphism 2-group from example 1.2.51.

Observation 1.2.56. For every 2-group coming from a crossed module $(G_1 \stackrel{\delta}{\to} G_0, \rho)$ there is a canonical morphism of 2-groups

$$(G_1 \to G_0) \to \mathrm{AUT}(G_1)$$

given by the commuting diagram of groups

$$\begin{array}{ccc} G_1 & \xrightarrow{\delta} & G_0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho} \\ G_1 & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(G_0) \end{array} .$$

Accordingly, every $(G_1 \to G_0)$-principal 2-bundle has an underlying G_1-gerbe, def. 1.2.55. But in general the passage to this underlying G_1-gerbe discards information.

Example 1.2.57. For G a simply connected and compact simple Lie group, let $\mathrm{String} \simeq (\hat{\Omega} G \to PG)$ be the corresopnding String 2-group from example 1.2.53. Then by observation 1.2.56 every String-principal 2-bundle has an underlying $\hat{\Omega}G$-gerbe. But there is more information in the String-2-bundle than in this gerbe underlying it.

Example 1.2.58. Let $G = (\mathbb{Z} \hookrightarrow \mathbb{R})$ be the crossed module that includes the additive group of integers into the additive group of real numbers, with trivial action. The canonical projection morphism

$$\mathbf{B}(\mathbb{Z} \to \mathbb{R}) \stackrel{\simeq}{\to} \mathbf{B}U(1)$$

is a weak equivalence, by the fact that locally every smooth $U(1)$-valued function is the quotient of a smooth $\mathbb{R}$-valued function by a (constant) $\mathbb{Z}$-valued function. This means in particular that up to equivalence, $(\mathbb{Z} \to \mathbb{R})$-2-bundles are the same as ordinary circle 1-bundles. But it means a bit more than that:

On a manifold X also $\mathbf{B}\mathbb{Z}$-principal 2-bundles have the same classification as $U(1)$-bundles. But the *morphisms* of $\mathbf{B}\mathbb{Z}$-principal 2-bundles are essentially different from those of $U(1)$-bundles. This means that the 2-groupoid $\mathbf{B}\mathbb{Z}\mathrm{Bund}(X)$ is not, in general equivalent to $U(1)\mathrm{Bund}(X)$. But we do have an equivalence of 2-groupoids

$$(\mathbb{Z} \to U(1))\mathrm{Bund}(X) \simeq U(1)\mathrm{Bund}(X) \,.$$

Example 1.2.59. Let $\hat{G} \to G$ be a central extension of Lie groups by an abelian group A. This induces the crossed module $(A \to \hat{G})$. There is a canonical 2-anafunctor

$$\begin{array}{ccc} \mathbf{B}(A \to \hat{G}) & \xrightarrow{c} & \mathbf{B}(A \to 1) = \mathbf{B}^2 A \\ \downarrow{\simeq} & & \\ \mathbf{B}G & & \end{array}$$

from $\mathbf{B}G$ to $\mathbf{B}^2 A$. This can be seen to be the *characteristic class* that classifies the extension (see 1.2.12.1 below): $\mathbf{B}\hat{G} \to \mathbf{B}G$ is the A-principal 2-bundle classified by this cocycle.

Accordingly, the collection of all $(A \to \hat{G})$-principal 2-bundles is, up to equivalence, the same as that of plain G-1-bundles. But they exhibit the natural projection to $\mathbf{B}A$-2-bundles. Fixing that projection gives *twisted G-1-bundles*.

more in detail: the above 2-anafunctor indiuces a 2-anafunctor on cocycle 2-groupoid

$$\begin{array}{ccc} (A \to \hat{G})\mathrm{Bund}(X) & \xrightarrow{c} & \mathbf{B}A\mathrm{Bund}(X) \\ \downarrow{\simeq} & & \\ G\mathrm{Bund}(X) & & \end{array} .$$

If we fix a $\mathbf{B}A$-2-bundle g we can consider the fiber of the characteristic class c over g, hence the pullback $G\mathrm{Bund}_{[g]}(X)$ in

$$\begin{array}{ccc} G\mathrm{Bund}_{[g]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow{g} \\ (A \to \hat{G})\mathrm{Bund}(X) & \xrightarrow{c} & \mathbf{B}A\mathrm{Bund}(X) \\ \downarrow{\simeq} & & \\ G\mathrm{Bund}(X) & & \end{array} .$$

This is the groupoid of *$[g]$-twisted G-bundles*. The principal 2-bundle classfied by g is also called the *lifting gerbe* of the G-principal bundles underlying the $[g]$-twisted $\hat{G}$-bundle: because this is the obstruction to lifting the former to a genuine $\hat{G}$-principal bundle.

If g is given by a Čech cocycle $\{g_{ijk} \in C^\infty(U_{ijk}, A)\}$ then $[g]$-twisted G-bundles are given by data $\{h_{ij} \in C^\infty(U_{ij}, G)\}$ which does not quite satisfy the usual cocycle condition, but instead a modification by g:

$$h_{ik} = \delta(g_{ijk}) h_{jk} h_{ij} .$$

For instance for the extension $U(1) \to U(n) \to PU(n)$ the corresponding twisted bundles are those that model *twisted K-theory* with n-torsion twists (4.4.8).

1.2.5.3 Principal 3-bundles and twisted 2-bundles

As one passes beyond (smooth) 2-groups and their 2-principal bundles, one needs more sophisticated tools for presenting them. While the crossed modules from def. 1.2.45 have convenient higher analogs – called *crossed complexes* – the higher analog of remark 1.2.49 does not hold for these: not every (smooth) 3-group is presented by them, much less every n-group for $n > 3$. Therefore below in 1.2.5.4 we switch to a different tool for the general situation: simplicial groups.

However, it so happens that a wide range of relevant examples of (smooth) 3-groups and generally of smooth n-groups does have a presentation by a crossed complex after all, as do the examples which we shall discuss now.

Definition 1.2.60. A *crossed complex of groupoids* is a diagram

$$C_\bullet = \left(\begin{array}{ccccccccc} \cdots & \xrightarrow{\delta} & C_3 & \xrightarrow{\delta} & C_2 & \xrightarrow{\delta} & C_1 & \underset{\delta_s}{\overset{\delta_t}{\rightrightarrows}} & C_0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta_s} & & \downarrow{\scriptstyle =} \\ \cdots & \xrightarrow[=]{} & C_0 & \xrightarrow[=]{} & C_0 & \xrightarrow[=]{} & C_0 & \xrightarrow[=]{} & C_0 \end{array} \right),$$

where $C_1 \underset{\delta_s}{\overset{\delta_t}{\rightrightarrows}} C_0$ is equipped with the structure of a 1-groupoid, and where $C_k \longrightarrow C_0$, for all $k \geq 2$, are bundles of groups, abelian for $k \geq 2$; and equipped with an action ρ of the groupoid C_1, such that

1. the maps δ_k, $k \geq 2$ are morphisms of groupoids over C_0 compatible with the action by C_1;
2. $\delta_{k-1} \circ \delta_k = 0$; k ≥ 3;
3. $\mathrm{im}(\delta_2) \subset C_1$ acts by conjugation on C_2 and trivially on C_k, $k \geq 3$.

Surveys of standard material on crossed complexes of groupoids are in [BrHiSi11][Por]. We discuss sheaves of crossed complexes, hence *cohesive crossed complexes* in more detail below in 2.2.6. As mentioned there, the key aspect of crossed complexes is that they provide an equivalent encoding of precisely those ∞-groupoids that are called *strict*.

Definition 1.2.61. A *crossed complex of groups* is a crossed complex $C_\bullet$ of groupoids with $C_0 = *$. If the complex of groups is constant on the trivial group beyond C_n, we say this is a *strict n-group*.

Explicitly, a *crossed complex of groups* is a complex of groups of the form

$$\cdots \xrightarrow{\delta_2} G_2 \xrightarrow{\delta_1} G_1 \xrightarrow{\delta_0} G_0$$

with $G_{k\geq 2}$ abelian (but G_1 and G_0 not necessarily abelian), together with an action ρ_k of G_0 on G_k for all $k \in \mathbb{N}$, such that

1. ρ_0 is the adjoint action of G_0 on itself;
2. $\rho_1 \circ \delta_0$ is the adjoint action of G_1 on itself;
3. $\rho_k \circ \delta_0$ is the trivial action of G_1 on G_k for $k > 1$;
4. all δ_k respect the actions.

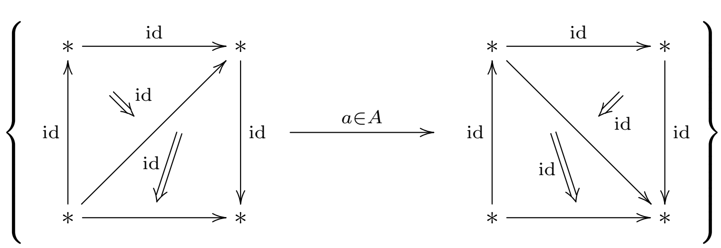
A morphism of crossed complexes of groups is a sequence of morphisms of component groups, respecting all this structure.

For $n = 2$ this reproduces the notion of *crossed module* and *strict 2-group*, def. 1.2.45. If furthermore G_1 and G_0 here are abelian and the action of G_0 is trivial, then this is an ordinary *complex of abelian groups* as considered in homological algebra. Indeed, all of homological algebra may be thought of as the study of this presentation of abelian ∞-groups. (More on this in 2.2.6 below.)

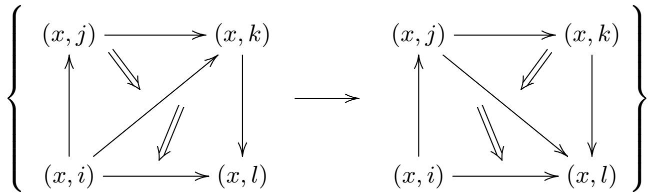
We consider now examples of strict 3-groups and of the corresponding principal 3-bundles.

Example 1.2.62. For A an abelian group, the delooping of the 3-group given by the complex $(A \to 1 \to 1)$ is the one-object 3-groupoid that looks like

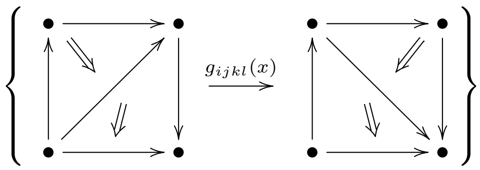
$\mathbf{B}^3 A =$



Therefore an ∞-anafunctor $X \overset{\simeq}{\leftarrow} C(\{U_i\}) \overset{g}{\to} \mathbf{B}^3U(1)$ sends 3-simplices in the Čech groupoid

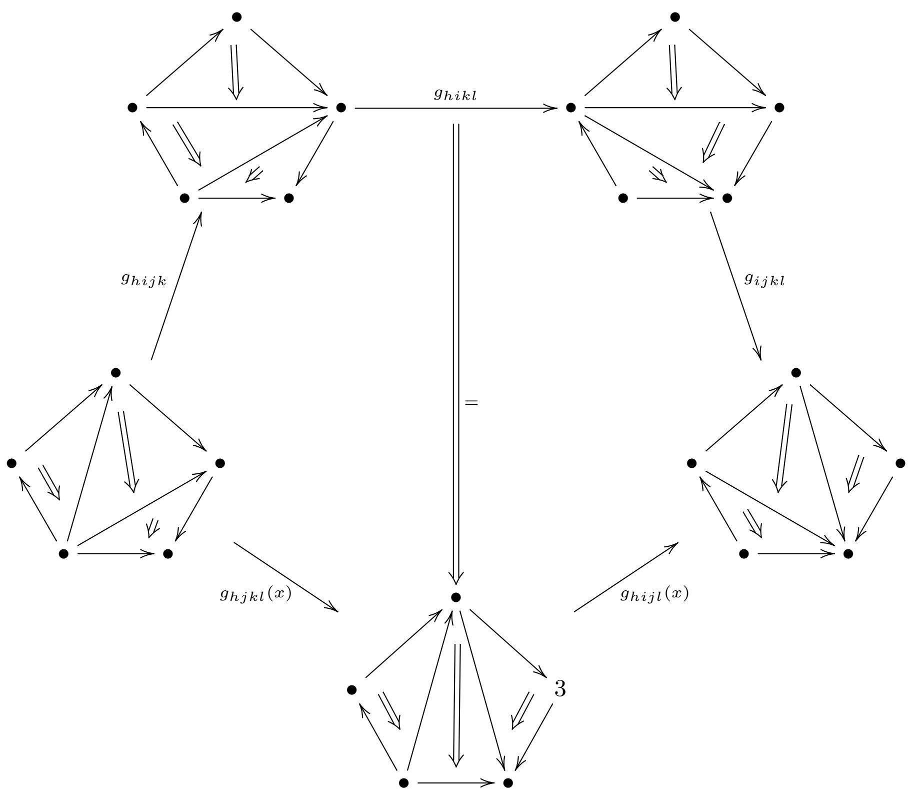


to 3-morphisms in $\mathbf{B}^3U(1)$ labeled by group elements $g_{ijkl}(x) \in U(1)$



(where all 1-morphisms and 2-morphisms in $\mathbf{B}^3U(1)$ are necessarily identities).

The 3-functoriality of this assignment is given by the following identity on all Čech 4-simplices (x,(h,i,j,k,l)):



This means that the cocycle data $\{g_{ijkl}(x)\}$ has to satisfy the equations

$$g_{hijk}(x)g_{hikl}(x)g_{ijkl}(x) = g_{hjkl}(x)g_{hijl}(x)$$

for all (h, i, j, k, l) and all $x \in U_{hijkl}$. Since $U(1)$ is abelian this can equivalently be rearranged to

$$g_{hijk}(x)g_{hijl}(x)^{-1}g_{hikl}(x)g_{hjkl}(x)^{-1}g_{ijkl}(x) = 1\,.$$

This is the usual form in which a Čech 3-cocycles with coefficients in $U(1)$ are written.

Definition 1.2.63. Given a cocycle as above, the total space object $\tilde{P}$ given by the pullback

$$\begin{array}{ccc} \tilde{P} & \longrightarrow & \mathbf{E}\mathbf{B}^2U(1) \\ \downarrow & & \downarrow \\ C(U) & \xrightarrow{g} & \mathbf{B}^3U(1) \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array}$$

is the corresponding *circle principal 3-bundle.*

In direct analogy to the argument that leads to observation 1.2.44 we find:

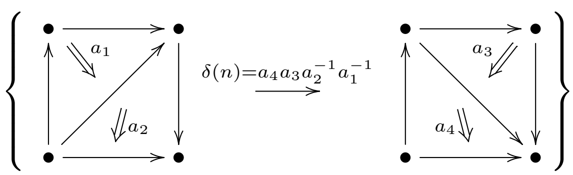
Observation 1.2.64. The structures known as *bundle 2-gerbes* [St01] are presentations of the 2-groupoids that are total spaces of circle principal 2-bundles, as above.

Again, notice that, despite a close relation, this is different from the original notion of *2-gerbe*. More discussion of this point is below in 4.3.10.

The next example is still abelian, but captures basics of the central mechanism of twistings of principal 2-bundles by principal 3-bundles.

Example 1.2.65. Consider a morphism $\delta : N \to A$ of abelian groups and the corresponding shifted crossed complex $(N \to A \to 1)$. The corresponding delooped 3-group looks like

$$\mathbf{B}(N \to A \to 1) \quad = $$



A cocycle for a $(N \to A \to 1)$-principal 3-bundle is given by data

$$\{a_{ijk} \in C^\infty(U_{ijk}, A),\, n_{ijkl} \in C^\infty(U_{ijkl}, N)\}$$

such that

1. $a_{jkl}a_{ijk}^{-1}a_{ijk}a_{ikl}^{-1} = \delta(n_{ijkl})$
2. $n_{hijk}(x)n_{hikl}(x)n_{ijkl}(x) = n_{hjkl}(x)n_{hijl}(x)$.

The first equation on the left is the cocycle for a 2-bundle as in observation 1.2.42. But the extra term n_{ijkl} on the right "twists" the cocycle. This twist itself satisfies a higher order cocycle condition.

Notice that there is a canonical projection

$$\mathbf{B}(N \to A \to 1) \to \mathbf{B}(N \to 1 \to 1) = \mathbf{B}^3N\,.$$

Therefore we can consider the higher analog of the notion of twisted bundles in example 1.2.59:

Definition 1.2.66. Let $N \to A$ be an inclusion and consider a fixed $\mathbf{B}^2N$-principal 3-bundle with cocycle g, let $\mathbf{B}(A/N)\mathrm{Bund}_{[g]}(X)$ be the pullback in

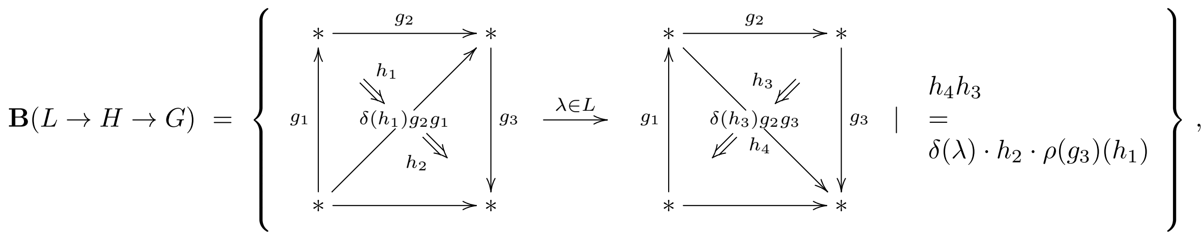
$$\begin{array}{ccc} \mathbf{B}(A/N)\mathrm{Bund}_{[g]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{B}(N \to A)\mathrm{Bund}(X) & \longrightarrow & \mathbf{B}^2N\mathrm{Bund}(X) \\ \downarrow{\scriptstyle \simeq} & & \\ \mathbf{B}(A/N)\mathrm{Bund}(X) & & \end{array} .$$

We say an object in this 2-groupoid is a $[g]$-*twisted* $\mathbf{B}(A/N)$-*principal 2-bundle*.

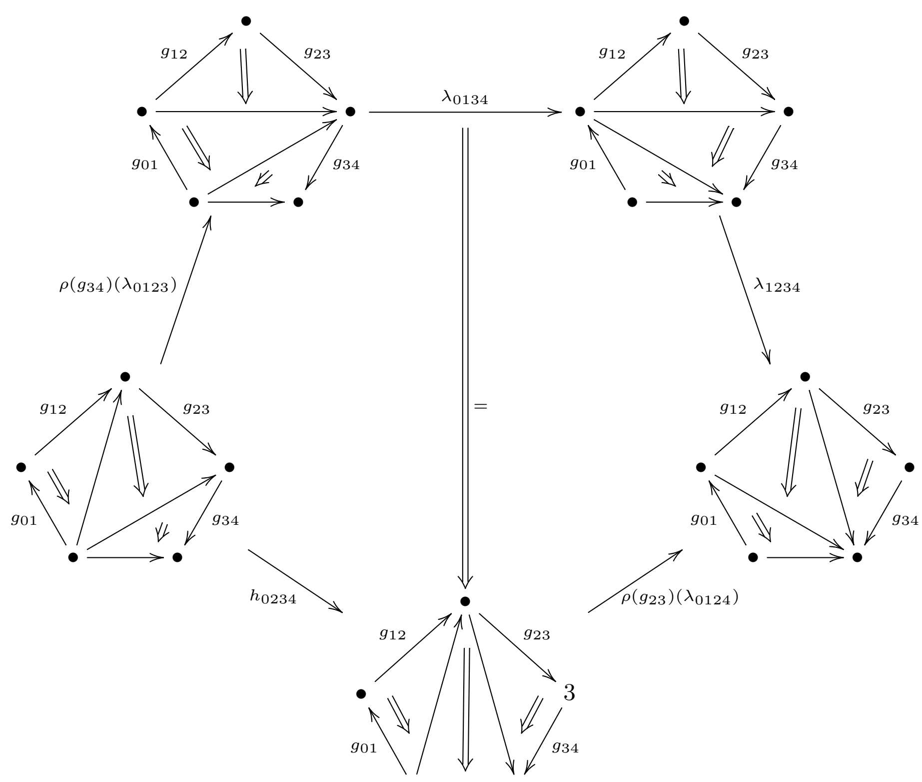
Below in example 1.2.107 we discuss this and its relation to characteristic classes of 2-bundles in more detail.

We now turn to the most general 3-group that is presented by a crossed complex.

Observation 1.2.67. For $(L \xrightarrow{\delta} H \xrightarrow{\delta} G)$ an arbitrary strict 3-group, def. 1.2.61, the delooping 3-groupoid looks like



with the 4-cells – the composition identities – being



If follows that a cocycle

$$X \stackrel{\simeq}{\leftarrow} C(U_i) \stackrel{(\lambda,h,g)}{\to} \mathbf{B}(L \to H \to G)$$

for a $(L \to H \to G)$-principal 3-bundle is a collection of functions

$$\{g_{ij} \in C^\infty(U_{ij}, G),\ h_{ijk} \in C^\infty(U_{ijk}, H),\ \lambda_{ijkl} \in C^\infty(U_{ijkl}, L)\}$$

satisfying the cocycle conditions

$$g_{ik} = \delta(h_{ijk}) g_{jk} g_{ij} \quad \text{on } U_{ijk}$$
$$h_{ijl} h_{jkl} = \delta(\lambda_{ijkl}) \cdot h_{ikl} \cdot \rho(g_3)(h_{ijk}) \quad \text{on } U_{ijkl}$$
$$\lambda_{ijkl} \lambda_{hikl} \rho(g_{kl})(\lambda_{hijk}) = \rho(g_{jk}) \lambda_{hijl} \lambda_{hjkl} \quad \text{on } U_{hijkl}\,.$$

Definition 1.2.68. Given such a cocycle, the pullback 3-groupoid P we call the corresponding *principal* $(L \to H \to G)$*-3-bundle*

$$\begin{array}{ccc} P & \longrightarrow & \mathbf{EB}(L \to H \to G) \\ \downarrow & & \downarrow \\ C(U_i) & \xrightarrow{(\lambda,h,g)} & \mathbf{B}(L \to H \to G) \\ \downarrow{\simeq} & & \\ X & & \end{array}$$

We can now give the next higher analog of the notion of twisted bundles, def. 1.2.59.

Definition 1.2.69. Given a 3-anafunctor

$$\begin{array}{ccccc} \mathbf{B}(L \to H \to G) & \longrightarrow & \mathbf{B}(L \to 1 \to 1) & = & \mathbf{B}^3 L \\ \downarrow{\simeq} & & & & \\ \mathbf{B}(H/L \to G) & & & & \end{array}$$

then for g the cocycle for an $\mathbf{B}^2 L$-principal 3-bundle we say that the pullback $(H \to G)\mathrm{Bund}_g(X)$ in

$$\begin{array}{ccc} (H \to G)\mathrm{Bund}_g(X) & \longrightarrow & * \\ \downarrow & & \downarrow{g} \\ (L \to H \to G)\mathrm{Bund}(X) & \longrightarrow & \mathbf{B}^3 L\mathrm{Bund}(X) \end{array}$$

is the 3-groupoid of g*-twisted* $(H \to G)$*-principal 2-bundles* on X.

Example 1.2.70. Let G be a compact and simply connected simple Lie group. By example 1.2.53 we have associated with this the *string 2-group* crossed module $\hat{\Omega} G \to PG$, where

$$U(1) \to \hat{\Omega} G \to \Omega G$$

is the Kac-Moody central extension of level 1 of the based loop group of G. Accordingly, there is an evident crossed complex

$$U(1) \to \hat{\Omega} G \to PG\,.$$

The evident projection

$$\mathbf{B}(U(1) \to \hat{\Omega} G \to PG) \stackrel{\simeq}{\to} \mathbf{B} G$$

is a weak equivalence. This means that $(U(1) \to \hat{\Omega} G \to PG)$-principal 3-bundles are equivalent to G-1-bundles. For fixed projection g to a $\mathbf{B}^2 U(1)$-3-bundle a $(U(1) \to \hat{\Omega} G \to PG)$-principal 3-bundles may hence be thought of as a g-twisted string-principal 2-bundle.

One finds that these serve as a resolution of G-1-bundles in attempts to lift to string-2-bundles (discussed below in 5.1).

1.2.5.4 A model for principal ∞-bundles We have seen above that the theory of ordinary smooth principal bundles is naturally situated within the context of Lie groupoids, and then that the theory of smooth principal 2-bundles is naturally situated within the theory of Lie 2-groupoids. This is clearly the beginning of a pattern in higher category theory where in the next step we see smooth 3-groupoids and so on. Finally the general theory of principal ∞-bundles deals with smooth ∞-groupoids. A comprehensive discussion of such smooth ∞-groupoids is given in section 4.4. In this introduction here we will just briefly describe principal ∞-bundles in this model.

Recall the discussion of ∞-groupoids from 1.2.3.2.1, in terms of Kan simplicial sets. Consider an object $\mathbf{B}G \in [C^{\mathrm{op}}, \mathrm{sSet}]$ which is an ∞-groupoid with a single object, so that we may think of it as the delooping of an ∞-group G. Let $*$ be the point and $* \to \mathbf{B}G$ the unique inclusion map. The *good replacement* of this inclusion morphism is the *universal G-principal ∞-bundle* $\mathbf{E}G \to \mathbf{B}G$ given by the pullback diagram

$$
\begin{array}{ccc}
\mathbf{E}G & \longrightarrow & * \\
\downarrow & & \downarrow \\
(\mathbf{B}G)^{\Delta[1]} & \longrightarrow & \mathbf{B}G \\
\downarrow & & \\
\mathbf{B}G & &
\end{array}
\quad .
$$

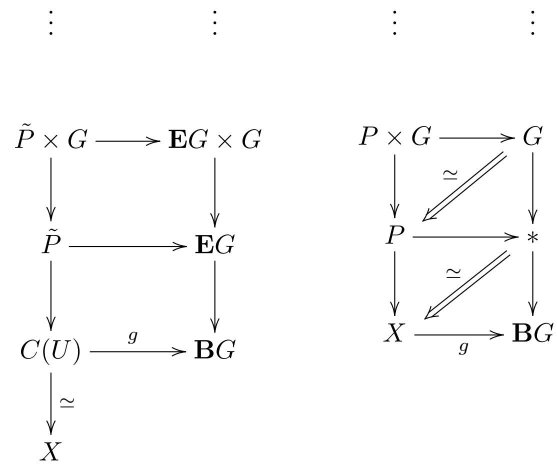
An ∞-anafunctor $X \stackrel{\simeq}{\leftarrow} \hat{X} \to \mathbf{B}G$ we call a *cocycle* on X with coefficients in G, and the ∞-pullback P of the point along this cocycle, which by the above discussion is the ordinary limit

$$
\begin{array}{ccccc}
P & \longrightarrow & \mathbf{E}G & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
 & & \mathbf{B}G^{\Delta[1]} & \longrightarrow & \mathbf{B}G \\
\downarrow & & \downarrow & & \\
\hat{X} & \stackrel{g}{\longrightarrow} & \mathbf{B}G & & \\
\downarrow{\scriptstyle \simeq} & & & & \\
X & & & &
\end{array}
$$

we call the principal ∞-bundle $P \to X$ *classified* by the cocycle.

Example 1.2.71. A detailed description of the 3-groupoid fibration that constitutes the universal principal 2-bundle $\mathbf{E}G$ for G any strict 2-group in given in [RoSc08].

It is now evident that our discussion of ordinary smooth principal bundles above is the special case of this for $\mathbf{B}G$ the nerve of the one-object groupoid associated with the ordinary Lie group G. So we find the complete generalization of the situation that we already indicated there, which is summarized in the following diagram:



in the model category in the ∞-topos

1.2.5.5 Higher fiber bundles We indicate here the natural notion of *principal bundle* in an ∞-topos and how it relates to the intrinsic notion of cohomology discussed above.

1.2.5.5.1 Ordinary principal bundles For G a group, a *G-principal bundle* over some space X is, roughly, a space $P \to X$ over X, which is equipped with a G-action over X that is fiberwise free and transitive ("principal"), hence which after a choice of basepoint in a fiber looks there like the canonical action of G on itself. A central reason why the notion of G-principal bundles is relevant is that it consistutes a "geometric incarnation" of the degree-1 (nonabelian) cohomology $H^1(X, G)$ of X with coefficients in G (with G regarded as the sheaf of G-valued functions on G): G-principal bundles are *classified* by $H^1(X, G)$. We will see that this classical statement is a special case of a natural and much more general fact, where *principal ∞-bundles* incarnate cocycles in the intrinsic cohomology of any ∞-topos. Before coming to that, here we briefly review aspects of the classical theory to set the scene.

Let G be a topological group and let X be a topological space.

Definition 1.2.72. A *topological G-principal bundle* over X is a continuous map $p : P \to X$ equipped with a continuous fiberwise G-action $\rho : P \times G \to G$

$$\begin{array}{c} P \times G \\ {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle \rho} \\ P \\ \downarrow{\scriptstyle p} \\ X \end{array}$$

which is *locally trivial*: there exists a cover $\phi : U \to X$ and an isomorphism of topological G-spaces

$$P|_U \simeq U \times G$$

between the restriction (pullback) of P to U and the trivial bundle $U \times G \to U$ equipped with the canonical G-action given by multiplication in G.

Observation 1.2.73. Let $P \to X$ be a topological G-principal bundle. An immediate consequence of the definition is

1. The base space X is isomorphic to the quotient of P by the G-action, and, moreover, under this identitfication $P \to X$ is the quotient projection $P \to P/G$.

2. The *principality condition* is satisfied: the *shear map*

$$(p_1, \rho) : P \times G \to P \times_X P$$

is an isomorphism.

Remark 1.2.74. Sometimes the quotient property of principal bundles has been taken to be the defining property. For instance [Cart50a, Cart50b] calls every quotient map $P \to P/G$ of a free topological group action a "G-principal bundle", *without* requiring it to be locally trivial. This is a strictly weaker definition: there are many examples of such quotient maps which are not locally trivial. To distinguish the notions, [Pa61] refers to the weaker definition as that of a *Cartan principal bundle*. Also for instance the standard textbook [Hus94] takes the definition via quotient maps as fundamental and explicitly adds the adjective "locally trivial" when necessary.

For our purposes the following two points are relevant.

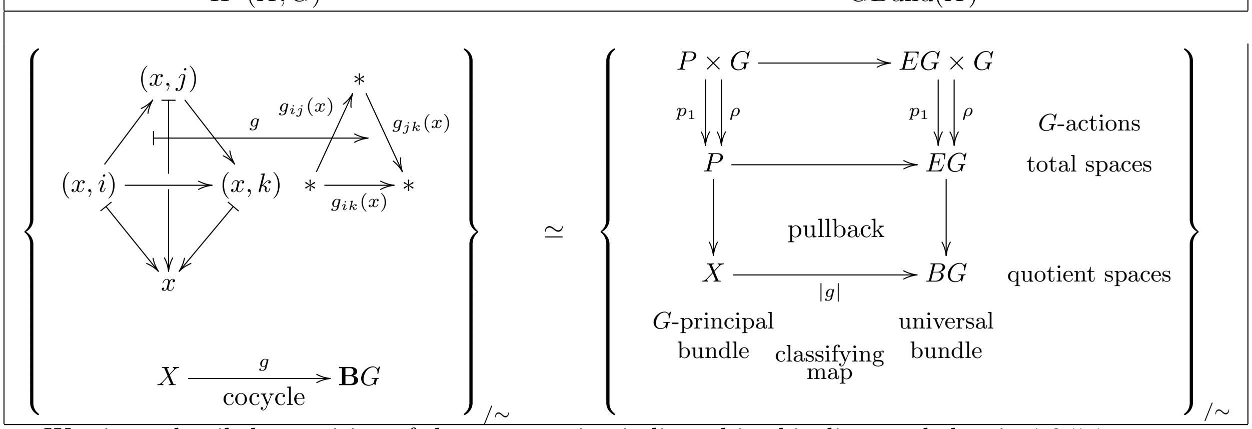
1. Local triviality is crucial for the classification of topological G-principal bundles by nonabelian sheaf cohomology to work, and so from this perspective a *Cartan principal bundle* may be pathological.

2. On the other hand, we see below that this problem is an artefact of considering G-principal bundles in the ill-suited context of the 1-category of topological spaces or manifolds. We find below that after embedding into an ∞-topos (for instance that of Euclidean topological ∞-groupoids, discussed in 4.3) both definitions in fact coincide.

   The reason is that the Yoneda embedding into the higher categorical context of an ∞-topos "corrects the quotients": those quotients of G-actions that are not locally trivial get replaced, while the "good quotients" are being preserved by the embedding. This statement we make precise in 3.6.10.4 below. See also the the discussion in 3.6.10.1 below.

It is a classical fact that for X a manifold and G a topological or Lie group, regarded as a sheaf of groups $C(-, G)$ on X, there is an equivalence of the following kind

algebraic data on X		**geometric data on X**
$\left\{ \begin{array}{c} \text{degree-1 nonabelian} \\ \text{sheaf cohomology} \end{array} \right\}$ $H^1(X, G)$	$\simeq$	$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{G-principal bundles over } X \end{array} \right\}$ $G\mathrm{Bund}(X)$



We give a detailed exposition of the construction indicated in this diagram below in 1.2.5.1.

1.2.5.5.2 Principal ∞-bundles Let now $\mathbf{H}$ be an ∞-topos, 1.2.3.2, and G a group object in $\mathbf{H}$, 1.2.3.2.3. Up to the technical issue of formulating homotopy coherence, the formulation in $\mathbf{H}$ of the definition of G-principal bundles, 1.2.5.5.1, in its version as *Cartan G-principal bundle*, remark 1.2.74, is immediate:
Definition. A **G-principal bundle** over $X \in \mathbf{H}$ is

- a morphism $P \to X$; with an ∞-action $\rho : P \times G \to P$;

- such that $P \to X$ is the ∞-quotient map $P \to P//G$.

In 3.6.10 below we discuss a precise formulation of this definition and the details of the following central statement about the relation between G-principal ∞-bundles and the intrinsic cohomology of $\mathbf{H}$ with coefficients in the delooping object $\mathbf{B}G$.

Theorem. There is equivalence of ∞-groupoids $G\mathrm{Bund}(X) \underset{\varinjlim}{\overset{\mathrm{hofib}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{H}(X, \mathbf{B}G)$, where

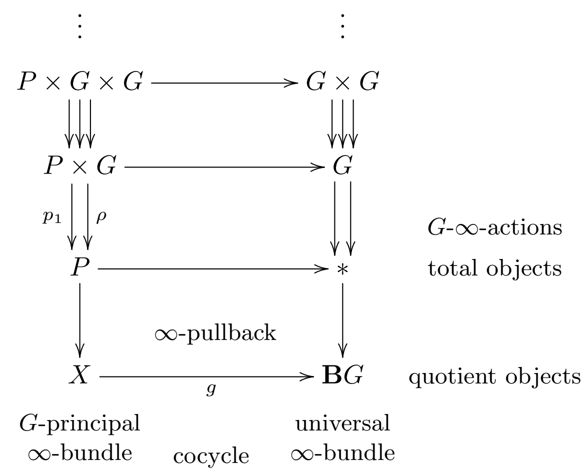
1. hofib sends a cocycle $X \to \mathbf{B}G$ to its homotopy fiber;
2. $\varinjlim$ sends an ∞-bundle to the map on ∞-quotients $X \simeq P//G \to *//G \simeq \mathbf{B}G$.

In particular, G-principal ∞-bundles are classified by the intrinsic cohomology of $\mathbf{H}$

$$G\mathrm{Bund}(X)/_\sim \simeq H^1(X, G) := \pi_0 \mathbf{H}(X, \mathbf{B}G)\,.$$

Idea of Proof. Repeatedly apply two of the *Giraud-Rezk-Lurie axioms*, def. 2.2.2, that characterize ∞-toposes:

1. every ∞-quotient is effective;
2. ∞-colimits are preserved by ∞-pullbacks. $\square$



This gives a general abstract theory of principal ∞-bundles in every ∞-topos. We also have the following explicit presentation. **Definition** For $G \in \mathrm{Grp}(\mathrm{sSh}(C))$, and $X \in \mathrm{sSh}(C)_{\mathrm{lfib}}$, a *weakly G-principal simplicial bundle* is a G-action ρ over X such that the *principality morphism* $(\rho, p_1) : P \times G \to P \times_X P$ is a stalkwise weak equivalence.

Below in 3.6.10.4 we discuss that this construction gives a presentation of the ∞-groupoid of G-principal bundles as the nerve of the ordinary category of weakly G-principal simplicial bundles.

$$\mathrm{Nerve}\left\{ \begin{array}{c} \text{weakly } G\text{-principal} \\ \text{simplicial bundles} \\ \text{over } X \end{array} \right\} \simeq G\mathrm{Bund}(X)\,.$$

For the special case that X is the terminal object over the site C and when restricted from cocycle ∞-groupoids to sets of cohomology classes, this reproduces the statement of [JaLu04]. For our applications in 5, in particular for applications in twisted cohomology, 3.6.12, it is important to have the general statement, where the base space of a principal ∞-bundle may be an arbitrary ∞-stack, and where we remember the ∞-groupoids of gauge transformations between them, instead of passing to their sets of equivalence classes.

The special case where the site C is trivial, $C = *$, leads to the notion of principal ∞-bundles in $\infty\mathrm{Grp}$. These are presented by certain bundles of simplicial sets. This we discuss below in 4.1.4.

1.2.5.5.3 Associated and twisted ∞-bundles The notion of G-principal bundle is a very special case of the following natural more general notion. For any F, an *F-fiber bundle* over some X is a space $E \to X$ over X such that there is a cover $U \to X$ over which it becomes equivalent as a bundle to the trivial F-bundle $U \times F \to U$.

Principal bundles themselves form but a small subclass of all possible fiber bundles over some space X. Even among G-fiber bundles the G-principal bundles are special, due to the constraint that the local trivialization has to respect the G-action on the fibers. However, every F-fiber bundle is *associated* to a G-principal bundle.

Given a representation $\rho : F \times G \to F$, the *ρ-associated* F-fiber bundle is the quotient $P \times_G F$ of the product $P \times F$ by the diagonal G-action. Conversely, using that the automorphism group $\mathrm{Aut}(F)$ of F canonically acts on F, it is immediate that every F-fiber bundle is associated to an $\mathrm{Aut}(F)$-principal bundle (a statement which, of course, crucially uses the local triviality clause).

All of these constructions and statements have their straightforward generalizations to higher bundles, hence to *associated ∞-bundles*. Moreover, just as the theory of principal bundles *improves* in the context of ∞-toposes, as discussed above, so does the theory of associated bundles.

For notice that by the above classification theorem of G-principal ∞-bundles, every G-∞-action ρ : $V \times G \to G$ has a *classifying map*, which we will denote by the same symbol:

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}G \end{array} .$$

One may observe now that this map $V//G \to \mathbf{B}G$ is the *universal ρ-associated V-∞-bundle*: for every F-fiber ∞-bundle $E \to X$ there is a morphism $X \to \mathbf{B}G$ such that $E \to X$ is the ∞-pullback of this map to X.

$$\begin{array}{ccc} E & \longrightarrow & V//G \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} .$$

One implication of this is, by the universal property of the ∞-pullback, that *sections* σ of the associated bundle

$$\begin{array}{c} E \\ {\scriptstyle \sigma}\uparrow\downarrow \\ X \end{array}$$

are equivalently lifts of its classifying map through the universal ρ-associated bundle

$$\Gamma_X(P \times_G V) := \left\{ \begin{array}{ccc} & & V//G \\ & {\scriptstyle \sigma}\nearrow & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} \right\} .$$

One observes that by local triviality and by the fact that V is, by the above, the homotopy fiber of $V//G \to \mathbf{B}G$, it follows that locally over a cover $U \to X$ such a section is identified with a V-valued map $U \to V$. Conversely, globally a section of a ρ-associated bundle may be regarded as a *twisted* V-valued function.

While this is an elementary and familiar statement for ordinary associated bundles, this is where the theory of associated ∞-bundles becomes considerably richer than that of ordinary ∞-bundles: because here V itself may be a higher stack, notably it may be a moduli ∞-stack $V = \mathbf{B}A$ for A-principal ∞-bundles. If so, maps $U \to V$ classify A-principal ∞-bundles locally over the cover U of X, and so conversely the section σ itself may globally be regarded as exhibiting a *twisted A-principal ∞-bundle* over X.

We can refine this statement by furthermore observing that the space of all sections as above is itself the hom-space in an ∞-topos, namely in the slice ∞-topos $\mathbf{H}_{/\mathbf{B}G}$. This means that such sections are themselves cocycles in a structured nonabelian cohomology theory:

$$\Gamma_X(P \times_G V) := \mathbf{G}_{/\mathbf{B}G}(g, \rho) .$$

This we may call the *g-twisted cohomology* of X relative to ρ. We discuss below in 5.4 how traditional notions of twisted cohomology are special cases of this general notion, as are many further examples.

Now ρ, regarded as an object of the slice $\mathbf{H}_{/\mathbf{B}G}$ is not in general a connected object. This means that it is not in general the moduli object for some principal ∞-bundles over the slice. But instead, we find that we can naturally identify geometric incarnations of such cocycles in the form of *twisted ∞-bundles*.

Theorem. The g-twisted cohomology $\mathbf{H}_{/\mathbf{B}G}(g, \rho)$ classifies *P-twisted ∞-bundles*: twisted G-equivariant ΩV-∞-bundles on P:

$$
\begin{array}{ccccccl}
Q & \longrightarrow & * & & & & P\text{-twisted } \Omega V\text{-principal } \infty\text{-bundle} \\
\downarrow & & \downarrow & & & & \\
P & \longrightarrow & V & \longrightarrow & * & & G\text{-principal } \infty\text{-bundle} \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \xrightarrow{\sigma} & V//G & \xrightarrow{\rho} & \mathbf{B}G & & \text{section of } \rho\text{-associated } V\text{-}\infty\text{-bundle}
\end{array}
\qquad (g = \rho\circ\sigma : X \to \mathbf{B}G)
$$

$$
\left\{ \begin{array}{c} \text{sections of} \\ \rho\text{-associated } V\text{-}\infty\text{-bundle} \end{array} \right\} \simeq \left\{ \begin{array}{c} g\text{-twisted } \Omega V\text{-cohomology} \\ \text{relative } \rho \end{array} \right\} \simeq \left\{ \begin{array}{c} \Omega V\text{-}\infty\text{-bundles} \\ \text{twisted by } P \end{array} \right\}
$$

A survey of classes of examples of twisted ∞-bundles classified by twisted cohomology is below in 5.4.1. Among them, in particular the classical notion of nonabelian *gerbe* [Gir71], and *2-gerbe* [Br94] is a special case.

Namely one see that a (nonabelian/Giraud-)*gerbe* on X is nothing but a connected and 1-truncated object in $\mathbf{H}_{/X}$. Similarly, a (nonabelian/Breen) *2-gerbe* over X is just a connected and 2-truncated object in $\mathbf{H}_{/X}$. Accordingly we may call a general connecte object in $\mathbf{H}_{/X}$ an *nonabelian ∞-gerbe* over X. We say that it is a *G-∞-gerbe* if it is an $\mathrm{Aut}(\mathbf{B}G)$-associated ∞-bundle. We say its *band* is the underlying $\mathrm{Out}(G)$-principal ∞-bundle. For 1-gerbes and 2-gerbes this reproduces the classical notions.

In terms of this, the above says that G-∞-gerbes *bound by a band* are classified by $(\mathbf{B}\mathrm{Aut}(\mathbf{B}G) \to \mathbf{B}\mathrm{Out}(G))$-twisted cohomology. This is the generalization of Giraud's original theorem. We discuss all this in detail below in 3.6.15.

1.2.6 Reduction of structure groups

(...)

1.2.7 Representations and associated bundles

(...)

1.2.8 Flat connection

(...)

1.2.9 de Rham coefficients

(...)

1.2.10 Maurer-Cartan forms

(...)

1.2.11 Principal connections

1.2.11.1 Parallel n-transport for low n With a decent handle on principal ∞-bundles as described above, we now turn to the description of *connections on ∞-bundles*. It will turn out that the above cocycle-description of G-principal ∞-bundles in terms of ∞-anafunctors $X \stackrel{\simeq}{\leftarrow} \hat{X} \stackrel{g}{\to} \mathbf{B}G$ has, under mild conditions, a natural generalization where $\mathbf{B}G$ is replaced by a (non-concrete) simplicial presheaf $\mathbf{B}G_{\mathrm{conn}}$, which we may think of as the ∞-groupoid of ∞-Lie algebra valued forms. This comes with a canonical map $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$ and an ∞-connection ∇ on the ∞-bundle classified by g is a lift ∇ of g in the diagram

$$\begin{array}{ccc} & & \mathbf{B}G_{\mathrm{conn}} \\ & {}^{\nabla}\nearrow & \downarrow \\ \hat{X} & \xrightarrow{g} & \mathbf{B}G \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array} .$$

In the language of ∞-stacks we may think of $\mathbf{B}G$ as the ∞-stack (on CartSp) or ∞-prestack (on SmoothMfd) $G\mathrm{TrivBund}(-)$ of *trivial* G-principal bundles, and of $\mathbf{B}G_{\mathrm{conn}}$ correspondingly as the object $G\mathrm{TrivBund}_{\nabla}(-)$ of trivial G-principal bundles with (non-trivial) connection. In this sense the statement that ∞-connections are cocycles with coefficients in some $\mathbf{B}G_{\mathrm{conn}}$ is a tautology. The real questions are:

1. What is $\mathbf{B}G_{\mathrm{conn}}$ in concrete formulas?
2. Why are these formulas what they are? What is the general abstract concept of an ∞-connection? What are its defining abstract properties?

A comprehensive answer to the second question is provided by the general abstract concepts discussed in section 3. Here in this introduction we will not go into the full abstract theory, but using classical tools we get pretty close. What we describe is a generalization of the concept of *parallel transport* to *higher parallel transport*. As we shall see, this is naturally expressed in terms of ∞-anafunctors out of path n-groupoids. This reflects how the full abstract theory arises in the context of an ∞-connected ∞-topos that comes canonically with a notion of fundamental ∞-groupoid.

Below we begin the discussion of ∞-connections by reviewing the classical theory of connections on a bundle in a way that will make its generalization to higher connections relatively straightforward. In an analogous way we can then describe certain classes of connections on a 2-bundle – subsuming the notion of connection on a bundle gerbe. With that in hand we then revisit the discussion of connections on ordinary bundles. By associating to each bundle with connection its corresponding *curvature 2-bundle with connection* we obtain a more refined description of connections on bundles, one that is naturally adapted to the construction of curvature characteristic forms in the Chern-Weil homomorphism. This turns out to be the kind of formulation of connections on an ∞-bundle that drops out of the general abstract theory. In classical terms, its full formulation involves the description of circle n-bundles with connection in terms of Deligne cohomology and the description of the ∞-groupoid of ∞-Lie algebra valued forms in terms of dg-algebra homomorphisms. The combination of these two aspects yields naturally an explicit model for the Chern-Weil homomorphism and its generalization to higher bundles.

Taken together, these constructions allow us to express a good deal of the general ∞-Chern-Weil theory with classical tools. As an example, we describe how the classical Čech-Deligne cocycle construction of the refined Chern-Weil homomorphism drops out from these constructions.

1.2.11.1.1 Connections on a principal bundle There are different equivalent definitions of the classical notion of a connection. One that is useful for our purposes is that a connection ∇ on a G-principal bundle $P \to X$ is a rule tra_{∇} for *parallel transport* along paths: a rule that assigns to each path $\gamma : [0,1] \to X$

a morphism $tra_\nabla(\gamma) : P_x \to P_y$ between the fibers of the bundle above the endpoints of these paths, in a compatible way:

$$\begin{array}{ccc} P_x \xrightarrow{\mathrm{tra}_\nabla(\gamma)} P_y \xrightarrow{\mathrm{tra}_\nabla(\gamma')} P_z & & P \\ & & \downarrow \\ x \xrightarrow{\gamma} y \xrightarrow{\gamma'} z & & X \end{array}$$

In order to formalize this, we introduce a (diffeological) Lie groupoid to be called the *path groupoid* of X. (Constructions and results in this section are from [ScWaI].

Definition 1.2.75. For X a smooth manifold let $[I, X]$ be the set of smooth functions $I = [0, 1] \to X$. For U a Cartesian space, we say that *a U-parameterized smooth family of points in* $[I, X]$ is a smooth map $U \times I \to X$. (This makes $[I, X]$ a diffeological space).

Say a path $\gamma \in [I, X]$ has *sitting instants* if it is constant in a neighbourhood of the boundary ∂I. Let $[I, P]_{\mathrm{si}} \subset [I, P]$ be the subset of paths with sitting instants.

Let $[I, X]_{\mathrm{si}} \to [I, X]_{\mathrm{si}}^{\mathrm{th}}$ be the projection to the set of equivalence classes where two paths are regarded as equivalent if they are cobounded by a smooth thin homotopy.

Say a U-parameterized smooth family of points in $[I, X]_{\mathrm{si}}^{\mathrm{th}}$ is one that comes from a U-family of representatives in $[I, X]_{\mathrm{si}}$ under this projection. (This makes also $[I, X]_{\mathrm{si}}^{\mathrm{th}}$ a diffeological space.)

The passage to the subset and quotient $[I, X]_{\mathrm{si}}^{\mathrm{th}}$ of the set of all smooth paths in the above definition is essentially the minimal adjustment to enforce that the concatenation of smooth paths at their endpoints defines the composition operation in a groupoid.

Definition 1.2.76. The *path groupoid* $\mathbf{P}_1(X)$ is the groupoid

$$\mathbf{P}_1(X) = ([I, X]_{\mathrm{si}}^{th} \overset{\to}{\to} X)$$

with source and target maps given by endpoint evaluation and composition given by concatenation of classes $[\gamma]$ of paths along any orientation preserving *diffeomorphism* $[0, 1] \to [0, 2] \simeq [0, 1] \coprod_{1,0} [0, 1]$ of any of their representatives

$$[\gamma_2] \circ [\gamma_1] : [0, 1] \overset{\simeq}{\to} [0, 1] \coprod_{1,0} [0, 1] \overset{(\gamma_2, \gamma_1)}{\to} X .$$

This becomes an internal groupoid in diffeological spaces with the above U-families of smooth paths. We regard it as a groupoid-valued presheaf, an object in $[CartSp^{op}, \mathrm{Grpd}]$:

$$\mathbf{P}_1(X) : U \mapsto (\mathrm{SmoothMfd}(U \times I, X)_{\mathrm{si}}^{\mathrm{th}} \overset{\to}{\to} \mathrm{SmoothMfd}(U, X)) .$$

Observe now that for G a Lie group and $\mathbf{B}G$ its delooping Lie groupoid discussed above, a smooth functor $\mathrm{tra} : \mathbf{P}_1(X) \to \mathbf{B}G$ sends each (thin-homotopy class of a) path to an element of the group G

$$\mathrm{tra} : (x \overset{[\gamma]}{\to} y) \mapsto (\bullet \overset{\mathrm{tra}(\gamma) \in G}{\to} \bullet)$$

such that composite paths map to products of group elements :

$$\mathrm{tra} : \left\{ \begin{array}{ccccc} & & y & & \\ & \overset{[\gamma]}{\nearrow} & \| & \overset{[\gamma']}{\searrow} & \\ x & & \xrightarrow[{[\gamma' \circ \gamma]}]{} & & z \end{array} \right\} \mapsto \left\{ \begin{array}{ccccc} & & * & & \\ & \overset{\mathrm{tra}(\gamma)}{\nearrow} & \| & \overset{\mathrm{tra}(\gamma')}{\searrow} & \\ * & & \xrightarrow[\mathrm{tra}(\gamma')\mathrm{tra}(\gamma)]{} & & * \end{array} \right\} .$$

and such that U-families of smooth paths induce smooth maps $U \to G$ of elements.

There is a classical construction that yields such an assignment: the *parallel transport* of a *Lie-algebra valued 1-form*.

Definition 1.2.77. Suppose $A \in \Omega^1(X, \mathfrak{g})$ is a degree-1 differential form on X with values in the Lie algebra $\mathfrak{g}$ of G. Then its parallel transport is the smooth functor

$$\mathrm{tra}_A : \mathbf{P}_1(X) \to \mathbf{B}G$$

given by

$$[\gamma] \mapsto P\exp(\int_{[0,1]} \gamma^* A) \ \in G\,,$$

where the group element on the right is defined to be the value at 1 of the unique solution $f : [0,1] \to G$ of the differential equation

$$d_{\mathrm{dR}} f + \gamma^* A \wedge f = 0$$

for the boundary condition $f(0) = e$.

Proposition 1.2.78. *This construction $A \mapsto \mathrm{tra}_A$ induces an equivalence of categories*

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd}](\mathbf{P}_1(X), \mathbf{B}G) \simeq \mathbf{B}G_{\mathrm{conn}}(X)\,,$$

where on the left we have the hom-groupoid of groupoid-valued presheaves, and where on the right we have the groupoid of Lie-algebra valued 1-forms*, whose*

- *objects are 1-forms $A \in \Omega^1(X, \mathfrak{g})$,*
- *morphisms $g : A_1 \to A_2$ are labeled by smooth functions $g \in C^\infty(X, G)$ such that $A_2 = g^{-1}Ag + g^{-1}dg$.*

This equivalence is natural in X, so that we obtain another smooth groupoid.

Definition 1.2.79. Define $\mathbf{B}G_{\mathrm{conn}} : \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Grpd}$ to be the (generalized) Lie groupoid

$$\mathbf{B}G_{\mathrm{conn}} : U \mapsto [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd}](\mathbf{P}_1(-), \mathbf{B}G)$$

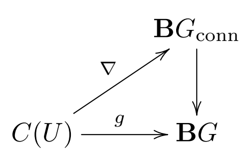
whose U-parameterized smooth families of groupoids form the groupoid of Lie-algebra valued 1-forms on U.

This equivalence in particular subsumes the classical facts that parallel transport $\gamma \mapsto P\exp(\int_{[0,1]} \gamma^* A)$

- is invariant under orientation preserving reparameterizations of paths;
- sends reversed paths to inverses of group elements.

Observation 1.2.80. There is an evident natural smooth functor $X \to \mathbf{P}_1(X)$ that includes points in X as constant paths. This induces a natural morphism $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$ that forgets the 1-forms.

Definition 1.2.81. Let $P \to X$ be a G-principal bundle that corresponds to a cocycle $g : C(U) \to \mathbf{B}G$ under the construction discussed above. Then a *connection* ∇ on P is a lift ∇ of the cocycle through $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$.



Observation 1.2.82. This is equivalent to the traditional definitions.

A morphism $\nabla : C(U) \to \mathbf{B}G_{\mathrm{conn}}$ is

- on each U_i a 1-form $A_i \in \Omega^1(U_i, \mathfrak{g})$;
- on each $U_i \cap U_j$ a function $g_{ij} \in C^\infty(U_i \cap U_j, G)$;

such that

- on each $U_i \cap U_j$ we have $A_j = g_{ij}^{-1}(A + d_{\mathrm{dR}})g_{ij}$;
- on each $U_i \cap U_j \cap U_k$ we have $g_{ij} \cdot g_{jk} = g_{ik}$.

Definition 1.2.83. Let $[I, X]^{\mathrm{th}}_{\mathrm{si}} \to [I, X]^h$ the projection onto the full quotient by smooth homotopy classes of paths. Write $\mathbf{\Pi}_1(X) = ([I, X]^h \rightrightarrows X)$ for the smooth groupoid defined as $\mathbf{P}_1(X)$, but where instead of thin homotopies, all homotopies are divided out.

Proposition 1.2.84. *The above restricts to a natural equivalence*

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd}](\mathbf{\Pi}_1(X), \mathbf{B}G) \simeq \flat\mathbf{B}G\,,$$

where on the left we have the hom-groupoid of groupoid-valued presheaves, and on the right we have the full sub-groupoid $\flat\mathbf{B}G \subset \mathbf{B}G_{\mathrm{conn}}$ *on those* $\mathfrak{g}$*-valued differential forms whose curvature 2-form* $F_A = d_{\mathrm{dR}}A + [A \wedge A]$ *vanishes.*

A connection ∇ *is* flat *precisely if it factors through the inclusion* $\flat\mathbf{B}G \to \mathbf{B}G_{\mathrm{conn}}$.

For the purposes of Chern-Weil theory we want a good way to extract the curvature 2-form in a general abstract way from a cocycle $\nabla : X \stackrel{\simeq}{\leftarrow} C(U) \to \mathbf{B}G_{\mathrm{conn}}$. In order to do that, we first need to discuss connections on 2-bundles.

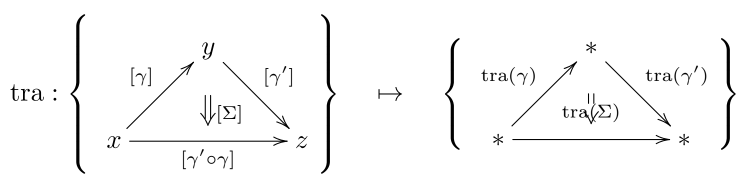
1.2.11.1.2 Connections on a principal 2-bundle There is an evident higher dimensional generalization of the definition of connections on 1-bundles in terms of functors out of the path groupoid discussed above. This we discuss now. We will see that, however, the obvious generalization captures not quite all 2-connections. But we will also see a way to recode 1-connections in terms of flat 2-connections. And that recoding then is the right general abstract perspective on connections, which generalizes to principal ∞-bundles and in fact which in the full theory follows from first principles.

(Constructions and results in this section are from [ScWaII], [ScWaIII].)

Definition 1.2.85. The path *path 2-groupoid* $\mathbf{P}_2(X)$ is the smooth strict 2-groupoid analogous to $\mathbf{P}_1(X)$, but with nontrivial 2-morphisms given by thin homotopy-classes of disks $\Delta^2_{Diff} \to X$ with sitting instants.

In analogy to the projection $\mathbf{P}_1(X) \to \mathbf{\Pi}_1(X)$ there is a projection to $\mathbf{P}_2(X) \to \mathbf{\Pi}_2(X)$ to the 2-groupoid obtained by dividing out full homotopy of disks, relative boundary.

We want to consider 2-functors out of the path 2-groupoid into connected 2-groupoids of the form $\mathbf{B}G$, for G a *2-group*, def. 1.2.45. A smooth 2-functor $\mathbf{\Pi}_2(X) \to \mathbf{B}G$ now assigns information also to surfaces



and thus encodes *higher parallel transport.*

Proposition 1.2.86. *There is a natural equivalence of 2-groupoids*

$$[\mathrm{CartSp}^{\mathrm{op}}, 2\mathrm{Grpd}](\mathbf{\Pi}_2(X), \mathbf{B}G) \simeq \flat\mathbf{B}G$$

where on the right we have the 2-groupoid of Lie 2-algebra valued forms] whose

- *objects are pairs* $A \in \Omega^1(X, \mathfrak{g}_1)$, $B \in \Omega^2(X, \mathfrak{g}_2)$ *such that the 2-form curvature*

  $$F_2(A, B) := d_{\mathrm{dR}}A + [A \wedge A] + \delta_* B$$

  and the 3-form curvature

  $$F_3(A, B) := d_{\mathrm{dR}}B + [A \wedge B]$$

  vanish.

- *morphisms* $(\lambda, a) : (A, B) \to (A', B')$ *are pairs* $a \in \Omega^1(X, \mathfrak{g}_2)$, $\lambda \in C^\infty(X, G_1)$ *such that* $A' = \lambda A \lambda^{-1} + \lambda d \lambda^{-1} + \delta_* a$ *and* $B' = \lambda(B) + d_{\mathrm{dR}} a + [A \wedge a]$
- *The description of 2-morphisms we leave to the reader (see [ScWaII]).*

As before, this is natural in X, so that we that we get a presheaf of 2-groupoids

$$\flat \mathbf{B}G : U \mapsto [\mathrm{CartSp}^{\mathrm{op}}, 2\mathrm{Grpd}](\mathbf{\Pi}_2(U), \mathbf{B}G)\,.$$

Proposition 1.2.87. *If in the above definition we use* $\mathbf{P}_2(X)$ *instead of* $\mathbf{\Pi}_2(X)$*, we obtain the same 2-groupoid, except that the 3-form curvature* $F_3(A, B)$ *is not required to vanish.*

Definition 1.2.88. Let $P \to X$ be a G-principal 2-bundle classified by a cocycle $C(U) \to \mathbf{B}G$. Then a structure of a *flat connection on a 2-bundle* ∇ on it is a lift

$$\begin{array}{ccc} & & \flat \mathbf{B}G \\ & \nearrow^{\nabla_{\mathrm{flat}}} & \downarrow \\ C(U) & \xrightarrow{g} & \mathbf{B}G \end{array}\,.$$

For $G = \mathbf{B}A$, a *connection on a 2-bundle* (not necessarily flat) is a lift

$$\begin{array}{ccc} & & [\mathbf{P}_2(-), \mathbf{B}^2 A] \\ & \nearrow^{\nabla_{\mathrm{flat}}} & \downarrow \\ C(U) & \xrightarrow{g} & \mathbf{B}G \end{array}\,.$$

We do not state the last definition for general Lie 2-groups G. The reason is that for general G 2-anafunctors out of $\mathbf{P}_2(X)$ do not produce the fully general notion of 2-connections that we are after, but yield a special case in between flatness and non-flatness: the case where precisely the 2-form curvature-components vanish, while the 3-form curvature part is unrestricted. This case is important in itself and discussed in detail below. Only for G of the form $\mathbf{B}A$ does the 2-form curvature necessarily vanish anyway, so that in this case the definition by morphisms out of $\mathbf{P}_2(X)$ happens to already coincide with the proper general one. This serves in the following theorem as an illustration for the toolset that we are exposing, but for the purposes of introducing the full notion of ∞-Chern-Weil theory we will rather focus on flat 2-connections, and then show below how using these one does arrive at a functorial definition of 1-connections that does generalize to the fully general definition of ∞-connections.

Proposition 1.2.89. *Let* $\{U_i \to X\}$ *be a good open cover, a cocycle* $C(U) \to [\mathbf{P}_2(-), \mathbf{B}^2 A]$ *is a cocycle in Čech-Deligne cohomology in degree 3.*

Moreover, we have a natural equivalence of bicategories

$$[\mathrm{CartSp}^{\mathrm{op}}, 2\mathrm{Grpd}](C(U), [\mathbf{P}_2(-), \mathbf{B}^2 U(1)]) \simeq U(1)\mathrm{Gerb}_\nabla(X)\,,$$

where on the right we have the bicategory of $U(1)$*-bundle gerbes with connection [Gaje97].*

In particular the equivalence classes of cocycles form the degree-3 ordinary differential cohomology of X*:*

$$H^3_{\mathrm{diff}}(X, \mathbb{Z}) \simeq \pi_0([C(U), [\mathbf{P}_2(-), \mathbf{B}^2 U(1))\,.$$

A cocycle as above naturally corresponds to a 2-anafunctor

$$\begin{array}{ccc} Q & \longrightarrow & \mathbf{B}^2 U(1) \\ \downarrow{\simeq} & & \\ \mathbf{P}_2(X) & & \end{array}$$

The value of this on 2-morphisms in $\mathbf{P}_2(X)$ is the higher parallel transport of the connection on the 2-bundle. This appears for instance in the action functional of the sigma model that describes strings charged under a Kalb-Ramond field.

The following example of a flat nonabelian 2-bundle is very degenerate as far as 2-bundles go, but does contain in it the seed of a full understanding of connections on 1-bundles.

Definition 1.2.90. For G a Lie group, its inner automorphism 2-group $\mathrm{INN}(G)$ is as a groupoid the universal G-bundle $\mathbf{E}G$, but regarded as a 2-group with the group structure coming from the crossed module $[G \stackrel{Id}{\to} G]$.

The depiction of the delooping 2-groupoid $\mathbf{B}\mathrm{INN}(G)$ is

$$\mathbf{B}\mathrm{INN}(G) = \left\{ \begin{array}{ccc} & * & \\ {}^{g_1}\nearrow & \Downarrow k & \searrow^{g_2} \\ * & \xrightarrow[k g_2 g_1]{} & * \end{array} \quad \middle| \; g_1, g_2 \in G, k \in G \right\}.$$

This is the Lie 2-group whose Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ is the one whose Chevalley-Eilenberg algebra is the Weil algebra of $\mathfrak{g}$.

Example 1.2.91. By the above theorem we have that there is a bijection of sets

$$\{\mathbf{\Pi}_2(X) \to \mathbf{B}\mathrm{INN}(G)\} \simeq \Omega^1(X, \mathfrak{g})$$

of flat $\mathrm{INN}(G)$-valued 2-connections and Lie-algebra valued 1-forms. Under the identifications of this theorem this identification works as follows:

- the 1-form component of the 2-connection is A;
- the vanishing of the 2-form component of the 2-curvature $F_2(A, B) = F_A + B$ identifies the 2-form component of the 2-connection with the curvature 2-form, $B = -F_A$;
- the vanishing of the 3-form component of the 3-curvature $F_3(A, B) = dB + [A \wedge B] = d_A + [A \wedge F_A]$ is the Bianchi identity satisfied by any curvature 2-form.

This means that 2-connections with values in $\mathrm{INN}(G)$ actually model 1-connections *and* keep track of their curvatures. Using this we see in the next section a general abstract definition of connections on 1-bundles that naturally supports the Chern-Weil homomorphism.

1.2.11.1.3 Curvature characteristics of 1-bundles We now describe connections on 1-bundles in terms of their *flat curvature 2-bundles* .

Throughout this section G is a Lie group, $\mathbf{B}G$ its delooping 2-groupoid and $\mathrm{INN}(G)$ its inner automorphism 2-group and $\mathbf{B}\mathrm{INN}(G)$ the corresponding delooping Lie 2-groupoid.

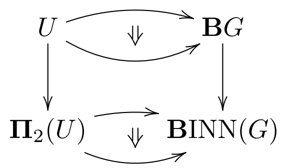
Definition 1.2.92. Define the smooth groupoid $\mathbf{B}G_{\mathrm{diff}} \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd}]$ as the pullback

$$\mathbf{B}G_{\mathrm{diff}} = \mathbf{B}G \times_{\mathbf{B}\mathrm{INN}(G)} \flat\mathbf{B}\mathrm{INN}(G)\,.$$

This is the groupoid-valued presheaf which assigns to $U \in \mathrm{CartSp}$ the groupoid whose objects are commuting diagrams

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{B}G \\ \downarrow & & \downarrow \\ \mathbf{\Pi}_2(U) & \longrightarrow & \mathbf{B}\mathrm{INN}(G) \end{array} \quad ,$$

where the vertical morphisms are the canonical inclusions discussed above, and whose morphisms are compatible pairs of natural transformations



of the horizontal morphisms.

By the above theorems, we have over any $U \in \mathrm{CartSp}$ that

- an object in $\mathbf{B}G_{\mathrm{diff}}(U)$ is a 1-form $A \in \Omega^1(U, \mathfrak{g})$;
- amorphism $A_1 \stackrel{(g,a)}{\to} A_2$ is labeled by a function $g \in C^\infty(U, G)$ and a 1-form $a \in \Omega^1(U, \mathfrak{g})$ such that

$$A_2 = g^{-1} A_1 g + g^{-1} dg + a \,.$$

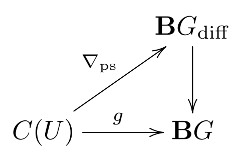
  Notice that this can always be uniquely solved for a, so that the genuine information in this morphism is just the data given by g.
- ther are *no* nontrivial 2-morphisms, even though $\mathbf{B}\mathrm{INN}(G)$ is a 2-groupoid: since $\mathbf{B}G$ is just a 1-groupoid this is enforced by the commutativity of the above diagram.

From this it is clear that

Proposition 1.2.93. *The projection* $\mathbf{B}G_{\mathrm{diff}} \stackrel{\simeq}{\to} \mathbf{B}G$ *is a weak equivalence.*

So $\mathbf{B}G_{\mathrm{diff}}$ is a resolution of $\mathbf{B}G$. We will see that it is the resoluton that supports 2-anafunctors out of $\mathbf{B}G$ which represent curvature characteristic classes.

Definition 1.2.94. For $X \stackrel{\simeq}{\leftarrow} C(U) \to \mathbf{B}U(1)$ a cocycle for a $U(1)$-principal bundle $P \to X$, we call a lift ∇_{ps} in



a *pseudo-connection* on P.

Pseudo-connections in themselves are not very interesting. But notice that every ordinary connection is in particular a pseudo-connection and we have an inclusion morphism of smooth groupoids

$$\mathbf{B}G_{\mathrm{conn}} \hookrightarrow \mathbf{B}G_{\mathrm{diff}} \,.$$

This inclusion plays a central role in the theory. The point is that while $\mathbf{B}G_{\mathrm{diff}}$ is such a boring extension of $\mathbf{B}G$ that it is actually equivalent to $\mathbf{B}G$, there is no inclusion of $\mathbf{B}G_{\mathrm{conn}}$ into $\mathbf{B}G$, but there is into $\mathbf{B}G_{\mathrm{diff}}$. This is the kind of situation that resolutions are needed for.

It is useful to look at some details for the case that G is an abelian group such as the circle group $U(1)$. In this abelian case the 2-groupoids $\mathbf{B}U(1)$, $\mathbf{B}^2U(1)$, $\mathbf{B}INN(U(1))$, etc., that so far we noticed are given by crossed complexes are actually given by ordinary chain complexes: we write

$$\Xi : \mathrm{Ch}_\bullet^+ \to s\mathrm{Ab} \to \mathrm{KanCplx}$$

for the Dold-Kan correspondence map that identifies chain complexes with simplicial abelian group and then considers their underlying Kan complexes. Using this map we have the following identifications of our 2-groupoid valued presheaves with complexes of group-valued sheaves

$$\mathbf{B}U(1) = \Xi[C^\infty(-, U(1)) \to 0]$$

$$\mathbf{B}^2 U(1) = \Xi[C^\infty(-, U(1)) \to 0 \to 0]$$

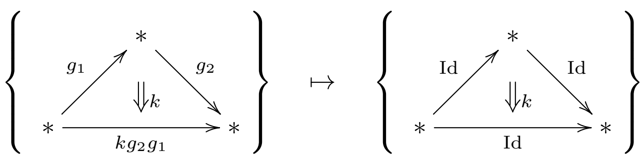
$$\mathbf{B}\mathrm{INN}U(1) = \Xi[C^\infty(-, U(1)) \stackrel{\mathrm{Id}}{\to} C^\infty(-, U(1)) \to 0]\,.$$

Observation 1.2.95. For $G = A$ an abelian group, in particular the circle group, there is a canonical morphism $\mathbf{B}\mathrm{INN}(U(1)) \to \mathbf{B}\mathbf{B}U(1)$.

On the level of chain complexes this is the evident chain map

$$\begin{array}{ccccc} [C^\infty(-, U(1)) & \stackrel{Id}{\longrightarrow} & C^\infty(-, U(1)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ [C^\infty(-, U(1)) & \longrightarrow & 0 & \longrightarrow & 0] \end{array} .$$

On the level of 2-groupoids this is the map that forgets the labels on the 1-morphisms



In terms of this map $\mathrm{INN}(U(1))$ serves to interpolate between the single and the double delooping of $U(1)$. In fact the sequence of 2-functors

$$\mathbf{B}U(1) \to \mathbf{B}\mathrm{INN}(U(1)) \to \mathbf{B}^2 U(1)$$

is a model for the universal $\mathbf{B}U(1)$-principal 2-bundle

$$\mathbf{B}U(1) \to \mathbf{E}\mathbf{B}U(1) \to \mathbf{B}^2 U(1)\,.$$

This happens to be an exact sequence of 2-groupoids. Abstractly, what really matters is rather that it is a fiber sequence, meaning that it is exact in the correct sense inside the ∞-category Smooth∞Grpd. For our purposes it is however relevant that this particular model is exact also in the ordinary sense in that we have an ordinary pullback diagram

$$\begin{array}{ccc} \mathbf{B}U(1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{INN}(U(1)) & \longrightarrow & \mathbf{B}^2 U(1) \end{array} ,$$

exhibitng $\mathbf{B}U(1)$ as the kernel of $\mathbf{B}\mathrm{INN}(U(1)) \to \mathbf{B}^2 U(1)$.

We shall be interested in the pasting composite of this diagram with the one defining $\mathbf{B}G_{\mathrm{diff}}$ over a domain U:

$$\begin{array}{ccccc} U & \longrightarrow & \mathbf{B}U(1) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{\Pi}_2(U) & \longrightarrow & \mathbf{B}\mathrm{INN}(U(1)) & \longrightarrow & \mathbf{B}^2 U(1) \end{array} ,$$

The total outer diagram appearing this way is a component of the following (generalized) Lie 2-groupoid.

Definition 1.2.96. Set
$$\flat_{\mathrm{dR}}\mathbf{B}^2U(1) := * \times_{\mathbf{B}^2U(1)} \flat\mathbf{B}^2U(1)\,.$$

Over any $U \in \mathrm{CartSp}$ this is the 2-groupoid whose objects are sets of diagrams
$$\begin{array}{ccc} U & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{\Pi}_2(U) & \longrightarrow & \mathbf{B}^2U(1) \end{array} \quad .$$

This are equivalently just morphisms $\mathbf{\Pi}_2(U) \to \mathbf{B}^2U(1)$, which by the above theorems we may identify with closed 2-forms $B \in \Omega^2_{\mathrm{cl}}(U)$.

The morphisms $B_1 \to B_2$ in $\flat_{\mathrm{dR}}\mathbf{B}^2U(1)$ over U are compatible pseudonatural transformations of the horizontal morphisms
$$\begin{array}{ccc} U & \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} & * \\ \downarrow & & \downarrow \\ \mathbf{\Pi}_2(U) & \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} & \mathbf{B}\mathrm{INN}(G) \end{array} \quad ,$$
which means that they are pseudonatural transformations of the bottom morphism whose components over the points of U vanish. These identify with 1-forms $\lambda \in \Omega^1(U)$ such that $B_2 = B_1 + d_{\mathrm{dR}}\lambda$. Finally the 2-morphisms would be modifications of these, but the commutativity of the above diagram constrais these to be trivial.

In summary this shows that

Proposition 1.2.97. *Under the Dold-Kan correspondence $\flat_{\mathrm{dR}}\mathbf{B}^2U(1)$ is the sheaf of truncated de Rham complexes*
$$\flat_{\mathrm{dR}}\mathbf{B}^2U(1) = \Xi[\Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^2_{\mathrm{cl}}(-)]\,.$$

Corollary 1.2.98. *Equivalence classes of 2-anafunctors*
$$X \to \flat_{\mathrm{dR}}\mathbf{B}^2U(1)$$
are canonically in bijection with the degree 2 de Rham cohomology of X.

Notice that – while every globally defined closed 2-form $B \in \Omega^2_{\mathrm{cl}}(X)$ defines such a 2-anafunctor – not every such 2-anafunctor comes from a globally defined closed 2-form. Some of them assign closed 2-forms B_i to patches U_1, that differ by differentials $B_j - B_i = d_{\mathrm{dR}}\lambda_{ij}$ of 1-forms λ_{ij} on double overlaps, which themselves satisfy on triple intersections the cocycle condition $\lambda_{ij} + \lambda_{jk} = \lambda_{ik}$. But (using a partition of unity) these non-globally defined forms are always equivalent to globally defined ones.

This simple technical point turns out to play a role in the abstract definition of connections on ∞-bundles: generally, for all $n \in \mathbb{N}$ the cocycles given by globally defined forms in $\flat_{\mathrm{dR}}\mathbf{B}^nU(1)$ constitute curvature characteristic forms of *genuine* connections. The non-globally defined forms *also* constitute curvature invariants, but of pseudo-connections. The way the abstract theory finds the genuine connections inside all pseudo-connections is by the fact that we may find for each cocycle in $\flat_{\mathrm{dR}}\mathbf{B}^nU(1)$ an equivalent one that does comes from a globally defined form.

Observation 1.2.99. There is a canonical 2-anafunctor $\hat{\mathbf{c}}_1^{\mathrm{dR}} : \mathbf{B}U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^2U(1)$
$$\begin{array}{ccc} \mathbf{B}U(1)_{\mathrm{diff}} & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}^2U(1) \\ \downarrow{\simeq} & & \\ \mathbf{B}U(1) & & \end{array} \quad ,$$
where the top morphism is given by forming the -composite with the universal $\mathbf{B}U(1)$-principal 2-bundle, as described above.

For emphasis, notice that this span is governed by a presheaf of diagrams that over $U \in CartSp$ is of the form

$$
\begin{array}{ccccc}
U & \longrightarrow & \mathbf{B}U(1) & & \text{transition function} \;. \\
\downarrow & & \downarrow & & \\
\mathbf{\Pi}_2(U) & \longrightarrow & \mathbf{B}\mathrm{INN}(U) & & \text{connection} \\
\downarrow & & \downarrow & & \\
\mathbf{\Pi}_2(U) & \longrightarrow & \mathbf{B}^2 U(1) & & \text{curvature}
\end{array}
$$

The top morphisms are the components of the presheaf $\mathbf{B}U(1)$. The top squares are those of $\mathbf{B}U(1)_{\mathrm{diff}}$. Forming the bottom square is forming the bottom morphism, which necessarily satifies the constraint that makes it a components of $\flat \mathbf{B}^2 U(1)$.

The interpretation of the stages is as indicated in the diagram:

1. the top morphism is the transition function of the underlying bundle;
2. the middle morphism is a choice of (pseudo-)connection on that bundle;
3. the bottom morphism picks up the curvature of this connection.

We will see that full ∞-Chern-Weil theory is governed by a slight refinement of presheaves of essentially this kind of diagram. We will also see that the three stage process here is really an incarnation of the computation of a connecting homomorphism, reflecting the fact that behind the scenes the notion of *curvature* is exhibited as the obstruction cocycle to lifts from bare bundles to flat bundles.

Observation 1.2.100. For $X \stackrel{\simeq}{\leftarrow} C(U) \stackrel{g}{\to} \mathbf{B}U(1)$ the cocycle for a $U(1)$-principal bundle as described above, the composition of 2-anafunctors of g with $\hat{\mathbf{c}}_1^{\mathrm{dR}}$ yields a cocycle for a 2-form $\hat{\mathbf{c}}_1^{\mathrm{dR}}(g)$

$$
\begin{array}{ccccc}
 & \stackrel{\nabla}{\nearrow} & \mathbf{B}U(1)_{\mathrm{conn}} & & \\
 & & \downarrow & & \\
C(V) & \longrightarrow & \mathbf{B}U(1)_{\mathrm{diff}} & \longrightarrow & \flat_{\mathrm{dR}} \mathbf{B}^2 U(1) \\
\downarrow \simeq & & \downarrow \simeq & & \\
C(U) & \stackrel{g}{\longrightarrow} & \mathbf{B}U(1) & & \\
\downarrow \simeq & & & & \\
X & & & &
\end{array}
\;.
$$

If we take $\{U_i \to X\}$ to be a good open cover, then we may assume $V = U$. We know we can always find a pseudo-connection $C(V) \to \mathbf{B}U(1)_{\mathrm{diff}}$ that is actually a genuine connection on a bundle in that it factors through the inclusion $\mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{diff}}$ as indicated.

The corresponding total map $c_1^{\mathrm{dR}}(g)$ represented by $\hat{\mathbf{c}}_1^{\mathrm{dR}}(\nabla)$ is the cocycle for the curvature 2-form of this connection. This represents the first Chern class of the bundle in de Rham cohomology.

For X, A smooth 2-groupoids, write $\mathbf{H}(X, A)$ for the 2-groupoid of 2-anafunctors between them.

Corollary 1.2.101. *Let $H^2_{\mathrm{dR}}(X) \to \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^2 U(1))$ be a choice of one closed 2-form representative for* *each degree-2 de Rham cohomology-class of X. Then the pullback groupoid $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}U(1))$ in*

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}U(1)) & \longrightarrow & H^2_{\mathrm{dR}}(X) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}U(1)_{\mathrm{diff}}) & \longrightarrow & \mathbf{H}(X, \flat_{\mathrm{dR}}\mathbf{B}^2 U(1)) \\ \downarrow \simeq & & \\ \mathbf{H}(X, \mathbf{B}U(1)) \simeq U(1)\mathrm{Bund}(X) & & \end{array}$$

is equivalent to disjoint union of groupoids of $U(1)$-bundles with connection whose curvatures are the chosen 2-form representatives.

1.2.11.1.4 Circle n-bundles with connection For A an abelian group there is a straightforward generalization of the above constructions to $(G = \mathbf{B}^{n-1}A)$-principal n-bundles with connection for all $n \in \mathbb{N}$. We spell out the ingredients of the construction in a way analogous to the above discussion. A first-principles derivation of the objects we consider here below in 4.4.16.

This is content that appeared partly in [SSS09c], [FSS10]. We restrict attention to the circle n-group $G = \mathbf{B}^{n-1}U(1)$.

There is a familiar traditional presentation of ordinary differential cohomology in terms of Cech-Deligne cohomology. We briefly recall how this works and then indicate how this presentation can be derived along the above lines as a presentation of circle n-bundles with connection.

Definition 1.2.102. For $n \in \mathbb{N}$ the *Deligne-Beilinson complex* is the chain complex of sheaves (on CartSp for our purposes here) of abelian groups given as follows

$$\mathbb{Z}(n+1)^\infty_D = \left[\begin{array}{ccccccccc} C^\infty(-, \mathbb{R}/\mathbb{Z}) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1(-) & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{n-1}(-) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n(-) \\ n & & n-1 & & \cdots & & 1 & & 0 \end{array} \right].$$

This definition goes back to [Del71] [Bel85]. The complex is similar to the n-fold shifted de Rham complex, up to two important differences.

- In degree n we have the sheaf of $U(1)$-valued functions, not of $\mathbb{R}$-valued functions (= 0-forms). The action of the de Rham differential on this is often written $d\log : C^\infty(-, U(1)) \to \Omega^1(-)$. But if we think of $U(1) \simeq \mathbb{R}/\mathbb{Z}$ then it is just the ordinary de Rham differential applied to any representative in $C^\infty(-, \mathbb{R})$ of an element in $C^\infty(-, \mathbb{R}/\mathbb{Z})$.
- In degree 0 we do not have closed differential n-forms (as one would have for the the de Rham complex shifted into non-negative degree), but all n-forms.

As before, we may use of the Dold-Kan correspondence $\Xi : \mathrm{Ch}^+_\bullet \stackrel{\simeq}{\to} \mathrm{sAb} \stackrel{U}{\to} \mathrm{sSet}$ to identify sheaves of chain complexes with simplicial sheaves. We write

$$\mathbf{B}^n U(1)_{\mathrm{conn}} := \Xi\, \mathbb{Z}(n+1)^\infty_D$$

for the simplicial presheaf corresponding to the Deligne complex.

Then for $\{U_i \to X\}$ a good open cover, the Deligne cohomology of X in degree $(n+1)$ is

$$H^{n+1}_{\mathrm{diff}}(X) = \pi_0[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}^n U(1)_{\mathrm{conn}})\,.$$

Further using the Dold-Kan correspondence, this is equivalently the cohomology of the Čech-Deligne double complex. A cocycle in degre $(n+1)$ then is a tuple

$$(g_{i_0,\cdots,i_n}, \cdots, A_{ijk}, B_{ij}, C_i)$$

with

- $C_i \in \Omega^n(U_i)$;
- $B_{ij} \in \Omega^{n-1}(U_i \cap U_j)$;
- $A_{ijk} \in \Omega^{n-2}(U_i \cap U_j \cap U_k)$
- and so on...
- $g_{i_0,\cdots,i_n} \in C^\infty(U_{i_0} \cap \cdots \cap U_{i_n}, U(1))$

satisfying the cocycle condition

$$(d_{\mathrm{dR}} + (-1)^{deg}\delta)(g_{i_0,\cdots,i_n}, \cdots, A_{ijk}, B_{ij}, C_i) = 0\,,$$

where $\delta = \sum_i (-1)^i p_i^*$ is the alternating sum of the pullback of forms along the face maps of the Čech nerve.

This is a sequence of conditions of the form

- $C_i - C_j = dB_{ij}$;
- $B_{ij} - B_{ik} + B_{jk} = dA_{ijk}$;
- and so on
- $(\delta g)_{i_0,\cdots,i_{n+1}} = 0$.

For low n we have seen these conditions in the dicussion of line bundles and of line 2-bundles (bundle gerbes) with connection above. Generally, for any $n \in \mathbb{N}$, this is Čech-cocycle data for a *circle n-bundle* with connection, where

- C_i are the local connection n-forms;
- $g_{i_0,\cdots,i_n}$ is the transition function of the circle n-bundle.

We now indicate how the Deligne complex may be derived from differential refinement of cocycles for circle n-bundles along the lines of the above discussions. To that end, write

$$\mathbf{B}^n U(1)_{\mathrm{ch}} := \Xi\, U(1)[n]\,,$$

for the simplicial presheaf given under the Dold-Kan correspondence by the chain complex

$$U(1)[n] = (C^\infty(-, U(1)) \to 0 \to \cdots \to 0)$$

with the sheaf represented by $U(1)$ in degree n.

Proposition 1.2.103. *For $\{U_i \to X\}$ an open cover of a smooth manifold X and $C(\{U_i\})$ its Čech nerve, ∞-anafunctors*

$$\begin{array}{ccc} C(\{U_i\}) & \xrightarrow{g} & \mathbf{B}^n U(1) \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array}$$

are in natural bijection with tuples of smooth functions

$$g_{i_0\cdots i_n} : U_{i_0} \cap \cdots \cap U_{i_n} \to \mathbb{R}/\mathbb{Z}$$

satisfying

$$(\partial g)_{i_0\cdots i_{n+1}} := \sum_{k=0}^{n} g_{i_0\cdots i_{k-1} i_k \cdot i_n} = 0\,,$$

that is, with cocycles in degree-n Čech cohomology on U with values in U(1).

Natural transformations

$$\begin{array}{ccc} C(\{U_i\}) \cdot \Delta^1 & \xrightarrow{(g \stackrel{\lambda}{\to} g')} & \mathbf{B}^n U(1) \\ \downarrow{\scriptstyle \simeq} & & \\ X \cdot \Delta^1 & & \end{array}$$

are in natural bijection with tuples of smooth functions

$$\lambda_{i_0\cdots i_{n-1}} : U_{i_0} \cap \cdots \cap U_{i_{n-1}} \to \mathbb{R}/\mathbb{Z}$$

such that

$$g'_{i_0\cdots i_n} - g_{i_0\cdots i_n} = (\delta\lambda)_{i_0\cdots i_n}\,,$$

that is, with Čech coboundaries.

The ∞-bundle $P \to X$ classified by such a cocycle according to 1.2.5.4 we call a *circle n-bundle*. For $n = 1$ this reproduces the ordinary $U(1)$-principal bundles that we considered before in 1.2.5.1, for $n = 2$ the bundle gerbes considered in 1.2.5.2 and for $n = 3$ the bundle 2-gerbes discussed in 1.2.5.3.

To equip these circle n-bundles with connections, we consider the differential refinements of $\mathbf{B}^n U(1)_{\mathrm{ch}}$ to be denoted $\mathbf{B}^n U(1)_{\mathrm{diff}}$, $\mathbf{B}^n U(1)_{\mathrm{conn}}$ and $\flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)$.

Definition 1.2.104. Write

$$\flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)_{\mathrm{chn}} := \Xi \left(\Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^2(-) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \stackrel{d_{\mathrm{dR}}}{\to} \Omega^n_{\mathrm{cl}}(-) \right)$$

– the *truncated de Rham complex* – and

$$\mathbf{B}^n U(1)_{\mathrm{diff}} = \left\{ \begin{array}{ccc} (-) & \longrightarrow & \mathbf{B}^n U(1) \\ \downarrow & & \downarrow \\ \mathbf{\Pi}(-) & \to & \mathbf{B}^n \mathrm{INN}(U(1)) \end{array} \right\} = \Xi \left(\begin{array}{ccccc} C^\infty(-, \mathbb{R}/\mathbb{Z}) & \to \Omega^1(-) & \xrightarrow{d_{\mathrm{dR}}} \cdots & \longrightarrow & \Omega^n(-) \\ \oplus & \nearrow_{\mathrm{Id}} & & \nearrow_{\mathrm{Id}} & \\ \Omega^1(-) & \xrightarrow[d_{\mathrm{dR}}]{} \cdots & \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-) & & \end{array} \right)$$

and

$$\mathbf{B}^n U(1)_{\mathrm{conn}} = \Xi \left(C^\infty(-, \mathbb{R}/\mathbb{Z}) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^2(-) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \stackrel{d_{\mathrm{dR}}}{\to} \Omega^n(-) \right)$$

– the *Deligne complex*, def. 1.2.102.

Observation 1.2.105. We have a pullback diagram

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1)_{\mathrm{diff}} & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^{n-1} U(1) \\ \downarrow{\scriptstyle \simeq} & & \\ \mathbf{B}^n U(1) & & \end{array}$$

in $[\mathrm{CartSp}^{op}, \mathrm{sSet}]$. This models an ∞-pullback

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1) & \longrightarrow & \flat_{\mathrm{dR}} \mathbf{B}^{n-1} U(1) \end{array}$$

in the ∞-topos Smooth∞Grpd, and hence for each smooth manifold X (in particular) a homotopy pullback

$$\begin{array}{ccc} \mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(X) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n U(1)) & \longrightarrow & \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n-1} U(1)) \end{array} .$$

We write

$$H^n_{\mathrm{diff}}(X) := \mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}})$$

for the group of cohomology classes on X with coefficients in $\mathbf{B}^n U(1)_{\mathrm{conn}}$. On these cohomology classes the above homotopy pullback diagram reduces to the commutative diagram

$$\begin{array}{ccccc} & & H^{n+1}_{\mathrm{diff}}(X) & & \\ & \swarrow & & \searrow & \\ H^{n+1}(X, \mathbb{Z}) & & & & \Omega^{n+1}_{\mathrm{cl}}(X) \\ & \searrow & & \swarrow & \\ & & H^{n+1}(X, \mathbb{R}) \simeq H^{n+1}_{\mathrm{dR}}(X) & & \end{array}$$

that had appeared above in 1.1.2. But notice that the homotopy pullback of the cocycle n-groupoids contains more information than this projection to cohomology classes.

Objects in $\mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}})$ are modeled by ∞-anafunctors $X \stackrel{\simeq}{\leftarrow} C(\{U_i\}) \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, and these are in natural bijection with tuples

$$\left(C_i, B_{i_0 i_1}, A_{i_0 i_1, i_2}, \cdots Z_{i_0 \cdots i_{n-1}}, g_{i_0 \cdots i_n}\right) ,$$

where $C_i \in \Omega^n(U_i)$, $B_{i_0 i_1} \in \Omega^{n-1}(U_{i_0} \cap U_{i_1})$, etc., such that

$$C_{i_0} - C_{i_1} = dB_{i_0 i_1}$$

and

$$B_{i_0 i_1} - B_{i_0 i_2} + B_{i_1 i_2} = dA_{i_0 i_1 i_2} ,$$

etc. This is a cocycle in Čech-Deligne cohomology. We may think of this as encoding a circle n-bundle with connection. The forms (C_i) are the local *connection n-forms*.

The definition of ∞-connections on G-principal ∞-bundles for nonabelian G may be reduced to this definition, by *approximating* every G-cocylce $X \stackrel{\simeq}{\leftarrow} C(\{U_i\}) \to \mathbf{B}G$ by abelian cocycles in all possible ways, by postcomposing with all possible *characteristic classes* $\mathbf{B}G \stackrel{\simeq}{\leftarrow} \widehat{\mathbf{B}G} \to \mathbf{B}^n U(1)$ to extract a circle n-bundle from it. This is what we turn to below in 1.2.12.1.

1.2.11.1.5 Holonomy and canonical action functionals We had started out with motivating differential refinements of bundles and higher bundles by the notion of higher parallel transport. Here we discuss aspects of this for the circle n-bundles

Let Σ be a compact smooth manifold of dimension n. For every smooth function $\Sigma \to X$ there is a corresponding pullback operation

$$H^{n+1}_{\mathrm{diff}}(X) \to H^{n+1}_{\mathrm{diff}}(\Sigma)$$

that sends circle n-connections on X to circle n-connections on Σ. But due to its dimension, the curvature $(n+1)$-form of any circle n-connection on Σ is necessarily trivial. From the definition of homotopy pullback one can show that this implies that every circle n-connection on Σ is equivalent to one which is given by a Cech-Deligne cocycle that involves a globally defined connection n-form ω. The integral of this form over Σ produces a real number. One finds that this is well-defined up to integral shifts. This gives an *n-volume holonomy* map

$$\int_\Sigma \;:\; \mathbf{H}(\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}}) \to U(1)\,.$$

For instance for $n = 1$ this is the map that sense an ordinary connection on an ordinary circle bundle over Σ to its ordinary parallel transport along Σ, its line holonomy.

For G any smooth (higher) group, any morphism

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

from the moduli stack of G-connections to that of circle n-connections therefore induces a canonical functional

$$\exp(iS_{\mathbf{c}}(-)) :\; \mathbf{H}(\Sigma, \mathbf{B}G_{\mathrm{conn}})) \xrightarrow{\mathbf{H}(\Sigma,\hat{\mathbf{c}})} \mathbf{H}(\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

from the ∞-groupoid of G-connections on Σ to $U(1)$.

1.2.11.2 Differential cohomology We now indicate how the combination of the *intrinsic cohomology* and the *geometric homotopy* in a locally ∞-connected ∞-topos yields a good notion of *differential cohomology in an ∞-topos.*

Using the defining adjoint ∞-functors $(\Pi \dashv \mathrm{Disc} \dashv \Gamma)$ we may reflect the fundamental ∞-groupoid $\Pi : \mathbf{H} \to \infty\mathrm{Grpd}$ from Top back into $\mathbf{H}$ by considering the composite endo-edjunction

$$(\mathbf{\Pi} \dashv \flat) := (\mathrm{Disc} \circ \Pi \dashv \mathrm{Disc} \circ \Gamma) :\; \mathbf{H} \rightleftarrows \mathbf{H}\;.$$

The $(\Pi \dashv \mathrm{Disc})$-unit $X \to \mathbf{\Pi}(X)$ may be thought of as the inclusion of X into its fundamental ∞-groupoid as the collection of constant paths in X.

As always, the boldface $\mathbf{\Pi}$ is to indicate that we are dealing with a cohesive refinement of the topological structure Π. The symbol "$\flat$" ("flat") is to be suggestive of the meaning of this construction:

For $X \in \mathbf{H}$ any cohesive object, we may think of $\Pi(X)$ as its cohesive fundamental ∞-groupoid. A morphism

$$\nabla : \mathbf{\Pi}(X) \to \mathbf{B}G$$

(hence a G-valued cocycle on $\mathbf{\Pi}(X)$) may be interpreted as assigning:

- to each point $x \in X$ the fiber of the corresponding G-principal ∞-bundle classified by the composite $g : X \to \mathbf{\Pi}(X) \stackrel{\nabla}{\to} \mathbf{B}G$;
- to each path in X an equivalence between the fibers over its endpoints;
- to each homotopy of paths in X an equivalence between these equivalences;
- and so on.

This in turn we may think as being the *flat higher parallel transport* of an ∞-connection on the bundle classified by $g : X \to \mathbf{\Pi}(X) \stackrel{\nabla}{\to} \mathbf{B}G$.

The adjunction equivalence allows us to identify $\flat\mathbf{B}G$ as the coefficient object for this flat differential G-valued cohomology on X:

$$H_{\mathrm{flat}}(X, G) := \pi_0 \mathbf{H}(X, \flat\mathbf{B}G) \simeq \pi_0 \mathbf{H}(\mathbf{\Pi}(X), \mathbf{B}G)\,.$$

In $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ and with $G \in \mathbf{H}$ an ordinary Lie group and $X \in \mathbf{H}$ an ordinary smooth manifold, we have that $H_{\mathrm{flat}}(X, G)$ is the set of equivalence classes of ordinary G-principal bundles on X with flat connections.

The $(\mathrm{Disc} \dashv \Gamma)$-counit $\flat\mathbf{B}G \to \mathbf{B}G$ provides the forgetful morphism

$$H_{\mathrm{flat}}(X, G) \to H(X, G)$$

form G-principal ∞-bundles with flat connection to their underlying principal ∞-bundles. Not every G-principal ∞-bundle admits a flat connection. The failure of this to be true - the obstruction to the existence of flat lifts - is measured by the homotopy fiber of the counit, which we shall denote $\flat_{\mathrm{dR}}\mathbf{B}G$, defined by the fact that we have a fiber sequence

$$\flat_{\mathrm{dR}}\mathbf{B}G \to \flat\mathbf{B}G \to \mathbf{B}G\,.$$

As the notation suggests, it turns out that $\flat_{\mathrm{dR}}\mathbf{B}G$ may be thought of as the coefficient object for nonabelian generalized de Rham cohomology. For instance for G an odinary Lie group regarded as an object in $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$, we have that $\flat_{\mathrm{dR}}\mathbf{B}G$ is presented by the sheaf $\Omega^1_{\mathrm{flat}}(-, \mathfrak{g})$ of Lie algebra valued differential forms with vanishing curvature 2-form. And for the circle Lie n-group $\mathbf{B}^{n-1}U(1)$ we find that $\flat_{\mathrm{dR}}\mathbf{B}^n U(1)$ is presented by the complex of sheaves whose abelian sheaf cohomology is de Rham cohomology in degree n. (More precisely, this is true for $n \geq 2$. For $n = 1$ we get just the sheaf of closed 1-forms. This is due to the obstruction-theoretic nature of $\flat_{\mathrm{dR}}$: as we shall see, in degree 1 it computes 1-form curvatures of groupoid principal bundles, and these are not quotiented by exact 1-forms.) Moreover, in this case our fiber sequence extends not just to the left but also to the right

$$\flat_{\mathrm{dR}}\mathbf{B}^n U(1) \to \flat\mathbf{B}^n U(1) \to \mathbf{B}^n U(1) \stackrel{\mathrm{curv}}{\to} \flat_{\mathrm{dR}}\mathbf{B}^{n+1} U(1)\,.$$

The induced morphism

$$\mathrm{curv}_X : \mathbf{H}(X, \mathbf{B}^n U(1)) \to \mathbf{H}(X, \flat_{\mathrm{dR}}\mathbf{B}^{n+1} U(1))$$

we may think of as equipping an $\mathbf{B}^{n-1}U(1)$-principal n-bundle (equivalently an $(n-1)$-bundle gerbe) with a connection, and then sending it to the higher curvature class of this connection. The homotopy fibers

$$\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n U(1)) \to \mathbf{H}(X, \mathbf{B}^n U(1)) \stackrel{\mathrm{curv}}{\to} \mathbf{H}(X, \flat_{\mathrm{dR}}\mathbf{B}^{n+1} U(1))$$

of this map therefore have the interpretation of being the cocycle ∞-groupoids of circle n-bundles with connection. This is the realization in $\mathrm{Smooth}\infty\mathrm{Grpd}$ of our general definition of ordinary differential cohomology in an ∞-topos.

All these definitions make sense in full generality for any locally ∞-connected ∞-topos. We used nothing but the existence of the triple of adjoint ∞-functors $(\Pi \dashv \mathrm{Disc} \dashv \Gamma) : \mathbf{H} \to \infty\mathrm{Grpd}$. We shall show for the special case that $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ and X an ordinary smooth manifold, that this general abstract definition reproduces ordinary differential cohomology over smooth manifolds as traditionally considered.

The advantage of the general abstract reformulation is that it generalizes the ordinary notion naturally to base objects that may be arbitrary smooth ∞-groupoids. This gives in particular the ∞-Chern-Weil homomorphism in an almost tautological form:

for $G \in \mathbf{H}$ any ∞-group object and $\mathbf{B}G \in \mathbf{H}$ its delooping, we may think of a morphism

$$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$$

as a representative of a characteristic class on G, in that this induces a morphism

$$[\mathbf{c}(-)] : H(X,G) \to H^n(X,U(1))$$

from G-principal ∞-bundles to degree-n cohomology-classes. Since the classification of G-principal ∞-bundles by cocycles is entirely general, we may equivalently think of this as the $\mathbf{B}^{n-1}U(1)$-principal ∞-bundle $P \to \mathbf{B}G$ given as the homotopy fiber of $\mathbf{c}$. A famous example is the Chern-Simons circle 3-bundle (bundle 2-gerbe) for G a simply connected Lie group.

By postcomposing further with the canonical morphism curv : $\mathbf{B}^nU(1) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ this gives in total a *differential characteristic class*

$$\mathbf{c}_{\mathrm{dR}} : \mathbf{B}G \stackrel{\mathbf{c}}{\to} \mathbf{B}^nU(1) \stackrel{\mathrm{curv}}{\to} \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$$

that sends a G-principal ∞-bundle to a class in de Rham cohomology

$$[\mathbf{c}_{\mathrm{dR}}] : H(X,G) \to H^{n+1}_{\mathrm{dR}}(X)\,.$$

This is the generalization of the plain Chern-Weil homomorphism.associated with the characteristic class c. In cases accessible by traditional theory, it is well known that this may be refined to what are called the assignment of *secondary characteristic classes* to G-principal bundles with connection, taking values in ordinary differential cohomology

$$[\hat{\mathbf{c}}] : H_{\mathrm{conn}}(X,G) \to H^{n+1}_{\mathrm{diff}}(X)\,.$$

We will discuss that in the general formulation this corresponds to finding objects $\mathbf{B}G_{\mathrm{conn}}$ that lift all curvature characteristic classes to their corresponding circle n-bundles with connection, in that it fits into the diagram

$$\begin{array}{ccccc}
\mathbf{H}(-,\mathbf{B}G_{\mathrm{conn}}) & \longrightarrow & \prod_i \mathbf{H}_{\mathrm{diff}}(-,\mathbf{B}^{n_i}U(1)) & \longrightarrow & \prod_i H^{n_i+1}_{\mathrm{dR}}(-) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{H}(-,\mathbf{B}G) & \longrightarrow & \prod_i \mathbf{H}(-,\mathbf{B}^{n_i}U(1)) & \stackrel{\mathrm{curv}}{\longrightarrow} & \prod_i \mathbf{H}(-,\flat_{\mathrm{dR}}\mathbf{B}^{n_i+1}U(1))
\end{array}$$

The cocycles in $\mathbf{H}_{\mathrm{conn}}(X,\mathbf{B}G) := \mathbf{H}(X,\mathbf{B}G_{\mathrm{conn}})$ we may identify with ∞-connections on the underlying principal ∞-bundles. Specifically for G an ordinary Lie group this captures the ordinary notion of connection on a bundle, for G Lie 2-group it captures the notion of connection on a 2-bundle/gerbe.

1.2.11.3 Higher geometric prequantization Observation. There is a canonical ∞-action γ of $\mathrm{Aut}_{\mathbf{H}_{/\mathbf{B}G}}(g)$ on the space of ∞-sections $\Gamma_X(P \times_G V)$.

Claim. Since $\mathrm{Sh}_\infty(\mathrm{SmthMfd})$ is cohesive, there is a notion of *differential refinement* of the above discussion, yielding *connections* on ∞-bundles.

Example. Let $\mathbb{C} \to \mathbb{C}//U(1) \to \mathbf{B}U(1)$ be the canonical complex-linear circle action. Then

- $g_{\mathrm{conn}} : X \to \mathbf{B}U(1)_{\mathrm{conn}}$ classifies a circle bundle with connection, a *prequantum line bundle* of its curvature 2-form;
- $\Gamma_X(P \times_{U(1)} \mathbb{C})$ is the corresponding space of smooth sections;
- γ is the exp(Poisson bracket)-group action of preqantum operators, containing the Heisenberg group action.

Example. Let $\mathbf{B}U \to \mathbf{B}\mathrm{PU} \to \mathbf{B}^2U(1)$ be the canonical 2-circle action. Then

- $g_{\mathrm{conn}} : X \to \mathbf{B}^2U(1)_{\mathrm{conn}}$ classifies a circle 2-bundle with connection, a *prequantum line 2-bundle* of its curvature 3-form;
- $\Gamma_X(P \times_{\mathbf{B}U(1)} \mathbf{B}U)$ is the corresponding groupoid of smooth sections = twisted bundles;
- γ is the exp(2-plectic bracket)-2-group action of 2-plectic geometry, containing the *Heisenberg 2-group* action.

1.2.12 Characteristic classes

1.2.12.1 Characteristic classes in low degree We discuss explicit presentations of *characteristic classes* of principal n-bundles for low values of n and for low degree of the characteristic class.

- General concept
- Examples
  - example 1.2.106 – First Chern class of unitary 1-bundles
  - example 1.2.107 – Dixmier-Douady class of circle 2-bundles (of bundle gerbes)
  - example 1.2.108 – Obstruction class of central extension
  - example 1.2.109 – First Stiefel-Whitney class of an O-principal bundle
  - example 1.2.110 – Second Stiefel-Whitney class of an SO-principal bundle
  - example 1.2.111 – Bockstein homomorphism
  - example 1.2.112 – Third integral Stiefel-Whitney class
  - example 1.2.113 – First Pontryagin class of Spin-1-bundles and twisted string-2-bundles

In the context of higher (smooth) groupoids the notion of characteristic class is conceptually very simple: for G some n-group and $\mathbf{B}G$ the corresponding one-object n-groupoid, a characteristic class of degree $k \in \mathbb{N}$ with coefficients in some abelian (Lie-)group A is presented simply by a morphism

$$c : \mathbf{B}G \to \mathbf{B}^n A$$

of cohesive ∞-groupoids. For instance if $A = \mathbb{Z}$ such a morphism represents a *universal integral characteristic class* on $\mathbf{B}G$. Then for

$$g : X \to \mathbf{B}G$$

any morphism of (smooth) ∞-groupoids that classifies a given G-principal n-bundle $P \to X$, as discussed above in 1.2.5, the corresponding characteristic class of P (equivalently of g) is the class of the composite

$$c(P) : \; X \xrightarrow{g} \mathbf{B}G \xrightarrow{c} \mathbf{B}^K A \;,$$

in the cohomology group $H^k(X, A)$ of the ambient ∞-topos.

In other words, in the abstract language of cohesive ∞-toposes the notion of characteristic classes of cohesive principal ∞-bundles is verbatim that of principal fibrations in ordinary homotopy theory. The crucial difference, though, is in the implementation of this abstract formalism.

Namely, as we have discussed previously, all the abstract morphisms $f : A \to B$ of cohesive ∞-groupoids here are presented by ∞-*anafunctors*, hence by spans of genuine morphisms of Kan-complex valued presheaves, whose left leg is a weak equivalence that exhibits a resolution of the source object.

This means that the characteristic map itself is presented by a span

$$\begin{array}{ccc} \widehat{\mathbf{B}G} & \xrightarrow{c} & \mathbf{B}^k A \;, \\ \downarrow{\simeq} & & \\ \mathbf{B}G & & \end{array}$$

as is of course the cocycle for the principal n-bundle

$$\begin{array}{ccc} C(U_i) & \xrightarrow{g} & \mathbf{B}G \\ \downarrow{\simeq} & & \\ X & & \end{array}$$

and the characteristic class $[c(P)]$ of the corresponding principal n-bundle is presented by a (any) span composite

$$\begin{array}{ccccc} C(T_i) & \xrightarrow{\hat{g}} & \widehat{\mathbf{B}G} & \xrightarrow{c} & \mathbf{B}^k A \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \\ C(U_i) & \xrightarrow{g} & \mathbf{B}G & & \\ \downarrow{\simeq} & & & & \\ X & & & & \end{array} \,,$$

where $C(T_i)$ is, if necessary, a refinement of the cover $C(U_i)$ over which the $\mathbf{B}G$-cocycle g lifts to a $\widehat{\mathbf{B}G}$-cocycle as indicated.

Notice the similarity of this situation to that of the discussion of twisted bundles in example 1.2.59. This is not a coincidence: every characteristic class induces a corresponding notion of *twisted n-bundles* and, conversely, every notion of twisted n-bundles can be understood as arising from the failure of a certain characteristic class to vanish.

We discuss now a list of examples.

Example 1.2.106 (first Chern class)**.** Let $N \in \mathbb{N}$. Consider the unitary group $U(N)$. By its definition as a matrix Lie group, this comes canonically equipped with the determinant function

$$\det : U(N) \to U(1)$$

and by the standard properties of the determinant, this is in fact a group homomorphism. Therefore this has a delooping to a morphism of Lie groupoids

$$\mathbf{B}\det : \mathbf{B}U(N) \to \mathbf{B}U(1)\,.$$

Under geometric realization this maps to a morphism

$$|\mathbf{B}\det| : BU(N) \to BU(1) \simeq K(\mathbb{Z}, 2)$$

of topological spaces. This is a characteristic class on the classifying space $BU(N)$: the ordinary *first Chern class*. Hence the morphism $\mathbf{B}\det$ on Lie groupoids is a *smooth refinement* of the ordinary first Chern class.

This smooth refinement acts on smooth $U(n)$-principal bundles as follows. Postcomposition of a Čech cocycle

$$P : \qquad \begin{array}{ccc} C(\{U_i\}) & \xrightarrow{(g_{ij})} & \mathbf{B}U(N) \\ \downarrow{\simeq} & & \\ X & & \end{array}$$

for a $U(N)$-principal bundle on a smooth manifold X with this characteristic class yields the cocycle

$$\det P : \qquad \begin{array}{ccccc} C(\{U_i\}) & \xrightarrow{(g_{ij})} & \mathbf{B}U(N) & \xrightarrow{\mathbf{B}\det} & \mathbf{B}U(1) \\ \downarrow{\simeq} & & & & \\ X & & & & \end{array}$$

for a circle bundle (or its associated line bundle) with transition functions $(\det(g_{ij}))$: the *determinant line bundle* of P.

We may easily pass to the *differential refinement* of the first Chern class along similar lines. By prop. 1.2.78 the differential refinement $\mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(n)$ of the moduli stack of $U(n)$-principal bundles is given by the groupoid-valued presheaf which over a test manifold U assigns

$$\mathbf{B}U(n)_{\mathrm{conn}} : U \mapsto \left\{ A \stackrel{g}{\to} A^g \mid A \in \Omega^1(U, \mathfrak{u}(n));\, g \in C^\infty(U, U(n)) \right\} .$$

One checks that $\mathbf{B}\det$ uniquely extends to a morphism of groupoid-valued presheaves $\mathbf{B}\det_{\mathrm{conn}}$

$$\begin{array}{ccc} \mathbf{B}U(n)_{\mathrm{conn}} & \xrightarrow{\mathbf{B}\det_{\mathrm{conn}}} & \mathbf{B}U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}U(n) & \xrightarrow{\mathbf{B}\det} & \mathbf{B}U(1) \end{array}$$

by sending $A \mapsto \mathrm{tr}(A)$. Here the trace operation on the matrix Lie algebra $\mathfrak{u}(n)$ is a unary *invariant polynomial* $\langle - \rangle : \mathfrak{u}(n) \to \mathfrak{u}(1) \simeq \mathbb{R}$.

Therefore, over a 1-dimensional compact manifold Σ (a disjoint union of circles) the canonical action functional, 1.2.11.1.5, induced by the first Chern class is

$$\exp(iS_{\mathbf{c}_1}) : \; \mathbf{H}(\Sigma, \mathbf{B}U(n)_{\mathrm{conn}}) \xrightarrow{\mathbf{H}(\Sigma, \mathbf{B}\det_{\mathrm{conn}})} \mathbf{H}(\Sigma, \mathbf{B}U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

sending

$$A \mapsto \exp(i \int_\Sigma \mathrm{tr}(A)) .$$

This is the action functional of 1-dimensional $U(n)$-Chern-Simons theory, discussed below in 5.7.4.

It is a basic fact that the cohomology class of line bundles can be identified within the second *integral cohomology* of X. For our purposes here it is instructive to rederive this fact in terms of anafunctors, *lifting gerbes* and twisted bundles.

To that end, consider from example 1.2.58 the equivalence of the 2-group $(\mathbb{Z} \hookrightarrow \mathbb{R})$ with the ordinary circle group, which supports the 2-anafunctor

$$\begin{array}{ccccc} \mathbf{B}(\mathbb{Z} \to \mathbb{R}) & \xrightarrow{c_1} & \mathbf{B}(\mathbb{Z} \to 1) & = & \mathbf{B}^2\mathbb{Z} \; . \\ \downarrow \simeq & & & & \\ \mathbf{B}U(1) & & & & \end{array}$$

We see now that this presents an integral characteristic class in degree 2 on $\mathbf{B}U(1)$. Given a cocycle $\{h_{ij} \in C^\infty(U_{ij}, U(1))\}$ for any circle bundle, the postcomposition with this 2-anafunctor amounts to the following:

1. refine the cover, if necessary, to a *good* open cover (where all non-empty $U_{i_0,\cdots,i_k}$ are contractible) – we shall still write $\{U_i\}$ now for this good cover;

2. choose on each U_{ij} a (any) lift of the circle-valued functor $h_{ij} : U_{ij} \to U(1)$ through the quotient map $\mathbb{R} \to U(1)$ to a function $\hat{h}_{ij} : U_{ij} \to \mathbb{R}$ – this is always possible over the contractible U_{ij};

3. compute the failures of the lifts thus chosen to constitute the cocycle for an $\mathbb{R}$-principal bundle: these are the elements
$$\lambda_{ijk} := \hat{h}_{ik}\hat{h}_{ij}^{-1}\hat{h}_{jk}^{-1} \in C^\infty(U_{ijk}, \mathbb{Z}) ,$$
which are indeed $\mathbb{Z}$-valued (hence constant) smooth functions due to the fact that the original $\{h_{ij}\}$ satisfied its cocycle law;

4. notice that by observation 1.2.54 this yields the construction of the cocycle for a $(\mathbb{Z} \to \mathbb{R})$-principal 2-bundle
$$\{\hat{h}_{ij} \in C^\infty(U_{ij}, \mathbb{R}), \lambda_{ijk} \in C^\infty(U_{ijk}, \mathbb{Z})\}\,,$$
which by example 1.2.59 we may also read as the cocycle for a twisted $\mathbb{R}$-1-bundle, with respect to the central extension $\mathbb{Z} \to \mathbb{R} \to U(1)$;

5. finally project out the cocycle for the "lifting $\mathbb{Z}$-gerbe" encoded by this, which is the the $\mathbf{B}\mathbb{Z}$-principal 2-bundle given by the $\mathbf{B}\mathbb{Z}$ cocycle
$$\{\lambda_{ijk} \in C^\infty(U_{ijk}, \mathbb{Z})\}\,,$$

This last cocycle is manifestly in degree-2 integral Čech cohomology, and hence indeed represents a class in $H^2(X, \mathbb{Z})$. This is the first Chern class of the circle bundle given by $\{h_{ij}\}$. If here $h_{ij} = \det g_{ij}$ is the determinant circle bundle of some unitary bundle, the this is also the first Chern class of that unitary bundle.

Example 1.2.107 (Dixmier-Douady class)**.** The discussion in example 1.2.106 of the first Chern class of a circle 1-bundle has an immediate generalization to an analogous canonical class of circle 2-bundles, def. 1.2.43, hence, by observation 1.2.44, to bundle gerbes. As before, while this amounts to a standard and basic fact, for our purposes it shall be instructive to spell this out in terms of ∞-anafunctors and twisted principal 2-bundles.

To that end, notice that by delooping the equivalence $\mathbf{B}(\mathbb{Z} \to \mathbb{R}) \xrightarrow{\simeq} \mathbf{B}U(1)$ yields
$$\mathbf{B}^2(\mathbb{Z} \to \mathbb{R}) \xrightarrow{\simeq} \mathbf{B}^2 U(1)\,.$$
This says that $\mathbf{B}U(1)$-principal 2-bundles/bundle gerbes are equivalent to $\mathbf{B}(\mathbb{Z} \to \mathbb{R})$-principal 3-bundles, def. 1.2.63.

As before, this supports a canonical integral characteristic class, now in degree 3, presented by the ∞-anafunctor
$$\begin{array}{ccccc} \mathbf{B}^2(\mathbb{Z} \to \mathbb{R}) & \longrightarrow & \mathbf{B}^2(\mathbb{Z} \to 1) & = & \mathbf{B}(\mathbb{Z} \to 1 \to 1) \\ \downarrow{\simeq} & & & & \\ \mathbf{B}^2 U(1) & & & & \end{array}\,.$$
The corresponding class in $H^3(\mathbf{B}U(1), \mathbb{Z})$ is the (smooth lift of) the *universal Dixmier-Douady class*.

Explicitly, for $\{g_{ijk} \in C^\infty(U_{ijk}, U(1))\}$ the Čech cocycle for a circle-2-bundle, def. 1.2.43, this class is computed as the composite of spans
$$\begin{array}{ccccc} C(U_i) & \xrightarrow{(\hat{g},\lambda)} & \mathbf{B}^2(\mathbb{Z} \to \mathbb{R}) & \longrightarrow & \mathbf{B}^3\mathbb{Z} \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \\ C(U_i) & \xrightarrow{g} & \mathbf{B}^2 U(1) & & \\ \downarrow{\simeq} & & & & \\ X & & & & \end{array}\,,$$
where we assume for simplicity of notation that the cover $\{U_i \to X\}$ already has be chosen (possibly after refining another cover) such that all patches and their non-empty intersections are contractible.

Here the lifted cocycle data $\{\hat{g}_{ijk} : U_{ijk} \to U(1)\}$ is through the quotient map $\mathbb{R} \to U(1)$ to real valued functions. These lifts will, in general, not satisfy the condition of a cocycle for a $\mathbf{B}\mathbb{R}$-principal 2-bundle. The failure is uniquely picked up by the functions
$$\lambda_{ijkl} := \hat{g}_{jkl} g_{ijk}^{-1} g_{ijl} g_{ikl}^{-1} \in C^\infty(U_{ijkl}, \mathbb{Z})\,.$$

By example 1.2.65 this data constitutes the cocycle for a $(\mathbb{Z} \to \mathbb{R} \to 1)$-principal 3-bundle or, by def. 1.2.66 that of a *twisted* $\mathbf{B}R$*-principal 2-bundle*.

The above composite of spans projects out the integral cocycle

$$\lambda_{ijkl} \in C^\infty(U_{ijkl}, \mathbb{Z}),$$

which manifestly gives a class in $H^3(X, \mathbb{Z})$. This is the Dixmier-Douady class of the original circle 3-bundle, the higher analog of the Chern-class of a circle bundle.

Example 1.2.108 (obstruction class of central extension)**.** For $A \to \hat{G} \to G$ a central extension of Lie groups, there is a long sequence of (deloopings of) Lie 2-groups

$$\mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G \stackrel{\mathbf{c}}{\to} \mathbf{B}^2 A,$$

where the characteristic class $\mathbf{c}$ is presented by the ∞-anafunctor

$$\begin{array}{ccc} \mathbf{B}(A \to \hat{G}) & \longrightarrow \mathbf{B}(A \to 1) = & \mathbf{B}^2 A \\ \downarrow \simeq & & \\ \mathbf{B}G & & \end{array}$$

with $(A \to \hat{G})$ the crossed module from example 1.2.52.

The proof of this is discussed below in prop. 4.4.38.

Example 1.2.109 (first Stiefel-Whitney class)**.** The morphism of groups

$$\mathrm{O}(n) \to \mathbb{Z}_2$$

which sends every element in the connected component of the unit element of $\mathrm{O}(n)$ to the unit element of $\mathbb{Z}_2$ and every other element to the non-trivial element of $\mathbb{Z}_2$ induces a morphism of delooping Lie groupoids

$$\mathbf{w}_1 : \mathbf{B}\mathrm{O}(n) \to \mathbf{B}\mathbb{Z}_2.$$

This represents the universal smooth *first Stiefel-Whiteney class*.

The relation of $\mathbf{w}_1$ to orientation structure is discussed below in 5.1.2.

Example 1.2.110 (second Stiefel-Whitney class)**.** The exact sequence that characterizes the Spin-group is

$$\mathbb{Z}_2 \to \mathrm{Spin} \to \mathrm{SO}$$

induces, by example 1.2.108, a long fiber sequence

$$\mathbf{B}\mathbb{Z}_2 \to \mathbf{B}\mathrm{Spin} \to \mathbf{B}\mathrm{SO} \stackrel{\mathbf{w}_2}{\to} \mathbf{B}^2\mathbb{Z}_2.$$

Here the the morphism $\mathbf{w}_2$ is presented by the ∞-anafunctor

$$\begin{array}{ccc} \mathbf{B}(\mathbb{Z}_2 \to \mathrm{Spin}) & \longrightarrow \mathbf{B}(\mathbb{Z}_2 \to 1) = & \mathbf{B}^2\mathbb{Z}_2 \\ \downarrow \simeq & & \\ \mathbf{B}\mathrm{SO} & & \end{array}.$$

This is a smooth incarnation of the *universal second Stiefel-Whitney class*. The $\mathbf{B}\mathbb{Z}_2$-principal 2-bundle associated by $\mathbf{w}_2$ to any $\mathrm{SO}(n)$-principal bundles is dicussed in [MuSi03] in terms of the corresponding bundle gerbe, via. observation 1.2.44.

Example 1.2.111 (Bockstein homomorphism)**.** The exact sequence

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2$$

induces, by example 1.2.108, for each $n \in \mathbb{N}$ a characteristic class

$$\beta_2 : \mathbf{B}^n \mathbb{Z}_2 \to \mathbf{B}^{n+1}\mathbb{Z}\,.$$

This is the *Bockstein homomorphism*.

Example 1.2.112 (third integral Stiefel-Whitney class)**.** The composite of the second Stiefel-Whitney class from example 1.2.110 with the Bockstein homomorphism from example 1.2.111 is the *third integral Stiefel-Whitney class*

$$W_3 : \mathbf{B}\mathrm{SO} \xrightarrow{\mathbf{w}_2} \mathbf{B}^2\mathbb{Z}_2 \xrightarrow{\beta_2} \mathbf{B}^3\mathbb{Z}\,.$$

This has a refined factorization through the universal Dixmier-Douady class from example 1.2.107:

$$\mathbf{W}_3 : \mathbf{B}\mathrm{SO} \to \mathbf{B}^2 U(1)\,.$$

This is discussed in lemma 5.4.71 below.

Example 1.2.113 (first Pontryagin class)**.** Let G be a compact and simply connected simple Lie group. Then the resolution from example 1.2.70 naturally supports a characteristic class presented by the 3-anafunctor

$$\begin{array}{ccccc} \mathbf{B}(U(1) \to \hat{\Omega} G \to PG) & \longrightarrow & \mathbf{B}(U(1) \to 1 \to 1) & = & \mathbf{B}^3 U(1) \\ \downarrow{\simeq} & & & & \\ \mathbf{B}G & & & & \end{array}\;.$$

For $G = \mathrm{Spin}$ the spin group, this presents one half of the universal *first Pontryagin class*. This we dicuss in detail in 5.1.

Composition with this class sends G-principal bundles to circle 2-bundles, 1.2.43, hence by 1.2.64 to bundle 2-gerbes. Our discussion in 5.1 shows that these are the *Chern-Simons 2-gerbes*.

The canonical action functional, 1.2.11.1.5, induced by $\frac{1}{2}\mathbf{p}_1$ over a compact 3-dimensional Σ

$$\exp(iS_{\frac{1}{2}\mathbf{p}_1}) : \; \mathbf{H}(\Sigma, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) \xrightarrow{\mathbf{H}(\Sigma, \frac{1}{2}\hat{\mathbf{p}}_1)} \mathbf{H}(\Sigma, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

is the action functional of ordinary 3-dimensional Chern-Simons theory, refined to the moduli stack of field configurations. This we discuss in 5.7.5.1.

1.2.13 Lie algebras

A Lie algebra is, in a precise sense, the infinitesimal approximation to a Lie group. This statement generalizes to *smooth n-groups* (the strict case of which we had seen in definition 1.2.60); their infinitesimal approximation are *Lie n-algebras* which for arbitrary n are known as *L_∞-algebras*. The statement also generalizes to *Lie groupoids* (discussed in 1.2.5); their infinitesimal approximation are *Lie algeboids*. Both these are special cases of a joint generalization; where smooth n-groupoids have *L_∞-algebroids* as their infinitesimal approximation.

The following is an exposition of basic L_∞-algebraic structures, their relation to smooth n-groupoids and the notion of connection data with coefficients in L_∞-algebras.

The following discussion proceeds by these topics:

- L_∞-algebroids;
- Lie integration;
- Characteristic cocycles from Lie integration;
- L_∞-algebra valued connections;
- Curvature characteristics and Chern-Simons forms;
- ∞-Connections from Lie integration;

1.2.13.1 L_∞-algebroids There is a precise sense in which one may think of a Lie algebra $\mathfrak{g}$ as the infinitesimal sub-object of the delooping groupoid $\mathbf{B}G$ of the corresponding Lie group G. Without here going into the details, which are discussed in detail below in 4.5.1, we want to build certain smooth ∞-groupoids from the knowledge of their infinitesimal subobjects: these subobjects are *L_∞-algebroids* and specifically *L_∞-algebras*.

For $\mathfrak{g}$ an $\mathbb{N}$-graded vector space, write $\mathfrak{g}[1]$ for the same underlying vector space with all degrees shifted up by one. (Often this is denoted $\mathfrak{g}[-1]$ instead). Then

$$\wedge^\bullet \mathfrak{g} = \mathrm{Sym}^\bullet(\mathfrak{g}[1])$$

is the *Grassmann algebra* on $\mathfrak{g}$; the free graded-commutative algebra on $\mathfrak{g}[1]$.

Definition 1.2.114. An *L_∞-algebra* structure on an $\mathbb{N}$-graded vector space $\mathfrak{g}$ is a family of multilinear maps

$$[-,\cdots,-]_k : \mathrm{Sym}^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$$

of degree -1, for all $k \in \mathbb{N}$, such that the *higher Jacobi identities*

$$\sum_{k+l=n+1} \sum_{\sigma \in \mathrm{UnSh}(l,k-1)} (-1)^\sigma t_{a_1}, \cdots, t_{a_l}], t_{a_{l+1}}, \cdots, t_{a_{k+l-1}}] = 0$$

are satisfied for all $n \in \mathbb{N}$ and all $\{t_{a_i} \in \mathfrak{g}\}$.

See [SSS09a] for a review and for references.

Example 1.2.115. If $\mathfrak{g}$ is concentrated in degree 0, then an L_∞-algebra structure on $\mathfrak{g}$ is the same as an ordinary Lie algebra structure. The only non-trivial bracket is $[-,-]_2 : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and the higher Jacobi identities reduce to the ordinary Jacobi identity.

We will see many other examples of L_∞-algebras. For identifying these, it turns out to be useful to have the following dual formulation of L_∞-algebras.

Proposition 1.2.116. *Let $\mathfrak{g}$ be a $\mathbb{N}$-graded vector space that is degreewise finite dimensional. Write $\mathfrak{g}^*$ for the degreewise dual, also $\mathbb{N}$-graded.*

Then dg-algebra structures on the Grassmann algebra $\wedge^\bullet \mathfrak{g}^* = \mathrm{Sym}^\bullet \mathfrak{g}[1]^*$ are in canonical bijection with L_∞-algebra structures on $\mathfrak{g}$, def. 1.2.114.

Here the sum is over all $(l, k-1)$*unshuffles*, which means all permutations $\sigma \in \Sigma_{k+l-1}$ that preserves the order within the first l and within the last $k-1$ arguments, respectively, and $(-1)^{\mathrm{sgn}}$ is the Koszul-sign of the permutation: the sign picked up by "unshuffling" $t^{a_1} \wedge \cdots, \wedge t^{a_{k+l-1}}$ according to σ.

Proof. Let $\{t_a\}$ be a basis of $\mathfrak{g}[1]$. Write $\{t^a\}$ for the dual basis of $\mathfrak{g}[1]^*$, where t^a is taken to be in the same degree as t_a.

A derivation $d : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$ of the Grassmann algebra is fixed by its value on generators, where it determines and is determined by a sequence of brackets graded-symmetric multilinear maps $\{[-,\cdots,-]_k\}_{k=1}^\infty$ by

$$d : t^a \mapsto -\sum_{k=1}^{\infty} \frac{1}{k!} [t_{a_1}, \cdots, t_{a_k}]^a \, t^{a_1} \wedge \cdots \wedge t^{a_k} \,,$$

where a sum over repeated indices is understood. This derivation is of degree +1 precisely if all the k-ary maps are of degree -1. It is straightforward to check that the condition $d \circ d = 0$ is equivalent to the higher Jacobi identities. □

Definition 1.2.117. The dg-algebra corresponding to an L_∞-algebra $\mathfrak{g}$ by prop. 1.2.116 we call the *Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$.

Example 1.2.118. For $\mathfrak{g}$ an ordinary Lie algebra, as in example 1.2.115, the notion of Chevalley-Eilenberg algebra from def. 1.2.117 coincides with the traditional notion.

Examples 1.2.119.

- A *strict* L_∞-algebra algebra is a dg-Lie algebra $(\mathfrak{g}, \partial, [-,-])$ with $(\mathfrak{g}^*, \partial^*)$ a cochain complex in non-negative degree. With $\mathfrak{g}^*$ denoting the degreewise dual, the corresponding CE-algebra is $CE(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{CE} = [-,-]^* + \partial^*$.

- We had already seen above the infinitesimal approximation of a Lie 2-group: this is a Lie 2-algebra. If the Lie 2-group is a smooth strict 2-group it is encoded equivalently by a crossed module of ordinary Lie groups, and the corresponding Lie 2-algebra is given by a differential crossed module of ordinary Lie algebras.

- For $n \in \mathbb{N}$, $n \geq 1$, the Lie n-algebra $b^{n-1}\mathbb{R}$ is the infinitesimal approximation to $\mathbf{B}^n U(\mathbb{R})$ and $\mathbf{B}^n \mathbb{R}$. Its CE-algebra is the dg-algebra on a single generators in degree n, with vanishing differential.

- For any ∞-Lie algebra $\mathfrak{g}$ there is an L_∞-algebra $\mathrm{inn}(\mathfrak{g})$ defined by the fact that its CE-algebra is the Weil algebra of $\mathfrak{g}$:

  $$\mathrm{CE}(\mathrm{inn}(\mathfrak{g})) = \mathrm{W}(\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), d_{\mathrm{W}}|_{\mathfrak{g}^*} = d_{CE} + \sigma) \,,$$

  where $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ is the grading shift isomorphism, extended as a derivation.

Example 1.2.120. For $\mathfrak{g}$ an L_∞-algebra, its *automorphism L_∞-algebra* $\mathfrak{der}(\mathfrak{g})$ is the dg-Lie algebra whose elements in degree k are the derivations

$$\iota : \mathrm{CE}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$$

of degree $-k$, whose differential is given by the graded commutator $[d_{\mathrm{CE}(\mathfrak{g})}, -]$ and whose Lie bracket is the commutator bracket of derivations.

In the context of rational homotopy theory, this is discussed on p. 312 of [Su77].

One advantage of describing an L_∞-algebra in terms of its dual Chevalley-Eilenberg algebra is that in this form the correct notion of morphism is manifest.

Definition 1.2.121. A morphism of L_∞-algebras $\mathfrak{g} \to \mathfrak{h}$ is a morphism of dg-algebras $\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{CE}(\mathfrak{h})$.

The category $L_\infty\mathrm{Alg}$ of L_∞-algebras is therefore the full subcatgeory of the opposite category of dg-algebras on those whose underlying graded algebra is free:

$$L_\infty\mathrm{Alg} \stackrel{\mathrm{CE}(-)}{\to} \mathrm{dgAlg}_{\mathbb{R}}^{\mathrm{op}} .$$

Replacing in this characterization the ground field $\mathbb{R}$ by an algebra of smooth functions on a manifold $\mathfrak{a}_0$, we obtain the notion of an L_∞*-algebroid* $\mathfrak{g}$ over $\mathfrak{a}_0$. Morphisms $\mathfrak{a} \to \mathfrak{b}$ of such ∞-Lie algebroids are dually precisely morphisms of dg-algebras $\mathrm{CE}(\mathfrak{a}) \leftarrow \mathrm{CE}(\mathfrak{b})$.

Definition 1.2.122. The category of L_∞*-algebroids* is the opposite category of the full subcategory of dgAlg

$$\infty\mathrm{LieAlgbd} \subset \mathrm{dgAlg}^{op}$$

on graded-commutative cochain dg-algebras in non-negative degree whose underlying graded algebra is an exterior algebra over its degree-0 algebra, and this degree-0 algebra is the algebra of smooth functions on a smooth manifold.

Remark 1.2.123. More precisely the above definition is that of *affine* C^∞-L_∞-algebroids. There are various ways to refine this to something more encompassing, but for the purposes of this introductory discussion the above is convenient and sufficient. A more comprehensive discussion is in 4.5.1 below.

Example 1.2.124.

- The *tangent Lie algebroid* TX of a smooth manifold X is the infinitesimal approximation to its fundamental ∞-groupoid. Its CE-algebra is the de Rham complex

  $\mathrm{CE}(TX) = \Omega^\bullet(X)$.

1.2.13.2 Lie integration We discusss *Lie integration*: a construction that sends an L_∞-algebroid to a smooth ∞-groupoid of which it is the infinitesimal approximation.

The construction we want to describe may be understood as a generalization of the following proposition. This is classical, even if maybe not reflected in the standard textbook literature to the extent it deserves to be.

Definition 1.2.125. For $\mathfrak{g}$ a (finite-dimensional) Lie algebra, let $\exp(\mathfrak{g}) \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ be the simplicial presheaf given by the assignment

$$\exp(\mathfrak{g}) : U \mapsto \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(U \times \Delta^\bullet)_{\mathrm{vert}}) ,$$

in degree k of dg-algebra homomorphisms from the Chevalley-Eilenberg algebra of $\mathfrak{g}$ to the dg-algebra of vertical differential forms with respect to the trivial bundle $U \times \Delta^k \to U$.

Shortly we will be considering variations of such assignments that are best thought about when writing out the hom-sets on the right here as sets of arrows; as in

$$\exp(\mathfrak{g}) : (U, [k]) \mapsto \left\{ \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) \stackrel{A_{\mathrm{vert}}}{\leftarrow} \mathrm{CE}(\mathfrak{g}) \right\}) .$$

For $\mathfrak{g}$ an ordinary Lie algebra it is an ancient and simple but important observation that dg-algebra morphisms $\Omega^\bullet(\Delta^k) \leftarrow \mathrm{CE}(\mathfrak{g})$ are in natural bijection with Lie-algebra valued 1-forms that are *flat* in that their curvature 2-forms vanish: the 1-form itself determines precisely a morphism of the underlying graded algebras, and the respect for the differentials is exactly the flatness condition. It is this elementary but similarly important observation that historically led Eli Cartan to Cartan calculus and the algebraic formulation of Chern-Weil theory.

One finds that it makes good sense to generally, for $\mathfrak{g}$ any ∞-Lie algebra or even ∞-Lie algebroid, think of $\mathrm{Hom}_{dgAlg}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Delta^k))$ as the set of ∞-Lie algebroid valued differential forms whose curvature forms (generally a whole tower of them) vanishes.

Proposition 1.2.126. *Let G be the simply-connected Lie group integrating $\mathfrak{g}$ according to Lie's three theorems and $\mathbf{B}G \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd}]$ its delooping Lie groupoid regarded as a groupoid-valued presheaf on* CartSp. *Write $\tau_1(-)$ for the truncation operation that quotients out 2-morphisms in a simplicial presheaf to obtain a presheaf of groupoids.*

We have an isomorphism

$$\mathbf{B}G = \tau_1 \exp(\mathfrak{g}) .$$

To see this, observe that the presheaf $\exp(\mathfrak{g})$ has as 1-morphisms U-parameterized families of $\mathfrak{g}$-valued 1-forms A_{vert} on the interval, and as 2-morphisms U-parameterized families of *flat* 1-forms on the disk, interpolating between these. By identifying these 1-forms with the pullback of the Maurer-Cartan form on G, we may equivalently think of the 1-morphisms as based smooth paths in G and 2-morphisms smooth homotopies relative endpoints between them. Since G is simply-connected this means that after dividing out 2-morphisms only the endpoints of these paths remain, which identify with the points in G.

The following proposition establishes the Lie integration of the shifted 1-dimensional abelian L_∞-algebras $b^{n-1}\mathbb{R}$.

Proposition 1.2.127. *For $n \in \mathbb{N}$, $n \geq 1$. Write*

$$\mathbf{B}^n\mathbb{R}_{\mathrm{ch}} := \Xi\mathbb{R}[n]$$

for the simplicial presheaf on CartSp *that is the image of the sheaf of chain complexes represented by $\mathbb{R}$ in degree n and 0 in other degrees, under the Dold-Kan correspondence $\Xi : \mathrm{Ch}_\bullet^+ \to \mathrm{sAb} \to \mathrm{sSet}$.*

Then there is a canonical morphism

$$\int_{\Delta^\bullet} : \exp(b^{n-1}\mathbb{R}) \stackrel{\simeq}{\to} \mathbf{B}^n\mathbb{R}_{ch}$$

given by fiber integration of differential forms along $U \times \Delta^n \to U$ and this is an equivalence (a global equivalence in the model structure on simplicial presheaves).

The proof of this statement is discussed in 4.4.14.

This statement will make an appearance repeatedly in the following discussion. Whenever we translate a construction given in terms $\exp(-)$ into a more convenient chain complex representation.

1.2.13.3 Characteristic cocycles from Lie integration

We now describe characteristic classes and curvature characteristic forms on G-bundles in terms of these simplicial presheaves. For that purpose it is useful for a moment to ignore the truncation issue – to come back to it later – and consider these simplicial presheaves untruncated.

To see characteristic classes in this picture, write $\mathrm{CE}(b^{n-1}\mathbb{R})$ for the commutative real dg-algebra on a single generator in degree n with vanishing differential. As our notation suggests, this we may think as the Chevalley-Eilenberg algebra of a *higher Lie algebra* – the ∞-Lie algebra $b^{n-1}\mathbb{R}$ – which is an Eilenberg-MacLane object in the homotopy theory of ∞-Lie algebras, representing ∞-Lie algebra cohomology in degree n with coefficients in $\mathbb{R}$.

Restating this in elementary terms, this just says that dg-algebra homomorphisms

$$\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{CE}(b^{n-1}\mathbb{R}) : \mu$$

are in natural bijection with elements $\mu \in \mathrm{CE}(\mathfrak{g})$ of degree n, that are closed, $d_{\mathrm{CE}(\mathfrak{g})}\mu = 0$. This is the classical description of a cocycle in the Lie algebra cohomology of $\mathfrak{g}$.

Definition 1.2.128. Every such ∞-Lie algebra cocycle μ induces a morphism of simplicial presheaves

$$\exp(\mu) : \exp(\mathfrak{g}) \to \exp(b^n\mathbb{R})$$

given by postcomposition

$$\Omega^\bullet_{\mathrm{vert}}(U \times \Delta^l) \stackrel{A_{\mathrm{vert}}}{\leftarrow} \mathrm{CE}(\mathfrak{g}) \stackrel{\mu}{\leftarrow} \mathrm{CE}(b^n\mathbb{R}) .$$

Example 1.2.129. Assume $\mathfrak{g}$ to be a semisimple Lie algebra, let $\langle -, - \rangle$ be the Killing form and $\mu = \langle -, [-,-] \rangle$ the corresponding 3-cocycle in Lie algebra cohomology. We may assume without restriction that this cocycle is normalized such that its left-invariant continuation to a 3-form on G has integral periods. Observe that since $\pi_2(G)$ is trivial we have that the 3-coskeleton (see around def. 3.6.28 for details on coskeleta) of $\exp(\mathfrak{g})$ is equivalent to $\mathbf{B}G$. By the inegrality of μ, the operation of $\exp(\mu)$ on $\exp(\mathfrak{g})$ followed by integration over simplices descends to an ∞-anafunctor from $\mathbf{B}G$ to $\mathbf{B}^3 U(1)$, as indicated on the right of this diagram in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$

$$
\begin{array}{ccccc}
 & & \exp(\mathfrak{g}) & \xrightarrow{\exp(\mu)} & \exp(b^{n-1}\mathbb{R}) \ . \\
 & & \downarrow & & \downarrow^{\int_{\Delta^\bullet}} \\
C(V) & \xrightarrow{\hat g} & \mathbf{cosk}_3 \exp(\mathfrak{g}) & \xrightarrow{\int_{\Delta^\bullet} \mathbf{cosk}_3 \exp(\mu)} & \mathbf{B}^3 \mathbb{R}/\mathbb{Z} \\
\downarrow^{\simeq} & & \downarrow^{\simeq} & & \\
C(U) & \xrightarrow{g} & \mathbf{B}G & & \\
\downarrow^{\simeq} & & & & \\
X & & & &
\end{array}
$$

Precomposing this – as indicated on the left of the diagram – with another ∞-anafunctor $X \stackrel{\simeq}{\leftarrow} C(U) \stackrel{g}{\to} \mathbf{B}G$ for a G-principal bundle, hence a collection of transition functions $\{g_{ij} : U_i \cap U_j \to G\}$ amounts to choosing (possibly on a refinement V of the cover U of X)

- on each $V_i \cap V_j$ a lift $\hat g_{ij}$ of g_{ij} to a familly of smooth based paths in G – $\hat g_{ij} : (V_i \cap V_j) \times \Delta^1 \to G$ – with endpoints g_{ij};
- on each $V_i \cap V_j \cap V_k$ a smooth family $\hat g_{ijk} : (V_i \cap V_j \cap V_k) \times \Delta^2 \to G$ of disks interpolating between these paths;
- on each $V_i \cap V_j \cap V_k \cap V_l$ a a smooth family $\hat g_{ijkl} : (V_i \cap V_j \cap V_k \cap V_l) \times \Delta^3 \to G$ of 3-balls interpolating between these disks.

On this data the morphism $\int_{\Delta^\bullet} \exp(\mu)$ acts by sending each 3-cell to the number

$$
\hat g_{ijkl} \mapsto \int_{\Delta^3} \hat g_{ijkl}^* \mu \mod \mathbb{Z} \,,
$$

where μ is regarded in this formula as a closed 3-form on G.

We say this is *Lie integration of Lie algebra cocycles.*

Proposition 1.2.130. *For $G = \mathrm{Spin}$, the Čech cohomology cocycle obtained this way is the first fractional* Pontryagin class *of the G-bundle classified by G.*

We shall show this below, as part of our L_∞-algebraic reconstruction of the above motivating example. In order to do so, we now add differential refinement to this Lie integration of characteristic classes.

1.2.13.4 L_∞-algebra valued connections In 1.2.5 we described ordinary connections on bundles as well as connections on 2-bundles in terms of parallel transport over paths and surfaces, and showed how such is equivalently given by cocycles with coefficients in Lie-algebra valued differential forms and Lie 2-algebra valued differential forms, respectively.

Notably we saw for the case of ordinary $U(1)$-principal bundles, that the connection and curvature data on these is encoded in presheaves of diagrams that over a given test space $U \in \mathrm{CartSp}$ look like

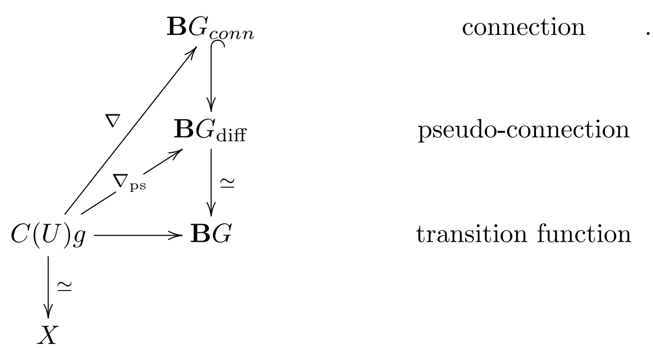
$$
\begin{array}{ccccc}
U & \longrightarrow & \mathbf{B}U(1) & & \text{transition function} \\
\downarrow & & \downarrow & & \\
\mathbf{\Pi}(U) & \longrightarrow & \mathbf{B}\mathrm{INN}(U) & & \text{connection} \\
\downarrow & & \downarrow & & \\
\mathbf{\Pi}(U) & \longrightarrow & \mathbf{B}^2U(1) & & \text{curvature}
\end{array}
$$

together with a constraint on the bottom morphism.

It is in the form of such a kind of diagram that the general notion of connections on ∞-bundles may be modeled. In the full theory in 3 this follows from first principles, but for our present introductory purpose we shall be content with taking this simple situation of $U(1)$-bundles together with the notion of Lie integration as sufficient motivation for the constructions considered now.

So we pass now to what is to some extent the reverse construction of the one considered before: we define a notion of L_∞-algebra valued differential forms and show how by a variant of Lie integration these integrate to coefficient objects for connections on ∞-bundles.

1.2.13.5 Curvature characteristics and Chern-Simons forms For G a Lie group, we have described above connections on G-principal bundles in terms of cocycles with coefficients in the Lie-groupoid of Lie-algebra valued forms $\mathbf{B}G_{\mathrm{conn}}$



In this context we had *derived* Lie-algebra valued forms from the parallel transport description $\mathbf{B}G_{\mathrm{conn}} = [\mathbf{P}_1(-), \mathbf{B}G]$. We now turn this around and use Lie integration to construct parallel transport from Lie-algebra valued forms. The construction is such that it generalizes verbatim to ∞-Lie algebra valued forms. For that purpose notice that another classical dg-algebra associated with $\mathfrak{g}$ is its *Weil algebra* $\mathrm{W}(\mathfrak{g})$.

Proposition 1.2.131. *The Weil algebra* $\mathrm{W}(\mathfrak{g})$ *is the free dg-algebra on the graded vector space* $\mathfrak{g}^*$*, meaning that there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{dgAlg}}(W(\mathfrak{g}), A) \simeq \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{Z}}}(\mathfrak{g}^*, A)\,,$$

which is singled out among the isomorphism class of dg-algebras with this property by the fact that the projection of graded vector spaces $\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \to \mathfrak{g}^*$ *extends to a dg-algebra homomorphism*

$$\mathrm{CE}(\mathfrak{g}) \leftarrow W(\mathfrak{g}) : i^*\,.$$

(Notice that general the dg-algebras that we are dealing with are *semi-free* dg-algebras in that only their underlying graded algebra is free, but not the differential).

The most obvious realization of the free dg-algebra on $\mathfrak{g}^*$ is $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$ equipped with the differential that is precisely the degree shift isomorphism $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ extended as a derivation. This is not the Weil algebra on the nose, but is of course isomorphic to it. The differential of the Weil algebra on $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$ is given on the unshifted generators by the sum of the CE-differential with the shift isomorphism

$$d_{W(\mathfrak{g})}|_{\mathfrak{g}^*} = d_{\mathrm{CE}(\mathfrak{g})} + \sigma \,.$$

This uniquely fixes the differential on the shifted generators – a phenomenon known (at least after mapping this to differential forms, as we discuss below) as the *Bianchi identity*.

Using this, we can express also the presheaf $\mathbf{B}G_{\mathrm{diff}}$ from above in diagrammatic fashion

Observation 1.2.132. For G a simply connected Lie group, the presheaf $\mathbf{B}G_{\mathrm{diff}} \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd}]$ is isomorphic to

$$\mathbf{B}G_{\mathrm{diff}} = \tau_1 \left(\exp(\mathfrak{g})_{\mathrm{diff}} : (U,[k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \end{array} \right\} \right)$$

where on the right we have the 1-truncation of the simplicial presheaf of diagrams as indicated, where the vertical morphisms are the canonical ones.

Here over a given U the bottom morphism in such a diagram is an arbitrary $\mathfrak{g}$-valued 1-form A on $U \times \Delta^k$. This we can decompose as $A = A_U + A_{\mathrm{vert}}$, where A_U vanishes on tangents to Δ^k and A_{vert} on tangents to U. The commutativity of the diagram asserts that A_{vert} has to be such that the curvature 2-form $F_{A_{\mathrm{vert}}}$ vanishes when both its arguments are tangent to Δ^k.

On the other hand, there is in the above no further constraint on A_U. Accordingly, as we pass to the 1-truncation of $\exp(\mathfrak{g})_{\mathrm{diff}}$ we find that morphisms are of the form $(A_U)_1 \xrightarrow{g} (A_U)_2$ with $(A_U)^i$ arbitrary. This is the definition of $\mathbf{B}G_{\mathrm{diff}}$.

We see below that it is not a coincidence that this is reminiscent to the first condition on an Ehresmann connection on a G-principal bundle, which asserts that restricted to the fibers a connection 1-form on the total space of the bundle has to be flat. Indeed, the simplicial presheaf $\mathbf{B}G_{\mathrm{diff}}$ may be thought of as the ∞-sheaf of pseudo-connections on *trivial* ∞-bundles. Imposing on this also the second Ehresmann condition will force the pseudo-connection to be a genuine connection.

We now want to lift the above construction $\exp(\mu)$ of characteristic classes by Lie integration of Lie algebra cocycles μ from plain bundles classified by $\mathbf{B}G$ to bundles with (pseudo-)connection classified by $\mathbf{B}G_{\mathrm{diff}}$. By what we just said we therefore need to extend $\exp(\mu)$ from a map on just $\exp(\mathfrak{g})$ to a map on $\exp(\mathfrak{g})_{\mathrm{diff}}$. This is evidently achieved by completing a square in dgAlg of the form

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{W}(\mathfrak{g}) & \xleftarrow{cs} & \mathrm{W}(b^{n-1}\mathbb{R}) \end{array}$$

and defining $\exp(\mu)_{\mathrm{diff}} : \exp(\mathfrak{g})_{\mathrm{diff}} \to \exp(b^{n-1}\mathbb{R})_{\mathrm{diff}}$ to be the operation of forming pasting composites with this.

Here $\mathrm{W}(b^{n-1}\mathbb{R})$ is the Weil algebra of the Lie n-algebra $b^{n-1}\mathbb{R}$. This is the dg-algebra on two generators c and k, respectively, in degree n and $(n+1)$ with the differential given by $d_{\mathrm{W}(b^{n-1}\mathbb{R})} : c \mapsto k$. The commutativity of this diagram says that the bottom morphism takes the degree-n generator c to an element $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ whose restriction to the unshifted generators is the given cocycle μ.

As we shall see below, any such choice cs will extend the characteristic cocycle obtained from $\exp(\mu)$ to a characteristic differential cocycle, exhibiting the ∞-Chern-Weil homomorphism. But only for special nice choices of cs will this take genuine ∞-connections to genuine ∞-connections – instead of to pseudo-connections. As we discuss in the full ∞-Chern-Weil theory, this makes no difference in cohomology. But in practice it is useful to fine-tune the construction such as to produce nice models of the ∞-Chern-Weil homomorphism given by genuine ∞-connections. This is achieved by imposing the following additional constraint on the choice of extension cs of μ:

Definition 1.2.133. For $\mu \in \mathrm{CE}(\mathfrak{g})$ a cocycle and $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ a lift of μ through $\mathrm{W}(\mathfrak{g}) \leftarrow \mathrm{CE}(\mathfrak{g})$, we say that $d_{\mathrm{W}(\mathfrak{g})}$ is an invariant polynomial *in transgression* with μ if $d_{\mathrm{W}(\mathfrak{g})}$ sits entirely in the shifted generators, in that $d_{\mathrm{W}(\mathfrak{g})} \in \wedge^\bullet \mathfrak{g}^*[1] \hookrightarrow W(\mathfrak{g})$.

Definition 1.2.134. Write $\mathrm{inv}(\mathfrak{g}) \subset \mathrm{W}(\mathfrak{g})$ (or $\mathrm{W}(\mathfrak{g})_{\mathrm{basic}}$) for the sub-dg-algebra on invariant polynomials.

Observation 1.2.135. We have $\mathrm{W}(b^{n-1}\mathbb{R}) \simeq \mathrm{CE}(b^n\mathbb{R})$.

Using this, we can now encode the two conditions on the extension cs of the cocycle μ as the commutativity of this double square diagram

$$\begin{array}{ccccc}
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathbb{R}) & & \text{cocycle} \\
\uparrow & & \uparrow & & \\
\mathrm{W}(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(b^{n-1}\mathbb{R}) & & \text{Chern-Simons element} \\
\uparrow & & \uparrow & & \\
\mathrm{inv}(\mathfrak{g}) & \xrightarrow{\langle - \rangle} & \mathrm{inv}(b^{n-1}\mathbb{R}) & & \text{invariant polynomial}
\end{array} .$$

Definition 1.2.136. In such a diagram, we call cs the *Chern-Simons element* that exhibits the transgression between μ and $\langle - \rangle$.

We shall see below that under the ∞-Chern-Weil homomorphism, Chern-Simons elements give rise to the familiar Chern-Simons forms – as well as their generalizations – as local connection data of secondary characteristic classes realized as circle nn-bundles with connection.

Observation 1.2.137. What this diagram encodes is the construction of the connecting homomorphism for the long exact sequence in cohomology that is induced from the short exact sequence

$$\ker(i^*) \to \mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$$

subject to the extra constraint of basic elements.

$$\begin{array}{ccccc}
 & & \langle - \rangle & \longleftarrow\!\!\shortmid & \langle - \rangle \\
 & & {\scriptstyle d_W}\uparrow & & \\
\mu & \longleftarrow\!\!\shortmid & \mathrm{cs} & & \\
 & & & & \\
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{i^*} & \mathrm{W}(\mathfrak{g}) & \longleftarrow & \mathrm{inv}(\mathfrak{g})
\end{array} .$$

To appreciate the construction so far, recall the following classical fact

Fact 1.2.138. *For G a compact Lie group, the rationalization $BG \otimes k$ of the classifying space BG is the rational space whose Sullivan model is given by the algebra* $\mathrm{inv}(\mathfrak{g})$ *of invariant polynomials on the Lie algebra* $\mathfrak{g}$.

So we have obtained the following picture:

delooped ∞-group	$\mathbf{B}G$	$\mathfrak{g}$	$\mathrm{CE}(\mathfrak{g})$	Chevalley-Eilenberg algebra
	$\downarrow$	$\downarrow$	$\uparrow$	
delooped groupal universal ∞-bundle	$\mathbf{BE}G$	$\mathrm{inn}(\mathfrak{g})$	$\mathrm{W}(\mathfrak{g}) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g}))$	Weil algebra
	$\downarrow$	$\downarrow$	$\uparrow$	
rationalized classifying space	$\prod_i \mathbf{B}^{n_i}\mathbb{R}$	$\prod_i b^{n_i-1}\mathbb{R}$	$\mathrm{inv}(\mathfrak{g})$	algebra of invariant polynomials

$$\xleftarrow{\text{Lie integration}}$$

Example 1.2.139. For $\mathfrak{g}$ a semisimple Lie algebra, $\langle -,- \rangle$ the Killing form invariant polynomial, there is a Chern-Simons element $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ witnessing the transgression to the cocycle $\mu = -\frac{1}{6}\langle -, [-,-]\rangle$. Under a $\mathfrak{g}$-valued form $\Omega^\bullet(X) \leftarrow W(\mathfrak{g}) : A$ this maps to the ordinary degree 3 Chern-Simons form

$$\mathrm{cs}(A) = \langle A \wedge dA \rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle \,.$$

1.2.13.6 ∞-Connections from Lie integration

For $\mathfrak{g}$ an L_∞-algebroid we have seen above the object $\exp(\mathfrak{g})_{\mathrm{diff}}$ that represents pseudo-connections on $\exp(\mathfrak{g})$-principal ∞-bundles and serves to support the ∞-Chern-Weil homomorphism. We now discuss the genuine ∞-connections among these pseudo-connections. A derivation from first principles of the following construction is given below in 4.4.17.

The construction is due to [SSS09c] and [FSS10].

Definition 1.2.140. Let X be a smooth manifold and $\mathfrak{g}$ an L_∞-algebra algebra or more generally an L_∞-algebroid.

An *L_∞-algebroid valued differential form* on X is a morphism of dg-algebras

$$\Omega^\bullet(X) \leftarrow \mathrm{W}(\mathfrak{g}) : A$$

from the Weil algebra of $\mathfrak{g}$ to the de Rham complex of X. Dually this is a morphism of L_∞-algebroids

$$A : TX \to \mathrm{inn}(\mathfrak{g})$$

from the inner automorphism ∞-Lie algebra.

Its *curvature* is the composite of morphisms of graded vector spaces

$$\Omega^\bullet(X) \xleftarrow{A} \mathrm{W}(\mathfrak{g}) \xleftarrow{F_{(-)}} \mathfrak{g}^*[1] : F_A \,.$$

Precisely if the curvatures vanish does the morphism factor through the Chevalley-Eilenberg algebra

$$(F_A = 0) \quad \Leftrightarrow \quad \left(\begin{array}{ccc} & & \mathrm{CE}(\mathfrak{g}) \\ & \swarrow^{\exists A_{\mathrm{flat}}} & \uparrow \\ \Omega^\bullet(X) & \xleftarrow{A} & W(\mathfrak{g}) \end{array} \right)$$

in which case we call A *flat*.

Remark 1.2.141. For $\{x^a\}$ a coordinate chart of an L_∞-algebroid $\mathfrak{a}$ and

$$A^a := A(x^a) \in \Omega^{\deg(x^a)}(X)$$

the differential form assigned to the generator x^a by the $\mathfrak{a}$-valued form A, we have the curvature components

$$F_A^a = A(\mathbf{d}x^a) \in \Omega^{\deg(x^a)+1}(X)\,.$$

Since $d_{\mathrm{W}} = d_{\mathrm{CE}} + \mathbf{d}$, this can be equivalently written as

$$F_A^a = A(d_{\mathrm{W}}x^a - d_{\mathrm{CE}}x^a)\,,$$

so the *curvature* of A precisely measures the "lack of flatness" of A. Also notice that, since A is required to be a dg-algebra homomorphism, we have

$$A(d_{\mathrm{W}(\mathfrak{a})}x^a) = d_{\mathrm{dR}}A^a\,,$$

so that

$$A(d_{\mathrm{CE}(\mathfrak{a})}x^a) = d_{\mathrm{dR}}A^a - F_A^a\,.$$

Assume now A is a degree 1 $\mathfrak{a}$-valued differential form on the smooth manifold X, and that cs is a Chern-Simons element transgressing an invariant polynomial $\langle - \rangle$ of $\mathfrak{a}$ to some cocycle μ, by def. 1.2.133. We can then consider the image $A(\mathrm{cs})$ of the Chern-Simons element cs in $\Omega^\bullet(X)$. Equivalently, we can look at cs as a map from degree 1 $\mathfrak{a}$-valued differential forms on X to ordinary (real valued) differential forms on X.

Definition 1.2.142. In the notations above, we write

$$\Omega^\bullet(X) \xleftarrow{A} \mathrm{W}(\mathfrak{a}) \xleftarrow{\mathrm{cs}} \mathrm{W}(b^{n+1}\mathbb{R}) \ : \mathrm{cs}(A)$$

for the differential form associated by the Chern-Simons element cs to the degree 1 $\mathfrak{a}$-valued differential form A, and call this the *Chern-Simons differential form* associated with A.

Similarly, for $\langle - \rangle$ an invariant polynomial on $\mathfrak{a}$, we write $\langle F_A \rangle$ for the evaluation

$$\Omega^\bullet_{\mathrm{closed}}(X) \xleftarrow{A} \mathrm{W}(\mathfrak{a}) \xleftarrow{\langle - \rangle} \mathrm{inv}(b^{n+1}\mathbb{R}) \ : \langle F_A \rangle\,.$$

We call this the *curvature characteristic forms* of A.

Definition 1.2.143. For U a smooth manifold, the ∞*-groupoid of* $\mathfrak{g}$*-valued forms* is the Kan complex

$$\exp(\mathfrak{g})_{\mathrm{conn}}(U) : [k] \mapsto \left\{ \Omega^\bullet(U \times \Delta^k) \stackrel{A}{\leftarrow} \mathrm{W}(\mathfrak{g}) \ \mid \ \forall v \in \Gamma(T\Delta^k) : \iota_v F_A = 0 \right\}$$

whose k-morphisms are $\mathfrak{g}$-valued forms A on $U \times \Delta^k$ with sitting instants, and with the property that their curvature vanishes on vertical vectors.

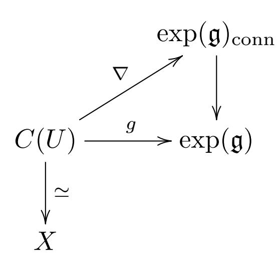
The canonical morphism

$$\exp(\mathfrak{g})_{\mathrm{conn}} \to \exp(\mathfrak{g})$$

to the untruncated Lie integration of $\mathfrak{g}$ is given by restriction of A to vertical differential forms (see below).

Here we are thinking of $U \times \Delta^k \to U$ as a trivial bundle.

The *first* Ehresmann condition can be identified with the conditions on lifts ∇ in ∞-anafunctors



that define connections on ∞-bundles.

1.2.13.6.1 Curvature characteristics

Proposition 1.2.144. *For $A \in \exp(\mathfrak{g})_{\mathrm{conn}}(U,[k])$ a $\mathfrak{g}$-valued form on $U \times \Delta^k$ and for $\langle - \rangle \in \mathrm{W}(\mathfrak{g})$ any invariant polynomial, the corresponding curvature characteristic form $\langle F_A \rangle \in \Omega^\bullet(U \times \Delta^k)$ descends down to U.*

To see this, it is sufficient to show that for all $v \in \Gamma(T\Delta^k)$ we have

1. $\iota_v \langle F_A \rangle = 0$;
2. $\mathcal{L}_v \langle F_A \rangle = 0$.

The first condition is evidently satisfied if already $\iota_v F_A = 0$. The second condition follows with Cartan calculus and using that $d_{\mathrm{dR}}\langle F_A \rangle = 0$:

$$\mathcal{L}_v \langle F_A \rangle = d\iota_v \langle F_A \rangle + \iota_v d \langle F_A \rangle = 0\,.$$

Notice that for a general ∞-Lie algebra $\mathfrak{g}$ the curvature forms F_A themselves are not generally closed (rather they satisfy the more Bianchi identity), hence requiring them to have no component along the simplex does not imply that they descend. This is different for abelian ∞-Lie algebras: for them the curvature forms themselves are already closed, and hence are themselves already curvature characteristics that do descent.

It is useful to organize the $\mathfrak{g}$-valued form A, together with its restriction A_{vert} to vertical differential forms and with its curvature characteristic forms in the commuting diagram

$$\begin{array}{ccccc}
\Omega^\bullet(U \times \Delta^k)_{\mathrm{vert}} & \stackrel{A_{\mathrm{vert}}}{\longleftarrow} & \mathrm{CE}(\mathfrak{g}) & \quad & \text{gauge transformation} \\
\uparrow & & \uparrow & & \\
\Omega^\bullet(U \times \Delta^k) & \stackrel{A}{\longleftarrow} & \mathrm{W}(\mathfrak{g}) & & \mathfrak{g}\text{-valued form} \\
\uparrow & & \uparrow & & \\
\Omega^\bullet(U) & \stackrel{\langle F_A \rangle}{\longleftarrow} & \mathrm{inv}(\mathfrak{g}) & & \text{curvature characteristic forms}
\end{array}$$

in dgAlg. The commutativity of this diagram is implied by $\iota_v F_A = 0$.

Definition 1.2.145. Write $\exp(\mathfrak{g})_{\mathrm{CW}}(U)$ for the ∞-groupoid of $\mathfrak{g}$-valued forms fitting into such diagrams.

$$\exp(\mathfrak{g})_{CW}(U) : [k] \mapsto \left\{ \begin{array}{ccc}
\Omega^\bullet(U \times \Delta^k)_{\mathrm{vert}} & \stackrel{A_{\mathrm{vert}}}{\longleftarrow} & \mathrm{CE}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(U \times \Delta^k) & \stackrel{A}{\longleftarrow} & \mathrm{W}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(U) & \stackrel{\langle F_A \rangle}{\longleftarrow} & \mathrm{inv}(\mathfrak{g})
\end{array} \right\}\,.$$

We call this the coefficient for $\mathfrak{g}$-valued ∞*-connections*

1.2.13.6.2 1-Morphisms: integration of infinitesimal gauge transformations The 1-morphisms in $\exp(\mathfrak{g})(U)$ may be thought of as *gauge transformations* between $\mathfrak{g}$-valued forms. We unwind what these look like concretely.

Definition 1.2.146. Given a 1-morphism in $\exp(\mathfrak{g})(X)$, represented by $\mathfrak{g}$-valued forms

$$\Omega^\bullet(U \times \Delta^1) \leftarrow \mathrm{W}(\mathfrak{g}) : A$$

consider the unique decomposition

$$A = A_U + (A_{\mathrm{vert}} := \lambda \wedge dt) \quad ,$$

with A_U the horizonal differential form component and $t : \Delta^1 = [0,1] \to \mathbb{R}$ the canonical coordinate.

We call λ the *gauge parameter*. This is a function on Δ^1 with values in 0-forms on U for $\mathfrak{g}$ an ordinary Lie algebra, plus 1-forms on U for $\mathfrak{g}$ a Lie 2-algebra, plus 2-forms for a Lie 3-algebra, and so forth.

We describe now how this encodes a gauge transformation

$$A_0(s=0) \stackrel{\lambda}{\to} A_U(s=1)\,.$$

Observation 1.2.147. By the nature of the Weil algebra we have

$$\frac{d}{ds} A_U = d_U \lambda + [\lambda \wedge A] + [\lambda \wedge A \wedge A] + \cdots + \iota_s F_A\,,$$

where the sum is over all higher brackets of the ∞-Lie algebra $\mathfrak{g}$.

In the Cartan calculus for the case that $\mathfrak{g}$ an ordinary one writes the corresponding *second Ehremsnn condition* $\iota_{\partial_s} F_A = 0$ equivalently

$$\mathcal{L}_{\partial_s} A = \mathrm{ad}_\lambda A\,.$$

Definition 1.2.148. Define the *covariant derivative of the gauge parameter* to be

$$\nabla \lambda := d\lambda + [A \wedge \lambda] + [A \wedge A \wedge \lambda] + \cdots\,.$$

Remark 1.2.149. In this notation we have

- the general identity

$$\frac{d}{ds} A_U = \nabla \lambda + (F_A)_s$$

- the *horizontality* constraint or *second Ehresmann condition* $\iota_{\partial_s} F_A = 0$, the differential equation

$$\frac{d}{ds} A_U = \nabla \lambda\,.$$

This is known as the equation for *infinitesimal gauge transformations* of an ∞-Lie algebra valued form.

Observation 1.2.150. By Lie integration we have that A_{vert} – and hence λ – defines an element $\exp(\lambda)$ in the ∞-Lie group that integrates $\mathfrak{g}$.

The unique solution $A_U(s=1)$ of the above differential equation at $s = 1$ for the initial values $A_U(s=0)$ we may think of as the result of acting on $A_U(0)$ with the gauge transformation $\exp(\lambda)$.

1.2.13.7 Examples of ∞-connections We discuss some examples of ∞-groupoids of ∞-connections obtained by Lie integration, as discussed in 1.2.13.6 above.

- 1.2.13.7.1 – Connections on ordinary principal bundles
- 1.2.13.7.2

1.2.13.7.1 Connections on ordinary principal bundles Let $\mathfrak{g}$ be an ordinary Lie algebra and write G for the simply connected Lie group integrating it. Write $\mathbf{B}G_{\mathrm{conn}}$ the groupoid of Lie algebra-valued forms from prop. 1.2.78.

Proposition 1.2.151. *The 1-truncation of the object* $\exp(\mathfrak{g})_{\mathrm{conn}}$ *from def. 1.2.143 is equivalent to the coefficient object for G-principal connections from prop. 1.2.78. We have an equivalence*

$$\tau_1 \exp(\mathfrak{g})_{\mathrm{conn}} = \mathbf{B}G_{\mathrm{conn}}$$

Proof. To see this, first note that the sheaves of objects on both sides are manifestly isomorphic, both are the sheaf of $\Omega^1(-,\mathfrak{g})$. For morphisms, observe that for a form $\Omega^\bullet(U \times \Delta^1) \leftarrow \mathrm{W}(\mathfrak{g}) : A$ which we may decompose into a horizontal and a verical piece as $A = A_U + \lambda \wedge dt$ the condition $\iota_{\partial_t} F_A = 0$ is equivalent to the differential equation

$$\frac{\partial}{\partial t} A = d_U \lambda + [\lambda, A]\,.$$

For any initial value $A(0)$ this has the unique solution

$$A(t) = g(t)^{-1}(A + d_U) g(t)\,,$$

where $g : [0,1] \to G$ is the parallel transport of λ:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(g_(t)^{-1}(A + d_U) g(t)\right) \\ =& g(t)^{-1}(A + d_U)\lambda g(t) - g(t)^{-1}\lambda(A + d_U) g(t) \end{aligned}$$

(where for ease of notation we write actions as if G were a matrix Lie group).

In particular this implies that the endpoints of the path of $\mathfrak{g}$-valued 1-forms are related by the usual cocycle condition in $\mathbf{B}G_{conn}$

$$A(1) = g(1)^{-1}(A + d_U) g(1)\,.$$

In the same fashion one sees that given 2-cell in $\exp(\mathfrak{g})(U)$ and any 1-form on U at one vertex, there is a unique lift to a 2-cell in $\exp(\mathfrak{g})_{conn}$, obtained by parallel transporting the form around. The claim then follows from the previous statement of Lie integration that $\tau_1 \exp(\mathfrak{g}) = \mathbf{B}G$. □

1.2.13.7.2 $\mathfrak{string}$-2-connections We discuss the $\mathfrak{string}$ Lie 2-algebra and local differential form data for $\mathfrak{string}$-2-connections. A detailed discussion of the corresponding String-principal 2-bundles is below in 5.1.4, more discussion of the 2-connections and their twisted generalization is in 5.4.7.3.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Write $\langle -,-\rangle : \mathfrak{g}^{\otimes 2} \to \mathbb{R}$ for its Killing form and

$$\mu = \langle -, [-,-]\rangle : \mathfrak{g}^{\otimes 3} \to \mathbb{R}$$

for the canonical 3-cocycle.

We discuss two very different looking, but nevertheless equivalent Lie 2-algebras.

Definition 1.2.152 (skeletal version of $\mathfrak{string}$)**.** Write $\mathfrak{g}_\mu$ for the Lie 2-algebra whose underlying graded vector space is

$$\mathfrak{g}_\mu = \mathfrak{g} \oplus \mathbb{R}[-1]\,,$$

and whose nonvanishing brackets are defined as follows.

- The binary bracket is that of $\mathfrak{g}$ when both arguments are from $\mathfrak{g}$ and 0 otherwise.
- The trinary bracket is the 3-cocycle

$$[-,-,-]_{\mathfrak{g}_\mu} := \langle -, [-,-]\rangle : \mathfrak{g}^{\otimes 3} \to \mathbb{R}\,.$$

Definition 1.2.153 (strict version of $\mathfrak{string}$)**.** Write $(\hat{\Omega}\mathfrak{g} \to P_*\mathfrak{g})$ for the Lie 2-algebra coming from the differential crossed module, def. 1.2.46, whose underlying vector space is

$$(\hat{\Omega}\mathfrak{g} \to P\mathfrak{g}) = P_*\mathfrak{g} \oplus (\Omega\mathfrak{g} \oplus \mathbb{R})[-1]\,,$$

where $P_*\mathfrak{g}$ is the vector space of smooth maps $\gamma : [0,1] \to \mathfrak{g}$ such that $\gamma(0) = 0$, and where $\Omega\mathfrak{g}$ is the subspace for which also $\gamma(1) = 0$, and whose non-vanishing brackets are defined as follows

- $[-]_1 = \partial := \Omega\mathfrak{g} \oplus \mathbb{R} \to \Omega\mathfrak{g} \hookrightarrow P_*\mathfrak{g}$;
- $[-,-] : P_*\mathfrak{g} \otimes P_*\mathfrak{g} \to P_*\mathfrak{g}$ is given by the pointwise Lie bracket on $\mathfrak{g}$ as

$$[\gamma_1, \gamma_2] = (\sigma \mapsto [\gamma_1(\sigma), \gamma_2(\sigma)])\,;$$

- $[-,-] : P_*\mathfrak{g} \otimes (\Omega\mathfrak{g} \oplus \mathbb{R}) \to \Omega\mathfrak{g} \oplus \mathbb{R}$ is given by pairs

$$[\gamma, (\ell, c)] := \left([\gamma, \ell],\ 2\int_0^1 \langle \gamma(\sigma), \frac{d\ell}{d\sigma}(\sigma)\rangle d\sigma \right)\,, \tag{1.1}$$

  where the first term is again pointwise the Lie bracket in $\mathfrak{g}$.

Proposition 1.2.154. *The linear map*

$$P_*\mathfrak{g} \oplus (\Omega\mathfrak{g} \oplus \mathbb{R})[-1] \to \mathfrak{g} \oplus \mathbb{R}[-1]\,,$$

which in degree 0 is evaluation at the endpoint

$$\gamma \mapsto \gamma(1)$$

and which in degree 1 is projection onto the $\mathbb{R}$-summand, induces a weak equivalence of L_∞ algebras

$$\mathfrak{string} \simeq (\hat{\Omega}\mathfrak{g} \to P_*\mathfrak{g}) \simeq \mathfrak{g}_\mu$$

Proof. This is theorem 30 in [BCSS07]. □

Definition 1.2.155. We write $\mathfrak{string}$ for the *string Lie 2-algebra* if we do not mean to specify a specific presentation such as so_μ or $(\hat{\Omega}\mathfrak{so} \to P_*\mathfrak{so})$.

In more technical language we would say that $\mathfrak{string}$ is defined to be the homotopy fiber of the morphism of L_∞-algebras $\mu_3 : \mathfrak{so} \to b^2\mathbb{R}$, well defined up to weak equivalence.

Remark 1.2.156. Proposition 1.2.154 says that the two Lie 2-algebras $(\hat{\Omega}\mathfrak{g} \to P_*\mathfrak{g})$ and $\mathfrak{g}_\mu$, which look quite different, are actually equivalent. Therefore also the local data for a String-2 connection can take two very different looking but nevertheless equivalent forms.

Let U be a smooth manifold. The data of $(\hat{\Omega}\mathfrak{g} \to P_*\mathfrak{g})$-valued forms on X is a triple

1. $A \in \Omega^1(U, P\mathfrak{g})$;
2. $B \in \Omega^2(U, \Omega\mathfrak{g})$;
3. $\hat{B} \in \Omega^2(U, \mathbb{R})$.

consisting of a 1-form with values in the path Lie algebra of $\mathfrak{g}$, a 2-form with values in the loop Lie algebra of $\mathfrak{g}$, and an ordinary real-valued 2-form that contains the central part of $\hat{\Omega}\mathfrak{g} = \Omega\mathfrak{g} \oplus \mathbb{R}$. The curvature data of this is

1. $F = dA + \frac{1}{2}[A \wedge A] + B \in \Omega^2(U, P\mathfrak{g})$;
2. $H = d(B + \hat{B}) + [A \wedge (B + \hat{B})] \in \Omega^3(U, \Omega\mathfrak{g} \oplus \mathbb{R})$, ,

where in the last term we have the bracket from (1.1). Notice that if we choose a basis $\{t_a\}$ of $\mathfrak{g}$ such that we have structure constant $[t_b, t_c] = f^a{}_{bc} t_a$, then for instance the first equation is

$$F^a(\sigma) = dA^a(\sigma) + \frac{1}{2} f^a{}_{bc} A^b(\sigma) \wedge A^c(\sigma) + B^a(\sigma) \,.$$

On the other hand, the data of forms in the equation Lie algebra $\mathfrak{g}_\mu$ on U is a tuple

1. $A \in \Omega^1(U, \mathfrak{g})$;
2. $\hat{B} \in \Omega^2(U, \mathbb{R})$,

consisting of a $\mathfrak{g}$-valued form and a real-valued 2-form. The curvature data of this is

1. $F = dA + [A \wedge A] \in \Omega^2(\mathfrak{g})$;
2. $H = d\hat{B} + \langle A \wedge [A \wedge A] \rangle \in \Omega^3(U)$.

While these two sets of data look very different, proposition 1.2.154 implies that under their respective higher gauge transformations they are in fact equivalent.

Notice that in the first case the 2-form is valued in a nonabelian Lie algebra, whereas in the second case the 2-form is abelian, but, to compensate this, a trilinear term appears in the formula for the curvatures. By the discussion in section 1.2.13.6 this means that a $\mathfrak{g}_\mu$-2-connection looks simpler on a single patch than an $(\hat{\Omega}\mathfrak{g} \to P_*\mathfrak{g})$-2-connection, it has relatively more complicated behavious on double intersections.

Moreover, notice that in the second case we see that one part of Chern-Simons term for A occurs, namely $\langle A \wedge [A \wedge A] \rangle$. The rest of the Chern-Simons term appears in this local formula after passing to yet another equivalent version of $\mathfrak{string}$, one which is well-adapted to the discussion of twisted String 2-connections. This we discuss in the next section.

The equivalence of the skeletal and the strict presentation for $\mathfrak{string}$ corresponds under Lie integration to two different but equivalent models of the smooth String-2-group.

Proposition 1.2.157. *The degeewise Lie integration of $\hat{\Omega}\mathfrak{so} \to P_*\mathfrak{so}$ yields the strict Lie 2-group* ($\hat{\Omega}\mathrm{Spin} \to P_*\mathrm{Spin}$)*, where* $\hat{\Omega}\mathrm{Spin}$ *is the level-1 Kac-Moody central extension of the smooth loop group of* Spin*.*

Proof. The nontrivial part to check is that the action of $P_*\mathfrak{so}$ on $\hat{\Omega}\mathfrak{so}$ lifts to a compatible action of $P_*\mathrm{Spin}$ on $\hat{\Omega}\mathrm{Spin}$. This is shown in [BCSS07]. □

Below in 5.1.4 we show that there is an equivalence of smooth n-stacks

$$\mathbf{B}(\hat{\Omega}\mathrm{Spin} \to P_*\mathrm{Spin}) \simeq \tau_2 \exp(\mathfrak{g}_\mu) \,.$$

1.2.14 The Chern-Weil homomorphism

We now come to the discussion the Chern-Weil homomorphism and its generalization to the ∞-Chern-Weil homomorphism.

We have seen in 1.2.5 G-principal ∞-bundles for general smooth ∞-groups G and in particular for abelian groups G. Naturally, the abelian case is easier and more powerful statements are known about this case. A general strategy for studying nonabelian ∞-bundles therefore is to *approximate* them by abelian bundles. This is achieved by considering characteristic classes. Roughly, a characteristic class is a map that functorially sends G-principal ∞-bundles to $\mathbf{B}^n K$-principal ∞-bundles, for some n and some abelian group K. In some cases such an assignment may be obtained by integration of infinitesimal data. If so, then the assignment refines to one of ∞-bundles with connection. For G an ordinary Lie group this is then what is called the *Chern-Weil homomorphism*. For general G we call it the ∞-*Chern-Weil homomorphism*.

The material of this section is due to [SSS09a] and [FSS10].

1.2.14.1 Motivating examples A simple motivating example for characteristic classes and the Chern-Weil homomorphism is the construction of determinant line bundles from example 1.2.106. This construction directly extends to the case where the bundles carry connections. We give an exposition of this *differential refinement* of the *universal first Chern class*, example 1.2.106. A more formal discussion of this situation is below in 5.4.7.1.

We may canonically identify the Lie algebra $\mathfrak{u}(n)$ with the matrix Lie algebra of skew-hermitian matrices on which we have the trace operation

$$\mathrm{tr} : \mathfrak{u}(n) \to \mathfrak{u}(1) = i\mathbb{R} .$$

This is the differential version of the determinant in that when regarding the Lie algebra as the infinitesimal neighbourhood of the neutral element in $U(N)$ the determinant becomes the trace under the exponential map

$$\det(1 + \epsilon A) = 1 + \epsilon \mathrm{tr}(A)$$

for $\epsilon^2 = 0$. It follows that for $\mathrm{tra}_\nabla : \mathbf{P}_1(U_i) \to \mathbf{B}U(N)$ the parallel transport of a connection on P locally given by a 1-forms $A \in \Omega^1(U_i, \mathfrak{u}(N))$ by

$$\mathrm{tra}_\nabla(\gamma) = \mathcal{P}\exp \int_{[0,1]} \gamma^* A$$

the determinant parallel transport

$$\det(\mathrm{tra}_\nabla =: \mathbf{P}_1(U_i) \stackrel{\mathrm{tra}_\nabla}{\to} \mathbf{B}U(N) \stackrel{\det}{\to} \mathbf{B}U(1)$$

is locally given by the formula

$$\det(\mathrm{tra}_\nabla(\gamma)) = \mathcal{P}\exp \int_{[0,1]} \gamma^* \mathrm{tr} A ,$$

which means that the local connection forms on the determinant line bundle are obtained from those of the unitary bundle by tracing.

$$(\det, \mathrm{tr}) : \{(g_{ij}), (A_i)\} \mapsto \{(\det g_{ij}), (\mathrm{tr} A_i)\} .$$

This construction extends to a functor

$$(\hat{\mathbf{c}}_1) := (\det, \mathrm{tr}) : U(N)\mathrm{Bund}_{\mathrm{conn}}(X) \to U(1)\mathrm{Bund}_{\mathrm{conn}}(X)$$

natural in X, that sends $U(n)$-principal bundles with connection to circle bundles with connection, hence to cocycles in degree-2 ordinary differential cohomology.

This assignment remembers of a unitary bundle one inegral class and its differential refinement:

- the integral class of the determinant bundle is the first Chern class the $U(N)$-bundle

$$[\hat{\mathbf{c}}_1(P)] = c_1(P) ;$$

- the curvature 2-form of its connection is a representative in de Rham cohomology of this class

$$[F_{\nabla_{\hat{\mathbf{c}}_1(P)}}] = c_1(P)_{\mathrm{dR}} .$$

$$
\begin{array}{ccccccc}
 & H^2_{\mathrm{diff}}(X) & & & & \hat{\mathbf{c}}_1(P) & \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
H^2(X,\mathbb{Z}) & & \Omega^2_{\mathrm{cl}}(X) & & c_1(P) & & \mathrm{tr}(F_\nabla)
\end{array}
$$

Equivalently this assignment is given by postcomposition of cocycles with a morphism of smooth ∞-groupoids

$$\hat{\mathbf{c}}_1 : \mathbf{B}U(N)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}} \,.$$

We say that $\hat{\mathbf{c}}_1$ is a *differential characteristic class*, the differential refinement of the first Chern class.

In [BrMc96b] an algorithm is given for contructing differential characteristic classes on Čech cocycles in this fashion for more general Lie algebra cocycles.

For instance these authors give the following construction for the diffrential refinement of the first Pontryagin class [BrMc93].

Let $N \in \mathbb{N}$, write $\mathrm{Spin}(N)$ for the Spin group and consider the canonical Lie algebra cohomology 3-cocycle

$$\mu = \langle -, [-,-] \rangle : \mathfrak{so}(n) \to \mathbf{b}^2\mathbb{R}$$

on semisimple Lie algebras, where $\langle -,- \rangle$ is the Killing form invariant polynomial. Let $(P \to X, \nabla)$ be a $\mathrm{Spin}(N)$-principal bundle with connection. Let $A \in \Omega^1(P, \mathfrak{so}(N))$ be the Ehresmann connection 1-form on the total space of the bundle.

Then construct a Čech cocycle for Deligne cohomology in degree 4 as follows:

1. pick an open cover $\{U_i \to X\}$ such that there is a choice of local sections $\sigma_i : U_i \to P$. Write

   $$(g_{ij}, A_i) := (\sigma_i^{-1}\sigma_j, \sigma_i^* A)$$

   for the induced Čech cocycle.

2. Choose a lift of this cocycle to an assignment

   - of based paths in $\mathrm{Spin}(N)$ to double intersections

     $$\hat{g}_{ij} : U_{ij} \times \Delta^1 \to Spin(N) \,,$$

     with $\hat{g}_{ij}(0) = e$ and $\hat{g}_{ij}(1) = g_{ij}$;

   - of based 2-simplices between these paths to triple intersections

     $$\hat{g}_{ijk} : U_{ijk} \times \Delta^2 \to \mathrm{Spin}(N) \,;$$

     restricting to these paths in the obvious way;

   - similarly of based 3-simplices between these paths to quadruple intersections

     $$\hat{g}_{ijkl} : U_{ijkl} \times \Delta^3 \to \mathrm{Spin}(N) \,.$$

   Such lifts always exists, because the Spin group is connected (because already $SO(N)$ is), simply connected (because $Spin(N)$ is the universal cover of $SO(N)$) and also has $\pi_2(Spin(N)) = 0$ (because this is the case for every compact Lie group).

3. Define from this a Deligne-cochain by setting

   $$\frac{1}{2}\hat{\mathbf{p}}_1(P) := (g_{ijkl}, A_{ijk}, B_{ij}, C_i) := \left(\begin{array}{l} \int_{\Delta^3} (\sigma_i \cdot \hat{g}_{ijkl})^* \mu(A) mod \mathbb{Z}, \\ \int_{\Delta^2} (\sigma_i \cdot \hat{g}_{ijk})^* \mathrm{cs}(A), \\ \int_{\Delta^1} (\sigma_i \cdot \hat{g}_{ij})^* \mathrm{cs}(A), \\ \sigma_i^* \mu(A) \end{array} \right) ,$$

   where $\mathrm{cs}(A) = \langle A \wedge F_A \rangle + c\langle A \wedge [A \wedge A] \rangle$ is the Chern-Simons form of the connection form A with respect to the cocyle $\mu(A) = \langle A \wedge [A \wedge A] \rangle$.

They then prove:

1. This is indeed a Deligne cohomology cocycle;
2. it represents the differential refinement of the first fractional Pontryagin class of P.

$$\begin{array}{ccc} & H^4_{\mathrm{diff}}(X) & \\ \swarrow & & \searrow \\ H^4(X,\mathbb{Z}) & & \Omega^4_{\mathrm{cl}}(X) \end{array} \qquad \begin{array}{ccc} & \frac{1}{2}\hat{\mathbf{p}}_1(P) & \\ \swarrow & & \searrow \\ \frac{1}{2}p_1(P) & & d\mathrm{cs}(A) \end{array} .$$

In the form in which we have (re)stated this result here the second statement amounts, in view of the first statement, to the observation that the curvature 4-form of the Deligne cocycle is proportional to

$$d\mathrm{cs}(A) \propto \langle F_A \wedge F_A \rangle \in \Omega^4_{\mathrm{cl}}(X)$$

which represents the first Pontryagin class in de Rham cohomology. Therefore the key observation is that we have a Deligne cocycle at all. This can be checked directly, if somewhat tediously, by hand.

But then the question remains: where does this successful *Ansatz* come from? And is it *natural*? For instance: does this construction extend to a morphism of smooth ∞-groupoids

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}(N)_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

from Spin-principal bundles with connection to circle 3-bundles with connection?

In the following we give a natural presentation of the ∞-Chern-Weil homomorphism by means of Lie integration of L_∞-algebraic data to simplicial presheaves. Among other things, this construction yields an understanding of why this construction is what it is and does what it does.

The construction proceeds in the following broad steps

1. The infinitesimal analog of a characteristic class $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ is an L_∞-algebra cocycle

   $$\mu : \mathfrak{g} \to b^{n-1}\mathbb{R} .$$

2. There is a formal procedure of universal Lie integration which sends this to a morphism of smooth ∞-groupoids

   $$\exp(\mu) : \exp(\mathfrak{g}) \to \exp(b^{n-1}\mathbb{R}) \simeq \mathbf{B}^n\mathbb{R}$$

   presented by a morphism of simplicial presheaves on CartSp.

3. By finding a Chern-Simons element cs that witnesses the transgression of μ to an invariant polynomial on $\mathfrak{g}$ this construction has a differential refinement to a morphism

   $$\exp(\mu, \mathrm{cs}) : \exp(\mathfrak{g})_{\mathrm{conn}} \to \mathbf{B}^n\mathbb{R}_{\mathrm{conn}}$$

   that sends L_∞-algebra valued connections to line n-bundles with connection.

4. The n-truncation $\mathbf{cosk}_{n+1}\exp(\mathfrak{g})$ of the object on the left produces the smooth ∞-groups on interest – $\mathbf{cosk}_{n+1}\exp(\mathfrak{g}) \simeq \mathbf{B}G$ – and the corresponding truncation of $\exp((\mu, cs))$ carves out the lattice Γ of periods in G of the cocycle μ inside $\mathbb{R}$. The result is the differential characteristic class

   $$\exp(\mu, \mathrm{cs}) : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n\mathbb{R}/\Gamma_{\mathrm{conn}} .$$

   Typically we have $\Gamma \simeq \mathbb{Z}$ such that this then reads

   $$\exp(\mu, \mathrm{cs}) : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}} .$$

1.2.14.2 The ∞-Chern-Weil homomorphism In the full ∞-Chern-Weil theory the ∞-Chern-Weil homomorphism is conceptually very simple: for every n there is canonically a morphism of smooth ∞-groupoids $\mathbf{B}^n U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ where the object on the right classifies ordinary de Rham cohomology in degree $n+1$. For G any ∞-group and any characteristic class $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+1}U(1)$, the ∞-Chern-Weil homomorphism is the operation that takes a G-principal ∞-bundle $X \to \mathbf{B}G$ to the composite $X \to \mathbf{B}G \to \mathbf{B}^n U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$.

All the construction that we consider here in this introduction serve to *mode* this abstract operation. The ∞-connections that we considered yield resolutions of $\mathbf{B}^n U(1)$ and $\mathbf{B}G$ in terms of which the abstract morphisms are modeled as ∞-anafunctors.

1.2.14.2.1 ∞-Chern-Simons functionals If we express G by Lie integration of an ∞-Lie algebra $\mathfrak{g}$, then the basic ∞-Chern-Weil homomorphism is modeled by composing an ∞-connection $(A_{\mathrm{vert}}, A, \langle F_A\rangle)$ with the transgression of an invariant polynomial $(\mu, \mathrm{cs}, \langle - \rangle)$ as follows

$$
\left(
\begin{array}{ccc}
\Omega^\bullet(U\times\Delta^k)_{\mathrm{vert}} \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) & & \text{Čech cocycle} \\
\uparrow \qquad\qquad \uparrow & & \\
\Omega^\bullet(U\times\Delta^k) \xleftarrow{A} \mathrm{W}(\mathfrak{g}) & & \text{connection} \\
\uparrow \qquad\qquad \uparrow & & \\
\Omega^\bullet(U) \xleftarrow{\langle F_A\rangle} \mathrm{inv}(\mathfrak{g}) & & \begin{array}{c}\text{curvature}\\ \text{characteristic forms}\end{array}
\end{array}
\right)
\circ
\left(
\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}) \xleftarrow{\mu} \mathrm{CE}(b^{n-1}\mathbb{R}) & & \text{cocycle} \\
\uparrow \qquad\qquad \uparrow & & \\
\mathrm{W}(\mathfrak{g}) \xleftarrow{\mathrm{cs}} \mathrm{W}(b^{n-1}\mathbb{R}) & & \begin{array}{c}\text{Chern-Simons}\\ \text{element}\end{array} \\
\uparrow \qquad\qquad \uparrow & & \\
\mathrm{inv}(\mathfrak{g}) \xleftarrow{\langle - \rangle} \mathrm{inv}(b^{n-1}\mathbb{R}) & & \begin{array}{c}\text{invariant}\\ \text{polynomial}\end{array}
\end{array}
\right)
$$

$$
= \left(
\begin{array}{ccccc}
\Omega^\bullet(U\times\Delta^k)_{\mathrm{vert}} \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \xleftarrow{\mu} \mathrm{CE}(b^{n-1}\mathbb{R}) & & : \mu(A_{\mathrm{vert}}) & & \text{characteristic class} \\
\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow & & & & \\
\Omega^\bullet(U\times\Delta^k) \xleftarrow{A} \mathrm{W}(\mathfrak{g}) \xleftarrow{\mathrm{cs}} \mathrm{W}(b^n - 1\mathbb{R}) & & : \mathrm{cs}_\mu(A) & & \text{Chern-Simons form} \\
\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow & & & & \\
\Omega^\bullet(U) \xleftarrow{\langle F_A\rangle} \mathrm{inv}(\mathfrak{g}) \xleftarrow{\langle - \rangle} \mathrm{inv}(b^{n-1}\mathbb{R}) & & : \langle F_A\rangle_\mu & & \begin{array}{c}\text{curvature}\\ \text{characteristic forms}\end{array}
\end{array}
\right) .
$$

This evidently yields a morphism of simplicial presheaves

$$\exp(\mu)_{\mathrm{conn}} : \exp(\mathfrak{g})_{\mathrm{conn}} \to \exp(b^{n-1}\mathbb{R})_{\mathrm{conn}}$$

and, upon restriction to the top two horizontal layers, a morphism

$$\exp(\mu)_{\mathrm{diff}} : \exp(\mathfrak{g})_{\mathrm{diff}} \to \exp(b^{n-1}\mathbb{R})_{\mathrm{diff}} .$$

Projection onto the third horizontal component gives the map to the curvature classes

$$\exp(b^{n-1}\mathbb{R})_{\mathrm{diff}} \to \flat_{\mathrm{dR}} \exp(b^n\mathbb{R})_{\mathrm{simp}} ,$$

In total, this constitutes an ∞-anafunctor

$$
\begin{array}{ccccc}
\exp(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{\exp(\mu)_{\mathrm{diff}}} & \exp(b^{n-1}\mathbb{R})_{\mathrm{diff}} & \longrightarrow & \flat_{\mathrm{dR}} b^n\mathbb{R} \\
\downarrow{\simeq} & & & & \\
\exp(\mathfrak{g}) & & & &
\end{array}
$$

Postcomposition with this is the simple ∞-Chern-Weil homomorphism: it sends a cocycle

$$\begin{array}{ccc} C(U) & \longrightarrow & \exp(\mathfrak{g}) \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array}$$

for an $\exp(\mathfrak{g})$-principal bundle to the curvature form represented by

$$\begin{array}{ccccccc} C(V) & \xrightarrow{(g,\nabla)} & \exp(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{\exp(\mu)_{\mathrm{diff}}} & \exp(b^{n-1}\mathbb{R})_{\mathrm{diff}} & \longrightarrow & \flat_{\mathrm{dR}} b^n\mathbb{R} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & & & \\ C(U) & \xrightarrow{g} & \exp(\mathfrak{g}) & & & & \\ \downarrow{\scriptstyle \simeq} & & & & & & \\ X & & & & & & \end{array} .$$

Proposition 1.2.158. *For $\mathfrak{g}$ an ordinary Lie algebra with simply connected Lie group G, the image under $\tau_1(-)$ of this diagram constitutes the ordinary Chern-Weil homomorphism in that:*

for g the cocycle for a G-principal bundle, any ordinary connection on a bundle constitutes a lift (g, ∇) to the tip of the anafunctor and the morphism represented by that is the Čech-hypercohomology cocycle on X with values in the truncated de Rham complex given by the globally defined curvature characteristic form $\langle F_\nabla \wedge \cdots \wedge F_\nabla \rangle$.

But evidently we have more information available here. The ordinary Chern-Weil homomorphism refines from a map that assigns curvature characteristic forms, to a map that assigns secondary characteristic classes in the sense that it assigns circle n-bundles with connection whose curvature is this cuvature characteristic form. The local connection forms of these circle bundles are given by the middle horizontal morphisms. These are the Chern-Simons forms

$$\Omega^\bullet(U) \xleftarrow{A} \mathrm{W}(\mathfrak{g}) \xleftarrow{cs} \mathrm{W}(b^{n-1}\mathbb{R}) : \mathrm{cs}(A) \,.$$

1.2.14.2.2 Secondary characteristic classes So far we discussed the untruncated coefficient object $\exp(\mathfrak{g})_{\mathrm{conn}}$ of $\mathfrak{g}$-valued ∞-connections. The real object of interest is the k-truncated version $\tau_k \exp(\mathfrak{g})_{\mathrm{conn}}$ where $k \in \mathbb{N}$ is such that $\tau_k \exp)\mathfrak{g} \simeq \mathbf{B}G$ is the delooping of the ∞-Lie group in question.

Under such a truncation, the integrated ∞-Lie algebra cocycle $exp(\mu) : exp(\mathfrak{g}) \to exp(b^{n-1}\mathbb{R})$ will no longer be a simplicial map. Instead, the periods of μ will cut out a lattice Γ in $\mathbb{R}$, and $\exp(\mu)$ does descent to the quotient of $\mathbb{R}$ by that lattice

$$\exp(\mu) : \tau_k \exp(\mathfrak{g}) \to \mathbf{B}^n\mathbb{R}/\Gamma \,.$$

We now say this again in more detail.

Suppose $\mathfrak{g}$ is such that the $(n+1)$-coskeleton $\mathbf{cosk}_{n+1}\exp(\mathfrak{g}) \xrightarrow{\simeq}\simeq \mathbf{B}G$ for the desired G. Then the periods of μ over $(n+1)$-balls cut out a lattice $\Gamma \subset \mathbb{R}$ and thus we get an ∞-anafunctor

$$\begin{array}{ccccc} \mathbf{cosk}_{n+1}\exp(\mathfrak{g})_{\mathrm{diff}} & \longrightarrow & \mathbf{B}^n\mathbb{R}/\Gamma_{\mathrm{diff}} & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}/\Gamma \\ \downarrow{\scriptstyle \simeq} & & & & \\ \mathbf{B}G & & & & \end{array}$$

This is *curvature characteristic class*. We may always restrict to genuine ∞-connections and refine

$$
\begin{array}{ccccc}
\mathbf{cosk}_{n+1}\exp(\mathfrak{g})_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^n\mathbb{R}/\Gamma_{\mathrm{conn}} & & \\
\downarrow & & \downarrow & & \\
\mathbf{cosk}_{n+1}\exp(\mathfrak{g})_{\mathrm{diff}} & \longrightarrow & \mathbf{B}^n\mathbb{R}/\Gamma_{\mathrm{diff}} & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}/\Gamma \\
\downarrow{\simeq} & & & & \\
\mathbf{B}G & & & &
\end{array}
$$

which models the refined ∞-Chern-Weil homomorphism with values in ordinary differential cohomology

$$H_{\mathrm{conn}}(X,G) \to \mathbf{H}^{n+1}_{conn}(X,\mathbb{R}/\Gamma)\,.$$

Example 1.2.159. Applying this to the discussion of the Chern-Simons circle 3-bundle above, we find a differential refinement

$$
\begin{array}{ccccc}
 & & \exp(\mathfrak{g})_{\mathrm{diff}}\exp(\mu)_{\mathrm{diff}} & \longrightarrow & \exp(b^{n-1}\mathbb{R})_{\mathrm{diff}}\;. \\
 & & \downarrow & & \downarrow{\textstyle\int_{\Delta^\bullet}} \\
C(V) & \xrightarrow{(\hat{g},\hat{\nabla})} & \mathbf{cosk}_3\exp(\mathfrak{g})_{\mathrm{diff}} & \longrightarrow & \mathbf{B}^3U(1)_{\mathrm{diff}} \\
\downarrow{\simeq} & & \downarrow & & \\
C(U) & \xrightarrow{(g,\nabla)} & \mathbf{B}G_{\mathrm{diff}} & & \\
\downarrow{\simeq} & & & & \\
X & & & &
\end{array}
$$

Chasing components through this composite one finds that this descibes the cocycle in Deligne cohomology given by

$$(CS(\sigma_i^*\nabla),\ \int_{\Delta^1} CS(\hat{g}_{ij}^*\nabla), \int_{\Delta^2} CS(\hat{g}_{ijk}^*\nabla), \int_{\Delta^3} \hat{g}_{ijkl}^*\mu)\,.$$

This is the cocycle for the circle n-bundle with connection.

This is precisely the form of the Čech-Deligne cocycle for the first Pontryagin class given in [BrMc96b], only that here it comes out automatically normalized such as to represent the fractional generator $\frac{1}{2}\mathbf{p}_1$.

By feeding in more general transgressive ∞-Lie algebra cocycles through this machine, we obtain cocycles for more general differential characteristic classes. For instance the next one is the second fractional Pontryagin class of String-2-bundles with connection [FSS10]. Moreover, these constructions naturally yield the full cocycle ∞-groupoids, not just their cohomology sets. This allows to form the homotopy fibers of the ∞-Chern-Weil homomorphism and thus define *differential string structures* etc. and *twisted* differential string structures etc. [SSS09c].

1.2.15 3d Chern-Simons theory

For G a simply connected compact simple Lie group, the above construction of the refined Chern-Weil homomorphism yields a differential characteristic map of moduli stacks

$$\hat{\mathbf{c}}:\ \mathbf{B}G_{\mathrm{conn}} \qquad \mathbf{B}^3U(1)_{\mathrm{conn}}$$

which is the smooth and differential refinement of the universal characteristic class $[c] \in H^4(BG,\mathbb{Z})$.

We discuss now how this serves as the *extended* Lagrangian for 3d Chern-Simons theory in that its *transgression* to mapping stacks out of k-dimensional manifolds yields all the "geometric prequantum" data of Chern-Simons theory in the corresponding dimension, in the sense of geometric quantization. For the purpose of this exposition we use terms such as "prequantum n-bundle" freely without formal definition. We expect the reader can naturally see at least vaguely the higher prequantum picture alluded to here. A more formal survey of these notions is in section 1.2.16.

The following paragraphs have been written jointly with Domenico Fiorenza. They are taken from [FiSaScV].

If X is a compact oriented manifold without boundary, then there is a fiber integration in differential cohomology lifting fiber integration in integral cohomology [HoSi05]:

$$\begin{array}{ccc} \hat{H}^{n+\dim X}(X \times Y; \mathbb{Z}) & \xrightarrow{\int_X} & \hat{H}^n(Y;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^{n+\dim X}(X \times Y; \mathbb{Z}) & \xrightarrow{\int_X} & H^n(Y;\mathbb{Z}) \end{array} .$$

In [**?**] Gomi and Terashima describe an explicit lift of this at the level of Čech-Deligne cocycles; see also [**?**]. Such a lift has a natural interpretation as a morphism

$$\mathrm{hol}_X : \mathbf{Maps}(X, \mathbf{B}^{n+\dim X}U(1)_{\mathrm{conn}}) \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

from the $(n+\dim X)$-stack of moduli of $U(1)$-$(n+\dim X)$-bundles with connection over X to the n-stack of $U(1)$-n-bundles with connection (sectin 2.4 of [**?**]). Therefore, if Σ_k is a compact oriented manifold of dimension k with $0 \leq k \leq 3$, we have a composition

$$\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_k, \hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_k, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_k}} \mathbf{B}^{3-k} U(1)_{\mathrm{conn}} .$$

This is the canonical $U(1)$-$(3-k)$-bundle with connection over the moduli space of principal G-bundles with connection over Σ_k induced by $\hat{\mathbf{c}}$: the *transgression* of $\hat{\mathbf{c}}$ to the mapping space. Composing on the right with the curvature morphism we get the underlying canonical closed $(4-k)$-form

$$\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \to \Omega^{4-k}(-;\mathbb{R})_{\mathrm{cl}}$$

on this moduli space. In other words, the moduli stack of principal G-bundles with connection over Σ_k carries a canonical *pre-$(3-k)$-plectic structure* (the higher order generalization of a symplectic structure, [Rog11]) and, moreover, this is equipped with a canonical geometric prequantization: the above $U(1)$-$(3-k)$-bundle with connection.

Let us now investigate in more detail the cases $k = 0, 1, 2, 3$.

1.2.15.1 $k = 0$: the universal Chern-Simons 3-connection $\hat{\mathbf{c}}$ The connected 0-manifold Σ_0 is the point and, by definition of **Maps**, one has a canonical identification

$$\mathbf{Maps}(*, \mathbf{S}) \cong \mathbf{S}$$

for any (higher) stack $\mathbf{S}$. Hence the morphism

$$\mathbf{Maps}(*, \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(*, \hat{\mathbf{c}})} \mathbf{Maps}(*, \mathbf{B}^3 U(1)_{\mathrm{conn}})$$

is nothing but the universal differential characteristic map $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$ that refines the universal characteristic class c. This map modulates a circle 3-bundle with connection (bundle 2-gerbe) on the universal moduli stack of G-principal connections. For $\nabla : X \longrightarrow \mathbf{B}G_{\mathrm{conn}}$ any given G-principal connection on some X, the pullback

$$\hat{\mathbf{c}}(\nabla) : \; X \xrightarrow{\nabla} \mathbf{B}G_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}} \mathbf{B}^3U(1)_{\mathrm{conn}}$$

is a 3-bundle (bundle 2-gerbe) on X which is sometimes in the literature called the *Chern-Simons 2-gerbe* of the given connection ∇. Accordingly, $\hat{\mathbf{c}}$ modulates the *universal* Chern-Simons bundle 2-gerbe with universal 3-connection. From the point of view of higher geometric quantization, this is the *prequantum 3-bundle* of extended prequantum Chern-Simons theory.

This means that the prequantum $U(1)$-$(3-k)$-bundles associated with k-dimensional manifolds are all determined by the the prequantum $U(1)$-3-bundle associated with the point, in agreement with the formulation of fully extended topological field theories [1]. We will denote by the symbol $\omega^{(4)}_{\mathbf{B}G_{\mathrm{conn}}}$ the pre-3-plectic 4-form induced on $\mathbf{B}G_{\mathrm{conn}}$ by the curvature morphism.

1.2.15.2 $k = 1$**: the Wess-Zumino-Witten bundle gerbe** We now come to the transgression of the extended Chern-Simons Lagrangian to the closed connected 1-manifold, the circle $\Sigma_1 = S^1$. Here we find a higher analog of the construction described in section **??**. Notice that, on the one hand, we can think of the mapping stack $\mathbf{Maps}(\Sigma_1, \mathbf{B}G_{\mathrm{conn}}) \simeq \mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{conn}})$ as a kind of moduli stack of G-connections on the circle – up to a slight subtlety, which we explain in more detail below in section **??**. On the other hand, we can think of that mapping stack as the *free loop space* of the universal moduli stack $\mathbf{B}G_{\mathrm{conn}}$.

The subtlety here is related to the differential refinement, so it is instructive to first discard the differential refinement and consider just the smooth characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)$ which underlies the extended Chern-Simons Lagrangian and which modulates the universal circle 3-bundle on $\mathbf{B}G$ (without connection). Now, for every pointed stack $* \to \mathbf{S}$ we have the corresponding (categorical) *loop space* $\Omega\mathbf{S} := * \times_{\mathbf{S}} *$, which is the homotopy pullback of the point inclusion along itself. Applied to the moduli stack $\mathbf{B}G$ this recovers the Lie group G, identified with the sheaf (i.e, the 0-stack) of smooth functions with target G: $\Omega\mathbf{B}G \simeq \underline{G}$. This kind of looping/delooping equivalence is familiar from the homotopy theory of classifying spaces; but notice that since we are working with smooth (higher) stacks, the loop space $\Omega\mathbf{B}G$ also knows the smooth structure of the group G, i.e. it knows G as a Lie group. Similarly, we have

$$\Omega\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1)$$

and so forth in higher degrees. Since the looping operation is functorial, we may also apply it to the characteristic map $\mathbf{c}$ itself to obtain a map

$$\Omega\mathbf{c} : \underline{G} \to \mathbf{B}^2U(1)$$

which modulates a $\mathbf{B}U(1)$-principal 2-bundle on the Lie group G. This is also known as the *WZW-bundle gerbe*; see [**?**, **?**]. The reason, as discussed there and as we will see in a moment, is that this is the 2-bundle that underlies the 2-connection with surface holonomy over a worldsheet given by the Wess-Zumino-Witten action functional. However, notice first that there is more structure implied here: for any pointed stack $\mathbf{S}$ there is a natural equivalence $\Omega\mathbf{S} \simeq \mathbf{Maps}_*(\mathbf{\Pi}(S^1), \mathbf{S})$, between the loop space object $\Omega\mathbf{S}$ and the moduli stack of *pointed maps* from the categorical circle $\mathbf{\Pi}(S^1) \simeq \mathbf{B}\mathbb{Z}$ to $\mathbf{S}$. Here $\mathbf{\Pi}$ denotes the *path ∞-groupoid* of a given (higher) stack.[2] On the other hand, if we do not fix the base point then we obtain the *free loop space object* $\mathcal{L}\mathbf{S} \simeq \mathbf{Maps}(\mathbf{\Pi}(S^1), \mathbf{S})$. Since a map $\mathbf{\Pi}(\Sigma) \to \mathbf{B}G$ is equivalently a map $\Sigma \to \flat\mathbf{B}G$, i.e., a flat G-principal connection on Σ, the free loop space $\mathcal{L}\mathbf{B}G$ is equivalently the moduli stack of flat G-principal connections on S^1. We will come back to this perspective in section **??** below. The homotopies that do not fix the base point act by conjugation on loops and hence we have, for any smooth (higher) group, that

$$\mathcal{L}\mathbf{B}G \simeq \underline{G}//_{\mathrm{Ad}}\underline{G}$$

[2] The existence and functoriality of the path ∞-groupoids is one of the features characterizing the higher topos of higher smooth stacks as being *cohesive*, see [**?**].

is the (homotopy) quotient of the adjoint action of G on itself; see [NSSa] for details on homotopy actions of smooth higher groups. For G a Lie group this is the familiar adjoint action quotient stack. But the expression holds fully generally. Notably, we also have

$$\mathcal{L}\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1)//_{\mathrm{Ad}}\mathbf{B}^2U(1)$$

and so forth in higher degrees. However, in this case, since the smooth 3-group $\mathbf{B}^2U(1)$ is abelian (it is a groupal E_∞-algebra) the adjoint action splits off in a direct factor and we have a projection

$$\mathcal{L}\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1) \times (*//\mathbf{B}^2U(1)) \xrightarrow{p_1} \mathbf{B}^2U(1) \ .$$

In summary, this means that the map $\Omega\mathbf{c}$ modulating the WZW 2-bundle over G descends to the adjoint quotient to the map

$$p_1 \circ \mathcal{L}\mathbf{c} : \underline{G}//_{\mathrm{Ad}}\underline{G} \to \mathbf{B}^2U(1)\,,$$

and this means that the WZW 2-bundle is canonically equipped with the structure of an ad_G-*equivariant* bundle gerbe, a crucial feature of the WZW bundle gerbe [**?**, **?**].

We emphasize that the derivation here is fully general and holds for any smooth (higher) group G and any smooth characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^nU(1)$. Each such pair induces a WZW-type $(n-1)$-bundle on the smooth (higher) group G modulated by $\Omega\mathbf{c}$ and equipped with G-equivariant structure exhibited by $p_1 \circ \mathcal{L}\mathbf{c}$. We discuss such higher examples of higher Chern-Simons-type theories with their higher WZW-type functionals further below in section **??**.

We now turn to the differential refinement of this situation. In analogy to the above construction, but taking care of the connection data in the extended Lagrangian $\hat{\mathbf{c}}$, we find a homotopy commutative diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}})} & \mathbf{Maps}(S^1;\mathbf{B}^3U(1)_{\mathrm{conn}}) & & \\ & & \downarrow{\scriptstyle \mathrm{hol}} & & \downarrow{\scriptstyle \mathrm{hol}} & & \\ \underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G} & \xrightarrow{\mathbf{wzw}} & \mathbf{B}^2U(1)_{\mathrm{conn}}//_{\mathrm{Ad}}\mathbf{B}^2U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^2U(1)_{\mathrm{conn}} \end{array} ,$$

where the vertical maps are obtained by forming holonomies of (higher) connections along the circle. The lower horizontal row is the differential refinement of $\Omega\mathbf{c}$: it modulates the Wess-Zumino-Witten $U(1)$-bundle gerbe with connection

$$\mathbf{wzw} : \underline{G} \to \mathbf{B}^2U(1)_{\mathrm{conn}} \ .$$

That $\mathbf{wzw}$ is indeed the correct differential refinement can be seen, for instance, by interpreting the construction by Carey-Johnson-Murray-Stevenson-Wang in [CJMSW05] in terms of the above diagram. That is, choosing a basepoint x_0 in S^1 one obtains a canonical lift of the leftmost vertical arrow:

$$\begin{array}{ccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) \\ & \nearrow^{(P_{x_0},\nabla_{x_0})} & \downarrow{\scriptstyle \mathrm{hol}} \\ \underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G} \end{array} ,$$

where $(P_{x_0}\nabla_{x_0})$ is the principal G-bundle with connection on the product $G \times S^1$ characterized by the property that the holonomy of ∇_{x_0} along $\{g\} \times S^1$ with starting point (g, x_0) is the element g of G. Correspondingly, we have a homotopy commutative diagram

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}})} & \mathbf{Maps}(S^1;\mathbf{B}^3U(1)_{\mathrm{conn}}) & & \\ & \nearrow^{(P_{x_0},\nabla_{x_0})} & \downarrow{\scriptstyle \mathrm{hol}} & & \downarrow{\scriptstyle \mathrm{hol}} & \searrow^{\mathrm{hol}_{S^1}} & \\ \underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G} & \xrightarrow{\mathbf{wzw}} & \mathbf{B}^2U(1)_{\mathrm{conn}}//_{\mathrm{Ad}}\mathbf{B}^2U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^2U(1)_{\mathrm{conn}} \end{array} .$$

Then Proposition 3.4 from [CJMSW05] identifies the upper path (and hence also the lower path) from $\underline{G}$ to $\mathbf{B}^2U(1)_{\mathrm{conn}}$ with the Wess-Zumino-Witten bundle gerbe.

Passing to equivalence classes of global sections, we see that **wzw** induces, for any smooth manifold X, a natural map $C^\infty(X;G) \to \hat{H}^2(X;\mathbb{Z})$. In particular, if $X = \Sigma_2$ is a compact Riemann surface, we can further integrate over X to get

$$wzw : C^\infty(\Sigma_2;G) \to \hat{H}^2(X;\mathbb{Z}) \xrightarrow{\int_{\Sigma_2}} U(1) \,.$$

This is the *topological term* in the Wess-Zumino-Witten model; see [**?**, **?**, **?**]. Notice how the fact that **wzw** factors through $\underline{G}//_{\mathrm{Ad}}\underline{G}$ gives the conjugation invariance of the Wess-Zumino-Witten bundle gerbe, and hence of the topological term in the Wess-Zumino-Witten model.

1.2.15.3 $k = 2$**: the symplectic structure on the moduli space of flat connections on Riemann surfaces** For Σ_2 a compact Riemann surface, the transgression of the extended Lagrangian $\hat{\mathbf{c}}$ yields a map

$$\mathbf{Maps}(\Sigma_2;\mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_2,\hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_2;\mathbf{B}^3U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_2}} \mathbf{B}U(1)_{\mathrm{conn}} \,,$$

modulating a circle-bundle with connection on the moduli space of gauge fields on Σ_2. The underlying curvature of this connection is the map obtained by composing this with

$$\mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F_{(-)}} \Omega^2(-;\mathbb{R})_{\mathrm{cl}} \quad ,$$

which gives the canonical presymplectic 2-form

$$\omega : \ \mathbf{Maps}(\Sigma_2;\mathbf{B}G_{\mathrm{conn}}) \longrightarrow \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$$

on the moduli stack of principal G-bundles with connection on Σ_2. Equivalently, this is the transgression of the invariant polynomial $\langle - \rangle : \ \mathbf{B}G_{\mathrm{conn}} \longrightarrow \Omega^4_{\mathrm{cl}}$ to the mapping stack out of Σ_2. The restriction of this 2-form to the moduli stack $\mathbf{Maps}(\Sigma_2;\flat\mathbf{B}G_{\mathrm{conn}})$ of flat principal G-bundles on Σ_2 induces a canonical symplectic structure on the moduli space

$$\mathrm{Hom}(\pi_1(\Sigma_2),G)/_{\mathrm{Ad}}G$$

of flat G-bundles on Σ_2. Such a symplectic structure seems to have been first made explicit in [**?**] and then identified as the phase space structure of Chern-Simons theory in [Wi97b]. Observing that differential forms on the moduli stack, and hence de Rham cocycles $\mathbf{B}G \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$, may equivalently be expressed by simplicial forms on the bar complex of G, one recognizes in the above transgression construction a stacky refinement of the construction of [**?**].

To see more explicitly what this form ω is, consider any test manifold $U \in \mathrm{CartSp}$. Over this the map of stacks ω is a function which sends a G-principal connection $A \in \Omega^1(U \times \Sigma_2)$ (using that every G-principal bundle over $U \times \Sigma_2$ is trivializable) to the 2-form

$$\int_{\Sigma_2} \langle F_A \wedge F_A \rangle \in \Omega^2(U) \,.$$

Now if A represents a field in the phase space, hence an element in the concretification of the mapping stack, then it has no "leg" [3] along U, and so it is a 1-form on Σ_2 that depends smoothly on the parameter U: it is a U-parameterized *variation* of such a 1-form. Accordingly, its curvature 2-form splits as

$$F_A = F_A^{\Sigma_2} + d_U A \,,$$

[3] That is, when written in local coordinates (u,σ) on $U \times \Sigma_2$, then $A = A_i(u,\sigma)du^i + A_j(u,\sigma)d\sigma^j$ reduces to the second summand.

where $F_A^{\Sigma_2} := d_{\Sigma_2} A + \frac{1}{2}[A \wedge A]$ is the U-parameterized collection of curvature forms on Σ_2. The other term is the *variational differential* of the U-collection of forms. Since the fiber integration map $\int_{\Sigma_2} : \Omega^4(U \times \Sigma_2) \to \Omega^2(U)$ picks out the component of $\langle F_A \wedge F_A \rangle$ with two legs along Σ_2 and two along U, integrating over the former we have that

$$\omega|_U = \int_{\Sigma_2} \langle F_A \wedge F_A \rangle = \int_{\Sigma_2} \langle d_U A \wedge d_U A \rangle \in \Omega^2_{\mathrm{cl}}(U) \,.$$

In particular if we consider, without loss of generality, $(U = \mathbb{R}^2)$-parameterized variations and expand

$$d_U A = (\delta_1 A) du^1 + (\delta_2 A) du^2 \in \Omega^2(\Sigma_2 \times U) \,,$$

then

$$\omega|_U = \int_{\Sigma_2} \langle \delta_1 A, \delta_2 A \rangle \,.$$

In this form the symplectic structure appears, for instance, in prop. 3.17 of [**?**] (in [Wi97b] this corresponds to (3.2)).

In summary, this means that the circle bundle with connection obtained by transgression of the extended Lagrangian $\hat{\mathbf{c}}$ is a *geometric prequantization* of the phase space of 3d Chern-Simons theory. Observe that traditionally prequantization involves an arbitrary *choice*: the choice of prequantum bundle with connection whose curvature is the given symplectic form. Here we see that in *extended* prequantization this choice is eliminated, or at least reduced: while there may be many differential cocycles lifting a given curvature form, only few of them arise by transgression from a higher differential cocycles in top codimension. In other words, the restrictive choice of the single geometric prequantization of the invariant polynomial $\langle -, - \rangle : \mathbf{B}G_{\mathrm{conn}} \to \Omega^4_{\mathrm{cl}}$ by $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$ down in top codimension induces canonical choices of prequantization over all Σ_k in all lower codimensions $(n - k)$.

1.2.15.4 $k = 3$**: the Chern-Simons action functional** Finally, for Σ_3 a compact oriented 3-manifold without boundary, transgression of the extended Lagrangian $\hat{\mathbf{c}}$ produces the morphism

$$\mathbf{Maps}(\Sigma_3; \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_3, \hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_3; \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_3}} \underline{U}(1) \,.$$

Since the morphisms in $\mathbf{Maps}(\Sigma_3; \mathbf{B}G_{\mathrm{conn}})$ are *gauge transformations* between field configurations, while $\underline{U}(1)$ has no non-trivial morphisms, this map necessarily gives a *gauge invariant* $U(1)$-valued function on field configurations. Indeed, evaluating over the point and passing to isomorphism classes (and hence to gauge equivalence classes), this induces the *Chern-Simons action functional*

$$S_{\hat{\mathbf{c}}} : \{G\text{-bundles with connection on } \Sigma_3\}/\mathrm{iso} \to U(1) \,.$$

It follows from the description of $\hat{\mathbf{c}}$ given in section **??** that if the principal G-bundle $P \to \Sigma_3$ is trivializable then

$$S_{\hat{\mathbf{c}}}(P, \nabla) = \exp 2\pi i \int_{\Sigma_3} \mathrm{CS}_3(A) \,,$$

where $A \in \Omega^1(\Sigma_3, \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on Σ_3 representing the connection ∇ in a chosen trivialization of P. This is actually always the case, but notice two things: first, in the stacky description one does not need to know a priori that every principal G-bundle on a 3-manifold is trivializable; second, the independence of $S_{\hat{\mathbf{c}}}(P, \nabla)$ on the trivialization chosen is automatic from the fact that $S_{\hat{\mathbf{c}}}$ is a morphism of stacks read at the level of equivalence classes.

Furthermore, if (P, ∇) can be extended to a principal G-bundle with connection $(\tilde{P}, \tilde{\nabla})$ over a compact 4-manifold Σ_4 bounding Σ_3, one has

$$S_{\hat{\mathbf{c}}}(P, \nabla) = \exp 2\pi i \int_{\Sigma_4} \tilde{\varphi}^* \omega^{(4)}_{\mathbf{B}G_{\mathrm{conn}}} = \exp 2\pi i \int_{\Sigma_4} \langle F_{\tilde{\nabla}}, F_{\tilde{\nabla}} \rangle \,,$$

where $\tilde{\varphi} : \Sigma_4 \to \mathbf{B}G_{\mathrm{conn}}$ is the morphism corresponding to the extended bundle $(\tilde{P}, \tilde{\nabla})$. Notice that the right hand side is independent of the extension chosen. Again, this is always the case, so one can actually take the above equation as a definition of the Chern-Simons action functional, see, e.g., [**?**, **?**]. However, notice how in the stacky approach we do not need a priori to know that the oriented cobordism ring is trivial in dimension 3. Even more remarkably, the stacky point of view tells us that there would be a natural and well-defined 3d Chern-Simons action functional even if the oriented cobordism ring were nontrivial in dimension 3 or that not every G-principal bundle on a 3-manifold were trivializable. An instance of checking a nontrivial higher cobordism group vanishes can be found in [KS05], allowing for the application of the construction of Hopkins-Singer [HoSi05].

1.2.15.5 The Chern-Simons action functional with Wilson loops To conclude our exposition of the examples of 1d and 3d Chern-Simons theory in higher geometry, we now briefly discuss how both unify into the theory of 3d Chern-Simons gauge fields with Wilson line defects. Namely, for every embedded knot

$$\iota : S^1 \hookrightarrow \Sigma_3$$

in the closed 3d worldvolume and every complex linear representation $R : G \to \mathrm{Aut}(V)$ one can consider the *Wilson loop observable* $W_{\iota,R}$ mapping a gauge field $A : \Sigma \to \mathbf{B}G_{\mathrm{conn}}$, to the corresponding "Wilson loop holonomy"

$$W_{\iota,R} : A \mapsto \mathrm{tr}_R(\mathrm{hol}(\iota^* A)) \in \mathbb{C}\,.$$

This is the trace, in the given representation, of the parallel transport defined by the connection A around the loop ι (for any choice of base point). It is an old observation[4] that this Wilson loop $W(C, A, R)$ is itself the *partition function* of a 1-dimensional topological σ-model quantum field theory that describes the topological sector of a particle charged under the nonabelian background gauge field A. In section 3.3 of [Wi97b] it was therefore emphasized that Chern-Simons theory with Wilson loops should really be thought of as given by a single Lagrangian which is the sum of the 3d Chern-Simons Lagrangian for the gauge field as above, plus that for this topologically charged particle.

We now briefly indicate how this picture is naturally captured by higher geometry and refined to a single *extended* Lagrangian for coupled 1d and 3d Chern-Simons theory, given by maps on higher moduli stacks. In doing this, we will also see how the ingredients of Kirillov's orbit method and the Borel-Weil-Bott theorem find a natural rephrasing in the context of smooth differential moduli stacks. The key observation is that for $\langle \lambda, - \rangle$ an integral weight for our simple, connected, simply connected and compact Lie group G, the contraction of $\mathfrak{g}$-valued differential forms with λ extends to a morphism of smooth moduli stacks of the form

$$\langle \lambda, - \rangle \; : \; \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \to \mathbf{B}U(1)_{\mathrm{conn}}\,,$$

where $T_\lambda \hookrightarrow G$ is the maximal torus of G which is the stabilizer subgroup of $\langle \lambda, - \rangle$ under the coadjoint action of G on $\mathfrak{g}^*$. Indeed, this is just the classical statement that exponentiation of $\langle \lambda, - \rangle$ induces an isomorphism between the integral weight lattice $\Gamma_{\mathrm{wt}}(\lambda)$ realtive to the maximal torus T_λ and the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathrm{Grp}}(T_\lambda, U(1))$ and that under this isomorphism a gauge transformation of a $\mathfrak{g}$-valued 1-form A turns into that of the $\mathfrak{u}(1)$-valued 1-form $\langle \lambda, A \rangle$.

Comparison with the discussion in section **??** shows that this is the extended Lagrangian of a 1-dimensional Chern-Simons theory. In fact it is just a slight variant of the trace-theory discussed there: if we realize $\mathfrak{g}$ as a matrix Lie algebra and write $\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha \cdot \beta)$ as the matrix trace, then the above Chern-Simons 1-form is given by the "λ-shifted trace"

$$\mathrm{CS}_\lambda(A) := \mathrm{tr}(\lambda \cdot A) \in \Omega^1(-; \mathbb{R})\,.$$

Then, clearly, while the "plain" trace is invariant under the adjoint action of all of G, the λ-shifted trace is invariant only under the subgroup T_λ of G that fixes λ.

[4]This can be traced back to [**?**]; a nice modern review can be found in section 4 of [**?**].

Notice that the domain of $\langle\lambda, -\rangle$ naturally sits inside $\mathbf{B}G_{\mathrm{conn}}$ by the canonical map

$$\Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \to \Omega^1(-,\mathfrak{g})//\underline{G} \simeq \mathbf{B}G_{\mathrm{conn}}\,.$$

One sees that the homotopy fiber of this map to be the *coadjoint orbit* $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ of $\langle\lambda,-\rangle$, equipped with the map of stacks

$$\theta : \mathcal{O}_\lambda \simeq \underline{G}//\underline{T}_\lambda \to \Omega^1(-,\mathfrak{g})//\underline{T}_\lambda$$

which over a test manifold U sends $g \in C^\infty(U,G)$ to the pullback $g^*\theta_G$ of the Maurer-Cartan form. Composing this with the above extended Lagrangian $\langle\lambda,-\rangle$ yields a map

$$\langle\lambda,\theta\rangle :\ \mathcal{O}_\lambda \xrightarrow{\ \theta\ } \Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \xrightarrow{\ \langle\lambda,-\rangle\ } \mathbf{B}U(1)_{\mathrm{conn}}$$

which modulates a canonical $U(1)$-principal bundle with connection on the coadjoint orbit. One finds that this is the canonical prequantum bundle used in the orbit method [**?**]. In particular its curvature is the canonical symplectic form on the coadjoint orbit.

So far this shows how the ingredients of the orbit method are incarnated in smooth moduli stacks. This now immediately induces Chern-Simons theory with Wilson loops by considering the map $\Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \to \mathbf{B}G_{\mathrm{conn}}$ itself as the target[5] for a field theory defined on knot inclusions $\iota : S^1 \hookrightarrow \Sigma_3$. This means that a field configuration is a diagram of smooth stacks of the form

$$\begin{array}{ccc} S^1 & \xrightarrow{(\iota^*A)^g} & \Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \\ {\scriptstyle\iota}\downarrow & \swarrow_{g} & \downarrow \\ \Sigma_3 & \xrightarrow[A]{} & \mathbf{B}G_{\mathrm{conn}} \end{array},$$

i.e., that a field configuration consists of

- a gauge field A in the "bulk" Σ_3;
- a G-valued function g on the embedded knot

such that the restriction of the ambient gauge field A to the knot is equivalent, via the gauge transformation g, to a $\mathfrak{g}$-valued connection on S^1 whose local $\mathfrak{g}$-valued 1-forms are related each other by local gauge transformations taking values in the torus T_λ. Moreover, a gauge transformation between two such field configurations (A,g) and (A',g') is a pair (t_{Σ_3}, t_{S^1}) consisting of a G-gauge transformation t_{Σ_3} on Σ_3 and a T_λ-gauge transformation t_{S^1} on S^1, intertwining the gauge transformations g and g'. In particular if the bulk gauge field on Σ_3 is held fixed, i.e., if $A = A'$, then t_{S^1} satisfies the equation $g' = g\,t_{S^1}$. This means that the Wilson-line components of gauge-equivalence classes of field configurations are naturally identified with smooth functions $S^1 \to G/T_\lambda$, i.e., with smooth functions on the Wilson loop with values in the coadjoint orbit. This is essentially a rephrasing of the above statement that G/T_λ is the homotopy fiber of the inclusion of the moduli stack of Wilson line field configurations into the moduli stack of bulk field configurations.

We may postcompose the two horizontal maps in this square with our two extended Lagrangians, that for 1d and that for 3d Chern-Simons theory, to get the diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{(\iota^*A)^g} & \Omega^1(-,\mathfrak{g})//T & \xrightarrow{\langle\lambda,-\rangle} & \mathbf{B}U(1)_{\mathrm{conn}} \\ {\scriptstyle\iota}\downarrow & \swarrow_{g} & \downarrow & & \\ \Sigma_3 & \xrightarrow[]{A} & \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^3U(1)_{\mathrm{conn}} \end{array}.$$

[5] This means that here we are secretely moving from the topos of (higher) stacks on smooth manifolds to its *arrow topos*, see section **??** below.

Therefore, writing $\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right)$ for the moduli stack of field configurations for Chern-Simons theory with Wilson lines, we find two action functionals as the composite top and left morphisms in the diagram

$$\begin{array}{ccccc}
\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right) & \longrightarrow & \mathbf{Maps}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathrm{hol}_{\Sigma_3}\mathbf{Maps}(\Sigma_3, \hat{\mathbf{c}})} & \underline{U}(1) \\
\downarrow & & \downarrow & & \\
\mathbf{Maps}(S^1, \Omega^1(-,\mathfrak{g})//T_\lambda) & \longrightarrow & \mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{con}}) & & \\
\Big\downarrow{\scriptstyle \mathrm{hol}_{S^1}\mathbf{Maps}(S^1, \langle\lambda,-\rangle)} & & & & \\
\underline{U}(1) & & & &
\end{array}$$

in $\mathbf{H}$, where the top left square is the homotopy pullback that characterizes maps in $\mathbf{H}^{(\Delta^1)}$ in terms of maps in $\mathbf{H}$. The product of these is the action functional

$$\begin{array}{ccccc}
\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right) & \longrightarrow & \mathbf{Maps}(\Sigma_3, \mathbf{B}^3U(1)_{\mathrm{conn}}) \times \mathbf{Maps}(S^1, \mathbf{B}U(1)_{\mathrm{conn}}) & & \\
& & \downarrow & & \\
& & \underline{U}(1) \times \underline{U}(1) & \xrightarrow{\cdot} & \underline{U}(1)\,.
\end{array}$$

where the rightmost arrow is the multiplication in $U(1)$. Evaluated on a field configuration with components (A, g) as just discussed, this is

$$\exp\left(2\pi i\left(\int_{\Sigma_3} \mathrm{CS}_3(A) + \int_{S^1}\langle\lambda, (\iota^*A)^g\rangle\right)\right)\,.$$

This is indeed the action functional for Chern-Simons theory with Wilson loop ι in the representation R correspponding to the integral weight $\langle\lambda,-\rangle$ by the Borel-Weil-Bott theorem, as reviewed for instance in Section 4 of [**?**].

Apart from being an elegant and concise repackaging of this well-known action functional and the quantization conditions that go into it, the above reformulation in terms of stacks immediately leads to prequantum line bundles in Chern-Simons theory with Wilson loops. Namely, by considering the codimension 1 case, one finds the the symplectic structure and the canonical prequantization for the moduli stack of field configurations on surfaces with specified singularities at specified punctures [Wi97b]. Moreover, this is just the first example in a general mechanism of (extended) action functionals with defect and/or boundary insertions. Another example of the same mechanism is the gauge coupling action functional of the open string. This we discuss in section 1.2.16.4 below.

1.2.16 Higher prequantum field theory

We give an introduction and survey to some aspects of the formulation of higher prequantum field thery in a cohesive ∞-topos. (Parts of this is taken from [FiSaScV].)

One of the pleasant consequences of formulating the geometry of (quantum) field theory in terms of higher stacks, hence in terms of higher topos theory, is that a wealth of constructions find a natural and unified formulation, which subsumes varied traditional constructions and generalizes them to higher geometry. In this last part here we give an outlook of the scope of field theoretic phenomena that the theory naturally captures or exhibits in the first place.

In the following we write $\mathbf{H}$ for the collection of higher stacks under consideration. The reader may want to think of the special case that was discussed in the previous sections, where $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ is the collection of *smooth ∞-groupoids*, hence of higher stacks on the site of smooth manifolds, or, equivalently, its dense subsite of Cartesian spaces. But one advantage of speaking in terms of higher topos theory is that essentially every construction considered in the following makes sense much more generally if only $\mathbf{H}$ is any higher topos that satisfies a small set of axioms called (differential) *cohesion*. This allows to transport all considerations across various kinds of geometries. Notably we can speak of higher *supergeometry*, hence of fermionic quantum fields, simply by refining the site of definition to be that of supermanifolds: also the higher topos $\mathbf{H} = \mathrm{SmoothSuper}\infty\mathrm{Grpd}$ is differentially cohesive.

Therefore we speak in the following in generality of *cohesive maps* when we refer to maps with geometric structure, be it topological, smooth, analytic, supergeometric or otherwise. Throughout, this geometric structure is *higher geoemtric* which we will sometimes highlight by adding the "∞-"-prefix as in *cohesive ∞-group*, but which we will often notationally suppress for brevity. Similarly, *all* of the diagrams appearing in the following are filled with homotopies, but only sometimes we explicitly display them (as double arrows) for emphasis or in order to label them.

The special case of *geometrically discrete* cohesion is exhibited by the ∞-topos $\infty\mathrm{Grpd}$ of bare ∞-groupoids or *homotopy types*. This is the context of traditional homotopy theory, presented by topological spaces regarded up to weak homotopy equivalences ("whe"s): $\infty\mathrm{Grpd} \simeq L_{\mathrm{whe}}\mathrm{Top}$. One of the axioms satisfied by a cohesive ∞-topos $\mathbf{H}$ is that the inclusion $\mathrm{Disc} : \infty\mathrm{Grpd} \hookrightarrow \mathbf{H}$ of bare ∞-groupoids as cohesive ∞-groupoids equipped with discrete cohesive structure has not only a right adjoint ∞-functor $\Gamma : \mathbf{H} \to \infty\mathrm{Grpd}$ – the functor that forgets the cohesive structure and remembers only the underlying bare ∞-groupoid – but also a left adjoint $|-| : \mathbf{H} \to \infty\mathrm{Grpd}$. This is the *geometric realization* of cohesive ∞-groupoids.

We discuss first the general notion of (quantum) fields, then that of Lagrangians and action functionals on spaces of fields and the corresponding *phase spaces*, and finally we discuss the geometric prequantum theory of such data.

- 1.2.16.1 – Fields
- 1.2.16.2 – Phase spaces
- 1.2.16.3 – Prequantum geometry

The following disucssion is based on and in part reviews previous work such as [SSS09c, FiSaScIV]. Lecture notes that provide an exposition of this material with an emphasis on fields as twisted (differential) cocycles are in [Sc12a].

1.2.16.1 Fields We discuss now how a plethora of species of (quantum) fields are naturally and precisely expressed by constructions in the higher topos $\mathbf{H}$. In fact, it is the *universal moduli stacks* **Fields** of a given species of fields which are naturally expressed: those objects such that maps $\phi : X \to \mathbf{Fields}$ into them are equivalently quantum fields of the given species on X. This has three noteworthy effects on the formulation of the corresponding field theory.

First of all it means that every quantum field theory thus expressed is formally analogous to a σ-model – the "target space" is a higher moduli stack – which brings about a unified treatment of varied types of QFTs.

Second it means that a differential cocycle on **Fields** of degree $(n+1)$ – itself modulated by a map

$$\mathbf{L} : \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

to the moduli stack n-form connections – serves as an *extended* Lagrangian of a field theory, in the sense that it expresses a QFT fully locally by Lagrangian data in arbitrary codimenion: for every closed oriented worldvolume Σ_k of dimension $k \leq n$ there is a *transgressed* Lagrangian

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{L}]) : \ \mathbf{Fields}(\Sigma_k) \xrightarrow{[\Sigma_k, \mathbf{L}]} [\Sigma_k, \mathbf{B}^n \mathbb{C}_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k}\mathbb{C}^\times_{\mathrm{conn}}$$

which itself is a differential $(n-k)$-form connection on the space of fields on Σ_k. In particular, when $n = k$ then $\mathbf{B}^0 U(1)_{\mathrm{conn}} \simeq U(1)$ and the transgressed Lagragian in codimension 0 is the (exponentiated) *action functional* of the theory, $\exp(iS(-)) : \mathbf{Fields}(\Sigma_n) \to U(1)$. On the other hand, the $(n-k)$-connections in higher codimension are higher (off-shell) *prequantum bundles* of the theory. This we discuss further below in 1.2.16.3.

Third, it means that the representation of fields by their higher moduli stacks in a higher topos identifies the notion of quantum field entirely with that of *cocycle* in general *cohomology*. This we turn to now in 1.2.16.1.1.

1.2.16.1.1 Cocycles: generalized, parameterized, twisted We discuss general aspects of cocycles and cohomology in an ∞-topos, as a general blueprint for all of the discussion to follow. The reader eager to see explicit structure genuinely related to (quantum) physics may want, on first reading, to skip ahead to 1.2.16.1.2 and come back here only as need be.

In higher topos theory the notion of *cocycle* c on some space X with coefficients in some object A and with some *cohomology* class $[c]$ is identified simply with that of a map (a morphism) $c : X \to A$ with equivalence class

$$[c] \in H(X, A) := \pi_0 \mathbf{H}(X, A)\,.$$

This is traditionally familiar for the case of discrete geometric structure hence bare homotopy theory $\mathbf{H} = \infty\mathrm{Grpd}$, where for any Eilenberg-Steenrod-*generalized cohomology theory* the object E is the corresponding spectrum, as given by the Brown representability theorem. That over non-trivial sites the same simple formulation subsumes all of *sheaf cohomology* ("parameterized cohomology") is known since [Br73], but it appears in the literature mostly in a bit of disguise in terms of some explicit model of a *derived global section functor*, computed by means of suitable projective/injective resolutions.)

If here $A = \mathbf{Fields}$ is interpreted as the moduli stack of certain *fields*, then such a cocycle *is* a field configuration on X. This is familiar for the case that we think of $A = X$ as the target space of a *σ-model*. But for instance for $G \in \mathrm{Grp}(\mathbf{H})$ a (higher) group and $A := \mathbf{B}G_{\mathrm{conn}}$ a differential refinement of the universel moduli stack of G-principal ∞-bundles, a map $c : X \to \mathbf{B}G_{\mathrm{conn}}$ is on the one hand a cocycle in (nonabelian) differential G-cohomology on X, and on the other hand equivalently a *G-gauge field* on X. In particular this means that in higher topos theory gauge field theories are unified with σ-models: an (untwisted) gauge field is a σ-modelfield whose target space is a universal differential moduli stack $\mathbf{B}G_{\mathrm{conn}}$.

Indeed, the kinds of fields which are identified as σ-model fields in higher topos theory, hence with cocycles in some geometric cohomology theory, is considerable richer, still. The reason for this is that with $B \in \mathbf{H}$ any object, the *slice* $\mathbf{H}_{/B}$ is itself again a higher topos. This slice topos is the collection of morphisms of $\mathbf{H}$ into B, where a map between two such morphisms $f_{1,2} : X_{1,2} \to B$ is

1. a map $\phi : X_1 \to X_2$ in $\mathbf{H}$

2. a homotopy $\eta : \; f_1 \xrightarrow{\simeq} f_2 \circ \phi$,

hence a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X_2 \\ & {}_{f_1}\searrow \;\; \swArrow_{\eta} \;\; \swarrow_{f_2} & \\ & B & \end{array}$$

. We are particularly interested in the case that $B = \mathbf{B}G$ is a moduli stack of G-principal ∞-bundles (or a differential refinement thereof). The fact that $\mathbf{H}$ is *cohesive* implies in particular that every morphism $g : X \to \mathbf{B}G$ has a unique global homotopy fiber $P \to X$. This is the *G-principal bundle* over X modulated by g, sitting in a long homotopy fiber sequence of the form

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & X & \xrightarrow{g} & \mathbf{B}G \end{array}\,.$$

In particular this means that there is an action of G on P (precisely: a *homotopy coherent* or A_∞-action) and that

$$P \to P//G \simeq X$$

is the quotient map of this action. Moreover, conversely every action of G on any object $V \in \mathbf{H}$ arises this way and is modulated by a morphism $V//G \xrightarrow{\rho} \mathbf{B}G$, sitting in a homotopy fiber sequence of the form

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}G \end{array} .$$

(This and the following facts about G-principal ∞-bundles in ∞-toposes and the representation theory and twisted cohomology of cohesive ∞-groups is due to [NSSa], an account in the present context is in section 3.6 here.) This fiber sequence exhibits $V//G \to \mathbf{B}G$ as the universal V-fiber bundle which is ρ-associated to the universal G-principal bundle over $\mathbf{B}G$. For instance the fiber sequence $G \to * \to \mathbf{B}G$ which defines the delooping of G corresponds to the action of G on itself by right (or left) multiplication; the fiber sequence $V \longrightarrow V \times \mathbf{B}G \xrightarrow{p_2} \mathbf{B}G$ corresponds to the trivial action on any V, and the fiber sequence $G \longrightarrow \mathcal{L}\mathbf{B}G \longrightarrow \mathbf{B}G$ of the free loop space object of $\mathbf{B}G$ corresponds to the adjoint action of G on itself.

Another case of special interest is that where $V \simeq \mathbf{B}A$ and $V///G \simeq \mathbf{B}\hat{G}$ are themselves deloopings of ∞-groups. In this case the above fiber sequence reads

$$\mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \longrightarrow \mathbf{B}G$$

and exhibits an *extension* $\hat{G}$ of G by A. The implied action of G on $\mathbf{B}A$ via $\mathbf{Aut}(\mathbf{B}G) \simeq \mathbf{Aut}_{\mathrm{Grp}}(G)//\mathrm{ad}$ is the datum known from traditional *Schreier theory* of general (nonabelian) group extensions. Now the previous discussion implies that if A is equipped with sufficient abelian structure in that also $\mathbf{B}A$ is equipped with ∞-group structure (a "braided ∞-group") and such that $\mathbf{B}\hat{G} \to \mathbf{B}G$ is the quotient projection of a $\mathbf{B}A$-action, then the extension is classified by an *∞-group cocycle* $\mathbf{c} : \mathbf{B}G \longrightarrow \mathbf{B}^2 A$ in ∞-group cohomology $[\mathbf{c}] \in H^2_{\mathrm{grp}}(G, A)$. Notice that this is *cohesive* group cohomology in that it does respect and reflect the geometric structure on G and A. Notaby in smooth cohesion and for G a Lie group and $A = \mathbf{B}^n K$ the n-fold delooping of an abelian Lie group, this reproduces not the naive Lie group cohomology but the refined Segal-Brylinski Lie group cohomology (this is shown in section 4.4.6.2 here). This implies that for G a compact Lie group and $A = \mathbf{B}^n U(1)$ we have an equivalence

$$H^n_{\mathrm{Grp}}(G, U(1)) \simeq H^{n+1}(BG, \mathbb{Z})$$

between the refined cohesive group cohomology with coefficients in the circle group and the ordinary integral cohomology of the clasifying space $BG \simeq |\mathbf{B}G|$ in one degree higher. In other words this means that every *universal characteristic class* $c : BG \longrightarrow K(\mathbb{Z}, n+1)$ is cohesively refined essentially uniquely to (the instanton sector of) a higher gauge field: a cohesive circle n-bundle (bundle $(n-1)$-gerbe) on the universal moduli stack $\mathbf{B}G$. The "universality" of this higher gauge field is reflected in the fact that this is really the (twisting structure underlying) an *extended action function for higher Chern-Simons theory* controld by the given universal class. This we come back to below in 1.2.16.1.3.

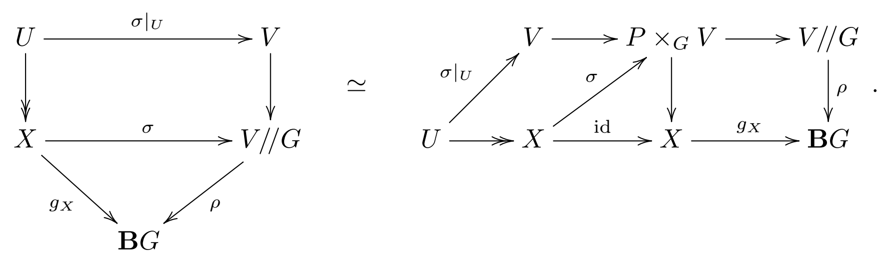
From this higher bundle theory, higher group theory and higher representation theory, we obtain a finer interpretation of maps in the slice $\mathbf{H}_{/\mathbf{B}G}$. First of all one finds that

$$\mathbf{H}_{/\mathbf{B}G} \simeq G\mathbf{Act}$$

is indeed the ∞-category of G-actions and G-action homomorphisms. In particular the base change functors $(\mathbf{G}\phi)_*$ and $(\mathbf{B}\phi)_!$ along maps $\mathbf{B}\phi : \mathbf{B}G \to \mathbf{B}G'$ corresponds to the (co)induction functors from G-representations to G'-representations along a group homomorphism ϕ. Since all this is homotopy-theoretic ("derived") the space of maps in the slice from the trivial representation to any given representation (V, ρ) (hence the *derived invariants* of (V, ρ)) is the cocycle ∞-groupoid of the *group cohomology* of G *with coefficients* in V:

$$H_{\mathrm{Grp}}(G, V) \simeq \pi_0 \mathbf{H}_{/\mathbf{B}G}(\mathrm{id}_{\mathbf{B}G}, \rho)\,.$$

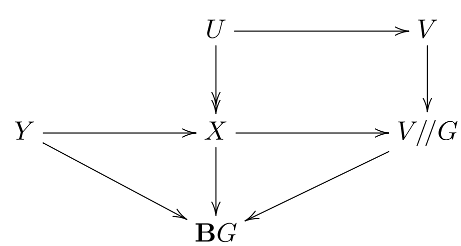
We are interested in the generalizations of this to the case where the univeral G-principal ∞-bundle modulated by $\mathrm{id}_{\mathbf{B}G}$ is replaced by any G-principal bundle modulated by a map $g_X : X \to \mathbf{B}G$. To see what general cocycles in $\mathbf{H}_{/\mathbf{B}G}(g_X, \rho)$ are like, notice that every G-principal ∞-bundle over a given X locally trivializes over a cover $U \twoheadrightarrow X$ (an *effective epimorphism* in $\mathbf{H}$) in that the modulating map becomes null-homotopic on U: $g_X|_U \simeq \mathrm{pt}_{\mathbf{B}G}$. But by the universal property of homotopy fibers this means that a cocycle $\sigma : g_X \to \rho$ in $\mathbf{H}_{/\mathbf{B}G}$ is *locally* a cocycle $\sigma|_U : U \to V$ in $\mathbf{H}$ with coefficients in the given G-module V, as shown on the left of the following diagram:

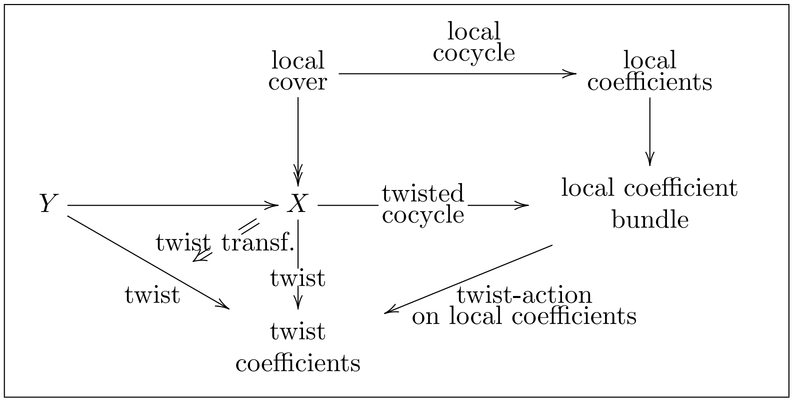


This means that σ is a cocycle with *local coefficients* in V, which however globally vary as controlled by g_X: it is *twisted* by g_X. On the right hand of the above diagram the same situation is displayed in an equivalent alternative perspective: since $\rho : V//G \to \mathbf{B}G$ is also the univeral ρ-associated V-fiber bundle, it follows that the V-fiber bundle $P \times_G V \to X$ associated to $P \to X$ is its pullback along g_X and then using again the universal property of the homotopy pullback it follows that σ is equivalently a *section* of this associated bundle. This is the traditional perspective of *g_X-twisted V-cohomology* as familiar notably from twisted K-theory, as well as from modern formulations of ordinary cohomology with local coefficients.

The perspective of twisted cohomology as cohomology in slice ∞-topos $\mathbf{H}_{/\mathbf{B}G}$ makes it manifest that what acts on twisted cocycle spaces are *twist homomorphisms*, hence maps $(\phi, \eta) : g_Y \to g_X$ in $\mathbf{H}_{/\mathbf{B}G}$. In particular for g_X and given twist its automorphism ∞-group $\mathrm{Aut}_{/\mathbf{B}G}(g_X)$ acts on the twisted cohomology $\mathbf{H}_{/\mathbf{B}G}(g_X, \rho)$ by precomposition in the slice.

In conclusion we find that cocycles and fields in the slice slice ∞-topos $\mathbf{H}_{/\mathbf{B}G}$ of a cohesive ∞-topos over the delooping of an ∞-group are structures with components as summarized in the following diagram:





In the following we list a wide variety of classes of examples of this unified general abstract picture.

1.2.16.1.2 Fields of gravity: special and generalized geometry As special cases of the above general discussion, we now discuss moduli ∞-stacks of *fields of gravity* and their generalizations as found in higher dimensional (super)gravity.

For $X \in \mathrm{Mfd}_n \hookrightarrow \mathbf{H}$ a manifold of dimension n, we may naturally regard it as an object in the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$ by way of the canonical map $\tau_X : X \to \mathbf{B}\mathrm{GL}(n)$ that modulates its frame bundle, the principal $\mathrm{GL}(n)$-bundle to which the tangent bundle TX is associated. A map $(\phi, \eta) : \tau_X \to \tau_Y$ in $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$ between two manifolds X, Y embedded in this way is equivalently a *local diffeomorphism* $\phi : X \to Y$ equipped with an explicit choice $\eta : \phi^* \tau_Y \simeq \tau_X$ of identification of the pullback tangent bundle with that of X.

The slice topos $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$ allows to express phyiscal fields which may not be restricted along arbitrary morphisms of manifolds (or morphisms of whatever kind of test geometries $\mathbf{H}$ is modeled on), but only along local diffeomorphism, such as *metric/vielbein* fields or symplectic structures.

For let $\mathbf{OrthStruc}_n : \mathbf{B}O(n) \to \mathbf{B}\mathrm{GL}(n)$ be the morphism of moduli stacks induced from the canonical inclusion of the orthogonal group into the general linear group, regarded as an object of the slice, $\mathbf{OrthStruc}_n \in \mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$. Then a cocycle/field

$$(o_X, e) : \tau_X \to \mathbf{OrthStruc}_n$$

is equivalently

1. an *orthogonal structure* o_X on X (a choice of *Lorentz frame bundle*);

2. a *vielbein* field $e : \mathbf{OrthStruc}_n \circ o_X \longrightarrow \tau_X$ which equips the frame bundle with that orthogonal structure.

Together this is equivalently a *Riemannian metric* field on X, hence a field of Euclidean gravity, and $\mathbf{OrthStruc}_n \in \mathbf{H}_{/\mathbf{B}\mathrm{GL}_n}$ is the universal moduli stack of Riemannian metrics in dimension n. Notice that this defines a notion of Riemannian metric for any object in $\mathbf{H}$ as soon as it is equipped with a $\mathrm{GL}(n)$-principal bundle. We obtain actual pseudo-Riemannian metrics by considering instead the delooped inclusion of $O(n-1, 1)$ into $\mathrm{GL}(n)$ and obtain dS-geometry, AdS-geometry etc. by further varying the signture.

This notion of $\mathbf{OrthStruc}_n$-structure in smooth stacks is of course closely related to the notion of orthogonal structure as considered in traditional homotopy theory. But there is a crucial difference, which we highlight now. First notice that there is a canonical ∞-functor

$$|-| \; : \; \mathbf{H} \to \infty\mathrm{Grpd} \simeq L_{\mathrm{whe}}\mathrm{Top}$$

which sends every cohesive ∞-groupoid/∞-stack to its *geometric realization*. Under certain conditions on the cohesive ∞-group G, in particular for Lie groups as considered here, this takes the moduli stack $\mathbf{B}G$ to the traditional *classifying space* BG. So under this map a choice of vielbein turns into a homotopy lift as shown on the right of

$$\begin{array}{ccc} & & \mathbf{B}O(n) \\ & {}^{o_X}\nearrow & \downarrow \\ X & \xrightarrow{\tau_X} & \mathbf{B}\mathrm{GL}(n) \end{array} \quad \overset{|-|}{\longmapsto} \quad \begin{array}{ccc} & & BO(n) \\ & {}^{|o_X|}\nearrow & \downarrow \simeq \\ X & \xrightarrow{|\tau_X|} & B\mathrm{GL}(n) \end{array} .$$

But since $O(n) \to \mathrm{GL}(n)$ is the inclusion of a maximal compact subgroup, it is a homotopy equivalence of the underlying topological spaces. Hence under $|-|$ a choice of $\mathbf{OrthStruc}_n$-structure is no choice at all, up to equivalence, there is no information encoded in this choice. This is of course the familiar statement that every vector bundle *admits* an orthogonal structure. But only in the context of cohesive stacks is the *choice* of this orthogonal structure actually equivalent to geometric data, to a choice of Riemannian metric.

Also notice that the homotopy fiber of $\mathbf{OrthStruc}_n$ is the cohesive coset $\mathrm{GL}(n)/\mathrm{O}(n)$ (the coset equipped with its smooth manifold structure) in that we have a fiber sequence

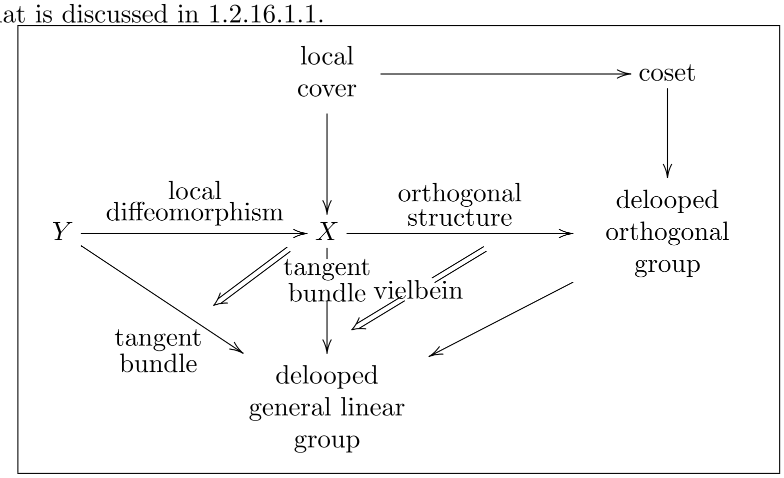
$$\mathrm{GL}(n)/O(n) \longrightarrow \mathbf{B}\mathrm{O}(n) \xrightarrow{\mathbf{OrthStruc}_n} \mathbf{B}\mathrm{GL}(n)$$

.

in $\mathbf{H}$, and by the discussion in 1.2.16.1.1 above a metric field $(o_X, e) : \tau_X \longrightarrow \mathbf{OrthStruc}_n$ is equivalently a τ_X-*twisted* $\mathrm{GL}(n)/O(n)$-*cocycle*. This reproduces the traditional statement that the space of choices of vielbein fields is locally the space of maps into the coset $\mathrm{GL}(n)/O(n)$ and fails to be globally so to the extent that the tangent bundle is non-trivial.

Moreover, by the general discussion in 1.2.16.1.1 we find that a twist transformation that may act on orthogonal structures is a morphism $\tau_Y \to \tau_X$ in the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$. This is equivalently a cohesive map $\phi : Y \to X$ in $\mathbf{H}$ equipped with an equivalence $\eta : \phi^*\tau_X \xrightarrow{\simeq} \tau_X$ from the pullback of the tangent bundle on X to that on Y. But such an isomorphism precisely witnesses ϕ as a *local diffeomorphism*. Hence it is the local diffeomorphisms that act as twist morphisms on tangent bundles regarded as twists for $\mathrm{GL}(n)/O(n)$-structures. This statement of course reproduces the traditional fact that metrics pull back along local diffeomorphisms (but not along more general cohesive maps). Abstractly it is reflected in the fact that the moduli stack $\mathbf{OrthStruc}_n$ for metrics in n dimensions is an object not of the base ∞-topos $\mathbf{H}$, but of the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$.

In conclusion, the following diagram summarizes the components of the formulation of metric fields as cocycles in the slice over $\mathbf{B}\mathrm{GL}(n)$, displayed as a special case of the general diagram for twisted cocycles that is discussed in 1.2.16.1.1.



This discussion of metric structure and vielbein fields of gravity is but a special case of *generalized vielbein fields* obtained from *reduction of structure groups*. If $\mathbf{c} : K \to G$ is any morphism of groups in $\mathbf{H}$ (typically taken to be a subgroup inclusion if one is speaking of structure group *reduction*, but not necessarily so in general, as for instance the example of the *generalized tangent bundle*, discussed in a moment, shows), and if $\tau_X : X \to \mathbf{B}G$ is the map modulating a given G-structure on X, then a map $(\phi, \eta) : \tau_X \to \mathbf{c}$ in $\mathbf{H}_{/\mathbf{B}G}$ is a generalized vielbein field on X which exhibits the reduction of the structure group from G to H along $\mathbf{c}$. These $\mathbf{c}$-*geometries* are compatible with pullback along along twist transformations $\eta : \tau_Y \to \tau_X$, namely along maps $\phi : Y \to X$ in $\mathbf{H}$ which are *generalized local diffeomorphisms* in that they are equipped with an equivalence $\eta : \ \phi^*\mathbf{c} \xrightarrow{\simeq} \tau_X$.

Of relevance in the T-duality covariant formulation of type II supergravity ("doubled field theory") is the reduction along the inclusion of the maximal compact subgroup into the the orthogonal group $O(n,n)$ (where $n = 10$ for full type II supergravity), whose delooping in $\mathbf{H}$ we write

$$\mathbf{typeII} : \ \mathbf{B}\left(O\left(n\right) \times O\left(n\right)\right) \longrightarrow \mathbf{B}O\left(n,n\right) \ .$$

A spacetime X that is to carry a $\mathbf{typeII}$-field accordingly must carry an $O(n,n)$-structure in the first place in that it must be equipped with a lift of its tangent bundle $\tau_X \in \mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$ in the slice over $\mathbf{B}\mathrm{GL}(n)$, as discussed above, to an object τ_X^{gen} in the slice $\mathbf{H}_{/\mathbf{B}O(n,n)}$. Since there is no suitable homomorphism from $O(n,n)$ to $\mathrm{GL}(n)$, this lift needs to be through a subgroup of $O(n,n)$ that does map to $\mathrm{GL}(n)$. The maximal such group is called the *geometric subgroup* $G_{\mathrm{geom}}(n) \overset{\iota}{\hookrightarrow} \mathrm{GL}(n)$. We write

$$\begin{array}{ccc} \mathbf{B}G_{\mathrm{geom}}(n) & \xrightarrow{\mathbf{B}\iota} & \mathbf{B}O(n,n) \\ \downarrow{\scriptstyle \mathbf{genTan}_n} & & \\ \mathbf{B}\mathrm{GL}(n) & & \end{array}$$

in $\mathbf{H}$. Then for $X \in \mathrm{Mfd} \hookrightarrow \mathbf{H}$ a spacetime, a map $(\tau_X^{\mathrm{gen}}, \eta) : \ \tau_X \longrightarrow \mathbf{genTan}_n$ in $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$, hence a diagram

$$\begin{array}{ccc} X & \overset{\tau_X^{\mathrm{gen}}}{\dashrightarrow} & \mathbf{B}G_{\mathrm{geom}}(n) \\ & {\scriptstyle \tau_X} \searrow \ \overset{\eta}{\Leftarrow} \ \swarrow {\scriptstyle \mathbf{genTan}_n} & \\ & \mathbf{B}\mathrm{GL}(n) & \end{array}$$

in $\mathbf{H}$, is called a choice of *generalized tangent bundle* for X. Given such, a map

$$(o_X^{\mathrm{gen}}, e^{\mathrm{gen}}) : \mathbf{B}\iota \circ \tau_X^{\mathrm{gen}} \to \mathbf{typeII}$$

in the slice $\mathbf{H}_{/\mathbf{B}O(n,n)}$ is equivalent to what is called a *generalized vielbein field* for *type II geometry* on X. This is a model for the generalized fields of gravity in the T-duality-covariant formulation of type II supergravity backgounds. (See for instance section 2 of [GMPW08] for a review and see section 4 here for discussion in the present context.) So $\mathbf{typeII} \in \mathbf{H}_{/\mathbf{B}O(n,n)}$ is the moduli stack for T-duality covariant *type II gravity* fields.

Similarly, if X is a manifold of even dimension $2n$ equipped with a generalized tangent bundle, then a map $\tau_X^{\mathrm{gen}} \longrightarrow \mathbf{genComplStruc}$ in the slice with coefficients in the canonical morphism

$$\mathbf{genComplStruc} : \ \mathbf{B}U(n,n) \longrightarrow \mathbf{B}O(2n,2n)$$

in a *generalized complex structures* on τ_X. Such $\mathbf{genComplStruc}$-fields appear in compactifications of supergravity on *generalized Calabi-Yau manifolds*, such that a global $N = 1$ supersymmetry is preserved.

Notice that the homotopy fiber sequence of the local coefficient bundle $\mathbf{typeII}$ is

$$O(n)\backslash O(n,n)/O(n) \longrightarrow \mathbf{B}O(n) \times O(n) \xrightarrow{\mathbf{typeII}} \mathbf{B}O(n,n)$$

in $\mathbf{H}$. The coset fiber on the left is the familiar local moduli spaces of generalized geometries known from the literature on T-duality and generalized geometry.

Notice also that the theory automatically determines what replaces the notion of *local diffeomorphism* in these generalized type II geometries: the generalized tangent bundles τ_X^{gen} now are the twists, and and a twist transformation $(\phi, \eta) : \tau_Y^{\mathrm{gen}} \to \tau_X^{\mathrm{gen}}$ in $\mathbf{H}_{/\mathbf{B}G_{\mathrm{geom}}(n)}$ is therefore a cohesive map $\phi : Y \to X$ equipped with an equivalence $\eta : \phi^* \tau_X^{\mathrm{gen}} \xrightarrow{\simeq} \tau_Y^{\mathrm{gen}}$ in $\mathbf{H}$ between the pullback of the generalized tangent bundle of Y and that of Y.

One can consider this setup for moduli objects being arbitrary group homomorphisms genGeom : $\mathbf{B}K \to \mathbf{B}G$ regarded as objects in the slice $\mathbf{H}_{/\mathbf{B}G}$. For instance the delooped inclusion

$$\mathbf{SuGraCompt}_n : \mathbf{B}K_n \longrightarrow \mathbf{B}E_{n(n)}$$

of the maxiomal compact subgroup of the the exceptional Lie groups produces the moduli object for U-duality covariant fields of supergravity compactified on an n-dimensional fiber. A map $\tau_X^{\mathrm{gen}} \longrightarrow \mathbf{SuGraCompt}_n$ is a generalized vielbein field in *exceptional generalized geometry* [Hull07]. Another type of exceptional geometry, that we will come back to below in 1.2.16.3, is that induced by the delooping

$$\mathbf{G}_2\mathbf{Struc} : \mathbf{B}G_2 \longrightarrow \mathbf{B}\mathrm{GL}(7)$$

of the defining inclusion of the exceptional Lie group G_2 as the subgroup of those linear transformations of $\mathbb{R}^7$ which preserves the "associative 3-form" $\langle -, (-) \times (-) \rangle$. For X a manifold of dimension 7, a field $\phi : \tau_X \to \mathbf{G}_2\mathbf{Struc}$ is a *G_2-structure* on X.

So far all the groups in the examples have been ordinary cohesive (Lie) groups, hence *0-truncated* cohesive ∞-group objects in $\mathbf{H}$. More generally we have "reduction" of structure groups for general ∞-groups exhibited by "higher vielbein fields" which are maps into moduli objects in a slice ∞-topos.

One degree higher, the first example comes from central extensions

$$A \longrightarrow \hat{G} \longrightarrow G$$

of ordinary groups. These induce long fiber sequences

$$A \longrightarrow \hat{G} \longrightarrow G \xrightarrow{\Omega \mathbf{c}} \mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \longrightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^2 A$$

in $\mathbf{H}$. Here $\mathbf{c}$ is the (cohesive) group 2-cocycle that classifies the extension, exhibited as a $\mathbf{B}A$-2-bundle $\mathbf{B}\hat{G} \to \mathbf{B}G$. Generally an object $(X, \phi_X) \in \mathbf{H}_{/\mathbf{B}}$ is an object $X \in \mathbf{H}$ equipped with a $\mathbf{B}A$-2-bundle (an A-bundle gerbe) modulated by a map $\phi_X : X \to \mathbf{B}^2 A$. A field $(\sigma, \eta) : \phi_X \to \mathbf{c}$ in $\mathbf{H}_{/\mathbf{B}^2 A}$ is a choice σ of a G-principal bundle on X together with an equivalence[6] $\eta : \sigma^* \mathbf{c} \xrightarrow{\simeq} \phi_X$.

Of particular relevance for physics is of course the example of this which is given by the Spin-extension of the special orthogonal grouop

$$\mathbf{B}\mathbb{Z}_2 \longrightarrow \mathbf{B}\mathrm{Spin} \xrightarrow{\mathbf{SpinStruc}} \mathbf{B}\mathrm{SO} \xrightarrow{\mathbf{w}_2} \mathbf{B}^2\mathbb{Z}_2 \ ,$$

which is classified by the universal second Stiefel-Whitney class $\mathbf{w}_2$. (From now on we notationally suppress, for convenience, the dimension n when displaying these groups.) For $o_X : X \to \mathbf{B}\mathrm{SO}$ an orientation structure on a manifold X, a map

$$o_X \longrightarrow \mathbf{SpinStruc}$$

in $\mathbf{H}_{/\mathbf{B}\mathrm{SO}}$ is equivalently a choice of Spin-structure on o_X. Alternatively, if $\phi : X \longrightarrow \mathbf{B}^2\mathbb{Z}_2$ is the map modulating a given $\mathbb{Z}_2$-2-bundle ($\mathbb{Z}_2$-bundle gerbe) over X, then a map $\phi_X \longrightarrow \mathbf{w}_2$ covering o_X is a

[6]This has now been called a "relative field" in [**?**].

ϕ-twisted spin structure on o_X. An important special case of this is where $\phi = \mathbf{c_1}(E) \operatorname{mod} 2$ is the mod-2 reduction of the Chern class of a given $U(1)$-principal bundle/complex line bundle on X: a $\mathbf{c}_1(E)$-twisted spin structure is equivalently a Spin^c-structure on X whose underlying $U(1)$-principal bundle is E. More generally, E itself is taken to be part of the field content and so we consider the universal Chern-class

$$\mathbf{c_1} : \mathbf{B}U(1) \longrightarrow \mathbf{B}^2\mathbb{Z}$$

of the universal $U(1)$-principal bundle. There is a diagram

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin}^c & \longrightarrow & \mathbf{B}U(1) \\ {\scriptstyle \mathbf{Spin^cStruc}}\downarrow & & \downarrow {\scriptstyle \mathbf{c}_1 \operatorname{mod} 2} \\ \mathbf{B}\mathrm{SO} & \xrightarrow{\mathbf{w}_2} & \mathbf{B}^2\mathbb{Z}_2 \end{array}$$

in $\mathbf{H}$ which exhibits the moduli stack of Spin^c-principal bundles as the homotopy fiber product of $\mathbf{c}_1$ with $\mathbf{w}_2$. With this, maps

$$o_X \longrightarrow \mathbf{Spin^cStruc}$$

in $\mathbf{H}_{\mathbf{B}\mathrm{SO}}$ are equivalently Spin^c-structures on X (for arbitrary underlying $U(1)$-principal bundle). Notice that the formalism of twist transformations again tells us what the right kind of transformations is along which Spin-structures and Spin^c-structures may be pulled back: these are maps $o_Y \longrightarrow o_X$ in $\mathbf{H}_{/\mathbf{B}\mathrm{SO}}$ and hence *orientation-preserving* local diffeomorphisms.

All of this is just a low-degree step in a whole tower of *higher* Spin-*structures* and *higher* Spin^c-*structure* that appear as fields in the effective higher supergravity theories underlying superstring theory. This tower is the *Whitehead tower* of $\mathbf{B}O$. Its smooth lift through $|-|$ to a tower of higher moduli stacks has been constructed in [FSS10] (an interpreted in physics as discussed now in [SSS09c], reviewed in the broader context of cohesive ∞-toposes in section 4 here):

$$\begin{array}{ccc} \vdots & & \\ \mathbf{B}\mathrm{Fivebrane} & & \\ {\scriptstyle \mathbf{FivebraneStruc}}\downarrow & & \\ \mathbf{B}\mathrm{String} & \xrightarrow{\frac{1}{6}\mathbf{p}_2} & \mathbf{B}^7U(1) \\ {\scriptstyle \mathbf{StringStruc}}\downarrow & & \\ \mathbf{B}\mathrm{Spin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \\ {\scriptstyle \mathbf{SpinStruc}}\downarrow & & \\ \mathbf{B}\mathrm{SO} & \xrightarrow{\mathbf{w}_2} & \mathbf{B}^2\mathbb{Z}_2 \\ {\scriptstyle \mathbf{OrientStruc}}\downarrow & & \\ \mathbf{B}\mathrm{O} & \xrightarrow{\mathbf{w}_1} & \mathbf{B}\mathbb{Z}_2 \\ {\scriptstyle \mathbf{OrthStruc}}\downarrow & & \\ \mathbf{B}\mathrm{GL} & & \end{array}$$

All of these structures can be further twisted. For instance we have the higher analog of Spin^c given by the delooping 2-group of the homotopy fiver product

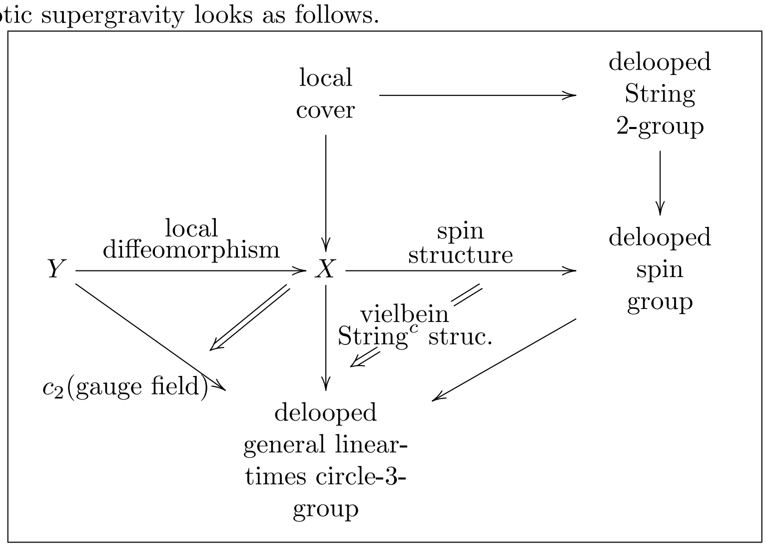
$$\begin{array}{ccc} \mathbf{B}\mathrm{String}^{c_2} & \longrightarrow & \mathbf{B}(\mathrm{E}_8 \times E_8) \\ {}^{\mathbf{String}^{c_2}\mathbf{Struc}}\downarrow & & \downarrow{}^{\mathbf{c}_2} \\ \mathbf{B}\mathrm{Spin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3 U(1) \end{array}$$

of $\frac{1}{2}\mathbf{p}_1$ with the smooth universal second Chern class $\mathbf{c}_2 : \mathbf{B}(E_8 \times E_8) \longrightarrow \mathbf{B}^3 U(1)$. On manifolds X equipped with a Spin-structure $s_X : X \to \mathbf{B}\mathrm{Spin}$, a field

$$s_X \longrightarrow \mathbf{String}^{c_2}\mathbf{Struc}$$

in $\mathbf{H}_{/\mathbf{B}\mathrm{Spin}}$ is a choice of String^{c_2}-structure, equivalently a choice of $(E_8 \times E_8)$-principal bundle and an equivalence between its Chern-Simons circle 3-bundle and the Chern-Simons circle 3-bundle of the Spin-structure. This is the quantum-anomaly-free instanton sector of a gauge field in the effective heterotic supergravity underlying the heterotic string [SSS09c]. Below in 1.2.16.1.3 we discuss how the differential refinement of String^{c_2}-structures capture the dynamical field of gravity and the gauge field in heterotic supergravity.

In summary, the specialization of the diagram of 1.2.16.1.1 to the anomaly-free instanton-sector of heterotic supergravity looks as follows.



There are further variants of all these examples and other further cases of gravity-like fields in physics given by maps in slice toposes. But for the present discussion we leave it at this and now turn to the other fundamental kind of fields in physics besides gravity: gauge fields.

1.2.16.1.3 Gauge fields: higher, twisted, non-abelian The other major kind of (quantum) fields besides the (generalized) fields of gravity that we discussed above are of course *gauge fields*. A seminal result of Dirac's old argument about electric/magnetic *charge quantization* is that a configuration of the plain *electromagnetic field* is mathematically a *connection* on a $U(1)$-principal bundle. Similarly the Yang-Mills field of quantum chromodynamics is mathematically a connection on a G-principal bundle, where G is the corresponding gauge group. The connection itself is locally the *gauge potential* traditonally denoted A, while the class of the underlying global bundle is the *magnetic background charge* for the case of electromagnetism and is the *instanton sector* for the case of $G = \mathrm{SU}(n)$.

Analogously, it has long been known that the background B-field to which the string couples is mathematically a connection on a $U(1)$-principal *2-bundle* (often presented as $U(1)$-bundle gerbe), hence a bundle that is principal under the higher group (2-group) $\mathbf{B}U(1)$. Together with the case of ordinary $U(1)$-principal bundles these are the first two (or three) degrees of what are known as cocycles in *ordinary differential cohomology*, a refinement of cocyles modulated in the coefficient stack $\mathbf{B}^n U(1)$ by *curvature twists* controled by smooth differential form data. A general formalization of this based on the underlying topological classifying spaces $K(\mathbb{Z}, n+1) \simeq |\mathbf{B}^n U(1)|$, or in fact any infinite loop space $|\mathbf{B}\mathbb{G}|$ representing a generalized cohomology theory, has been given in [HoSi05]. Here we refine this construction to the cohesive higher topos case and obtain higher cohesive moduli stacks $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ such that maps $X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ with coefficients in these are differential $\mathbb{G}$-cocycles and hence equivalently (higher) *gauge fields* on X for the (higher, cohesive) gauge group $\mathbb{G}$.

An ∞-group $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ is *abelian* or E_∞ if it is equipped with an n-fold delooping $\mathbf{B}^n\mathbb{G} \in \mathbf{H}$ for all $n \in \mathbb{N}$. If it is equipped at least with a second delooping $\mathbf{B}^2\mathbb{G}$, then we say it is a *braided* ∞*-group*. Equivalently this means that the single delooping object $\mathbf{B}\mathbb{G}$ is itself equipped with the structure of an ∞-group. For example the full subcategory of any braided monoidal ∞-category on the objects that are invertible under the tensor product is a braided ∞-group, hence the name.

For a braided ∞-group $\mathbb{G}$ in a cohesive ∞-topos, the axioms of cohesion induce a canonical map

$$\mathrm{curv}_{\mathbb{G}} : \mathbf{B}\mathbb{G} \longrightarrow \flat_{\mathrm{dR}} \mathbf{B}^2\mathbb{G}$$

to the *de Rham coefficient objects* of the group $\mathbf{B}\mathbb{G}$. On the one hand this may be interpreted as the *Maurer-Cartan form* on th cohesive group $\mathbf{B}\mathbb{G}$. Equivalently, one finds that this is the *universal curvature characteristic* of $\mathbb{G}$-principal ∞-bundles: the map can be seen to proceed by equipping a $\mathbb{G}$-principal ∞-bundle with a *pseudo-connection* and then sending that to the coresponding curvature in the de Rham hypercohomology with coefficients in the ∞-Lie algebra of $\mathbb{G}$.

In order to pick among those (higher) pseudo-connections with curvature in hypercohomology those that are genuine (higher) connections characterized by having globally well defined curvature differential form data, let $\Omega_{\mathrm{cl}}(-, \mathbb{G}) \in \mathbf{H}$ be a 0-truncated object equipped with a map $\Omega_{\mathrm{cl}}(-, \mathbb{G}) \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$ which has the following property: for every manifold Σ the induced map

$$[\Sigma, \Omega_{\mathrm{cl}}(-, \mathbb{G})] \longrightarrow [\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}]$$

is 1-epimorphism (an effective epimorphism, hence an epimorphism in the sheaf topos under 0-truncation). This expresses the fact that $\Omega_{\mathrm{cl}}(-, \mathbb{G})$ is a sheaf of flat Lie($\mathbb{G}$)-valued differential forms, in that every de Rham cohomology class over a manifold is represented by such a form.

(More generally one considers a suitable filtration $\Omega^\bullet_{\mathrm{cl}}(-, \mathbb{G}) \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$, hence a kind of *universal mixed Hodge structure on* $\mathbb{G}$*-cohomology*).

Then the moduli object $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ for *differential* $\mathbb{G}$*-cocycles* is the homotopy pullback in

$$\begin{array}{ccc} \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \longrightarrow & \Omega^n_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \mathbf{B}\mathbb{G} & \xrightarrow{\mathrm{curv}_{\mathbb{G}}} & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \end{array} .$$

For example if $\mathbb{G} \simeq \mathbf{B}^{n-1}U(1)$ in smooth ∞-groupoids, then the object $\mathbf{B}^n U(1)_{\mathrm{conn}}$ defined this way is the n-stack which is presenented under the Dold-Kan correspondence by the *Deligne-complex* of sheaves. It modulates ordinary differential cohomology.

A configuration of the electromagnetic field on a space X is a map $X \to \mathbf{B}U(1)_{\mathrm{conn}}$. A configuration of the B-field background gauge field of the bosonic string is a map $X \to \mathbf{B}^2U(1)_{\mathrm{conn}}$. (For the superstring the situation is a bit more refined, discussed below.) A configuration of the C-field background gauge field of M-theory involves (among other data) a map $X \to \mathbf{B}^3U(1)_{\mathrm{conn}}$.

(...)

twisted differential string and fivebrane and ninebrane structures

(...)

Differential T-duality and B_n-geometry

Above we have seen that the *extended* Lagrangian $\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ for $G = \mathrm{Spin}, \mathrm{SU}$-Chern-Simons 3d gauge field theory also serves as the twist that defines the moduli stack $\mathbf{B}\mathrm{String}^{c_2}_{\mathrm{conn}}$ of Green-Scharz anomaly-free heterotic background gauge field configurations. In view of this it is natural to ask: does the extended Lagrangian of $U(1)$-Chern-Simons theory similarly play a role as part of the background gauge field structure for superstrings? Indeed this turns out to be the case: the extended $U(1)$-Chern-Simons Lagrangian encodes the twist that defines *differential T-duality structures* and *B_n-geometry*.

To see this, we observe by direct inspection that what in [KaVa10] is called a *differential T-duality structure* on a pair of circle-bundles $S^1 \to X_1, X_2 \to Y$ over some base Y and equipped with connections ∇_1 and ∇_2, is a trivialization of the corresponding cup-product circle 3-bundle, hence of the extended Chern-Simons Lagrangian of two-species $U(1)$-Chern-Simons theory pulled back along the map that modulates the two circle bundles.

We now say this again in more detail. Let T^1 be a circle and $\tilde{T}^1 := \mathrm{Hom}(T^1, U(1))$ the dual circle, with the canonical pairing denoted $\langle -,- \rangle : T^1 \times \tilde{T}^1 \to U(1)$. Then the first spacetime $X_1 \to Y$ is modulated by a map $\mathbf{c}_1 : Y \longrightarrow \mathbf{B}T^1_{\mathrm{conn}}$, and its T-dual $\tilde{c}_1 : X_2 \to Y$ by a map $\tilde{\mathbf{c}_1} : Y \to \mathbf{B}\tilde{T}^1_{\mathrm{conn}}$.

Now the pairing and the cup product together form a universal characteristic map of moduli stacks

$$\langle - \cup - \rangle : \mathbf{B}(T^1 \times \tilde{T}^1) \longrightarrow \mathbf{B}^3U(1) \ .$$

By the above discussion, this has a differential refinement

$$\langle - \cup - \rangle : \mathbf{B}(T^1 \times \tilde{T}^1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^3U(1)_{\mathrm{conn}}$$

which is the extended Lagrangian of $U(1)$-Chern-Simons theory in 3d. If instead we regard the same map as a 3-cocycle, it modulates a higher group extension $\mathrm{String}(T^1 \times \tilde{T}^1) \to T^1 \times \tilde{T}^1$, sitting in a long fiber sequence of higher moduli stacks of the form

$$\cdots \longrightarrow \mathbf{B}U(1) \longrightarrow \mathrm{String}(T^1 \times \tilde{T}^1) \longrightarrow (T^1 \times \tilde{T}^1) \longrightarrow \mathbf{B}^2U(1) \longrightarrow \mathbf{B}\mathrm{String}(T^1 \times \tilde{T}^1) \longrightarrow \mathbf{B}(T^1 \times \tilde{T}^1) \longrightarrow \mathbf{B}^3$$

One sees from this that a *differential T-duality structure* on (X_1, X_2) as considered in def. 2.1 of [KaVa10] is equivalently – when refined to the context of smooth higher geometry – a lift of $(\mathbf{c}_1, \tilde{\mathbf{c}}_1)$ through the left vertical projection in the homotopy pullback square

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(T^1 \times \tilde{T}^1)_{\mathrm{conn}} & \longrightarrow & \Omega^{4 \leq \bullet \leq 3}_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathbf{B}(T^1 \times \tilde{T}^1)_{\mathrm{conn}} & \xrightarrow{\langle - \cup - \rangle} & \mathbf{B}^3U(1)_{\mathrm{conn}} \end{array} ,$$

hence is a map in the slice over $\mathbf{B}^3U(1)_{\mathrm{conn}}$, hence is a *differential* $\mathrm{String}(T^1 \times \tilde{T}^1)$*-structure* on the given data. Along the lines of the discussion in [FSS10] one finds, as for the twisted differential String-structures discussed above, that such a lift locally corresponds to a choice of 3-form H satisfying

$$d_{\mathrm{dR}}H = \langle F_{A_1} \wedge F_{A_2} \rangle \, ,$$

where A_1, A_2 are the local connection forms of the two circle bundles. This is the local structure that has been referred to as *B_n-geometry*, see the corresponding discussion and references given in [FiSaScIV].

Observe that by the universal property of homotopy fibers, the underlying trivialization of the cup product circle 3-bundle corresponds to a choice of factorization of $(\mathbf{c}_1, \tilde{\mathbf{c}}_1)$ as shown on the bottom of the following diagram

$$\begin{array}{ccccc} X_1 \times_Y X_2 & \xrightarrow{\kappa} & \mathbf{B}^2U(1) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{B}\mathrm{String}(T^1 \times \tilde{T}^1) & \longrightarrow & \mathbf{B}(T^1 \times \tilde{T}^1) \end{array} .$$

Forming the consecutive homotopy pullback of the point inclusion as given by these two squares, the map $X_1 \times_Y X_2 \to \mathbf{B}^2 U(1)$ induced by the universal property of the homotopy pullback modulates a circle 2-bundle ($U(1)$-bundle gerbe) on the correspondence space. This is the bundle gerbe on the correspondence space considered in 2.2, 2.3 of [KaVa10]. Notice that this is just a special case of the general phenomenon of twisted higher bundles, as laid out in [NSSa].

(...)

1.2.16.1.4 Gauge invariance, equivariance and general covariance

The notion of *gauge transformation* and *gauge invariance* is built right into higher geometry. Any object $X \in \mathbf{H}$ in general contains not just (local) points, but also gauge equivalences between these, gauge-of-gauge equivalences between those, and so on. A map $\exp(iS(-)) : \mathbf{Fields} \to U(1)$ is automatically a *gauge invariant function* with respect to whatever gauge transformations the species of fields encoded by the moduli object **Fields** encodes.

Specifically, if an ∞-group G acts on some Y, then a G-*equivariance* structure on a map $Y \to A$ is an extension

$$\begin{array}{ccc} Y & \longrightarrow & A \\ \downarrow & \nearrow & \\ Y//G & & \end{array}$$

along the canonical quotient projection.

If A here is a 0-truncated object such that $U(1)$, then the existence of such an extension is just a property. But if A has itself gauge equivalences, say if $A = \mathbf{B}^n U(1)_{\mathrm{conn}}$ for positive n-then a choice of such an extension is genuine extra structure. For $n = 1$ this is the familiar structure on an *equivariant bundle*. For higher n it is a suitable higher order generalization of this notion.

Equivariance is preserved by transgression. If $\mathbf{L} : \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ is an extended Lagrangian, hence equivalently a equivariant n-connection on the space of fields, then for Σ_k any object the mapping space $[\Sigma_k, \mathbf{Fields}]$ contains the gauge equivalences of the given field species on Σ and accordingly the transgressed Lagrangian

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{L}]) \; : \; [\Sigma_k, \mathbf{Fields}] \to \mathbf{B}^{n-k} U(1)_{\mathrm{conn}}$$

is gauge invariant (precisely: carries gauge-equivariant structure).

A particular kind of gauge equivalence/equivariance is the *diffeomorphism equivariance* of a *generally covariant field theory*. In such a field theory two fields $\phi_1, \phi_2 : \Sigma \to \mathbf{Fields}$ are to be regarded as gauge equivalent if there is a diffeomorphism, hence an automorphism $\alpha : \Sigma \xrightarrow{\simeq} \Sigma$ in $\mathbf{H}$, such that $\alpha^* \phi_2 \simeq \phi_1$.

Formally this means that for generally covariant field theries the field space $[\Sigma, \mathbf{Fields}]$ over a given worldvolume Σ is to be formed in the slice $\mathbf{H}_{/\mathbf{BAut}}(\Sigma) \simeq \mathbf{Aut}(\Sigma)\mathrm{Act}$, with Σ understood as equipped with the defining $\mathbf{Aut}(\Sigma)$-action and with **Fields** equipped with the trivial $\mathbf{Aut}(\Sigma)$-action, we write

$$[\Sigma, \mathbf{Fields}]_{/\mathbf{BAut}(\Sigma)} \in \mathbf{H}_{/\mathbf{BAut}(\Sigma)}$$

for emphasis. To see this one observes that generally for $(V_1, \rho_1), (V_2, \rho_2) \in \mathbf{GAct}$ two objects equipped with G-action, their mapping space $[V_1, V_2]_{/\mathbf{B}G}$ formed in the slice is the absolute mapping space $[V_1, V_2]$ formed in $\mathbf{H}$ and equipped with the *conjugation action* of G, under which an element $g \in G$ acts on an element $f : V_1 \to V_2$ by sending it to $\rho_2(g)^{-1} \circ f \circ \rho_1(g)$.

Hence the mapping space $[\Sigma, \mathbf{Fields}]_{/\mathbf{BAut}(\Sigma)}$ formed in the slice corresponds in $\mathbf{H}$ to the fiber sequence

$$\begin{array}{ccc} \Sigma & \longrightarrow & \mathbf{Aut}(\Sigma)\backslash\backslash[\Sigma, \mathbf{Fields}] \\ & & \downarrow \\ & & \mathbf{BAut}(\Sigma) \end{array}$$

and a *generally covariant field theory* for the given species of fields is one whose configuration spaces are $\mathbf{Aut}(\Sigma)\backslash\backslash[\Sigma, \mathbf{Fields}]$, the action groupoids of the ∞-groupoid of field configurations on Σ by the diffeomorphism action on Σ.

Ordinary 3d Chern-Simons theory is strictly speaking to be regared as a generally covarnat field theory, but this is often not made explicit, due to a special property of 3d Chern-Simons theory: if two *on-shell* field configurations are related by a diffeomorphism (connected to the identity), then they are already gauge equivalent also by a gauge transformation in $[\Sigma, \mathbf{B}G_{\mathrm{conn}}]$. This holds in fact also for all higher Chern-Simons theories that come from *binary* invariant polynomials, but it does not hold fully generally. Even when this is the case, supporessing the general covariance is a dubious move, since while the gauge equivalence classes may coincide, $\tau_0[\Sigma, \mathbf{Fields}]_{\mathrm{onshell}} \simeq \tau_0\mathbf{Aut}(\Sigma)\backslash\backslash[\Sigma, \mathbf{Fields}]_{\mathrm{onshell}}$, the two full homotopy types still need not be equivalent and hence the corresponding quantum field theories may not be equivalent.

1.2.16.2 Phase spaces Traditionally, the *phase space* of a physical system which is given by an action functional $\exp(iS) : \mathbf{Fields}(\Sigma) \longrightarrow U(1)$ is the *variational critical locus* of $\exp(iS)$: the subspace of field configurations $\mathbf{Fields}(\Sigma)$ on some manifold Σ with boundary, such that the variation $\mathbf{d}S$ of the action (with fields on $\partial\Sigma$ held fixed) vanishes when restricted to this subspace. One also calls this the space of solutions of the *Euler-Lagrange equations* of the system. Often one considers the special case where $\Sigma = \Sigma_{\mathrm{in}} \times [0,1]$ is the cylinder over a closed manifold Σ_{in} and under suitable conditions on S, solutions to the Euler-Lagrange equations are fixed by their value and first derivative on Σ_{in}, in which case the phase space may be identified with the cotangent bundle $T^*\mathbf{Field}(\Sigma_{\mathrm{in}})$. This simple special case is sometimes regarded as the definition of the notion of phase space, and in order to distinguish the general notion from this special case one calls the space of solutions of the Euler-Lagrange equations also the *covariant phase space*. For S a local action functional this space is canonically equipped with a presymplectic form. Quotienting out the (gauge) symmetries makes this a genuine symplectic form on what is called the *reduced phase space*. But as with all quotients, this quotient makes good sense in general only when formed in a suitable homotopy-theoretic sense, hence in higher geometry. The physics literature knows a formalism for dealing with this as the *BV-BRST formalism*.

In the following we discuss these issues in differential cohesive higher geometry, for the prequantum theory of n-dimensional field theories defined by extended Lagrangians $\mathbf{L} : \mathbf{Fields} \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}}$, which induce action functionals as above after transgression to the mapping space out of some Σ.

1.2.16.2.1 Variational calculus, critical loci, Euler-Langrange equations Let $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ be a cohesive ∞-group. By the discussion in 1.2.16.1.3 above there is a canonical de Rham cocycle on $\mathbb{G}$, the *∞-Maurer-Cartan form*

$$\theta : \mathbb{G} \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G} \ .$$

A little reflection shows that in the context of higher stacks, this form is also the *universal differential* for $\mathbb{G}$-valued functions, in that for

$$S : X \longrightarrow \mathbb{G}$$

any map, the composite

$$S^{-1}\mathbf{d}S : X \xrightarrow{S} \mathbb{G} \xrightarrow{\theta} \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}$$

which corresponds to a $\mathrm{Lie}(\mathbb{G})$-valued differential cocycle on X, is the normalized differential of S.

If here $X = \mathbf{Fields}(\Sigma)$ is a space of fields on some manifold Σ with bounday $\partial\Sigma \hookrightarrow \Sigma$, then the *variational differential* is the restriction of this differential to variations which keep the field configurations over the boundary $\partial\Sigma$ fixed. This restriction is given by precomposition with the top horizontal morphism in the following homotopy pullback diagram

$$\begin{array}{ccc} \mathbf{Fields}(\Sigma)_{\partial\Sigma} & \xrightarrow{\iota} & \mathbf{Fields}(\Sigma) \\ \downarrow & & \downarrow \\ \flat\mathbf{Fields}(\partial\Sigma) & \longrightarrow & \mathbf{Fields}(\partial\Sigma) \end{array} .$$

Here $\flat : \mathbf{H} \to \mathbf{H}$ is the *flat modality* given by the cohesion of $\mathbf{H}$. In summary, the *variational differential* of a map $S : \mathbf{Fields} \longrightarrow \mathbb{G}$ is the composite

$$S^{-1}\mathbf{d}_{\mathrm{var}}S : \mathbf{Fields}(\Sigma)_{\partial\Sigma} \xrightarrow{\iota} \mathbf{Fields}(\Sigma) \xrightarrow{S} \mathbb{G} \xrightarrow{\theta} \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G} .$$

Now the *phase space* or *variational critical locus* or *solution space of the Euler-Lagrange equations* of S is supposed to be the subobject of $\mathbf{Fields}(\Sigma)_{\partial\Sigma}$ "on which this differential vanishes". But one needs to be careful with how to interpret this. For instance the differential vanishes when the whole expression above is restricted to any point $* \to \mathbf{Fields}(\Sigma)_{\partial\Sigma}$, simply because any de Rham data on the point is trivial: there is only an essentially unique map $* \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}$: the 0-form. Therefore, what should really be meant by a point where the differential vanishes is a point such that the differential vanishes *on every infinitesimal neighbourhood* of it.

In other words, when testing whether $S^{-1}\mathbf{d}_{\mathrm{var}}S$ vanishes when restricted to a subspace $\phi : U \hookrightarrow \mathbf{Fields}(\Sigma)_{\partial\Sigma}$, we need to ensure that U is *infinitesimally spread out* or *infinitesimally open* in $\mathbf{Fields}(\Sigma)$. Such a *spread-out map* ϕ is commonly known by the French term as an *étale map*; and an *infinitesimally spread out* map is known as a *formally étale map* (with "formal" as in "formal power series" rings, which are the rings of functions on the infinitesimal neighbourhood of the origin in a linear space).

The differential cohesion of the ambient ∞-topos canonically induces a notion of such formally étale maps: the *infinitesimal path modality* $\mathbf{\Pi}_{\mathrm{inf}} : \mathbf{H} \longrightarrow \mathbf{H}$ sends an object X to what is sometimes called its *de Rham space* $\mathbf{\Pi}_{\mathrm{inf}}(X)$, in which infinitesimally close points are made equivalent. There is a natural inclusion $X \longrightarrow \mathbf{\Pi}_{\mathrm{inf}}(X)$ which may alternatively be thought of as the inclusion of the constant paths in X into the infinitesimal paths in X, or as the quotient map that quotients out the infinitesimal neighbourhood relation.

Now, a map $f : X \longrightarrow Y$ is *formally étale* if the naturality square of this inclusion

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbf{\Pi}_{\mathrm{inf}}(f)} \\ Y & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(Y) \end{array}$$

is a homotopy pullback square. This is a generalization to cohesive ∞-groupoids of the traditional fact that a map f between smooth manifolds is a *local diffeomorphism* precisely if the square of tangent bundle projections

$$\begin{array}{ccc} TX & \longrightarrow & X \\ \downarrow{\scriptstyle df} & & \downarrow{\scriptstyle f} \\ TY & \longrightarrow & Y \end{array}$$

is a pullback diagram of smooth manifolds. (To see how the general condition above relates to this one, let $D \hookrightarrow \mathbb{R}$ be the first order infinitesimal neighbourhood of the origin in the real line and observe that $X^D \simeq TX$, $f^D \simeq df$, but that $(\mathbf{\Pi}_{\mathrm{inf}}(X))^D \simeq \mathbf{\Pi}_{\mathrm{inf}}(X)$.)

For any object $X \in \mathbf{H}$ we then have the ∞-category

$$\mathrm{Sh}_{\mathbf{H}}(X) := (\mathbf{H}_{/X})^{\mathrm{fet}} \underset{\mathrm{Et}}{\overset{\hookrightarrow}{\leftarrow}} \mathbf{H}_{/X}$$

of formally étale maps into X. As the notation on the left indicates, this may be thought of as the *petit ∞-topos of ∞-sheaves on X*, in generalization of the classical fact of topos theory which identifies sheaves on a topological space with étale topological spaces over it.

The inclusion of formally étale maps into the entire slice ∞-topos $\mathbf{H}_{/X}$ (the *gros ∞-topos* of X) has a right adjoint reflector Et, as indicated above. This induces for any object $A \in \mathbf{H}$ the ∞-sheaf on X of *A-valued functions on X*:

$$\mathcal{O}_X(A) := \mathrm{Et}(X \times A \stackrel{p_1}{\to} X) \in \mathrm{Sh}_{\mathbf{H}}(X)\,.$$

In particular, for $\mathbb{G}$ as above we have the ∞-sheaf

$$\mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}) \in \mathrm{Sh}_{\mathbf{H}}(X)$$

of flat Lie($\mathbb{G}$)-valued differential forms on X. The 0-section $0 : * \to \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}$ induces a 0-section

$$0 : X \to \mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G})$$

in $\mathrm{Sh}_{\mathbf{H}}(X)$, and more generally any map $\omega : X \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}$ induces a section

$$\omega : X \to \mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G})$$

in $\mathrm{Sh}_{\mathbf{H}}(X)$.

But now since in $\mathrm{Sh}_{\mathbf{H}}(X)$ every subspace $U \hookrightarrow X$ is guaranteed to be formally étale, this is the right context to solve the Euler-Lagrange equations of an action functional: we say that the *critical locus* of $S : \mathbf{Fields}(\Sigma)_{\partial\Sigma} \longrightarrow \mathbb{G}$ is the homotopy fiber

$$\sum_{\phi \in \mathbf{Fields}(\Sigma)_{\partial\Sigma}} (S^{-1}(\phi)\mathbf{d}_{\mathrm{var}}S(\phi) \simeq 0) \in \mathrm{Sh}_{\mathbf{H}}(\mathbf{Fields}(\Sigma)_{\partial\Sigma})\,,$$

sitting in the homotopy pullback square

$$\begin{array}{ccc} \sum_{\phi \in \mathbf{Fields}(\Sigma)_{\partial\Sigma}} (S^{-1}(\phi)\mathbf{d}_{\mathrm{var}}S(\phi) \simeq 0) & \longrightarrow & \mathbf{Fields}(\Sigma)_{\partial\Sigma} \\ \downarrow & & \downarrow^{0} \\ \mathbf{Fields}(\Sigma)_{\partial\Sigma} & \xrightarrow{S^{-1}\mathbf{d}_{\mathrm{var}}S} & \mathcal{O}_{\mathbf{Fields}(\Sigma)_{\partial\Sigma}}(\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}) \end{array}$$

in $\mathrm{Sh}_{\mathbf{H}}(\mathbf{Fields}(\Sigma)_{\partial\Sigma})$.

This critical locus is known in traditional literature for the special case that $\mathbb{G} = \mathbb{R}$ is the additive Lie group of real numbers and in its *infinitesimal* approximation: the *∞-Lie algebroid* of the critical locus is known, dually as the *on-shell BRST complex* of the system (whereas the ∞-Lie algebroid of $\mathbf{Fields}(\Sigma)$ itself is the *off-shell BRST complex*). Moreover, if the ambient ∞-topos $\mathbf{H}$ is not 1-localic, and specifically if it has a site of definition given by formal duals of simplicial (smooth) algebras, then the critical locus as above is also called the *derived crititcal locus* for emphasis, and its ∞-Lie algrboid is dually known as the *BV-BRST complex* of the system. (For discussion of how the traditional formulation of BV-BRST complexes models homotopy pullbacks of the above form see [**?**].)

But with the general notion of critical loci in cohesive ∞-toposes, we obtain examples beyond those discussed in the literature whenever $\mathbb{G}$ is a *higher* group.

Notably when $\mathbb{G} := \mathbf{B}^n U(1)$ is the circle $(n+1)$-group, then the universal differential

$$\theta : \mathbf{B}^n U(1) \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$$

is equivalently, by the discussion in 1.2.16.1.3, the *universal curvature characteristic* for smooth circle n-bundles, and so there are accordingly higher order interpretations of phase spaces in *extended* prequantization.

For example, let $\mathbf{c} : \mathbf{B}G \longrightarrow \mathbf{B}^n U(1)$ be a universal characteristic map on the moduli stack of a cohesive ∞-group G. Then

$$S := p_1 \circ [\mathbf{\Pi}(S^1), \mathbf{c}] : G/\!/_{\mathrm{ad}} G \longrightarrow \mathbf{B}^{n-1}U(1)/\!/_{\mathrm{ad}}\mathbf{B}^{n-1}U(1) \longrightarrow \mathbf{B}^{n-1}U(1)$$

is the WZW-$(n-1)$-bundle (equiopped with its ad-equivariant structure) of the corresponding n-dimensional Chern-Simons theory. Regarding this as a $\mathbb{G}$-valued function, we find that its variational differential

$$S^{-1}\mathbf{d}_{\mathrm{var}}S : G/\!/_{\mathrm{ad}}G \xrightarrow{S} \mathbf{B}^{n-1}U(1) \xrightarrow{\theta} \flat_{\mathrm{dR}}\mathbf{B}^n U(1)$$

is the curvature, in de Rham hypercohomology, of the WZW-$(n-1)$-bundle.

1.2.16.2.2 Differential moduli stacks (...)

1.2.16.3 Prequantum geometry We had indicated in section **??** how a single extended Lagrangian, given by a map of universal higher moduli stacks $\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, induces, by transgression, circle $(n-k)$-bundles with connection

$$\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L}) : \mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

on moduli stacks of field configurations over each closed k-manifold Σ_k. In codimension 1, hence for $k = n-1$, this reproduces the ordinary *prequantum circle bundle* of the n-dimensional Chern-Simons type theory, as discussed in section 1.2.15.3. The space of sections of the associated line bundle is the space of *prequantum states* of the theory. This becomes the space of genuine quantum states after choosing a *polarization* (i.e., a decomposition of the moduli space of fields into *canonical coordinates* and *canonical momenta*) and restricting to polarized sections (i.e., those depending only on the canonical coordinates). But moreover, for each Σ_k we may regard $\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L})$ as a *higher prequantum bundle* of the theory in higher codimension and hence consider its prequantum geometry in higher codimension.

We discuss now some generalities of such a higher geometric prequantum theory and then show how this perspective sheds a useful light on the gauge coupling of the open string, as part of the transgression of prequantum 2-states of Chern-Simons theory in codimension 2 to prequantum states in codimension 1.

1.2.16.3.1 Higher prequantum states and prequantum operators We indicate here the basic concepts of higher extended prequantum theory and how they reproduce traditional prequantum theory.

Consider a (pre)-n-plectic form, given by a map

$$\omega : X \longrightarrow \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}$$

in $\mathbf{H}$. A *n-plectomorphism* of (X, ω) is an auto-equivalence of ω regarded as an object in the slice $\mathbf{H}_{/\Omega^{n+1}_{\mathrm{cl}}}$, hence a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\omega} & \\ & \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}} & \end{array} .$$

A *prequantization* of (X, ω) is a choice of prequantum line bundle, hence a choice of lift ∇ in

$$\begin{array}{ccc} & & \mathbf{B}^n U(1)_{\mathrm{conn}} \\ & {\scriptstyle\nabla}\nearrow & \downarrow{\scriptstyle F_{(-)}} \\ X & \xrightarrow{\omega} & \Omega^{n+1}_{\mathrm{cl}} \end{array} ,$$

modulating a circle n-bundle with connection on X. We write $\mathbf{c}(\nabla) : X \xrightarrow{\nabla} \mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ for the underlying $(\mathbf{B}^{n-1}U(1))$-principal n-bundle. An autoequivalence

$$\hat{O} : \nabla \xrightarrow{\simeq} \nabla$$

of the prequantum n-bundle regarded as an object in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}$, hence a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & \swarrow\!\!\!\!\Leftarrow_{O} & \\ {}_{\nabla}\searrow & & \swarrow_{\nabla} \\ & \mathbf{B}^n U(1)_{\mathrm{conn}} & \end{array}$$

is an (exponentiated) *prequantum operator* or *quantomorphism* or *regular contact transformation* of the prequantum geometry (X, ∇), forming an ∞-group in $\mathbf{H}$. The L_∞-algebra of this *quantomorphism ∞-group* is the higher *Poisson bracket* Lie algebra of the system. If X is equipped with abelian group structure then the quantomorphisms covering these translations form the *Heisenberg ∞-group*. The homotopy labeled O above diagram is the *Hamiltonian* of the prequantum operator. The image of the quantomorphisms in the symplectomorphisms (given by composition the above diagram with the curvature morphism $F_{(-)}$: $\mathbf{B}^n U(1)_{\mathrm{conn}} \to \Omega^{n+1}_{\mathrm{cl}}$) is the group of *Hamiltonian n-plectomorphisms*. A lift of an ∞-group action $G \to \mathbf{Aut}(X)$ on X from automorphisms of X (diffeomorphism) to quantomorphisms is a *Hamiltonian action*, infinitesimally (and dually) a *momentum map*.

To define higher prequantum states we fix a representation (V, ρ) of the circle n-group $\mathbf{B}^{n-1}U(1)$. By the general results in [NSSa] this is equivalent to fixing a homotopy fiber sequence of the form

$$\begin{array}{ccc} \underline{V} & \longrightarrow & \underline{V}/\!/\mathbf{B}^{n-1}U(1) \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}^n U(1) \end{array}$$

in $\mathbf{H}$. The vertical morphism here is the *universal ρ-associated V-fiber ∞-bundle* and characterizes ρ itself. Given such, a section of the V-fiber bundle which is ρ-associated to $\mathbf{c}(\nabla)$ is equivalently a map

$$\Psi : \mathbf{c}(\nabla) \longrightarrow \rho$$

in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)}$. This is a higher *prequantum state* of the prequantum geometry (X, ∇). Since every prequantum operator $\hat{O}$ as above in particular is an auto-equivalence of the underlying prequantum bundle $\hat{O} : \mathbf{c}(\nabla) \xrightarrow{\simeq} \mathbf{c}(\nabla)$ it canonically acts on prequantum states given by maps as above simply by precomposition

$$\Psi \mapsto \hat{O} \circ \Psi \,.$$

Notice also that from the perspective of section **??** all this has an equivalent interpretation in terms of twisted cohomology: a preqantum state is a cocycle in twisted V-cohomology, with the twist being the prequantum bundle. And a prequantum operator/quantomorphism is equivalently a twist automorphism (or "generalized local diffeomorphism").

For instance if $n = 1$ then ω is an ordinary (pre)symplectic form and ∇ is the connection on a circle bundle. In this case the above notions of prequantum operators, quantomorphism group, Heisenberg group and Poisson bracket Lie algebra reproduce exactly all the traditional notions if X is a smooth manifold, and generalize them to the case that X is for instance an orbifold or even itself a higher moduli stack, as we have seen. The canonical representation of the circle group $U(1)$ on the complex numbers yields a homotopy fiber sequence

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}U(1) \end{array} ,$$

where $\underline{\mathbb{C}}//\underline{U}(1)$ is the stack corresponding to the ordinary action groupoid of the action of $U(1)$ on $\mathbb{C}$, and where the vertical map is the canonical functor forgetting the data of the local $\mathbb{C}$-valued functions. This is the *universal complex line bundle* associated to the universal $U(1)$-principal bundle. One readily checks that a prequantum state $\Psi : \mathbf{c}(\nabla) \to \rho$, hence a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & \underline{\mathbb{C}}//\underline{U}(1) \\ & {\scriptstyle \mathbf{c}(\nabla)} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathbf{B}U(1) & \end{array}$$

in $\mathbf{H}$ is indeed equivalently a section of the complex line bundle canonically associated to $\mathbf{c}(\nabla)$ and that under this equivalence the pasting composite

$$\begin{array}{ccccc} X & \xrightarrow{\simeq} & X & \longrightarrow & \underline{\mathbb{C}}//\underline{U}(1) \\ & {\scriptstyle \mathbf{c}(\nabla)} \searrow \; \swarrow_{O} & \downarrow {\scriptstyle \mathbf{c}(\nabla)} & \swarrow {\scriptstyle \rho} & \\ & & \mathbf{B}U(1) & & \end{array}$$

is the result of the traditional formula for the action of the prequantum operator $\hat{O}$ on Ψ.

Instead of forgetting the connection on the prequantum bundle in the above composite, one can equivalently equip the prequantum state with a differential refinement, namely with its *covariant derivative* and then exhibit the prequantum operator action directly. Explicitly, let $\mathbb{C}//U(1)_{\mathrm{conn}}$ denote the quotient stack $(\underline{\mathbb{C}} \times \Omega^1(-,\mathbb{R}))//\underline{U}(1)$, with $U(1)$ acting diagonally. This sits in a homotopy fiber sequence

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & \underline{\mathbb{C}}//\underline{U}(1)_{\mathrm{conn}} \\ & & \downarrow {\scriptstyle \rho_{\mathrm{conn}}} \\ & & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$$

which may be thought of as the differential refinement of the above fiber sequence $\underline{\mathbb{C}} \to \underline{\mathbb{C}}//\underline{U}(1) \to \mathbf{B}U(1)$. (Compare this to section 1.2.15.5, where we had similarly seen the differential refinement of the fiber sequence $\underline{G}/\underline{T}_\lambda \to \mathbf{B}T_\lambda \to \mathbf{B}G$, which analogously characterizes the canonical action of G on the coset space G/T_λ.) Prequantum states are now equivalently maps

$$\widehat{\mathbf{\Psi}} : \nabla \longrightarrow \rho_{\mathrm{conn}}$$

in $\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$. This formulation realizes a section of an associated line bundle equivalently as a connection on what is sometimes called a groupoid bundle. As such, $\widehat{\mathbf{\Psi}}$ has not just a 2-form curvature (which is that of the prequantum bundle) but also a 1-form curvature: this is the covariant derivative $\nabla\sigma$ of the section.

Such a relation between sections of higher associated bundles and higher covariant derivatives holds more generally. In the next degree for $n = 2$ one finds that the quantomorphism 2-group is the Lie 2-group which integrates the *Poisson bracket Lie 2-algebra* of the underlying 2-plectic geometry as introduced in [Rog11]. In the next section we look at an example for $n = 2$ in more detail and show how it interplays with the above example under transgression.

The above higher prequantum theory becomes a genuine quantum theory after a suitable higher analog of a choice of *polarization*. In particular, for $\mathbf{L} : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ an extended Lagrangian of an n-dimensional quantum field theory as discussed in all our examples here, and for Σ_k any closed manifold, the polarized prequantum states of the transgressed prequantum bundle $\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L})$ should form the $(n-k)$-vector spaces of higher quantum states in codimension k. These states would be assigned to Σ_k by the *extended quantum field theory*, in the sense of [LurieTQFT], obtained from the extended Lagrangian $\mathbf{L}$ by extended geometric quantization. There is an equivalent reformulation of this last step for $n = 1$ given simply by the push-forward of the prequantum line bundle in K-theory (see section 6.8 of [**?**]) and so one would expect that accordingly the last step of higher geometric quantization involves similarly a push-forward of the associated V-fiber ∞-bundles above in some higher generalized cohomology theory. But this remains to be investigated.

1.2.16.4 Example: The anomaly-free gauge coupling of the open string As an example of these general phenomena, we close by briefly indicating how the higher prequantum states of 3d Chern-Simons theory in codimension 2 reproduce the *twisted Chan-Paton gauge bundles* of open string backgrounds, and how their transgression to codimension 1 reproduces the cancellation of the Freed-Witten-Kapustin anomaly of the open string.

By the above, the Wess-Zumino-Witten gerbe $\mathbf{wzw} : G \to \mathbf{B}^2U(1)_{\mathrm{conn}}$ as discussed in section 1.2.15.2 may be regarded as the *prequantum 2-bundle* of Chern-Simons theory in codimension 2 over the circle. Equivalently, if we consider the WZW σ-model for the string on G and take the limiting TQFT case obtained by sending the kinetic term to 0 while keeping only the gauge coupling term in the action, then it is the extended Lagrangian of the string σ-model: its transgression to the mapping space out of a *closed* worldvolume Σ_2 of the string is the topological piece of the exponentiated WZW σ-model action. For Σ_2 with boundary the situation is more interesting, and this we discuss now.

The canonical representation of the 2-group $BU(1)$ is on the complex K-theory spectrum, whose smooth (stacky) refinement is given by $\mathbf{B}U := \varinjlim_n \mathbf{B}U(n)$ in $\mathbf{H}$ (see section 5.4.3 of [?] for more details). On any component for fixed n the action of the smooth 2-group $\mathbf{B}U(1)$ is exhibited by the long homotopy fiber sequence

$$U(1) \longrightarrow U(n) \to \mathrm{PU}(n) \longrightarrow \mathbf{B}U(1) \longrightarrow \mathbf{B}U(n) \longrightarrow \mathbf{B}\mathrm{PU}(n) \xrightarrow{\mathbf{dd}_n} \mathbf{B}^2U(1)$$

in $\mathbf{H}$, in that $\mathbf{dd}_n$ is the universal $(\mathbf{B}U(n))$-fiber 2-bundle which is associated by this action to the universal $(\mathbf{B}U(1))$-2-bundle.[7] Using the general higher representation theory in $\mathbf{H}$ as developed in [NSSa], a local section of the $(\mathbf{B}U(n))$-fiber prequantum 2-bundle which is $\mathbf{dd}_n$-associated to the prequantum 2-bundle $\mathbf{wzw}$, hence a local prequantum 2-state, is, equivalently, a map

$$\mathbf{\Psi} : \mathbf{wzw}|_Q \longrightarrow \mathbf{dd}_n$$

in the slice $\mathbf{H}_{/\mathbf{B}^2U(1)}$, where $\iota_Q : Q \hookrightarrow G$ is some subspace. Equivalently (compare with the general discussion in section **??**), this is a map

$$(\mathbf{\Psi}, \mathbf{wzw}) : \iota_Q \longrightarrow \mathbf{dd}_n$$

in $\mathbf{H}^{(\Delta^1)}$, hence a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} Q & \xrightarrow{\mathbf{\Psi}} & \mathbf{B}\mathrm{PU}(n) \\ {\scriptstyle \iota_Q}\big\downarrow & \swarrow\!\!\swarrow & \big\downarrow {\scriptstyle \mathbf{dd}_n} \\ G & \xrightarrow[\mathbf{wzw}]{} & \mathbf{B}^2U(1)\,. \end{array}$$

One finds (section 5.4.3 of [?]) that this equivalently modulates a unitary bundle on Q which is *twisted* by the restriction of $\mathbf{wzw}$ to Q as in twisted K-theory (such a twisted bundle is also called a *gerbe module* if $\mathbf{wzw}$ is thought of in terms of bundle gerbes [?]). So

$$\mathbf{dd}_n \ \in \mathbf{H}_{/\mathbf{B}^2U(1)}$$

is the moduli stack for twisted rank-n unitary bundles. As with the other moduli stacks before, one finds a differential refinement of this moduli stack, which we write

$$(\mathbf{dd}_n)_{\mathrm{conn}} \ : \ (\mathbf{B}U(n)//\mathbf{B}U(1))_{\mathrm{conn}} \to \mathbf{B}^2U(1)_{\mathrm{conn}}\,,$$

and which modulates twisted unitary bundles with twisted connections (bundle gerbe modules with connection). Hence a differentially refined state is a map $\widehat{\mathbf{\Psi}} \ : \ \mathbf{wzw}|_Q \to (\mathbf{dd}_n)_{\mathrm{conn}}$ in $\mathbf{H}_{/\mathbf{B}^2U(1)_{\mathrm{conn}}}$; and this is

[7] The notion of $(\mathbf{B}U(n))$-fiber 2-bundle is equivalently that of nonabelian $U(n)$-*gerbes* in the original sense of Giraud, see [NSSa]. Notice that for $n = 1$ this is more general than then notion of $U(1)$-bundle gerbe: a G-gerbe has structure 2-group $\mathbf{Aut}(\mathbf{B}G)$, but a $U(1)$-bundle gerbe has structure 2-group only in the left inlcusion of the fiber sequence $\mathbf{B}U(1) \hookrightarrow \mathbf{Aut}(\mathbf{B}U(1)) \to \mathbb{Z}_2$.

precisely a twisted gauge field on a D-brane Q on which open strings in G may end. Hence these are the *prequantum 2-states* of Chern-Simons theory in codimension 2. Precursors of this perspective of Chan-Paton bundles over D-branes as extended prequantum 2-states can be found in [**?**, **?**].

Notice that by the above discussion, together the discussion in section **??**, an equivalence

$$\hat{O} : \; \mathbf{wzw} \xrightarrow{\simeq} \mathbf{wzw}$$

in $\mathbf{H}_{/\mathbf{B}^2 U(1)_{\mathrm{conn}}}$ has two different, but equivalent, important interpretations:

1. it is an element of the *quantomorphism 2-group* (i.e. the possibly non-linear generalization of the Heisenberg 2-group) of 2-prequantum operators;
2. it is a twist automorphism analogous to the generalized diffeomorphisms for the fields in gravity.

Moreover, such a transformation is locally a structure well familiar from the literature on D-branes: it is locally (on some cover) given by a transformation of the B-field of the form $B \mapsto B + d_{\mathrm{dR}} a$ for a local 1-form a (this is the *Hamiltonian 1-form* in the interpretation of this transformation in higher prequantum geometry) and its prequantum operator action on prequantum 2-states, hence on Chan-Paton gauge fields $\widehat{\mathbf{\Psi}} : \; \mathbf{wzw} \longrightarrow (\mathbf{dd}_n)_{\mathrm{conn}}$ (by precomposition) is given by shifting the connection on a twisted Chan-Paton bundle (locally) by this local 1-form a. This local gauge transformation data

$$B \mapsto B + da \; , \qquad\qquad A \mapsto A + a \; ,$$

is familiar from string theory and D-brane gauge theory (see e.g. [**?**]). The 2-prequantum operator action $\Psi \mapsto \hat{O}\Psi$ which we see here is the fully globalized refinement of this transformation.

1.2.16.5 Surface transport and the twisted bundle part of Freed-Witten-Kapustin anomalies. The map $\widehat{\mathbf{\Psi}} : (\iota_Q, \mathbf{wzw}) \to (\mathbf{dd}_n)_{\mathrm{conn}}$ above is the gauge-coupling part of the extended Lagrangian of the *open* string on G in the presence of a D-brane $Q \hookrightarrow G$. We indicate what this means and how it works. Note that for all of the following the target space G and background gauge field $\mathbf{wzw}$ could be replaced by any target space with any circle 2-bundle with connection on it.

The object ι_Q in $\mathbf{H}^{(\Delta^1)}$ is the target space for the open string. The worldvolume of that string is a smooth compact manifold Σ with boundary inclusion $\iota_{\partial\Sigma} : \partial\Sigma \to \Sigma$, also regarded as an object in $\mathbf{H}^{(\Delta^1)}$. A field configuration of the string σ-model is then a map

$$\phi : \iota_\Sigma \to \iota_Q$$

in $\mathbf{H}^{(\Delta^1)}$, hence a diagram

$$\begin{array}{ccc} \partial\Sigma & \longrightarrow & Q \\ {\scriptstyle \iota_{\partial\Sigma}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_Q} \\ \Sigma & \xrightarrow{\phi} & G \end{array}$$

in $\mathbf{H}$, hence a smooth function $\phi : \Sigma \to G$ subject to the constraint that the boundary of Σ lands on the D-brane Q. Postcomposition with the background gauge field $\widehat{\mathbf{\Psi}}$ yields the diagram

$$\begin{array}{ccccc} \partial\Sigma & \longrightarrow & Q & \xrightarrow{\widehat{\mathbf{\Psi}}} & (\mathbf{B}U(n)//U(1))_{\mathrm{conn}} \\ {\scriptstyle \iota_{\partial\Sigma}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_Q} & & \\ \Sigma & \xrightarrow{\phi} & G & \xrightarrow[\mathbf{wzw}]{} & \mathbf{B}^2 U(1)_{\mathrm{conn}} \; . \end{array}$$

Comparison with the situation of Chern-Simons theory with Wilson lines in section 1.2.15.5 shows that the total action functional for the open string should be the product of the fiber integration of the top composite morphism with that of the bottom composite morphisms. Hence that functional is the product of the surface parallel transport of the **wzw** B-field over Σ with the line holonomy of the twisted Chan-Paton bundle over $\partial\Sigma$.

This is indeed again true, but for more subtle reasons this time, since the fiber integrations here are *twisted*. For the surface parallel transport we mentioned this already at the end of section **??**: since Σ has a boundary, parallel transport over Σ does not yield a function on the mapping space out of Σ, but rather a section of the line bundle on the mapping space out of $\partial\Sigma$, pulled back to this larger mapping space.

Furthermore, the connection on a twisted unitary bundle does not quite have a well-defined traced holonomy in $\mathbb{C}$, but rather a well defined traced holonomy up to a coherent twist. More precisely, the transgression of the WZW 2-connection to maps out of the circle as in section **??** fits into a diagram of moduli stacks in $\mathbf{H}$ of the form

$$\begin{array}{ccc} \mathbf{Maps}(S^1, (\mathbf{B}U(n)//\mathbf{B}U(1))_{\mathrm{conn}}) & \xrightarrow{\mathrm{tr\,hol}_{S^1}} & \underline{\mathbb{C}}//\underline{U}(1)_{\mathrm{conn}} \\ \downarrow{\scriptstyle \mathbf{Maps}(S^1,(\mathbf{dd}_n)_{\mathrm{conn}})} & & \downarrow \\ \mathbf{Maps}(S^1, \mathbf{B}^2U(1)_{\mathrm{conn}}) & \xrightarrow{\mathrm{hol}_{S^1}} & \mathbf{B}U(1)_{\mathrm{conn}} \end{array} .$$

This is a transgression-compatibility of the form that we have already seen in section 1.2.15.2.

In summary, we obtain the transgression of the extended Lagrangian of the open string in the background of B-field and Chan-Paton bundles as the following pasting diagram of moduli stacks in $\mathbf{H}$ (all squares are filled with homotopy 2-cells, which are notationally suppressed for readability)

$$\begin{array}{ccccc} \mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma}) & \longrightarrow & \mathbf{Maps}(\Sigma, G) & \xrightarrow{\exp(2\pi i \int_\Sigma [\Sigma, \mathbf{wzw}])} & \underline{\mathbb{C}}//\underline{U}(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{Maps}(\iota_{\partial\Sigma}, G)} & & \downarrow \\ \mathbf{Maps}(S^1, Q) & \xrightarrow{\mathbf{Maps}(S^1, \iota_Q)} & \mathbf{Maps}(S^1, G) & & \downarrow \\ \downarrow{\scriptstyle \mathbf{Maps}(S^1, \widehat{\mathbf{\Psi}})} & & & \searrow{\scriptstyle \mathbf{Maps}(S^1, \mathbf{wzw})} & \downarrow \\ \mathbf{Maps}(S^1, (\mathbf{B}U(n)//\mathbf{B}U(1))_{\mathrm{conn}}) & & \xrightarrow{\mathbf{Maps}(S^1, (\mathbf{dd}_n)_{\mathrm{conn}})} & \mathbf{Maps}(S^1, \mathbf{B}^2U(1)_{\mathrm{conn}}) & \downarrow \\ \downarrow{\scriptstyle \mathrm{tr\,hol}_{S^1}} & & & \searrow{\scriptstyle \mathrm{hol}_{S^1}} & \downarrow \\ \underline{\mathbb{C}}//\underline{U}(1)_{\mathrm{conn}} & & \longrightarrow & & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$$

Here

- the top left square is the homotopy pullback square that computes the mapping stack $\mathbf{Maps}(\iota_{\partial\Sigma}, \iota_Q)$ in $\mathbf{H}^{(\Delta^1)}$, which here is simply the smooth space of string configurations $\Sigma \to G$ which are such that the string boundary lands on the D-brane Q;

- the top right square is the twisted fiber integration of the **wzw** background 2-bundle with connection: this exhibits the parallel transport of the 2-form connection over the worldvolume Σ with boundary S^1 as a section of the pullback of the transgression line bundle on loop space to the space of maps out of Σ;

- the bottom square is the above compatibility between the twisted traced holonomy of twisted unitary bundles and the trangression of their twisting 2-bundles.

The total diagram obtained this way exhibits a difference between two section of a single complex line bundle on $\mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma})$ (at least one of them non-vanishing), hence a map

$$\exp\left(2\pi i \int_{\Sigma} [\Sigma, \mathbf{wzw}]\right) \cdot \mathrm{tr}\,\mathrm{hol}_{S^1}([S^1, \widehat{\mathbf{\Psi}}]) \;:\; \mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma}) \longrightarrow \underline{\mathbb{C}}\,.$$

This is the well-defined action functional of the open string with endpoints on the D-brane $Q \hookrightarrow G$, charged under the background $\mathbf{wzw}$ B-field and under the twisted Chan-Paton gauge bundle $\widehat{\Psi}$.

Unwinding the definitions, one finds that this phenomenon is precisely the twisted-bundle-part, due to Kapustin [Ka99], of the Freed-Witten anomaly cancellation for open strings on D-branes, hence is the Freed-Witten-Kapustin anomaly cancellation mechanism either for the open bosonic string or else for the open type II superstring on Spin^c-branes. Notice how in the traditional discussion the existence of twisted bundles on the D-brane is identified just as *some* construction that happens to cancel the B-field anomaly. Here, in the perspective of extended quantization, we see that this choice follows uniquely from the general theory of extended prequantization, once we recognize that $\mathbf{dd}_n$ above is (the universal associated 2-bundle induced by) the canonical representation of the circle 2-group $\mathbf{B}U(1)$, just as in one codimension up $\mathbb{C}$ is the canonical representation of the circle 1-group $U(1)$.

2 Homotopy type theory

We discuss here aspects of *homotopy type theory*, the theory of locally cartesian closed ∞-*categories* and of ∞-*toposes*, that we need in the following. Much of this is a review of material available in the literature, we just add some facts that we will need and for which we did not find a citation. The reader at least roughly familiar with this theory can skip ahead to our main contribution, the discussion of *cohesive* ∞-*toposes* in 3. We will refer back to these sections here as needed.

2.1 ∞-Categories

The natural joint generalization of the notion of *category* and of *homotopy type* is that of ∞-*category*: a collection of objects, such that between any ordered pair of them there is a homotopy type of morphisms. We briefly survey key definitions and properties in the theory of ∞-categories.

2.1.1 Presentation by simplicial sets

Definition 2.1.1. An ∞-*category* is a simplicial set C such that all horns $\Lambda^i[n] \to C$ that are *inner*, in that $0 < i < n$, have an extension to a simplex $\Delta[n] \to C$.

A vertex $c \in C_0$ is an *object*, an edge $f \in C_1$ is a *morphism* in C.

An ∞-*functor* $f : C \to D$ between ∞-categories C and D is a morphism of the underlying simplicial sets.

This definition is due [Joyal].

Remark 2.1.2. For C an ∞-category, we think of C_0 as its collection of *objects*, and of C_1 as its collection of *morphisms* and generally of C_k as the collection of k-*morphisms*. The inner horn filling property can be seen to encode the existence of composites of k-morphisms, well defined up to coherent $(k+1)$-morphisms. It also implies that for $k > 1$ these k-morphisms are invertible, up to higher morphisms. To emphasize this fact one also says that C is an $(\infty, 1)$-*category*. (More generally an (∞, n)-*category* would have k morphisms for all k such that for $k > n$ these are equivalences.)

The power of the notion of ∞-categories is that it supports the higher analogs of all the crucial facts of ordinary category theory. This is a useful meta-theorem to keep in mind, originally emphasized by André Joyal and Charles Rezk.

Fact 2.1.3. *In general*

- ∞-*Category theory parallels category theory;*
- ∞-*Topos theory parallels topos theory.*

More precisely, essentially all the standard constructions and theorems have their ∞-analogs if only we replace *isomorphism* between objects and equalities between morphisms consistently by *equivalence*s and coherent higher equivalences in an ∞-category.

Proposition 2.1.4. *For C and D two ∞-categories, the internal hom of simplicial sets* $\mathrm{sSet}(C, D) \in \mathrm{sSet}$ *is an ∞-category.*

Definition 2.1.5. We write $\mathrm{Func}(C, D)$ for this ∞-category and speak of the ∞-*category of* ∞-*functors* between C and D.

Remark 2.1.6. The objects of $\mathrm{Func}(C, D)$ are indeed the ∞-functors from def. 2.1.1. The morphisms may be called ∞-*natural transformations*.

Definition 2.1.7. The *opposite* C^{op} of an ∞-category C is the ∞-category corresponding to the opposite of the corresponding sSet-category.

Definition 2.1.8. Let KanCplx $\subset$ sSet be the full subcategory of sSet on the Kan complexes, regarded naturally as an sSet-enriched category, in fact a Kan-complex enriched category. Below in 2.1.2 we recall the *homotopy coherent nerve* construction N_h that sends a Kan-complex enriched category to an ∞-category.

We say that

$$\infty\mathrm{Grpd} := N_h \mathrm{KanCplx}$$

is the ∞-*category of* ∞-*groupoids*.

Definition 2.1.9. For C an ∞-category, we write

$$\mathrm{PSh}_\infty(C) := \mathrm{Func}(C^{\mathrm{op}}, \infty\mathrm{Grpd})$$

and speak of the ∞-*category of* ∞-*presheaves* on C.

The following is the ∞-category theory analog of the Yoneda lemma.

Proposition 2.1.10. *For C an ∞-category, $U \in C$ any object, $j(U) \simeq C(-, U) : C^{\mathrm{op}} \to \infty\mathrm{Grpd}$ an ∞-presheaf represented by U we have for every ∞-presheaf $F \in \mathrm{PSh}_\infty(C)$ a natural equivalence of ∞-groupoids*

$$\mathrm{PSh}_\infty(C)(j(U), F) \simeq F(U) .$$

From this derives a notion of ∞-limits and of adjoint ∞-functors and they satisfy the expected properties. This we discuss below in 2.3.

2.1.2 Presentation by simplicially enriched categories

A convenient way of handling ∞-categories is via sSet-enriched categories: categories which for each ordered pair of objects has not just a set of morphisms, but a simplicial set of morphisms (see [Ke82] for enriched category theory in general and section A of [LuHTT] for sSet-enriched category theory in the context of ∞-category theory in particular):

Proposition 2.1.11. *There exists an adjunction between simplicially enriched categories and simplicial sets*

$$(|-| \dashv N_h) : \mathrm{sSetCat} \underset{N_h}{\overset{|-|}{\rightleftarrows}} \mathrm{sSet}$$

such that

- *if $S \in \mathrm{sSetCat}$ is such that for all objects $X, Y \in S$ the simplicial set $S(X, Y)$ is a Kan complex, then $N_h(S)$ is an ∞-category;*
- *the unit of the adjunction is an equivalence of ∞-categories (see def. 2.1.13 below).*

This is for instance prop. 1.1.5.10 in [LuHTT].

Remark 2.1.12. In particular, for C an ordinary category, regarded as an sSet-category with simplicially constant hom-objects, $N_h C$ is an ∞-category. A functor $C \to D$ is precisely an ∞-functor $N_h C \to N_h D$. In this and similar cases we shall often notationally suppress the N_h-operation. This is justified by the following statements.

Definition 2.1.13. For C an ∞-category, its *homotopy category* $\mathrm{Ho}(C)$ (or Ho_C) is the ordinary category obtained from $|C|$ by taking connected components of all simplicial hom-sets:

$$\mathrm{Ho}_C(X, Y) = \pi_0(|C|(X, Y)) .$$

A morphism $f \in C_1$ is called an *equivalence* if its image in $\mathrm{Ho}(C)$ is an isomorphism. Two objects in C connected by an equivalence are called *equivalent objects*.

Definition 2.1.14. An ∞-functor $F : C \to D$ is called an *equivalence of ∞-categories* if

1. It is *essentially sujective* in that the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(C) \to \mathrm{Ho}(D)$ is essentially surjective;
2. and it is *full and faithful* in that for all objects X, Y the induced morphism $f_{X,Y} : |C|(X, Y) \to |D|(X, Y)$ is a weak homotopy equivalence of simplicial sets.

For C an ∞-category and X, Y two of its objects, we write

$$C(X, Y) := |C|(X, Y)$$

and call this Kan complex the *hom-∞-groupoid* of C from X to Y.

The following assertion guarantees that sSet-categories are indeed a faithful presentation of ∞-categories.

Proposition 2.1.15. *For every ∞-category C the unit of the $(| - | \dashv N_h)$-adjunction from prop. 2.1.11 is an equivalence of ∞-categories*

$$C \xrightarrow{\simeq} N_h|C| .$$

This is for instance theorem 1.1.5.13 together with remark 1.1.5.17 in [LuHTT].

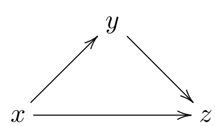
Definition 2.1.16. An ∞-groupoid is an ∞-category in which all morphisms are equivalences.

Proposition 2.1.17. *∞-groupoids in this sense are precisely Kan complexes.*

This is due to [Joyal02]. See also prop. 1.2.5.1 in [LuHTT].

A convenient way of constructing ∞-categories in terms of sSet-categories is via categories with weak equivalences.

Definition 2.1.18. A *category with weak equivalences* (C, W) is a category C equipped with a subcategory $W \subset C$ which contains all objects of C and such that W satsifies the *2-out-of-3 property*: for every commuting triangle



in C with two of the three morphisms in W, also the third one is in W.

Definition 2.1.19. The *simplicial localization* of a category with weak equivalences (C, W) is the sSet-category

$$L_W C \in \mathrm{sSetCat}$$

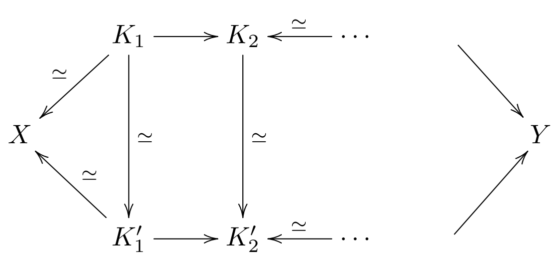
(or LC for short, when W is understood) given as follows: the objects are those of C; and for $X, Y \in C$ two objects, the simplicial hom-set $LC(X, Y)$ is the inductive limit over $n \in \mathbb{N}$ of the nerves of the following categories:

- objects are equivalence classes of zig-zags of length n of morphisms

  $$X \xleftarrow{\simeq} K_1 \longrightarrow K_2 \xleftarrow{\simeq} \cdots \longrightarrow Y$$

  in C, such that the left-pointing morphisms are in W;

- morphisms are equivalence classes of transformations of such zig-zags

  ,

  such that the vertical morphisms are in W;

- subject to the equivalence relation that identifies two such (transformations of) zig-zags if one is obtained from the other by discarding identity morphisms and then composing consecutive morphisms.

This simplicial "hammock localization" is due to [DwKa80a].

Proposition 2.1.20. *Let (C, W) be a category with weak equivalences and LC be its simplicial localization. Then its homotopy category in the sense of def. 2.1.13 is equivalent to the ordinary homotopy category $\mathrm{Ho}(C, W)$ (the category obtained from C by universally inverting the morphisms in W):*

$$\mathrm{Ho} L_W C \simeq \mathrm{Ho}(C, W) \,.$$

A convenient way of controlling simplicial localizations is via $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category structures (see section A.2 of [LuHTT] for a good discussion of all related issues).

Definition 2.1.21. A *model category* is a category with weak equivalences (C, W) that has all limits and colimits and is equipped with two further classes of morphisms, $\mathrm{Fib}, \mathrm{Cof} \subset \mathrm{Mor}(C)$ – the *fibrations* and *cofibrations* – such that $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ are two weak factorization systems on C. Here the elements in $\mathrm{Fib} \cap W$ are called *acyclic fibrations* and those in $\mathrm{Cof} \cap W$ are called *acyclic cofibrations*. An object $X \in C$ is called *cofibrant* if the canonical morphism $\emptyset \to X$ is a cofibration. It is called *fibrant* if the canonical morphism $X \to *$ is a fibration.

A *Quillen adjunction* between two model categories is a pair of adjoint functors between the underlying categories, such that the right adjoint preserves cofibrations and acyclic cofibrations, which equivalently means that the left adjoint preserves cofibrations and acyclic cofibrations.

Remark 2.1.22. The axioms on model categories directly imply that every object is weakly equivalent to a fibrant object, and to a cofibrant objects and in fact to a fibrant and cofibrant objects.

Example 2.1.23. The category of simplicial sets carries a model category structure, here denoted $\mathrm{sSet}_{\mathrm{Quillen}}$, whose weak equivalences are the weak homotopy equivalences, cofibrations are the monomorphisms, and fibrations and the Kan fibrations.

Definition 2.1.24. Let A, B, C be model categories. Then a functor

$$F : A \times B \to C$$

is a *left Quillen bifunctor* if

1. it preserves colimits separately in each argument;
2. for $i : a \to a'$ and $j : b \to b'$ two cofibrations in A and in B, respectively, the canonical induced morphism

   $$F(a', b) \coprod_{F(a,b)} F(a, b') \to F(a', b')$$

   is a cofibration and C and is in addition a weak equivalence if i or j is.

Remark 2.1.25. In particular, for $F : A \times B \to C$ a left Quillen bifunctor, if $a \in A$ is cofibrant then

$$F(a, -) : B \to C$$

is an ordinary left Quillen functor if F is a left Quillen bifunctor, as is

$$F(-, b) : A \to C$$

for b cofibrant.

Definition 2.1.26. A *monoidal model category* is a category equipped both with the structure of a model category and with the structure of a monoidal category, such that the tensor product functor of the monoidal structure is a left Quillen bifunctor, def. 2.1.24, with respect to the model category structure.

Example 2.1.27. The model category $\mathrm{sSet}_{\mathrm{Quillen}}$ is a monoidal model category with respect to its Cartesian monoidal structure.

Definition 2.1.28. For $\mathcal{V}$ a monoidal model category, an *$\mathcal{V}$-enriched model category* is a model category equipped with the structure of an $\mathcal{V}$-enriched category which is also $\mathcal{V}$-tensored and -cotensored, such that the $\mathcal{V}$-tensoring functor is a left Quillen bifunctor, def. 2.1.24.

Remark 2.1.29. An $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category is often called a *simplicial model category.* Notice that, while entirely standard, this use of terminology is imprecise: first, not every simplicial object in categories is a sSet-enriched category, and second, there are other and inequivalent model category structure on sSet that make it a monoidal model category with respect to its Cartesian monoidal structure.

Definition 2.1.30. For C an ($\mathrm{sSet}_{\mathrm{Quillen}}$-enriched) model category write

$$C^\circ \in \mathrm{sSetCat}$$

for the full sSet-subcategory on the fibrant and cofibrant objects.

Proposition 2.1.31. *Let C be an $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category. Then there is an equivalence of ∞-categories*

$$C^\circ \simeq LC\,.$$

This is corollary 4.7 with prop. 4.8 in [DwKa80b].

Proposition 2.1.32. *The hom-∞-groupoids $(N_h C^\circ)(X,Y)$ are already correctly given by the hom-objects in C from a cofibrant to a fibrant representative of the weak equivalence class of X and Y, respectively.*

In this way $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category structures constitute particularly convenient extra structure on a category with weak equivalences for constructing the corresponding ∞-category.

In terms of the presentation of ∞-categories by simplicial categories, 2.1.2, adjoint ∞-functors are presented by *simplicial Quillen adjunctions*, def. 2.1.21, between simplicial model categories: the restriction of a simplicial Quillen adjunction to fibrant-cofibrant objects is the sSet-enriched functor that presents the ∞-derived functor under the model of ∞-categories by simplicially enriched categories.

Proposition 2.1.33. *Let C and D be simplicial model categories and let*

$$(L \dashv R) :\ C \underset{R}{\overset{L}{\leftrightarrows}} D$$

be an sSet*-enriched adjunction whose underlying ordinary adjunction is a Quillen adjunction. Let C° and D° be the ∞-categories presented by C and D (the Kan complex-enriched full* sSet*-subcategories on fibrant-cofibrant objects). Then the Quillen adjunction lifts to a pair of adjoint ∞-functors*

$$(\mathbb{L}L \dashv \mathbb{R}R) :\ C^\circ \leftrightarrows D^\circ$$

On the decategorified level of the homotopy categories these are the total left and right derived functors, respectively, of L and R.

This is [LuHTT], prop 5.2.4.6.

The following proposition states conditions under which a simplicial Quillen adjunction may be detected already from knowing of the right adjoint only that it preserves fibrant objects (instead of all fibrations).

Proposition 2.1.34. *If C and D are simplicial model categories and D is a left proper model category, then for an* sSet*-enriched adjunction*

$$(L \dashv R) :\ C \leftrightarrows D$$

to be a Quillen adjunction it is already sufficient that L preserves cofibrations and R preserves fibrant objects.

This appears as [LuHTT], cor. A.3.7.2.

We will use this for finding simplicial Quillen adjunctions into left Bousfield localizations of left proper model categories: the left Bousfield localization preserves the left properness, and the fibrant objects in the Bousfield localized structure have a good characterization: they are the fibrant objects in the original model structure that are also local objects with respect to the set of morphisms at which one localizes. Therefore for D the left Bousfield localization of a simplicial left proper model category E at a class S of morphisms, for checking the Quillen adjunction property of $(L \dashv R)$ it is sufficient to check that L preserves cofibrations, and that R takes fibrant objects c of C to such fibrant objects of E that have the property that for all $f \in S$ the derived hom-space map $\mathbb{R}\mathrm{Hom}(f, R(c))$ is a weak equivalence.

2.2 ∞-Toposes

The natural context for discussing the geometry of spaces that are locally modeled on test spaces in some category C (and equipped with a notion of coverings) is the category called the *sheaf topos* $\mathrm{Sh}(C)$ over C [John03]. Analogously, the natural context for discussing the *higher* geometry of such spaces is the ∞-category called the ∞-*sheaf topos* $\mathbf{H} = \mathrm{Sh}_\infty(C)$.

The theory of ∞-toposes has been given a general abstract formulation in [LuHTT], using the ∞-category theory introduced by [Joyal] and building on [Re05] and [ToVe02]. One of the central results proven there is that the old homotopy theory of simplicial presheaves, originating around [Br73] and developed notably in [Jard87] and [Dugg01], is indeed a *presentation* of ∞-topos theory.

2.2.1 General abstract

Following [LuHTT], for us "∞-topos" means this:

Definition 2.2.1. An ∞-topos is an accessible ∞-geometric embedding

$$\mathbf{H} \underset{}{\overset{L}{\hookleftarrow \!\!\! \longrightarrow}} \mathrm{Func}(C^{\mathrm{op}}, \infty\mathrm{Grpd})$$

into an ∞-category of ∞-presheaves over some small ∞-category C.

We say this is an ∞-*category of* ∞-*sheaves* (as opposed to a hypercompletion of such) if $\mathbf{H}$ is the reflective localization at the covering sieves of a Grothendieck topology on the homotopy category of C (a *topological localization*), and then write $\mathbf{H} = \mathrm{Sh}_\infty(C)$ with the site structure on C understood.

More intrinsically, ∞-toposes are characterized as follows (we review the ingredients of the following statement in 2.3 and 3.6.7 below).

Definition 2.2.2 (Giraud-Rezk-Lurie axioms)**.** An ∞-*topos* is a presentable ∞-category $\mathbf{H}$ that satisfies the following properties.

1. **Coproducts are disjoint.** For every two objects $A, B \in \mathbf{H}$, the intersection of A and B in their coproduct is the initial object: in other words the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \coprod B \end{array}$$

is a pullback.

2. **Colimits are preserved by pullback.** For all morphisms $f \colon X \to B$ in $\mathbf{H}$ and all small diagrams $A \colon I \to \mathbf{H}_{/B}$, there is an equivalence

$$\varinjlim_i f^* A_i \simeq f^*(\varinjlim_i A_i)$$

between the pullback of the colimit and the colimit over the pullbacks of its components.

3. **Quotient maps are effective epimorphisms.** Every simplicial object $A_\bullet : \Delta^{\mathrm{op}} \to \mathbf{H}$ that satisfies the groupoidal Segal property (Definition 3.6.88) is the Čech nerve of its quotient projection:

$$A_n \simeq A_0 \times_{\varinjlim_n A_n} A_0 \times_{\varinjlim_n A_n} \cdots \times_{\varinjlim_n A_n} A_0 \quad (n \text{ factors})\,.$$

The equivalence of these two definitions is theorem 6.1.0.6 in [LuHTT].

An ordinary topos is famously characterized by the existence of a classifier object for monomorphisms, the *subobject classifier*. With hindsight, this statement already carries in it the seed of the close relation between topos theory and bundle theory, for we may think of a monomorphism $E \hookrightarrow X$ as being a *bundle of (-1)-truncated fibers* over X. The following axiomatizes the existence of arbitrary universal bundles

Definition 2.2.3. An ∞-*topos* $\mathbf{H}$ is a presentable ∞-category with the following properties.

1. **Colimits are preserved by pullback.**
2. **There are universal κ-small bundles.** For every sufficiently large regular cardinal κ, there exists a morphism $\widehat{\mathrm{Obj}}_\kappa \to \mathrm{Obj}_\kappa$ in $\mathbf{H}$ which *represents the core of the κ-small codomain fibration* in that for every object X, there is an equivalence

   $$\mathrm{name} : \mathrm{Core}(\mathbf{H}_{/_\kappa X}) \xrightarrow{\simeq} \mathbf{H}(X, \mathrm{Obj}_\kappa)$$

   between the ∞-groupoid of bundles (morphisms) $E \to X$ which are relatively κ-small over X and the ∞-groupoid of morphisms from X into Obj_κ, such that there are ∞-pullback squares

   $$\begin{array}{ccc} E & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{name}(E)} & \mathrm{Obj}_\kappa \end{array}\,.$$

These two characterizations of ∞-toposes, Definition 2.2.2 and Definition 2.2.3 are equivalent; this is due to Rezk and Lurie, appearing as Theorem 6.1.6.8 in [LuHTT]. We find that the second of these axioms gives the equivalence between V-fiber bundles and $\mathbf{Aut}(V)$-principal ∞-bundles in Proposition 3.6.213.

For $\mathbf{H}$ an ∞-topos we write $\mathbf{H}(X, Y)$ for its hom-∞-groupoid between objects X and Y and write $H(X, Y) = \pi_0 \mathbf{H}(X, Y)$ for the hom-set in the homotopy category.

The theory of cohesive ∞-toposes revolves around situations where the following fact has a refinement:

Proposition 2.2.4. *For every ∞-topos $\mathbf{H}$ there is an essentially unique geometric morphism to the ∞-topos* $\infty\mathrm{Grpd}$.

$$(\Delta \dashv \Gamma) : \mathbf{H} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \infty\mathrm{Grpd}$$

This is prop 6.3.41 in [LuHTT].

Proposition 2.2.5. *Here Γ forms global sections, in that $\Gamma(-) \simeq \mathbf{H}(*, -)$, and Δ forms constant ∞-sheaves – $\Delta(-) \simeq L\mathrm{Const}(-)$.*

Proof. By prop. 2.2.4 it is sufficient to exhibit an ∞-adjunction $(L\mathrm{Const}(-) \dashv \mathbf{H}(*, -))$ such that the left adjoint preserves finite ∞-limits. The latter follows since $\mathrm{Const} : \infty\mathrm{Grpd} \to \mathrm{PSh}_\infty(C)$ preserves all limits (for C some ∞-site of definition for $\mathbf{H}$) and $L : \mathrm{PSh}(C) \to \mathbf{H}$ by definition preserves finite ∞-limits.

To show the ∞-adjunction we use prop. 2.3.1, which says that every ∞-groupoid is the ∞-colimit over itself of the ∞-functor constant on the point: $S \simeq \lim_{\longrightarrow S} *$. From this we obtain the natural hom-equivalence

$$\begin{aligned}
\mathbf{H}(L\mathrm{Const}S, X) &\simeq \mathrm{PSh}_C(\mathrm{Const}S, X) \\
&\simeq \mathrm{PSh}(\mathrm{Const}\lim_{\longrightarrow S} *, X) \\
&\simeq \lim_{\longleftarrow S} \mathrm{Psh}(\mathrm{Const}*, X) \\
&\simeq \lim_{\longleftarrow S} \mathbf{H}(L\mathrm{Const}*, X) \\
&\simeq \lim_{\longleftarrow S} \mathbf{H}(*, X) \\
&\simeq \lim_{\longleftarrow S} \infty\mathrm{Grpd}(*, \mathbf{H}(*, X)) \\
&\simeq \infty\mathrm{Grpd}(\lim_{\longrightarrow S} *, \mathbf{H}(*, X)) \\
&\simeq \infty\mathrm{Grpd}(S, \mathbf{H}(*, X))\,.
\end{aligned}$$

Here and in the following "$*$" always denotes the terminal object in the corresponding ∞-category. We used that LConst preserves the terminal object (the empty ∞-limit.) □

2.2.2 The syntax of homotopy type theory

After ordinary toposes had been introduced by Grothendieck in the study of geometry, a central step in abstract topos theory was the later observation by Lawvere and Tierney, Joyal and Kripke, that toposes $\mathcal{T}$ have a rich *internal logic* obtained by regarding subobjects $\phi \hookrightarrow X$ in the toposes – hence (-1)-truncated objects in its slices $\mathcal{T}_{/X}$ – as *propositions* $\phi(t)$ about *terms* t of *type* X. In *type theory* as developed by Martin-Löf and others, the foundations of mathematics are entirely formalized by means of such a general notion of "types" (as opposed to "sets"). The flavor of the theory known as *extensive dependent type theory* can be shown to be precisely the *syntax* of which locally cartesian closed categories (such as toposes) are the *semantics*: a*judgement* in the logical theory is interpreted as a *morphism* in the category, and universal constructions in the category (such as pullback) correspond to *variable substitution* and *quantifiers* in the type theory.

This *internal logic* of toposes is famously *intuitionistic* (more general than classical logic in that it does not enforce the axiom of excluded middle). But moreover, the development of type theory was motivated by *constructive mathematics*: here a *proposition* ϕ about terms t of type X is regarded as true only if a *proof* of it can be "explicitly constructed" in the form of exhibiting a term of the type underlying the proposition, which in the categorical interpretation means the construction of a generalized element of ϕ regarded as an object in the slice $\mathcal{T}_{/X}$.

However, the extensive dependent type theory equivalent to locally cartesian closed categories fails to be *constructive* in this sense in one important aspect: it has an axiom that enforces that any two proofs, on this sense, of a proposition of the form "$t_1 = t_2$" (for two terms t_1, t_2 of some type X) are necessarily equal. This axiom is necessary to make the theory equivalent to the internal logical of locally cartesian closed categories. It is however non-constrcutive in that the truth of a statement (the equality of two proofs of the equality of two terms) is assumed without *proof*.

For this reason constructive type theorist tended and tend to discard this axiom and consider instead the *simpler* and formally much better behave type theory without that axiom. For decades this conceptually satisfying move seemed to come at the cost that the resulting theory, known as *intensional dependent type theory*, had no categorical semantics. While it had a nice syntax, it was unclear what intensional type theory *means*.

But with the hindsight of *higher* category theory this open question finally found its solution: the possible non-uniqueness of equivalences between two elements is precisely the characteristic property of *groupoids*, and hence of higher groupoids. Therefore the *constructive* notion of identity in type theory axiomatizes not classical identity as in classical set theory, but axiomatizes *equivalence* in higher category theory. This insight was finally obtained by Vladimir Voevodsky. One then shows that *intensional* dependent type theory no longer corresponds to just locally cartesian closed categories, but instead to locally cartesian closed ∞-categories: a *type* in intensional type theory is interpreted as a *homotopy type*, an object of an ∞-category.

But the convergence of formal logic and higher category theory goes further, even. Type theory is most natural when assuming a *type of types* denoted Type, for that allows to unify the *judgements* of the form "t is a term of type X" with the more basic judgement that "X is a type" in the first place. The formal symbols used for these judgements are

$$\vdash t : X$$

and

$$\vdash X : \mathrm{Type}\,.$$

More generally, *dependent* type theory owes its name to the fact that a type $E(x)$ is allowed to be parameterized over terms x of another type X, which is written as the *sequent*

$$x : X \vdash E(x) : \mathrm{Type}\,.$$

While intensive depdent type theory without such a type of types is interpreted in locally cartesian closed ∞-categories, the theory with such a type of types turns out to have an interpretation in ∞-toposes: the type of types is interpreted as the object classifier of def. 2.2.3: the judgement "$x : X \vdash E(x) : \mathrm{Type}$" corresponds to the *name* morphism $X \stackrel{\mathrm{name}(E)}{\longrightarrow} \mathrm{Obj}_\kappa$ of a morphism $E \to X$ in the ∞-topos, according to def. 2.2.3. If here we declare to abbreviate $(\vdash E) := \mathrm{name}(E)$ then this means we have the following disctionary between the symbols used to talk about objects of slices in ∞-toposes and equivalently dependent types in homotopy type theory.

morphisms to sequents:

notation in\ for	objects/types				elements/terms			
∞-topos theory	X	$\stackrel{\vdash E}{\longrightarrow}$		Obj_κ	X	$\stackrel{t}{\longrightarrow}_X$		E
homotopy type theory	$x : X$	$\vdash E(x)$	$:$	Type	$x : X$	$\vdash t(x)$	$:$	$E(x)$

2.2.3 Presentation by simplicial (pre-)sheaves

For computations it is useful to employ a generators-and-relations presentation of presentable ∞-categories in general and of ∞-toposes in particular, given by ordinary sSet-enriched categories equipped with the structure of combinatorial simplicial model categories. These may be obtained by left Bousfield localization of a model structure on simplicial presheaves (as reviewed in appendix 2 and 3 of [LuHTT]).

We discuss these presentations and then discuss various constructions in terms of these presentations that will be useful over and over again in the following. Much of this material is standard and our discussion serves to briefly collect the relevant pieces. But we also highlight a few points that are not usually discussed explicitly in the literature, but which we will need later on.

Definition 2.2.6. Let C be a small category.

- Write $[C^{\mathrm{op}}, \mathrm{sSet}]$ for the category of functors $C^{\mathrm{op}} \to \mathrm{sSet}$ to the category of simplicial sets. This is naturally equivalent to the category $[\Delta^{\mathrm{op}}, [C^{\mathrm{op}}, \mathrm{Set}$ of simplicial objects in the category of presheaves on C. Therefore one speaks of the *category of simplicial presheaves* over C.
- For $\{U_i \to U\}$ a covering family in the site C, write

$$C(\{U_i\}) \in [C^{\mathrm{op}}, \mathrm{sSet}] := \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} j(U_{i_0}) \times_{j(U)} \cdots \times_{j(U)} j(U_{i_k})$$

  for the corresponding *Čech nerve* simplicial presheaf. This is in degree k the disjoint union of the $(k+1)$-fold intersections of patches of the cover. It is canonically equipped with a morphism $C(\{U_i\}) \to j(U)$. (Here $j : C \to [C^{\mathrm{op}}, \mathrm{Set}]$ is the Yoneda embedding.)

- The category $[C^{\mathrm{op}}, \mathrm{sSet}]$ is naturally an sSet-enriched category. For any two objects $X, A \in [C^{\mathrm{op}}, \mathrm{sSet}]$ write $\mathrm{Maps}(X, A) \in \mathrm{sSet}$ for the simplicial hom-set.
- Write $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ for the category of simplicial presheaves equipped with the following choices of classes of morphisms (which are natural transformations between sSet-valued functors):
  - the *fibrations* are those morphisms whose component over each object $U \in C$ is a Kan fibration of simplicial sets;
  - the *weak equivalences* are those morphisms whose component over each object is a weak equivalence in the Quillen model structure on simplicial sets;
  - the *cofibrations* are the morphisms having the right lifting property against th morphisms that are both fibrations as well as weak equivalences.

  This makes $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ into a combinatorial simplicial model category.
- Write $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ for model category structure on simplicial presheaves which is the left Bousfield localization of $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ at the set of morphisms of the form $C(\{U_i\}) \to U$ for all covering families $\{U_i \to U\}$ of C.

  This is called the *projective* local model structure on simplicial presheaves [Dugg01].

Definition 2.2.7. The operation of forming objectwise simplicial homotopy groups extends to functors

$$\pi_0^{\mathrm{PSh}} : [C^{\mathrm{op}}, \mathrm{sSet}] \to [C^{\mathrm{op}}, \mathrm{Set}]$$

and for $n > 1$

$$\pi_n^{\mathrm{PSh}} : [C^{\mathrm{op}}, \mathrm{sSet}]_* \to [C^{\mathrm{op}}, \mathrm{Set}] \,.$$

These presheaves of homotopy groups may be sheafified. We write

$$\pi_0 : [C^{\mathrm{op}}, \mathrm{sSet}] \stackrel{\pi_0^{\mathrm{PSh}}}{\to} [C^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Sh}(C)$$

and for $n > 1$

$$\pi_n : [C^{\mathrm{op}}, \mathrm{sSet}]_* \stackrel{\pi_n^{\mathrm{PSh}}}{\to} [C^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Sh}(C) \,.$$

Proposition 2.2.8. *For $X \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ fibrant, the homotopy sheaves $\pi_n(X)$ from def. 2.2.7 coincide with the abstractly defined homotopy groups of $X \in \mathrm{Sh}_\infty(C)$ from [LuHTT].*

Proof. One may observe that the $\mathrm{sSet}_{\mathrm{Quillen}}$-powering of $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ does model the abstract ∞Grpd-powering of $\mathrm{Sh}_\infty(C)$. □

Definition 2.2.9. A site C has *enough points* if a morphism $(A \stackrel{f}{\to} B) \in \mathrm{Sh}(C)$ in its sheaf topos is an isomorphism precisely if for every *topos point*, hence for every geometric morphism

$$(x^* \dashv x_*) : \mathrm{Set} \underset{x_*}{\overset{x^*}{\rightleftarrows}} \mathrm{Sh}(C)$$

from the topos of sets we have that $x^*(f) : x^*A \to x^*B$ is an isomorphism.

Notice here that, by definition of geometric morphism, the functor i^* is left adjoint to i_* – hence preserves all colimits – and in addition preserves all *finite* limits.

Example 2.2.10. The following sites have enough points.

- The categories Mfd (SmoothMfd) of (smooth) finite-dimensional, paracompact manifolds and smooth functions between them;

- the category CartSp of Cartesian spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ and continuous (smooth) functions between them.

This is discussed in detail below in 4.3.1. We restrict from now on attention to this case.

Assumption 2.2.11. The site C has enough points.

Theorem 2.2.12. *For C a site with enough points, the weak equivalences in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ are precisely the stalkwise weak equivalences in* $\mathrm{sSet}_{\mathrm{Quillen}}$

Proof. By theorem 17 in [Ja96] and using our assumption 2.2.11 the statement is true for the local injective model structure. The weak equivalences there coincide with those of the local projective model structure. □

Definition 2.2.13. We say that a morphism $f : A \to B$ in $[C^{\mathrm{op}}, \mathrm{sSet}]$ is a *local fibration* or a *local weak equivalence* precisely if for all topos points x the morphism $x^* f : x^* A \to x^* B$ is a fibration of weak equivalence, respectively.

Warning. While by theorem 2.2.12 the local weak equivalences are indeed the weak equivalences in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, it is not true that the fibrations in this model structure are the local fibrations of def. 2.2.13.

Proposition 2.2.14. *Pullbacks in $[C^{\mathrm{op}}, \mathrm{sSet}]$ along local fibrations preserve local weak equivalences.*

Proof. Let

$$\begin{array}{ccccc} A & \longrightarrow & C & \longleftarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & C' & \longleftarrow & B' \end{array}$$

be a diagram where the vertical morphisms are local weak equivalences. Since the inverse image x^* of a topos point x preserves finite limits and in particular pullbacks, we have

$$x^*(A \times_C B \stackrel{f}{\to} A' \times_{C'} B') = (x^* A \times_{x^* C} x^* B \stackrel{x^* f}{\to} x^* A' \times_{x^* C'} x^* B')\,.$$

On the right the pullbacks are now by assumption pullbacks of simplicial sets along Kan fibrations. Since $\mathrm{sSet}_{\mathrm{Quillen}}$ is right proper, these are homotopy pullbacks and therefore preserve weak equivalences. So $x^* f$ is a weak equivalence for all x and thus f is a local weak equivalence. □

The following characterization of ∞-toposes is one of the central statements of [LuHTT]. For the purposes of our discussion here the reader can take this to be the *definition* of ∞-toposes.

Theorem 2.2.15. *For C a site with enough points, the ∞-topos over C is the simplicial localization, def. 2.1.19,*

$$\mathrm{Sh}_\infty(C) \simeq L([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

of the category of simplicial presheaves on C at the local weak equivalences.

In view of prop. 2.2.17 this is prop. 6.5.2.14 in [LuHTT].

2.2.4 Presentation by simplicial objects in the site

We will have use of the following different presentation of $\mathrm{Sh}_\infty(C)$.

Definition 2.2.16. Let C be a small site with enough points. Write $\bar{C} \subset [C^{\mathrm{op}}, \mathrm{sSet}]$ for the free coproduct completion.

Let $(\bar{C}^{\Delta^{\mathrm{op}}}, W)$ be the category of simplicial objects in $\bar{C}$ equipped with the stalkwise weak equivalences inherited from the canonical embedding

$$i : \bar{C}^{\Delta^{\mathrm{op}}} \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]\,.$$

Proposition 2.2.17. *The induced ∞-functor*

$$N_h L \bar{C}^{\Delta^{\mathrm{op}}} \to N_h L[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

is an equivalence of ∞-categories.

This is due to [NSSb]. We prove this after noticing the following fact.

Proposition 2.2.18. *Let C be a category and $\bar{C}$ its free coproduct completion.*

Every simplicial presheaf over C is equivalent in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ to a simplicial object in $\bar{C}$ (after the degreewise Yoneda embedding $j^{\Delta^{\mathrm{op}}} : \bar{C}^{\Delta^{\mathrm{op}}} \to [C^{\mathrm{op}}, \mathrm{sSet}]$).

If moreover C has pullbacks and sequential colimits, then the simplicial object in $\bar{C}$ can be taken to be globally Kan, hence fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. The first statement is prop. 2.8 in [Dugg01], which says that for every $X \in [C^{\mathrm{op}}, \mathrm{sSet}]$ the canonical morphism from the simplicial presheaf

$$(QX) : [k] \mapsto \coprod_{U_0 \to \cdots \to U_k \to X_k} j(U_0)\,,$$

where the coproduct runs over all sequences of morphisms between representables U_i as indicated and using the evident face and degeneracy maps, is a global weak equivalence

$$QX \stackrel{\simeq}{\to} X\,.$$

The second statement follows by postcomposing with Kan's fibrant replacement functor (see for instance section 3 in [Jard87])

$$\mathrm{Ex}^\infty : \mathrm{sSet} \to \mathrm{KanCplx} \hookrightarrow \mathrm{sSet}\,.$$

This functor forms new simplices by subdivision, which only involves forming iterated pullbacks over the spaces of the original simplices. □

Example 2.2.19. Let C be a category of *connected* topological spaces with given extra structure and properties (for instance smooth manifolds). Then $\bar{C}$ is the category of all such spaces (with arbitrary many connected components).

Then the statement is that every ∞-stack over C has a presentation by a simplicial object in $\bar{C}$. This is true with respect to any Grothendieck topology on C, since the weak equivalences in the global projective model structure that prop. 2.2.18 refers to remain weak equivalences in any left Bousfield localization.

If moreover C has all pullbacks (for instance for connected topological spaces, but not for smooth manifolds) then every ∞-stack over C even has a presentation by a globally Kan simplicial object in $\bar{C}$.

Proof of theorem 2.2.17. Let $Q : [C^{\mathrm{op}}, \mathrm{sSet}] \to \bar{C}^{\Delta^{\mathrm{op}}}$ be Dugger's replacement functor from the proof of prop. 2.2.18. In [Dugg01] it is shown that for all X the simplicial presheaf QX is cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and that the natural morphism $QX \to X$ is a weak equivalence. Since left Bousfield localization does not affect the cofibrations and only enlarges the weak equivalences, the same is still true in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Therefore we have a natural transformation

$$i \circ Q \to \mathrm{Id} : [C^{\mathrm{op}}, \mathrm{sSet}] \to [C^{\mathrm{op}}, \mathrm{sSet}]$$

whose components are weak equivalences. From this the claim follows by prop. 3.5 in [DwKa80a]. □

Remark 2.2.20. If the site C is moreover equipped with the structure of a *geometry* as in [Lurie09a] then there is canonically the notion of a C-*manifold*: a sheaf on C that is *locally* isomorphic to a representable in C. Write

$$\bar{C} \hookrightarrow C\mathrm{Mfd} \hookrightarrow [C^{\mathrm{op}}, \mathrm{Set}]$$

for the full subcategory of presheaves on the C-manifolds.

Then the above argument applies verbatim also to the category $C\mathrm{Mfd}^{\Delta^{\mathrm{op}}}$ of simplicial C-manifolds. Therefore we find that the ∞-topos over C is presented by the simplicial localization of simplicial C-manifolds at the stalkwise weak equivalences:

$$\mathrm{Sh}_\infty(C) \simeq N_h L C\mathrm{Mfd}^{\Delta^{\mathrm{op}}} .$$

Example 2.2.21. Let $C = \mathrm{CartSp}_{\mathrm{smooth}}$ be the full subcategory of the category SmthMfd of smooth manifolds on the Cartesian spaces, $\mathbb{R}^n$, for $n \in \mathbb{R}$. Then $\bar{C} \subset \mathrm{SmthMfd}$ is the full subcatgory on manifolds that are disjoint unions of Cartesian spaces and $C\mathrm{Mfd} \simeq \mathrm{SmthMfd}$. Therefore we have an equivalence of ∞-categories

$$\mathrm{Sh}_\infty(\mathrm{SmthMfd}) \simeq \mathrm{Sh}_\infty(\mathrm{CartSp}) \simeq L\ \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}} .$$

2.2.5 ∞-Sheaves and descent

We discuss some details of the notion of ∞-*sheaves* from the point of view of the presentations discussed above in 2.2.3.

By def. 2.2.1 we have, abstractly, that an ∞-sheaf over some site C is an ∞-presheaf that is in the essential image of a given reflective inclusion $\mathrm{Sh}_\infty(C) \hookrightarrow \mathrm{PSh}_\infty(C)$. By prop. 2.2.15 this reflective embedding is presented by the Quillen adjunction that exhibits the left Bousfield localization of the model category of simplicial presheaves at the Čech covers

$$\begin{array}{ccc} ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ & \underset{\mathbb{R}\mathrm{Id}}{\overset{\mathbb{L}\mathrm{Id}}{\rightleftarrows}} & ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}})^\circ \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Sh}_\infty(C) & \underset{}{\overset{L}{\hookleftarrow}} & \mathrm{PSh}_\infty(X) \end{array} .$$

Since the Quillen adjunction that exhibits left Bousfield localization is given by identity-1-functors, as indicated, the computation of ∞-sheafification (∞-stackification) L by deriving the left Quillen functor is all in the cofibrant replacement in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ followed by fibrant replacement in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Since the collection of cofibrations is preserved by left Bousfield localization, this simply amounts to cofibrant-fibrant replacement in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Since, finally, the derived hom space $\mathrm{Sh}_\infty(U, A)$ is computed in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ already on a fibrant resolution of A out of a cofibrant resolution of U, and since every representable is necessarily cofibrant, one may effectively identify the ∞-sheaf condition in $\mathrm{PSh}_\infty(C)$ with the fibrancy condition in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

We discuss aspects of this fibrancy condition.

Definition 2.2.22. For C a site, we say a covering family $\{U_i \to U\}$ is a *good cover* if the corresponding Čech nerve

$$C(U_i) := \int^{[k] \in \Delta} \coprod_{i_0, \cdots, i_k} j(U_{i_0}) \times_{j(U)} \cdots \times_{j(U)} j(U_k) \quad \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$$

(where $j : C \to [C^{\mathrm{op}}, \mathrm{sSet}]$ is the Yoneda embedding) is degreewise a coproduct of representables, hence if all non-empty finite intersections of the U_i are again representable:

$$j(U_{i_0, \cdots, i_k}) = U_{i_0} \times_U \cdots \times_U U_{i_k} .$$

Proposition 2.2.23. *The Čech nerve $C(U_i)$ of a good cover is cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ as well as in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. In the terminology of [DuHoIs04] the good-ness condition on a cover makes its Čech nerve a *split hypercover*. By the result of [Dugg01] this is cofirant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Since left Bousfield localization preserves cofibrations, it is also cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. ☐

Definition 2.2.24. For A a simplicial presheaf with values in Kan complexes and $\{U_i \to U\}$ a good cover in the site C, we say that

$$\mathrm{Desc}(\{U_i\}, A) := [C^{\mathrm{op}}, \mathrm{sSet}](C(U_i), A)\,,$$

where on the right we have the sSet-enriched hom of simplicial presheaves, is the *descent object* of A over $\{U_i \to U\}$.

Remark 2.2.25. By assumption A is fibrant and $C(U_i)$ is cofibrant (by prop. 2.2.23) in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Since this is a simplicial model category, it follows that $\mathrm{Desc}(\{U_i\}, A)$ is a Kan complex, an ∞-groupoid. We may also speak of the *descent ∞-groupoid*. Below we show that its objects have the interpretation of *gluing data* or *descent data* for A. See [DuHoIs04] for more details.

Proposition 2.2.26. *For C a site whose topology is generated from good covers, a simplicial presheaf A is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ precisely if it takes values in Kan complexes and if for each generating good cover $\{U_i \to U\}$ the canonical morphism*

$$A(U) \to \mathrm{Desc}(\{U_i\}, A)$$

is a weak equivalence of Kan complexes.

Proof. By standard results recalled in A.3.7 of [LuHTT] the fibrant objects in the local model structure are precisely those which are fibrant in the global model structure and which are *local* with respect to the morphisms at which one localizes: such that the derived hom out of these morphisms into the given object produces a weak equivalence.

By prop. 2.2.23 we have that $C(U_i)$ is cofibrant for $\{U_i \to U\}$ a good cover. Therefore the derived hom is computed already by the enriched hom as in the above statement. □

Remark 2.2.27. The above condition manifestly generalizes the *sheaf* condition on an ordinary sheaf [John03]. One finds that

$$(\pi_0^{\mathrm{PSh}}(C(U_i)) \to \pi_0^{\mathrm{PSh}}(U)) = (S(U_i) \hookrightarrow U)$$

is the (subfunctor corresponding to the) *sieve* associated with the cover $\{U_i \to U\}$. Therefore when A is itself just a presheaf of sets (of simplicially constant simplicial sets) the above condition reduces to the statement that

$$A(U) \to [C^{\mathrm{op}}, \mathrm{Set}](S(U_i), A)$$

is an isomorphism. This is the standard sheaf condition.

We discuss the descent object, def. 2.2.24, in more detail.

Definition 2.2.28. Write

$$\mathrm{coDesc}(\{U_i\}, A) \in \mathrm{sSet}^{\Delta}$$

for the cosimlicial simplicial set that in degree k is given by the value of A on the k-fold intersections:

$$\mathrm{coDesc}(\{U_i\}, A)_k = \prod_{i_0, \cdots, i_k} A(U_{i_0, \cdots, i_k})\,.$$

Proposition 2.2.29. *The descent object from def. 2.2.24 is the* totalization *of the codescent object:*

$$\begin{aligned} \mathrm{Desc}(\{U_i\}, A) &= \mathrm{tot}(\mathrm{coDesc}(\{U_i\}), A) \\ &:= \int_{[k] \in \Delta} \mathrm{sSet}(\Delta[k], \mathrm{coDesc}(\{U_i\}, A)_k) \end{aligned}$$

Here and in the following equality signs denote isomorphism (such as to distinguish from just weak equivalences of simplicial sets).
Proof. Using sSet-enriched category calculus for the sSet-enriched and sSet-tensored category of simplicial presheaves (for instance [Ke82] around (3.67)) we compute as follow

$$
\begin{aligned}
\mathrm{Desc}(\{U_i\}, A) &:= [C^{\mathrm{op}}, \mathrm{sSet}](C(U_i), A) \\
&= [C^{\mathrm{op}}, \mathrm{sSet}]\Big(\int^{[k]\in\Delta} \Delta[k] \cdot C(U_i)_k, A\Big) \\
&= \int_{[k]\in\Delta} [C^{\mathrm{op}}, \mathrm{sSet}](\Delta[k] \cdot C(U_i), A) \\
&= \int_{[k\in\Delta]} \mathrm{sSet}(\Delta[k], [C^{\mathrm{op}}, \mathrm{sSet}](C(U_i)_k), A) \\
&= \int_{[k\in\Delta]} \mathrm{sSet}(\Delta[k], A(C(U_i)_k)) \\
&= \mathrm{tot}(A(C(U_i)_\bullet)) \\
&= \mathrm{tot}\left(\mathrm{coDesc}(\{C(U_i)\}, A)\right) .
\end{aligned}
$$

Here we used in the first step that every simplicial set Y (hence every simplicial presheaf) is the realization of itself, in that

$$
Y = \int^{[k]\in\Delta} \Delta[k] \cdot Y_k \,,
$$

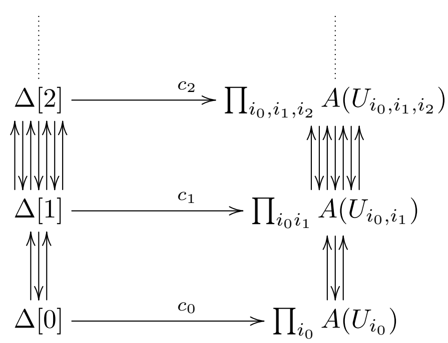
which is effectively a variant of the Yoneda-lemma. □

Remark 2.2.30. This provides a fairly explicit description of the objects in $\mathrm{Desc}(\{U_i\}, A)$ by what is called *nonabelian Čech hypercohomology*.

Notice that an element c of the end $\int_{[k]\in\Delta} \mathrm{sSet}(\Delta[k], \mathrm{coDesc}(\{U_i\}, A))$ is by definition of *end*s a collection of morphisms

$$
\{c_k : \Delta[k] \to \prod_{i_0,\cdots,i_k} A_k(U_{i_0,\cdots,i_k})\}
$$

that makes commuting all parallel diagrams in the following:

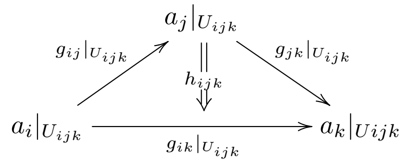


This says in words that c is

1. a collection of objects $a_i \in A(U_i)$ on each patch;
2. a collection of morphisms $\{g_{ij} \in A_1(U_{ij})\}$ over each double intersection, such that these go between the restrictions of the objects a_i and a_j, respectively

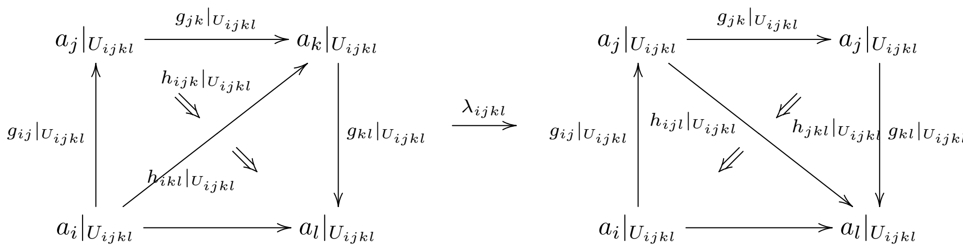
$$
a_i|_{U_{ij}} \xrightarrow{g_{ij}} a_j|_{U_{ij}}
$$

3. a collection of 2-morphisms $\{h_{ijk} \in A_2(U_{ijk})\}$ over triple intersections, which go between the corresponding 1-morphisms:



,

4. a collection of 3-morphisms $\{\lambda_{ijkl} \in A_3(U_{ijkl})\}$ of the form



;

5. and so on.

This recovers the cocycle diagrams that we have discussed more informally in 1.2.5 and generalizes them to arbitrary coefficient objects A.

2.2.6 ∞-Sheaves with values in chain complexes

Many simplicial presheaves appearing in practice are (equivalent to) objects in sub-∞-categories of $\mathrm{Sh}_\infty(C)$ of ∞-sheaves with values in abelian or at least in "strict" ∞-groupoids. These subcategories typically offer convenient and desireable contexts for formulating and proving statements about special cases of general simplicial presheaves.

One well-known such notion is given by the *Dold-Kan correspondence* (discussed for instance in [GoJa99]). This identifies chain complexes of abelian groups with strict and strictly symmetric monoidal ∞-groupoids.

Proposition 2.2.31. *Let* $\mathrm{Ch}^+_{\mathrm{proj}}$ *be the standard projective model structure on chain complexes of abelian groups in non-negative degree and let* $\mathrm{sAb}_{\mathrm{proj}}$ *be the standard projective model structure on simplicial abelian groups. Let* C *be any small category. There is a composite Quillen adjunction*

$$((N_\bullet F)_* \dashv \Xi) : \; [C^{\mathrm{op}}, Ch^+_{\mathrm{proj}}]_{\mathrm{proj}} \underset{\Gamma_*}{\overset{(N_\bullet)_*}{\underset{\simeq}{\rightleftarrows}}} [C^{\mathrm{op}}, \mathrm{sAb}_{\mathrm{proj}}]_{\mathrm{proj}} \underset{U_*}{\overset{F_*}{\leftrightarrows}} [C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}} \;,$$

where the first is given by postcomposition with the Dold-Puppe-Kan correspondence and the second by postcomposition with the degreewise free-forgetful adjunction for abelian groups over sets.

We also write $\mathrm{DK} := \Xi$ for this Dold-Kan map. Dropping the condition on symmetric monoidalness we obtain a more general such inclusion, a kind of non-abelian Dold-Kan correspondence: the identification of *crossed complexes*, def. 1.2.60, with strict ∞-groupoids (see [BrHiSi11][Por] for details).

Definition 2.2.32. A *globular set* X is a collection of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with functions $\{s_n, t_n : X_{n+1} \to X_n\}_{n\in\mathbb{N}}$ such that $\forall_{n\in\mathbb{N}}(s_n \circ s_{n+1} = s_n \circ t_{n+1})$ and $\forall_{n\in\mathbb{N}}(t_n \circ s_{n+1} = t_n \circ t_{n+1})$. (These relations ensure that for every pair $k_1 < k_2 \in \mathbb{N}$ there are uniquely defined functions $s, t : X_{k_2} \to X_{k_1}$.) A *strict* ∞*-groupoid* is a globular set $X_\bullet$ equipped for each $k \geq 1$ with the structure of a groupoid on $X_k \underset{t}{\overset{s}{\rightrightarrows}} X_0$ such that for all $k_1 < k_2 \in \mathbb{N}$ this induces the structure of a strict 2-groupoid on

$$X_{k_2} \underset{t}{\overset{s}{\rightrightarrows}} X_{k_1} \underset{t}{\overset{s}{\rightrightarrows}} X_0 \;.$$

Remark 2.2.33. We have a sequence of (non-full) inclusions

$$\begin{array}{ccccc} \mathrm{ChainComplex} & \longrightarrow & \mathrm{CrossedComplex} & \longrightarrow & \mathrm{KanComplex} \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{StrAbStr}\infty\mathrm{Grpd} & \longrightarrow & \mathrm{Str}\infty\mathrm{Grpd} & \longrightarrow & \infty\mathrm{Grpd} \end{array}$$

of strict ∞-groupoids into all ∞-groupoids, where in the top row we list the explicit presentation and in the bottom row the abstract notions.

We state a useful theorem for the computation of descent for presheaves, prop. 2.2.26, with values in strict ∞-groupoids.

Suppose that $\mathcal{A} : C^{\mathrm{op}} \to \mathrm{Str}\infty\mathrm{Grpd}$ is a presheaf with values in strict ∞-groupoids. In the context of strict ∞-groupoids the standard n-simplex is given by the nth *oriental* $O(n)$ [Stre04]. This allows to perform a construction that looks like a descent object in Str∞Grpd:

Definition 2.2.34 (Street 04)**.** The descent object for $\mathcal{A} \in [C^{\mathrm{op}}, \mathrm{Str}\infty\mathrm{Grpd}]$ relative to $Y \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is

$$\mathrm{Desc}_{\mathrm{Street}}(Y, \mathcal{A}) := \int_{[n]\in\Delta} \mathrm{Str}\infty\mathrm{Cat}(O(n), \mathcal{A}(Y_n)) \ \in \mathrm{Str}\infty\mathrm{Grpd}\,,$$

where the end is taken in Str∞Grpd.

This object had been suggested by Ross Street to be the right descent object for strict ∞-category-valued presheaves in [Stre04].

Canonically induced by the orientals is the ω-nerve

$$N : \mathrm{Str}\omega\mathrm{Cat} \to \mathrm{sSet}$$

Applying this to the descent object of prop. 2.2.34 yields the simplicial set $N\mathrm{Desc}(Y, \mathcal{A})$. On the other hand, applying the ω-nerve componentwise to $\mathcal{A}$ yields a simplicial presheaf $N\mathcal{A}$ to which the ordinary simplicial descent from def. 2.2.24 applies. The following theorem asserts that under certain conditions the ∞-groupoids presented by both these simplicial sets are equivalent.

Proposition 2.2.35 (Verity 09)**.** *If* $\mathcal{A} : C^{\mathrm{op}}, \mathrm{Str}\infty\mathrm{Grpd}$ *and* $Y : C^{\mathrm{op}} \to \mathrm{sSet}$ *are such that* $N\mathcal{A}(Y_\bullet) : \Delta \to$ *sSet is fibrant in the Reedy model structure* $[\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$*, then*

$$N\mathrm{Desc}_{\mathrm{Street}}(Y, \mathcal{A}) \stackrel{\simeq}{\to} \mathrm{Desc}(Y, N\mathcal{A})$$

is a weak homotopy equivalence of Kan complexes.

This is proven in [Veri09]. In our applications the assumptions of this theorem are usually satisfied:

Corollary 2.2.36. *If* $Y \in [C^{\mathrm{op}}, \mathrm{sSet}]$ *is such that* $Y_\bullet : \Delta \to [C^{\mathrm{op}}, \mathrm{Set}] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$ *is cofibrant in* $[\Delta, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{Reedy}}$ *then for* $\mathcal{A} : C^{\mathrm{op}} \to \mathrm{Str}\infty\mathrm{Grpd}$ *we have a weak equivalence*

$$N\mathrm{Desc}(Y, \mathcal{A}) \stackrel{\simeq}{\to} \mathrm{Desc}(Y, N\mathcal{A})\,.$$

Proof. If $Y_\bullet$ is Reedy cofibrant, then by definition the canonical morphisms

$$\lim_{\to}(([n] \stackrel{+}{\to} [k]) \mapsto Y_k) \to Y_n$$

are cofibrations in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Since the latter is an $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category and $N\mathcal{A}$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, it follows that the hom-functor $[C^{\mathrm{op}}, \mathrm{sSet}](-, N\mathcal{A})$ sends cofibrations to fibrations, so that

$$N\mathcal{A}(Y_n) \to \lim_{\leftarrow}([n] \stackrel{+}{\to} [k] \mapsto N\mathcal{A}(Y_k))$$

is a Kan fibration. But this says that $N\mathcal{A}(Y_\bullet)$ is Reedy fibrant, so that the assumption of prop. 2.2.35 is met. □

2.3 Universal constructions

We discuss some basic abstract properties and some presentations of universal constructions in ∞-category theory that we will refer to frequently.

2.3.1 General abstract

2.3.1.1 ∞-Colimits in $\infty\mathrm{Grpd}$ The following proposition says that every ∞-groupoid is the ∞-colimit over itself, regarded as a diagram, of the ∞-functor constant on the point in $\infty\mathrm{Grpd}$.

Proposition 2.3.1. *For $S \in \infty\mathrm{Grpd}$, the ∞-colimit of the ∞-functor $S \to \infty\mathrm{Grpd}$ constant on the terminal object is equivalent to S:*

$$\varinjlim_S * \simeq S \,.$$

This is essentially corollary 4.4.4.9 in [LuHTT].

2.3.1.2 ∞-Pullbacks We will have have ample application for the following immediate ∞-category theoretic generalization of a basic 1-categorical fact.

Proposition 2.3.2 (pasting law for ∞-pullbacks)**.** *Let*

$$\begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ d & \longrightarrow & e & \longrightarrow & f \end{array}$$

be a diagram in an ∞-category and suppose that the right square is an ∞-pullback. Then the left square is an ∞-pullback precisely if the outer rectangle is.

This appears as [LuHTT], lemma 4.4.2.1. Notice that here and in the following we do not explicitly display the 2-morphisms/homotopies that do fill these diagrams in the given ∞-category.

2.3.1.3 Effective epimorphisms We briefly record the definition and main properties of effective epimorphisms in an ∞-topos from [LuHTT], section 6.2.3.

Definition 2.3.3. A morphism $Y \to X$ in an ∞-topos is an *effective epimorphism* if it exhibits the ∞-colimit over the simplicial diagram that is its Čech nerve:

$$Y \simeq \varinjlim_n Y^{\times^n_X} \,.$$

See for instance below cor. 6.2.3.5 in [LuHTT].

Remark 2.3.4. In view of the discussion of groupoid objects below in 3.6.7 (see remark 3.6.91 there) we also speak of an effective epimorphism $U \longrightarrow\!\!\!\!\!\rightarrow X$ as being an *atlas*, or, more explicitly, as *exhibiting U as an atlas of X*.

Proposition 2.3.5. *Effective epimorphisms are preserved by ∞-pullback.*

This is prop. 6.2.3.15 in [LuHTT].

Proposition 2.3.6.

A morphism $p : X \to Y$ is an effective epimorphism precisely if its 0-truncation $\tau_0 p : \tau_0 X \to \tau_0 Y$, def. 3.6.22, is an effective epimorphism, hence an epimorphism, in the 1-topos of 0-truncated objects.

This is prop. 7.2.1.14 in [LuHTT].

Example 2.3.7. A morphism in $\infty\mathrm{Grpd}$ is effective epi precisely if it induces an epimorphism $\pi_0(X) \to \pi_0(Y)$ of sets of connected components.

2.3.2 Presentations

We discuss presentations of various classes of ∞-limits and ∞-colimits in an ∞-category by *homotopy limits* and *homotopy colimits* in categories with weak equivalences presenting them.

2.3.2.1 ∞-Pullbacks We discuss here tools for computing ∞-pullbacks in an ∞-category $\mathbf{H}$ in terms of homotopy pullbacks in a homotopical 1-category presenting it.

Proposition 2.3.8. *Let $A \to C \leftarrow B$ be a cospan diagram in a model category, def. 2.1.21. Sufficient conditions for the ordinary pullback $A \times_C B$ to be a homotopy pullback are*

- *one of the two morphisms is a fibration and all three objects are fibrant;*
- *one of the two morphisms is a fibration and the model structure is right proper.*

This appears for instance as prop. A.2.4.4 in [LuHTT].

It remains to have good algorithms for identifying fibrations and for resolving morphisms by fibrations. A standard recipe for constructing fibration resolutions is

Proposition 2.3.9 (factorization lemma)**.** *Let $B \to C$ be a morphism between fibrant objects in a model category and let $C \xrightarrow{\simeq} C^I \twoheadrightarrow C \times C$ be a path object for B. Then the composite vertical morphism in*

$$\begin{array}{ccc} C^I \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow \\ C^I & \longrightarrow & C \\ \downarrow & & \\ C & & \end{array}$$

* *

is a fibrantion replacement of $B \to C$.

This appears for instance on p. 4 of [Br73].

Corollary 2.3.10. *For $A \to C \leftarrow B$ a diagram of fibrant objects in a model category, its homotopy pullback is presented by the ordinary limit $A \times^h_C B$ in*

$$\begin{array}{ccccc} A \times^h_C B & \longrightarrow & C^I \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ & & C^I & \longrightarrow & C \\ \downarrow & & \downarrow & & \\ A & \longrightarrow & C & & \end{array},$$

which is, up to isomorphism, the same as the ordinary pullback in

$$\begin{array}{ccc} A \times^h_C B & \longrightarrow & C^I \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & C \times C \end{array}.$$

Remark 2.3.11. For the special case of "abelian" objects another useful way of constructing fibrations is via the *Dold-Kan correspondence*, wich we discuss in 2.2.6. As described there, a morphism between simplicial presheaves that arise from presheaves of chain complexes is a fibration (in the projective model structure on simplicial presheaves) if it arises from a degreewise surjection of chain complexes.

2.3.2.2 Finite ∞-limits of ∞-sheaves We discuss presentations for finite ∞-limits specifically in ∞-toposes.

Proposition 2.3.12. *Let C be a site with enough points, def. 2.2.9. Write $\mathbf{H} \simeq (\mathrm{Sh}(C)^{\Delta^{\mathrm{op}}}, W)$ for the hypercomplete ∞-topos over C, where W is the class of local weak equivalences, theorem 2.2.12.*

Then pullbacks in $\mathrm{Sh}(C)^{\Delta^{\mathrm{op}}}$ along local fibrations, def. 2.2.13, are homotopy pullbacks, hence present ∞-pullbacks in $\mathbf{H}$.

Proof. Let $A \stackrel{\mathrm{loc}}{\twoheadrightarrow} C \leftarrow B$ be a cospan with the left leg a local fibration. By the existence of the projective local model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ there exists a morphism of diagrams

$$\begin{array}{ccccc} A & \stackrel{\mathrm{loc}}{\twoheadrightarrow} & C & \leftarrow & B \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ A' & \twoheadrightarrow & C' & \leftarrow & B' \end{array} ,$$

where the bottom cospan is a fibrant diagram with respect to the projective local model structure, hence a cospan of genuine fibrations between fibrant objects, so that the ordinary pullback $A' \times_{C'} B'$ is a presentation of the homotopy pullback of the original diagram. Here the vertical morphisms are weak equivalences, and by theorem 2.2.12 this means that they are stalkwise weak equivalences of simplicial sets. Moreover, by the nature of left Bousfield localization, the genuine fibrations are in particular global projective fibrations, hence in particular are stalkwise fibrations.

Now for $p : \mathrm{Set} \to \mathrm{Sh}(C)$ any topos point, the stalk functor p^* preserves finite limits and hence preserves (the sheafification of) the above pullbacks. So by the asumption that $A \to C$ is a local fibration, the simplicial set $p^*(A \times_C B)$ is a pullback of simplicial sets along a Kan fibration, hence, by the right properness of $\mathrm{sSet}_{\mathrm{Quillen}}$, and using prop. 2.3.8, is a homotopy pullback there. Moreover, the induced morphism $p^*(A \times_C B) \to p^*(A' \times_{C'} B')$ is therefore a morphism of homotopy pullbacks along a weak equivalence of diagrams. This means that it is itself a weak equivalence. Since this is true for all topos points, it follows that $A \times_C B \to A' \times_{C'} B'$ is a stalkwise weak equivalence, hence a weak equivalence, hence that $A \times_C B$ is itself already a model for the homotopy pullback. $\square$

The following proposition establishes the model category analog of the statement that by left exactness of ∞-sheafification, finite ∞-limits of ∞-sheafified ∞-presheaves may be computed as the ∞-sheafification of the finite ∞-limit of the ∞-presheaves.

Proposition 2.3.13. *Let C be a site and $F : D \to [C^{\mathrm{op}}, \mathrm{sSet}]$ be a finite diagram.*

Write $\mathbb{R}_{\mathrm{glob}} \lim_{\leftarrow} F \in [C^{op}, \mathrm{sSet}]$ for (any representative of) the homotopy limit over F computed in the global model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, well defined up to isomorphism in the homotopy category.

Then $\mathbb{R}_{\mathrm{glob}} \lim_{\leftarrow} F \in [C^{\mathrm{op}}, \mathrm{sSet}]$ presents also the homotopy limit of F in the local model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Proof. By [LuHTT], theorem 4.2.4.1, we have that the homotopy limit $\mathbb{R} \lim_{\leftarrow}$ computes the corresponding ∞-limit. Since ∞-sheafification L is by definition a left exact ∞-functor it preserves these finite ∞-limits:

$$\begin{array}{ccc} ([D, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{inj}})^\circ & \stackrel{L_*}{\longleftarrow} & ([D, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{inj}})^\circ \\ \downarrow \mathbb{R}\lim_{\leftarrow} & & \downarrow \mathbb{R}\lim_{\leftarrow} \\ ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ & \stackrel{L \simeq \mathbb{L}\mathrm{Id}}{\longleftarrow} & ([C^{op}, \mathrm{sSet}]_{\mathrm{proj}})^\circ \end{array} .$$

Here $L \simeq \mathbb{L}\mathrm{Id}$ is the left derived functor of the identity for the left Bousfield localization. Therefore for F a finit diagram in simplicial presheaves, its homotopy limit in the local model structure $\mathbb{R}\lim_{\leftarrow} L_* F$ is equivalently computed by $\mathbb{L}\mathrm{Id}\mathbb{R}\lim_{\rightarrow} F$, with $\mathbb{R}\lim_{\leftarrow} F$ the homotopy limit in the global model structure. $\square$
Together with 2.3.2.1, this provides an efficient algorithm for computing presentations of ∞-pullbacks in a model structure on simplicial presheaves.

Remark 2.3.14. Taken together, prop. 2.3.13, prop. 2.3.8 and definition 2.2.6 imply that we may compute ∞-pullbacks in an ∞-topos by the following algorithm:

1. Present the ∞-topos by a local *projective* model structure on simplicial presheaves;
2. find a presentation of the morphisms to be pulled back such that one of them is over each object of the site a Kan fibration of simplicial sets;
3. then form the ordinary pullback of simplicial presheaves, which in turn is over each object the ordinary pullback of simplicial sets.

The resulting object presents the ∞-pullback of ∞-sheaves.

2.3.2.3 ∞-Colimits We collect some standard facts and tools concerning the computation of homotopy colimits.

Proposition 2.3.15. *Let C be a combinatorial model category and let J be a small category. Then the colimit over J-diagrams in C is a left Quillen functor for the projective model structure on functors on J:*

$$\varinjlim : [J, C]_{\mathrm{proj}} \to C\,.$$

Proof. For C combinatorial, the projective model structure exists by [LuHTT] prop. A.2.8.2. The right adjoint to the colimit

$$\mathrm{const} : C \to [J, C]_{\mathrm{proj}}$$

is manifestly right Quillen for the projective model structure. □

Example 2.3.16. Write

$$(\mathbb{N}, \leq) := \{\, 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \,\}$$

for the *cotower category.* A cotower $X_0 \to X_1 \to A_2 \to \cdots$ in a model category C is projectively cofibrant precisely if

1. every morphism $X_i \to X_{i+1}$ is a cofibration in C;
2. the first object X_0, and hence all objects X_i, are cofibrant in C.

Therefore a sequential ∞-colimit over a cotower is presented by the ordinary colimit of a presentation of this cotower where all morphisms are cofibrations and all objects are cofibrant.

This is a simple example, but since we will need details of this at various places, we spell out the proof for the record.

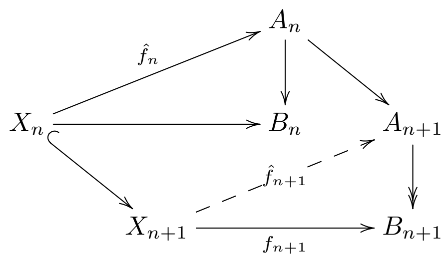
Proof. Given a cotower $X_\bullet$ with properties as stated, we need to check that for $p_\bullet : A_\bullet \to B_\bullet$ a morphism of cotowers such that for all $n \in \mathbb{N}$ the morphism $p_n : A_n \to B_n$ is an acyclic fibration in C, and for $f_\bullet : X_\bullet \to B_\bullet$ any morphism, there is a lift $\hat{f}_\bullet$ in

$$\begin{array}{ccc} & & A_\bullet \\ & \overset{\hat{f}_\bullet}{\nearrow} & \downarrow{\scriptstyle p_\bullet} \\ X_\bullet & \xrightarrow{f_\bullet} & B_\bullet \end{array}\,.$$

This lift we can construct by induction on n. For $n = 0$ we only need a lift in

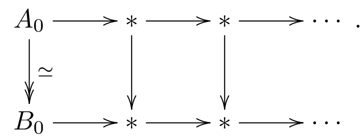
$$\begin{array}{ccc} & & A_0 \\ & \overset{\hat{f}_0}{\nearrow} & \downarrow{\scriptstyle p_0} \\ X_0 & \xrightarrow{f_0} & B_0 \end{array}\,,$$

which exists by assumption that X_0 is cofibrant. Assume then that a lift has been for $f_{\leq n}$. Then the next lift $\hat{f}_{n+1}$ needs to make the diagram

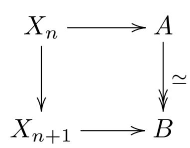


commute. Such a lift exists now by assumption that $X_n \to X_{n+1}$ is a cofibration.

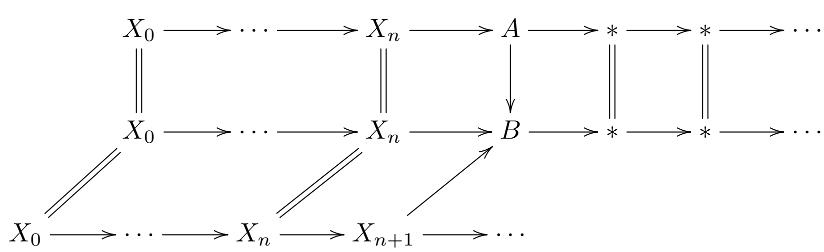
Conversely, assume that $X_\bullet$ is projectively cofibrant. Then first of all it has the left lifting property against all cotower morphisms of the form



Such a lift is equivalent to a lift of X_0 against $A_0 \xrightarrow{\simeq} B_0$ and hence X_0 is cofibrant in C. To see that every morphism $X_n \to X_{n+1}$ is a cofibration, notice that for every lifting problem in C of the form



the cotower lifting problem of the form



is equivalent. □

For less trivial diagram categories it quickly becomes hard to obtain projective cofibrant resolutions. In these cases it is often it is useful to compute the (homotopy) colimit instead as a special case of a (homotopy) coend.

Proposition 2.3.17. *Let $F : A \times B \to C$ be a Quillen bifunctor, def. 2.1.24, and let J be a Reedy category, then the coend over F (see [Ke82])*

$$\int^S F(-,-) : [J, A]_{\text{Reedy}} \times [J^{\text{op}}, B]_{\text{Reedy}} \to C$$

is a Quillen bifunctor from the product of the Reedy model categories on functors with values in A and B, respectively, to C.

Similarly, if A and B are combinatorial model categories and J is any small category, then the coend

$$\int^S F(-,-) : [J,A]_{\text{proj}} \times [J^{\text{op}}, B]_{\text{inj}} \to C$$

is a Quillen bifunctor.

This appears in [LuHTT] as prop. A.2.9.26 and remark A.2.9.27.

Proposition 2.3.18. *If $\mathcal{V}$ is a closed monoidal model category, C is a $\mathcal{V}$-enriched model category, and J is a small category which is Reedy, then the homotopy colimit of J-shaped diagrams in C is presented by the left derived functor of*

$$\int^J (-) \cdot Q_{\text{Reedy}}(I) : [J,C]_{\text{Reedy}} \to C\,,$$

where $Q_{\text{Reedy}}(I)$ is a cofibrant replacement of the functor constant in the tensor unit in $[J^{\text{op}}, \mathcal{V}]_{\text{Reedy}}$, and where

$$(-)\cdot(-) : C \times \mathcal{V} \to C$$

is the given $\mathcal{V}$-tensoring of C. Similarly, if J is not necessarily Reedy, but $\mathcal{V}$ and C are combinatorial, then the homotopy colimit is also given by the left derived functor of

$$\int^J (-) \cdot Q_{\text{proj}}(I) : [J,C]_{\text{inj}} \to C\,,$$

where now $Q_{\text{proj}}(I)$ is a cofibrant resolution of the tensor unit in $[J^{\text{op}}, \mathcal{V}]_{\text{proj}}$.

This is nicely discussed in [Gam10].

Proof. By definition of enriched category, the $\mathcal{V}$-tensoring operation is a left Quillen bifunctor. With this the statement follows from prop. 2.3.17. □

Various classical facts of model category theory are special cases of these formulas.

2.3.2.4 ∞-Colimits over simplicial diagrams

We discuss here a standard presentation of *homotopy colimits over simplical diagrams* given by the *diagonal simplicial set* or the *total simplicial set* associated with a bisimplicial set.

Proposition 2.3.19. *Write $[\Delta, \mathrm{sSet}]$ for the category of cosimplicial simplicial sets. For* sSet *equipped with its cartesian monoidal structure, the tensor unit is the terminal object $*$.*

- *The* simplex functor
$$\Delta : [n] \mapsto \Delta[n] := \Delta(-,[n])$$
is a cofibrant resolution of $*$ in $[\Delta, \mathrm{sSet}_{\text{Quillen}}]_{\text{Reedy}}$;
- *the* fat simplex functor
$$\mathbf{\Delta} : [n] \mapsto N(\Delta/[n])$$
is a cofibrant resolution of $*$ in $[\Delta, \mathrm{sSet}_{\text{Quillen}}]_{\text{proj}}$.

Proposition 2.3.20. *Let C be a simplicial model category and $F : \Delta^{\text{op}} \to C$ a simplicial diagram*

1. *If every monomorphism in C is a cofibration, then the homotopy colimit over F is given by the realization*
$$\mathbb{L}\lim_{\to} F \simeq \int^{[n]\in\Delta} F([n]) \cdot \Delta[n]\,.$$

2. *If F takes values in cofibrant objects, then the the homotopy colimit over F is given by the fat realization*

$$\mathbb{L}\lim_{\to} F \simeq \int^{[n]\in\Delta} F([n]) \cdot \mathbf{\Delta}[n]\,.$$

3. *If F is Reedy cofibrant, then the canonical morphism*

$$\int^{[n]\in\Delta} F([n]) \cdot \mathbf{\Delta}[n] \to \int^{[n]\in\Delta} F([n]) \cdot \Delta[n]$$

(the Bousfield-Kan map*) is a weak equivalence.*

Proof. If every monomorphism is a cofibration, then F is necessarily cofibrant in $[\Delta^{\mathrm{op}}, C]_{\mathrm{Reedy}}$. The first statement then follows from prop. 2.3.18 and the first item in prop. 2.3.19. On the other hand, if F takes values in cofibrant objects, then it is cofibrant in $[\Delta^{\mathrm{op}}, C]_{\mathrm{inj}}$, and so the second statement follows from prop. 2.3.18 and the second item in prop. 2.3.19.

Notice that projective cofibrancy implies Reedy cofibrancy, so that $\mathbf{\Delta}$ is also Reedy cofibrant. Therefore the morphism in the last item of the proposition is, by remark 2.1.25, the image under a left Quillen functor of a weak equivalence between cofibrant objects and therefore itself a weak equivalence. □

An important example of this general situation is the following.

Proposition 2.3.21. *Every simplicial set, and more generally every simplicial presheaf is the homotopy colimit over its simplicial diagram of cells. Precisely, let C be a small site, and let $[C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{inj,loc}}$ be the corresponding local injective model structure on simplicial presheaves. Then for any $X \in [C^{\mathrm{op}}, \mathrm{sSet}]$, with*

$$X_\bullet : \Delta^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{Set}] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]$$

its simplicial diagram of components, we have

$$X \simeq \mathbb{L}\lim_{\longrightarrow} X_\bullet\,.$$

Proof. By prop. 2.3.20 the homotopy colimit is given by the coend

$$\mathbb{L}\lim_{\longrightarrow} X_\bullet \simeq \int^{[n]\in\Delta} X_n \times \Delta[n]\,.$$

By basic properties of the coend, this is isomorphic to X. □

Proposition 2.3.22. *The homotopy colimit of a simplicial diagram in* $\mathrm{sSet}_{\mathrm{Quillen}}$*, or more generally of a simplicial diagram of simplicial presheaves, is given by the diagonal of the corresponding bisimplicial set / bisimplicial presheaf.*

More precisely, for

$$F : \Delta^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{inj,log}}$$

a simplicial diagram, its homotopy colimit is given by

$$\mathbb{L}\lim_{\longrightarrow} F_\bullet \simeq dF : ([n] \mapsto (F_n)_n)\,.$$

Proof. By prop. 2.3.20 the homotopy colimit is given by the coend

$$\mathbb{L}\lim_{\longrightarrow} F_\bullet \simeq \int^{[n]\in\Delta} F_n \cdot \Delta[n]\,.$$

By a standard fact (e.g. exercise 1.6 in [GoJa99]), this coend is in fact isomorphic to the diagonal. □

Definition 2.3.23. Write Δ_a for the *augmented simplex category*, which is the simplex category with an initial object adjoined, denoted $[-1]$.

This is a symmetric monoidal category with tensor product being the *ordinal sum* operation

$$[k], [l] \mapsto [k + l + 1] \,.$$

Write

$$\sigma : \Delta \times \Delta \to \Delta$$

for the restriction of this tensor product along the canonical inclusion $\Delta \hookrightarrow \Delta_a$. Write

$$\sigma^* : \mathrm{sSet} \to [\Delta^{\mathrm{op}}, \mathrm{sSet}]$$

for the operation of precomposition with this functor. By right Kan extension this induces an adjoint pair of functors

$$(\mathrm{Dec} \dashv T) : \; [\Delta^{\mathrm{op}}, \mathrm{sSet}] \underset{\sigma_*}{\overset{\sigma^*}{\rightleftarrows}} \mathrm{sSet} \;\; .$$

- $\mathrm{Dec} := \sigma^*$ is called the *total décalage* functor;
- $T := \sigma_*$ is called the *total simplicial set* functor.

The total simplicial set functor was introduced in [ArMa66]. Details are in [St11].

Remark 2.3.24. By definition, for $X \in [\Delta^{\mathrm{op}}, \mathrm{sSet}]$, its total décalage is the bisimplicial set given by

$$(\mathrm{Dec}X)_{k,l} = X_{k+l+1} \,.$$

Remark 2.3.25. For $X \in [\Delta^{\mathrm{op}}, \mathrm{sSet}]$, the simplicial set TX is in each degree given by an equalizer of maps between finite products of components of X. Hence forming T is compatible with sheafification and other processes that preserve finite limits.

See [St11], equation (2).

Proposition 2.3.26. *For every* $X \in [\Delta^{\mathrm{op}}, \mathrm{sSet}]$

- *the canonical morphism*

  $$dX \to TX$$

  from the diagonal to the total simplicial set is a weak equivalence in $\mathrm{sSet}_{\mathrm{Quillen}}$*;*
- *the adjunction unit*

  $$X \to T\mathrm{Dec}X$$

  is a weak equivalence in $\mathrm{sSet}_{\mathrm{Quillen}}$*.*

For every $X \in \mathrm{sSet}$

- *there is a natural isomorphism* $T\mathrm{const}X \simeq X$*.*

This is due to [CeRe][St11].

Corollary 2.3.27. *For*

$$F : \Delta^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{inj,loc}}$$

a simplicial object in simplicial presheaves, its homotopy colimit is given by applying objectwise over each $U \in C$ *the total simplicial set functor*

$$\mathbb{L}\varinjlim F \simeq (U \mapsto TF(U)) \,.$$

Proof. By prop. 2.3.26 this follows from prop. 2.3.22. □

Remark 2.3.28. The use of the total simplicial set instead of the diagonal simplicial set in the presentation of simplicial homotopy colimits is useful and reduces to various traditional notions in particular in the context of group objects and action groupoid objects. This we discuss below in 3.6.8.2 and 3.6.10.3.

2.3.2.5 Effective epimorphisms, atlases and décalage We discuss apsects of the presentation of effective epimorphisms, def. 2.3.3, with respect to presentations of the ambient ∞-topos by categories of simplicial presheaves, 2.2.3.

Observation 2.3.29. If the ∞-topos $\mathbf{H}$ is presented by a category of simplicial presheaves, 2.2.3, then for X a simplicial presheaf the canonical morphism of simplicial presheaves $\mathrm{const} X_0 \to X$ that includes the presheaf of 0-cells as a simplicially constant simplicial presheaf presents an effective epimorphism in $\mathbf{H}$.

Proof. By prop. 2.3.6. □

Remark 2.3.30. In practice the presentation of an ∞-stack by a simplicial presheaf is often taken to be understood, and then observation 2.3.29 induces also a canonical atlas.

We now discuss a fibration resolution of the canonical atlas. Let $\sigma : \Delta \times \Delta \to \Delta$ the functor from def. 2.3.23, defining *total décalage*.

Definition 2.3.31. Write

$$\mathrm{Dec}_0 : \mathrm{sSet} \to \mathrm{sSet}$$

for the functor given by precomposition with $\sigma(-,[0]) : \Delta \to \Delta$, and

$$\mathrm{Dec}^0 : \mathrm{sSet} \to \mathrm{sSet}$$

for the functor given by precomposition with $\sigma([0],-) : \Delta \to \Delta$. This is called the plain *décalage functor* or *shifting functor*.

This functor was introduced in [Il72]. A discussion in the present context is in section 2.2 of [St11].

Proposition 2.3.32. *The décalage of X is isomorphic to the simplicial set*

$$\mathrm{Dec}_0 X = \mathrm{Hom}(\Delta^\bullet \star \Delta[0], X)\,,$$

where $(-)\star(-) : \mathrm{sSet} \times \mathrm{sSet} \to \mathrm{sSet}$ is the join of simplicial sets. The canonical inclusions $\Delta[n], \Delta[0] \to \Delta[n] \star \Delta[0]$ induce two canonical morphisms

$$\begin{array}{ccc} \mathrm{Dec}_0 X & \twoheadrightarrow & X \\ \downarrow{\simeq} & & \\ \mathrm{const} X_0 & & \end{array} ,$$

where

- *the horizontal morphism is given in degree n by $d_{n+1} : X_{n+1} \to X_n$;*
- *the horizontal morphism is a Kan fibration;*
- *the vertical morphism is a weak homotopy equivalence;*
- *a weak homotopy inverse is given by the morphism that is degreewise given by the degeneracy morphisms in X.*

Proof. The relation to the join of simplicial sets is nicely discussed around page 7 of [RoSt12]. The weak homotopy equivalence is classical, see for instance [St11].

To see that $\mathrm{Dec}_0 X \to X$ is a Kan fibration, notice that for all $n \in \mathbb{N}$ we have $(\mathrm{Dec}_0 X)_n = \mathrm{Hom}(\Delta[c] \star \Delta[0], X)$, where $(-) \star (-) : \mathrm{sSet} \times \mathrm{sSet} \to \mathrm{sSet}$ is the join of simplicial sets. Therefore the lifting problem

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & \mathrm{Dec}_0 X \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & X \end{array}$$

is equivalently the lifting problem

$$\begin{array}{ccc} (\Lambda^i[n] \star \Delta[n]) \coprod_{\Lambda^i[n]} \Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta[n] \star \Delta[0] & \longrightarrow & * \end{array} .$$

Here the left moprhism is a anodyne morphism, in fact is an $(n+1)$-horn inclusion. Hence a lift exists if X is a Kan complex. (Alternatively, notice that $\mathrm{Dec}_0 X$ is the disjoint union of slices $X_{/x}$ for $x \in X_0$. By cor. 2.1.2.2 in [LuHTT] the projection $X_{/x} \to X$ is a left fibration if X is Kan fibrant, and by prop. 2.1.3.3 there this implies that it is a Kan fibration). □

Corollary 2.3.33. *For X in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ fibrant, a fibration resolution of the canonical effective epimorphism* $\mathrm{const} X_0 \to X$ *from observation 2.3.29 is given by the décalage morphism* $\mathrm{Dec}_0 X \to X$*, def. 2.3.31.*

Proof. It only remains to observe that we have a commuting diagram

$$\begin{array}{ccc} \mathrm{const} X_0 & \xrightarrow{s} & \mathrm{Dec}_0 X \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array} ,$$

where the top morphism, given degreewise by the degeneracy maps in X, is a weak homotopy equivalence by classical results. □

3 Cohesive and differential homotopy type theory

We discuss here the general abstract theory of *cohesive ∞-toposes* and of *differential cohesive ∞-toposes* and of the homotopical, cohomological, geometrical and differential structures internal to them.

Below in 4 we construct models of these axioms.

3.1 Introduction and survey

A topos or ∞-topos may be viewed both as a category or, respectively, ∞-category *of* generalized spaces – then also called a "*gros topos*" – or as a generalized space itself – then also called a "*petit topos*". The duality relation between these two perspectives is given by prop. 3.6.14, which says that every ∞-topos regarded as a generalized space is equivalent to the ∞-category of generalized étale spaces *over* it, while, conversely, every collection of generalized spaces encoded by an ∞-topos may be understood as being those generalized spaces equipped with local equivalences to a fixed generalized model space.

From this description it is intuitively clear that the "smaller" an ∞-topos is when regarded as a generalized space, the "larger" is the collection of generalized spaces locally modeled on it, and vice versa. If by "size" we mean "dimension", there are two notions of *dimension of an ∞-topos* $\mathbf{H}$ that coincide with the ordinary notion of dimension of a manifold X when $\mathbf{H} = \mathrm{Sh}_\infty(X)$, but which may be different in general. These are

- homotopy dimension (see def. 3.6.71 below);
- cohomology dimension ([LuHTT], section 7.2.2).

If by "size" we mean "nontriviality of homotopy groups", hence nontriviality of *shape* of a space, there is the notion of

- shape of an ∞-topos ([LuHTT], section 7.1.6);

which coincides with the topological shape of X in the case that $\mathbf{H} = \mathrm{Sh}_\infty(X)$, as above. Finally, if by "small size" we just mean *finite dimensional*, then the property of ∞-toposes reflecting that is

- hypercompleteness ([LuHTT], section 6.5.2).

For the description of higher geometry and higher differential geometry, we are interested in ∞-toposes that are "maximally *gros*" and"minimally *petit*": regarded as generalized spaces they should look like *fat points* or *contractible blobs* being the abstract blob of *geometry* that every object in them is supposed to be locally modeled on, but that otherwise do not make these objects be parameterized over a nontrivial space.

The following notions of *local ∞-topos*, *∞-connected ∞-topos*, *cohesive ∞-topos*, and *differential cohesive ∞-topos* describe extra properties of the global section geometric morphism of an ∞-topos that imply that some or all of the measures of "size" of the ∞-topos vanish, hence that make the ∞-topos be far from being a non-trivial generalized space itself, and instead be genuinely a collection *of* generalized spaces modeled on some notion of local geometry.

All these properties are equivalently encoded in terms of *idempotent ∞-(co)monads* on the ∞-topos $\mathbf{H}$

$$\Box, \Diamond : \mathbf{H} \to \mathbf{H}\,.$$

Internally, on the homotopy type theory language of $\mathbf{H}$, these are (higher) *closure operators* or *modalities* on the type system (more on this is below in 3.4.1.2). Externally, these structures correspond to adjunctions

$$(L \dashv R) : \mathbf{H} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{B}$$

such that L or R is a fully faithful ∞-functor, by $\Box \simeq L \circ R$ and $\Diamond \simeq R \circ L$, or the other way around.

Proposition 3.1.1. *Let* $(L \dashv R) : \mathcal{C} \underset{R}{\overset{L}{\rightrightarrows}} \mathcal{D}$ *be a pair of adjoint ∞-functors. Then*

1. *The left adjoint ∞-functor L is fully faithful precisely if the the adjunction unit is an equivalence* $\mathrm{id}_{\mathcal{D}} \xrightarrow{\simeq} R \circ L$.

2. *The right adjoint ∞-functor R is fully faithful precisely if the the adjunction counit is an equivalence* $L \circ R \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}$.

Proof. This is [LuHTT], p. 308 or follows directly from it. □

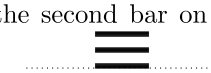
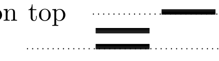
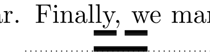
For encoding "gros" geometry in the above sense, here the comonadic $\Box$ is itself to be part of an adjunction with the monadic $\lozenge$, as $\Box \dashv \lozenge$ or $\lozenge \dashv \Box$. Such a situation corresponds externally to adjoint triples of ∞-functors

$$(f_! \dashv f^* \dashv f_*) : \ \mathbf{H} \underset{f_*}{\overset{f_!}{\underset{\xleftarrow{f^*}}{\rightrightarrows}}} \mathbf{B} \qquad \text{or} \qquad (f^* \dashv f_* \dashv f^!) : \ \mathbf{H} \underset{f^!}{\overset{f^*}{\underset{\xrightarrow{f_*}}{\leftleftarrows}}} \mathbf{B}$$

such that the middle functor or the two outer functors are fully faithful:

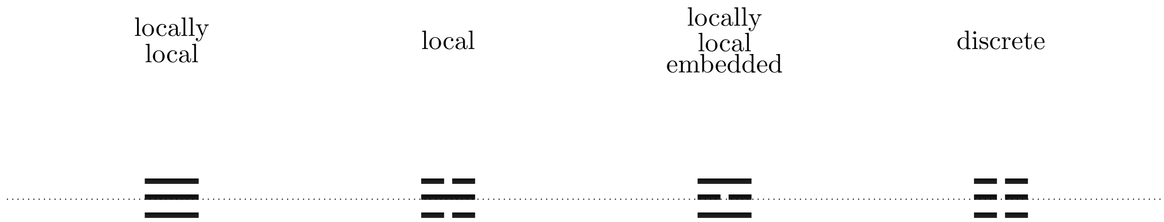
$$(\lozenge \dashv \Box) \simeq (f^* f_! \dashv f^* f_*) \quad \text{or} \quad (\Box \dashv \lozenge) \simeq (f^* f_* \dashv f^! f_*)\,.$$

All that matters for the nature of the induced modalities is in which direction these functors go and which of them are fully faithful. Moreover, both direction and fully faithfulness are necessarily alternating through the adjoint triple, so what really matters is only which functor we regard as the direct image, the number of adjoints it has to the left and to the right, and whether it is itself fully faithful or its adjoints are. To bring that basic information out more clearly it may be helpful to introduce the following condensed notation.

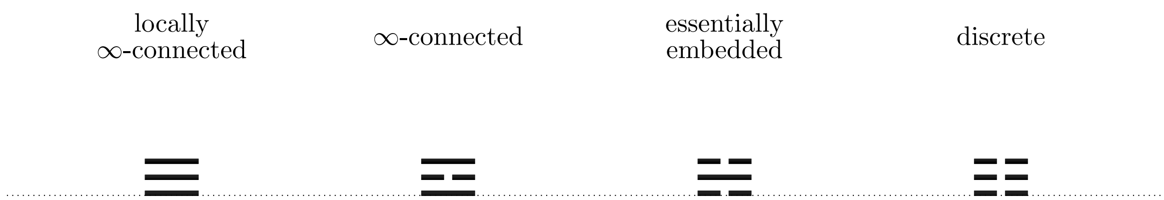
Let  stand for an adjoint pair where the direct image f_* points from $\mathbf{H}$ to $\mathbf{B}$, (this is the bar on the dotted baseline) and such that it has a single left adjoint f^* (the second bar on top). Accordingly, if there is a further left adjoint $f_!$ then we draw a further bar on top  . If there is a further right adjoint $f^!$ then we draw a further bar on the bottom  . And so forth: bars on top are left adjoint to bars below them, and the direction is left-to-right for the bar on the base line and for every second bar next to it, while it is right-to-left for every other bar. Finally, we mark the fully faithful functors by breaking the corresponding bar. For instance the notation  means that the inverse image is fully faithful, hence is shorthand for an adjunction of the form $\mathbf{H} \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\hookleftarrow}}} \mathbf{B}$, and so forth.

The following table lists, in the above notation, the possibilities for adjoint higher modalities together with the name of the corresponding attribute of **H** as an ∞-topos over the base **B**.

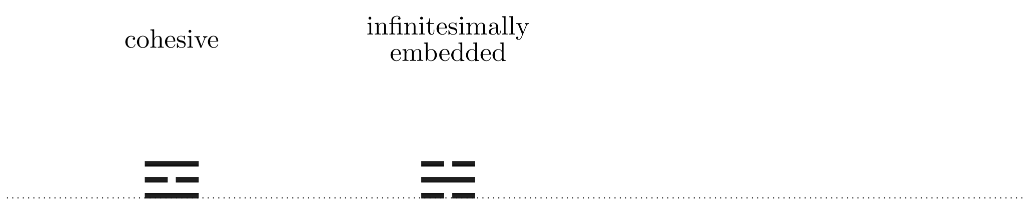
Locality ($\flat \dashv \sharp$) (section 3.2).



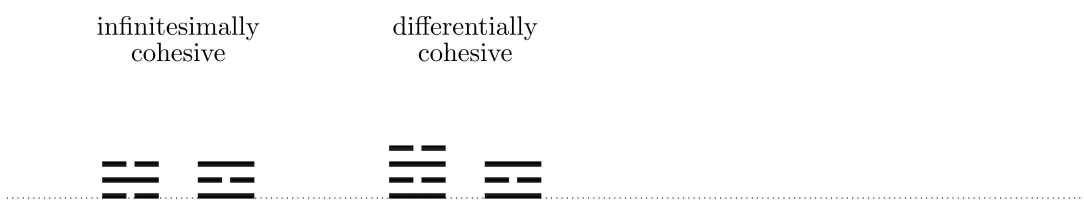
∞**-Connectedness** ($\mathbf{\Pi} \dashv \flat$) (section 3.3).



Cohesion ($\mathbf{\Pi} \dashv \flat \dashv \sharp$) (section 3.4).



Differential cohesion ($\mathbf{Red} \dashv \mathbf{\Pi}_{\text{inf}} \dashv \flat_{\text{inf}}$) (section 3.5).



We discuss a list of structures that may be formulated internal to such **H**:

- 3.6 – Structures in an ∞-topos;
- 3.7 – Structures in a local ∞-topos;
- 3.8 – Structures in an ∞-connected ∞-topos;
- 3.9 – Structures in a cohesive ∞-topos;
- 3.10 – Structures in a differential ∞-topos.

3.2 Local ∞-toposes

The following definition is the direct generalization of the notion of *local topos* [JoMo94].

Definition 3.2.1. An ∞-topos $\mathbf{H}$ is called *locally local* if the global section geometric morphism has a right adjoint.

$$\mathbf{H} \underset{\text{coDisc}}{\overset{\text{Disc}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\Gamma}}}}} \infty\text{Grpd} \ .$$

It is called *local* if that right adjoint is in addition fully faithful.

Proposition 3.2.2. *A local ∞-topos*

1. *has homotopy dimension 0 (see def. 3.6.71 below);*
2. *has cohomological dimension 0 ([LuHTT], section 7.2.2);*
3. *is hypercomplete.*

Proof. The first statement is cor. 3.6.77 below. The second is a consequence of the first by [LuHTT], cor. 7.2.2.30. The third follows from the second by [LuHTT], cor. 7.2.1.12. □

3.3 Locally ∞-connected ∞-toposes

We discuss ∞-toposes satisfying a higher geometric connectedness condition.

3.3.1 General abstract

The following definition is the direct generalization standard notion of a *locally/globally connected topos* [John03]: a topos whose terminal geometric morphism has an extra left adjoint that computes geometric connected components, hence a geometric notion of π_0. We will see in 3.8, that as we pass to ∞-toposes, the extra left adjoint provides a good definition of all geometric homotopy groups.

Definition 3.3.1. An ∞-topos $\mathbf{H}$ we call *locally ∞-connected* if the (essentially unique) global section ∞-geometric morphism from prop. 2.2.4 is an *essential ∞-geometric morphism* in that it has a further left adjoint Π:

$$(\Pi \dashv \Delta \dashv \Gamma) : \ \mathbf{H} \underset{\Gamma}{\overset{\Pi}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\Delta}}}}} \infty\text{Grpd} \ .$$

If in addition Δ is fully faithful, then we say that $\mathbf{H}$ is in addition an *∞-connected* or *globally ∞-connected* ∞-topos.

Remark 3.3.2. Meanwhile, a locally ∞-connected ∞-topos as above has been called an ∞-topos *of constant shape* in [Lurie11], section A.1. Some of the following statements now overlap with the discussion there.

Proposition 3.3.3. *For* $\mathbf{H}$ *a locally/globally ∞-connected ∞-topos, the underlying 1-topos* $\tau_{\leq 0}\mathbf{H}$ *of 0-truncated objects (def. 3.6.22) is a* locally/globally connected topos *(as in [John03] C1.5, C3.3).*

Proof. By prop. 2.2.5 and by the very definition of truncated objects Γ takes 0-truncated objects in $\mathbf{H}$ to 0-truncated objects in ∞Grpd, hence the restriction $\Gamma|_{\tau_\leq}$ factors through the inclusion $\text{Set} \simeq \tau_{\leq 0}\infty\text{Grpd} \hookrightarrow \infty\text{Grpd}$.

Similarly the restriction $\Delta|_{\leq 0}$ factors through the inclusion $\tau_{\leq 0}\mathbf{H} \hookrightarrow \mathbf{H}$: by definition this is the case if for all $S \in \text{Set}$ and all $X \in \mathbf{H}$ the hom-∞-groupoid $\mathbf{H}(X, \Delta S) \in \infty\text{Grpd}$ is equivalently a set. But by the defining right-adjointness of Δ this is equivalently

$$\mathbf{H}(X, \Delta S) \simeq \infty\text{Grpd}(\Pi(X), S) \simeq \text{Set}(\tau_{\leq 0}\Pi(X), S) \in \text{Set} \hookrightarrow \infty\text{Grpd} \, ,$$

which is a set by assumption that S is 0-truncated.

By uniqueness of adjoints and the fact that $\tau_{\leq 0} : \infty\mathrm{Grpd} \to \mathrm{Set}$ is left adjoint to the inclusion, this means that $\Delta|_{\leq 0} : \mathrm{Set} \hookrightarrow \infty\mathrm{Grpd} \xrightarrow{\Delta} \mathbf{H}$ has a left adjoint

$$\Pi_0 := \tau_{\leq} \circ \Pi \,.$$

Finally $\tau_{\leq 0}$ preserves finite products by [LuHTT], lemma 6.5.1.2. and if Π preserves the terminal object then so does Π_0. □

Proposition 3.3.4. *A locally ∞-connected topos $(\Pi \dashv \Delta \dashv \Gamma) : \mathbf{H} \to \infty\mathrm{Grpd}$ is globally ∞-connected precisely if the following euivalent conditions hold.*

1. *The inverse image Δ is a fully faithful ∞-functor.*

2. *The extra left adjoint Π preserves the terminal object.*

Proof. This follows verbatim the proof for the familiar statement about connected toposes, since all the required properties have ∞-analogs: we have that

- Δ is fully faithful precisely if the $(\Pi \dashv \Delta)$-adjunction unit is an equivalence, by prop. 3.1.1.
- every ∞-groupoid S is the ∞-colimit over itself of the ∞-functor constant on the point, by prop. 2.3.1:

$$S \simeq \varinjlim_{S} * \,.$$

Therefore if Δ is fully faithful, then

$$\begin{aligned}\Pi(*) &\simeq \Pi\Delta(*)\\ &\simeq *\end{aligned}$$

and hence Π preserves the terminal object. Conversely, if Π preserves the terminal object then for any $S \in \infty\mathrm{Grpd}$ we have that

$$\begin{aligned}\Pi\Delta S &\simeq \Pi\Delta \varinjlim_{S} * \\ &\simeq \varinjlim_{S} \Pi\Delta * \\ &\simeq \varinjlim_{S} * \\ &\simeq S\end{aligned} \,.$$

and hence Δ is fully faithful. □

Proposition 3.3.5. *A locally ∞-connected ∞-topos*

1. *has the shape of $\Pi(*)$;*

2. *hence has the shape of the point if it is globally ∞-connected.*

Proof. By inspection of the definitions. □

3.3.2 Presentations

We discuss presentations of locally and globally ∞-connected ∞-toposes, def. 3.3.1, by categories of simplicial presheaves over a suitable site of definition.

Definition 3.3.6. We call a site (a small category equipped with a coverage) *locally and globally ∞-connected* if

1. it has a terminal object $*$;
2. for every generating covering family $\{U_i \to U\}$ in C
   (a) $\{U_i \to U\}$ is a *good covering*, def. 2.2.22: the Čech nerve $C(\{U_i\}) \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is degreewise a coproduct of representables;
   (b) the colimit $\varinjlim : [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{sSet}$ of $C(\{U_i\})$ is weakly contractible

$$\varinjlim C(\{U_i\}) \stackrel{\simeq}{\to} * \,.$$

Proposition 3.3.7. *For C a locally and globally ∞-connected site, the ∞-topos $Sh_\infty(C)$ is locally and globally ∞-connected.*

We prove this after noting two lemmas.

Lemma 3.3.8. *For $\{U_i \to U\}$ a covering family in the ∞-connected site C, the Čech nerve $C(\{U_i\}) \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is a cofibrant resolution of U both in the global projective model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ as well as in the local model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. By assumption on C we have that $C(\{U_i\})$ is a split hypercover [DuHoIs04]. This implies that $C(\{U_i\})$ is cofibrant in the global model structure. By general properties of left Bousfield localization we have that the cofibrations in the local model structure as the same as in the global one. Finally that $C(\{U_i\}) \to U$ is a weak equivalence in the local model structure holds effectively by definition (since we are localizing at these morphisms). □

Proposition 3.3.9. *On a locally and globally ∞-connected site C, the global section ∞-geometric morphsm $(\Delta \dashv \Gamma) : \mathrm{Sh}_\infty(C) \to \infty\mathrm{Grpd}$ is presented under prop. 2.1.33 by the simplical Quillen adjunction*

$$(\mathrm{Const} \dashv \Gamma) : \ [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}} \underset{\Gamma}{\overset{\mathrm{Const}}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}} \ ,$$

where Γ is the functor that evaluates on the terminal object, $\Gamma(X) = X(*)$, and where Const *is the functor that assigns constant presheaves* $\mathrm{Const}S : U \mapsto S$*.*

Proof. That we have a 1-categorical adjunction $(\mathrm{Const} \dashv \Gamma)$ follows by noticing that since C has a terminal object we have that $\Gamma = \varprojlim$ is given by the limit operation.

To see that we have a Quillen adjunction first notice that we have a Quillen adjunction on the global model structure

$$(\mathrm{Const} \dashv \Gamma) : \ [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \underset{\Gamma}{\overset{\mathrm{Const}}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}} \ ,$$

since Γ manifestly preserves fibrations and acyclic fibrations there. Because $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ is left proper and has the same cofibrations as the global model structure, it follows with prop. 2.1.34 that for this to descend to a Quillen adjunction on the local model structure it is sufficient that Γ preserves locally fibrant objects. But every fibrant object in the local structure is in particular fibrant in the global structure, hence in particular fibrant over the terminal object of C.

The left derived functor $\mathbb{L}\mathrm{Const}$ of $\mathrm{Const} : \mathrm{sSet}_{\mathrm{Quillen}} \to [C^{\mathrm{op}}, \mathrm{sSet}]$ preserves ∞-limits (because ∞-limits in an ∞-category of ∞-presheaves are computed objectwise), and moreover ∞-stackification, being the left derived functor of $\mathrm{Id} : [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \to [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, is a left exact ∞-functor, therefore the left derived functor of $\mathrm{Const} : \mathrm{sSet}_{\mathrm{Quillen}} \to [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ preserves finite ∞-limits.

This means that our Quillen adjunction does model an ∞-geometric morphism $\mathrm{Sh}_\infty(C) \to \infty\mathrm{Grpd}$. By prop. 2.2.4 this is indeed a representative of the terminal geometric morphism as claimed. $\square$

Proof of theorem 3.3.7. By general abstract facts the sSet-functor $\mathrm{Const} : \mathrm{sSet} \to [C^{\mathrm{op}}, \mathrm{sSet}]$ given on $S \in \mathrm{sSet}$ by $\mathrm{Const}(S) : U \mapsto S$ for all $U \in C$ has an sSet-left adjoint

$$\Pi : X \mapsto \int^U X(U) = \varinjlim X$$

naturally in X and S, given by the colimit operation. Notice that since sSet is itself a category of presheaves (on the simplex category), these colimits are degreewise colimits in Set. Also notice that the colimit over a representable functor is the point (by a simple Yoneda lemma-style argument).

Regarded as a functor $\mathrm{sSet}_{\mathrm{Quillen}} \to [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ the functor Const manifestly preserves fibrations and acyclic fibrations and hence

$$(\Pi \dashv \mathrm{Const}) : \ [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \underset{\mathrm{Const}}{\overset{\varinjlim}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}}$$

is a Quillen adjunction, in particular $\Pi : [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \to \mathrm{sSet}_{\mathrm{Quillen}}$ preserves cofibrations. Since by general properties of left Bousfield localization the cofibrations of $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ are the same, also $\Pi : [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}} \to \mathrm{sSet}_{\mathrm{Quillen}}$ preserves cofibrations.

Since $\mathrm{sSet}_{\mathrm{Quillen}}$ is a left proper model category it follows with prop. 2.1.34 that for

$$(\Pi \dashv \mathrm{Const}) : \ [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}} \underset{\mathrm{Const}}{\overset{\varinjlim}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}}$$

to be a Quillen adjunction, it suffices now that Const preserves fibrant objects. This means that constant simplicial presheaves satisfy descent along covering families in the ∞-cohesive site C: for every covering family $\{U_i \to U\}$ in C and every simplicial set S it must be true that

$$[C^{\mathrm{op}}, \mathrm{sSet}](U, \mathrm{Const}S) \to [C^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathrm{Const}S)$$

is a homotopy equivalence of Kan complexes. (Here we use that U, being a representable, is cofibrant, that $C(\{U_i\})$ is cofibrant by the lemma 3.3.8 and that $\mathrm{Const}S$ is fibrant in the projective structure by the assumption that S is fibrant. So the simplicial hom-complexes in the above equaltion really are the correct derived hom-spaces.)

But that this is the case follows by the condition on the ∞-connected site C by which $\varinjlim C(\{U_i\}) \simeq *$: using this we have that

$$[C^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathrm{Const}S) = \mathrm{sSet}(\varinjlim C(\{U_i\}), S) \simeq \mathrm{sSet}(*, S) = S\,.$$

So we have established that $(\varinjlim \dashv \mathrm{Const})$ is also a Quillen adjunction on the local model structure.

It is clear that the left derived functor of $\varinjlim$ preserves the terminal object: since that is representable by assumption on C, it is cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, hence $\mathbb{L}\varinjlim * \simeq \varinjlim * = *$. $\square$

3.4 Cohesive ∞-toposes

We now combine the notions of local ∞-toposes and ∞-connected ∞-toposes to that of cohesive ∞-toposes

3.4.1 General abstract

We give the definition and basic properties of cohesive ∞-toposes first externally, in 3.4.1.1 in terms of properties of the global section geometric morphism, and then internally, in the language of the internal logic of an ∞-topos, in 3.4.1.2.

3.4.1.1 External formulation The following definition is the direct generalization of the main axioms in the definition of *topos of cohesion* from [Lawv07].

Definition 3.4.1. A *cohesive ∞-topos* $\mathbf{H}$ is

1. a locally and globally ∞-connected topos $\mathbf{H}$, def 3.3.1,
2. which in addition is a *local ∞-topos*, def. 3.2.1;
3. and such that the extra left adjoint preserves not just the terminal object, but all finite products.

Remark 3.4.2. The two conditions say in summary that an ∞-topos is cohesive precisely if it admits quadruple of adjoint ∞-functors

$$(\Pi \dashv \Delta \dashv \Gamma \dashv \nabla) : \mathbf{H} \xymatrix{ \ar@<+9pt>[r]^{\Pi} \ar@{<-^{)}}@<+3pt>[r]|{\Delta} \ar@<-3pt>[r]|{\Gamma} \ar@{<-^{)}}@<-9pt>[r]_{\nabla} & } \infty\mathrm{Grpd}$$

such that Π preserves finite products.

We may think of these axioms as encoding properties that characterize those ∞-toposes of ∞-groupoids that are equipped with extra *cohesive structure*. In order to reflect this geometric interpretation notationally we will from now on write

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) : \mathbf{H} \xymatrix{ \ar@<+9pt>[r]^{\Pi} \ar@{<-^{)}}@<+3pt>[r]|{\mathrm{Disc}} \ar@<-3pt>[r]|{\Gamma} \ar@{<-^{)}}@<-9pt>[r]_{\mathrm{coDisc}} & } \infty\mathrm{Grpd}$$

for the defining ∞-connected and ∞-local geometric morphism and say for $S \in \infty\mathrm{Grpd}$ that

- $\mathrm{Disc}S \in \mathbf{H}$ is a *discrete object* of $\mathbf{H}$ or a *discrete cohesive ∞-groupoid* obtained by equipping S with *discrete cohesive structure*;
- $\mathrm{coDisc}S \in \mathbf{H}$ is a *codiscrete object* of $\mathbf{H}$ or a *codiscrete cohesive ∞-groupoid*, obtained by equipping S with *indiscrete cohesive structure*;

and for $X \in \mathbf{H}$ that

- $\Gamma(X) \in \infty\mathrm{Grpd}$ is the *underlying ∞-groupoid* of X;
- $\Pi(X)$ is the *fundamental ∞-groupoid* or *geometric path ∞-groupoid* of X.

A simple but instructive toy example illustrating these interpretations is given by the *Sierpinski ∞-topos*, discussed below in example 4.2.2. A detailed discussion of these geometric interpretations in various models is in 4.

Every adjoint quadruple of functors induces an adjoint triple of endofunctors:

Definition 3.4.3. On any cohesive ∞-topos $\mathbf{H}$ define the adjoint triple of functors

$$(\boldsymbol{\Pi} \dashv \flat \dashv \sharp) : \mathbf{H} \xymatrix{ \ar@<+6pt>[r]^{\Pi} \ar@{<-^{)}}[r]|{\mathrm{Disc}} \ar@<-6pt>[r]_{\Gamma} & } \infty\mathrm{Grpd} \xymatrix{ \ar@{^{(}->}@<+6pt>[r]^{\mathrm{Disc}} \ar@{<-}[r]|{\Gamma} \ar@{^{(}->}@<-6pt>[r]_{\mathrm{coDisc}} & } \mathbf{H} \,.$$

The geometric interpretation of these three functors is discussed below in 3.8.3, 3.8.5 and 3.7.2, respectively:

- $\mathbf{\Pi}$ is the *geometric path* or *geometric homotopy* functor;
- for $A \in \mathbf{H}$ we may pronounce $\flat A$ as "flat A", it is the coefficient for *flat cohomology* with coefficients in A;
- for $A \in \mathbf{H}$ we may pronounce $\sharp A$ as "sharp A", it is the classifying object for "sharply varying" A-principal ∞-bundles, those that need not be geometric (not continuous).

For emphasis we record the following list of pointlike properties of a cohesive ∞-topos.

Proposition 3.4.4. *A cohesive ∞-topos*

1. *has homotopy dimension 0;*
2. *has cohomological dimension 0;*
3. *has the shape of the point;*
4. *is hypercomplete.*

Proof. By prop. 3.2.2 and prop. 3.3.5. □

The following captures further aspects of the notion of cohesion encoded by a cohesive ∞-topos.

Definition 3.4.5. Given an object $X \in \mathbf{H}$ of a cohesive ∞-topos over $\infty\mathrm{Grpd}$, we say that

1. *pieces have points* in X if the canonical morphism

$$(\Gamma X \to \Pi X) := (\,\Gamma X \xrightarrow{\Gamma\iota} \Gamma\,\mathrm{Disc}\,\Pi X \xrightarrow{\simeq} \Pi X\,)$$

   is an effective epimorphism, def. 2.3.3.

2. X has *one point per piece* if this morphism is an equivalence.

For the class of cohesive ∞-toposes discussed below in 3.4.2 it is true for all their objects that *pieces have points*. A class of (relative) cohesive ∞-toposes for which this is not the case is discussed in 4.2.1.

3.4.1.2 Internal formulation The above discussion of cohesion looks at an ∞-topos "from the outside", namely as an object of the ∞-category of all ∞-toposes, and characterizes it in terms of additional properties of functors defined *on* it. Since any ∞-topos $\mathbf{H}$ also serves as an ambient context for homotopical mathematics formulated *internal* to it, it is desireable to have an equivalent reformulation of cohesion entirely in the internal language of $\mathbf{H}$.

This we discuss now. This section draws from discussion with and ideas of Mike Shulman.

Theorem 3.4.6. *Let $\mathbf{H}$ be an ∞-topos. The inclusion of a full sub-∞-category*

$$\mathrm{Disc} : \mathbf{B}_{\mathrm{disc}} \hookrightarrow \mathbf{H}$$

– to be called the discrete objects *– and of a full sub-∞-category*

$$\mathrm{coDisc} : \mathbf{B}_{\mathrm{cod}} \hookrightarrow \mathbf{H}$$

– to be called the codiscrete objects *– satisfies $\mathbf{B}_{\mathrm{disc}} \simeq \mathbf{B}_{\mathrm{cod}}$ and extends to an adjoint quadruple of the form*

$$\mathbf{H} \underset{\underset{\mathrm{coDisc}}{\hookleftarrow}}{\overset{\overset{\Pi}{\longrightarrow}}{\underset{\Gamma}{\overset{\mathrm{Disc}}{\hookleftarrow}\atop\longrightarrow}}} \mathbf{B}$$

as in def. 3.4.1 precisely if for every object $X \in \mathbf{H}$

1. *there exists, with notation from def. 3.4.3,*
    (a) *a morphism $X \to \mathbf{\Pi} X$ to a discrete object;*
    (b) *a morphism $\flat X \to X$ from a discrete object;*
    (c) *a morphism $X \to \sharp X$ to codiscrete object;*
2. *such that for all discrete Y and codiscrete $\tilde{Y}$ the induced morphisms*
    (a) $\mathbf{H}(\mathbf{\Pi} X, Y) \to \mathbf{H}(X, Y)$;
    (b) $\mathbf{H}(Y, \flat X) \to \mathbf{H}(Y, X)$;
    (c) $\mathbf{H}(\sharp X, \tilde{Y}) \to \mathbf{H}(X, \tilde{Y})$;
    (d) $\sharp(\flat X \to X)$;
    (e) $\flat(X \to \sharp X)$

    are equivalences.

Finally, Π *preserves the terminal object if the morphism $* \to \mathbf{\Pi} *$ is an equivalence.*

Proof. Prop. 5.2.7.8 in [LuHTT] asserts that a full sub-∞-category $\mathbf{B} \hookrightarrow \mathbf{H}$ is reflectively embedded precisely if for every object $X \in \mathbf{H}$ there is a morphism

$$\mathrm{loc}_X : X \to \mathbf{L} X$$

to an object $\mathbf{L} X \in \mathbf{H} \hookrightarrow \mathbf{H}$ such that for all $Y \in \mathbf{B} \hookrightarrow \mathbf{H}$ the morphism

$$\mathbf{H}(\mathrm{loc}_X, Y) : \mathbf{H}(\mathbf{L} X, Y) \to \mathbf{H}(X, Y)$$

is an equivalence. In this case $\mathbf{L}$ is the composite of the embedding and its left adjoint. Accordingly, a dual statement holds for coreflective embeddings. This gives the structure and the first three properties of the above assertion. We identify therefore

$$(\mathbf{\Pi} \dashv \flat \dashv \sharp) := (\mathrm{Disc}\,\Pi \dashv \mathrm{Disc}\,\Gamma \dashv \mathrm{coDisc}\,\Gamma)\,.$$

It remains to show that the last two properties say precisely that the sub-∞-categories of discrete and codiscrete objects are equivalent and that under this equivalence their coreflective and reflective embedding, respectively, fits into a single adjoint triple. It is clear that if this is the case then the last two properties hold. We show the converse.

First notice that the two embeddings always combine into an adjunction of the form

$$\mathbf{B}_{\mathrm{disc}} \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \mathbf{H} \underset{\mathrm{coDisc}}{\overset{\tilde{\Gamma}}{\rightleftarrows}} \mathbf{B}_{\mathrm{cod}}\,.$$

The equivalence $\sharp(\flat X \to X)$ applied to $X := \mathrm{coDisc} A$ gives that coDisc applied to the counit of this composite adjunction is an equivalence

$$\mathrm{coDisc}\,\tilde{\Gamma}\,\mathrm{Disc}\,\Gamma\mathrm{coDisc} A \xrightarrow{\simeq} \mathrm{coDisc}\,\tilde{\Gamma}\,\mathrm{coDisc} A \xrightarrow{\simeq} \mathrm{coDisc} A$$

and since coDisc is full and faithful, so is the composite counit itself. Dually, the equivalence $\flat(X \to \sharp X)$ implies that the unit of this composite adjunction is an equivalence. Hence the adjunction itself is an equivalence, and so $\mathbf{B}_{\mathrm{disc}} \simeq \mathbf{B}_{\mathrm{cod}}$. Using this we obtain a composite equivalence

$$\mathrm{Disc}\,\tilde{\Gamma} X \xrightarrow{\simeq} \mathrm{Disc}\,\Gamma\mathrm{coDisc}\,\tilde{\Gamma} X \xrightarrow{\simeq} \mathrm{Disc}\,\Gamma X\,,$$

where the left morphism is the image under Disc of the ave composite adjunction on the codiscrete object $\tilde{\Gamma} X$, and where the second is a natural inverse of $\flat(X \to \sharp X)$. Since Disc is full and faithful, this implies that

$$\Gamma \simeq \tilde{\Gamma} .$$

□

This formulation of cohesion is not entirely internal yet, since it still refers to the external hom ∞-groupoids $\mathbf{H}$. But cohesion also implies that the external ∞-groupoids can be re-internalized.

Proposition 3.4.7. *The statement of theorem 3.4.6 remains true with items 2. a) - 2. b) replaced by*

2. (a') $\sharp[\mathbf{\Pi} X, Y] \to \sharp[X, Y]$;

2. (b') $\sharp[Y, \flat X] \to \sharp[Y, X]$;

2. (c') $[\sharp X, \tilde{Y}] \to [X, \tilde{Y}]$;

where $[-,-]$ denotes the internal hom in $\mathbf{H}$.

Proof. By prop. 3.7.2 we have for codiscrete $\tilde{Y}$ equivalences $[X, \tilde{Y}] \simeq \mathrm{coDisc}\mathbf{H}(X, \tilde{Y})$. Since coDisc is full and faithful, the morphism $\mathbf{H}(\sharp X, \tilde{Y}) \to \mathbf{H}(X, \tilde{Y})$ is an equivalence precisely if $[\sharp X, \tilde{Y}] \to [X, \tilde{Y}]$ is.

Generally, we have $\Gamma[X, Y] \simeq \mathbf{H}(X, Y)$. With the full and faithfulness of coDisc this similarly gives the remaining statements. □

3.4.2 Presentation

We discuss the presentation of cohesive ∞-toposes, in the sense of presentation of ∞-toposes as discussed in 2.2.3. In 3.4.2.1 we consider sites such that the ∞-topos of ∞-sheaves over them is cohesive. In 3.4.2.2 we analyze fibrancy and descent over these sites.

3.4.2.1 Presentation over ∞-cohesive sites We discuss a class of sites with the property that the ∞-toposes of ∞-sheaves over them (2.2.3) are cohesive, def. 3.4.1.

Definition 3.4.8. An ∞*-cohesive site* is a site such that

1. it has finite products;
2. every object $U \in C$ has at least one point: $C(*, U) \neq \emptyset$;
3. for every covering family $\{U_i \to U\}$ its Čech nerve $C(\{U_i\}) \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is degreewise a coproduct of representables
4. the canonical morphisms $C(\{U_i\}) \to U$ are taken to weak equivalences by both limit and colimit $[C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{sSet}$:

$$\begin{aligned} \varinjlim C(\{U_i\}) &\xrightarrow{\simeq} \varinjlim U_i \\ \varprojlim C(\{U_i\}) &\xrightarrow{\simeq} \varprojlim U_i \end{aligned} .$$

Notice that for the representable U we have $\lim_{\to} U \simeq *$ and that since C is assumed to have finite products and hence in particular a terminal object $\lim_{\leftarrow} U = C(*, U)$.

Proposition 3.4.9. *The ∞-sheaf ∞-topos over an ∞-cohesive site is a cohesive ∞-topos in which for all objects* pieces have points*, def. 3.4.5.*

Proof. Since an ∞-cohesive site is in particular a locally and globally ∞-connected site (def. 3.3.6) it follows with theorem 3.3.7 that Π exists and preserves the terminal object. Moreover, by the discussion there Π acts by sending a fibrant-cofibrant simplicial presheaf $F : C^{\mathrm{op}} \to \mathrm{sSet}$ to its colimit. Since C is assumed to have finite products, C^{op} has finite coproducts, hence is a sifted category. Therefore taking colimits of functors on C^{op} commutes with taking products of these functors. Since the ∞-product of ∞-presheaves is modeled by the ordinary product on fibrant simplicial presheaves, it follows that over an ∞-cohesive site Π indeed exhibits a strongly ∞-connected ∞-topos.

Using the notation and results of the proof of theorem 3.3.7, we show that the further right adjoint Δ exists by exhibiting a suitable right Quillen adjoint to $\Gamma : [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{sSet}$, which is given by evaluation on the terminal object. Its sSet-enriched right adjoint is given by

$$\nabla S : U \mapsto \mathrm{sSet}(\Gamma(U), S)$$

as confirmed by the following end/coend computation:

$$\begin{aligned}
(X, \nabla(S)) &= \int_{U \in C} \mathrm{sSet}(X(U), \mathrm{sSet}(\Gamma(U), S)) \\
&= \int_{U \in C} \mathrm{sSet}(X(U) \times \Gamma(U), S) \\
&= \mathrm{sSet}(\int^{U \in C} X(U) \times \Gamma(U), \quad S) \\
&= \mathrm{sSet}(\int^{U \in C} X(U) \times \mathrm{Hom}_C(*, U), \quad S) \\
&= \mathrm{sSet}(X(*), S) \\
&= \mathrm{sSet}(\Gamma(X), S)
\end{aligned} \quad ,$$

We have that

$$(\Gamma \dashv \nabla) : [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \underset{\nabla}{\overset{\Gamma}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}}$$

is a Quillen adjunction, since ∇ manifestly preserves fibrations and acyclic fibrations. Since $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ is a left proper model category, to see that this descends to a Quillen adjunction on the local model structure it is sufficient by prop. 2.1.34 to check that $\nabla : \mathrm{sSet}_{\mathrm{Quillen}} \to [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ preserves fibrant objects, in that for S a Kan complex we have that ∇S satisfies descent along Čech nerves of covering families.

This is implied by the second defining condition on the ∞-local site C, that $\underset{\longleftarrow}{\lim} C(\{U_i\}) = \mathrm{Hom}_C(*, C(\{U_i\})) \simeq \mathrm{Hom}_C(*, U) = \underset{\longleftarrow}{\lim} U$ is a weak equivalence. Using this we have for fibrant $S \in \mathrm{sSet}_{\mathrm{Quillen}}$ the descent weak equivalence

$$\begin{aligned}
[C^{\mathrm{op}}, \mathrm{sSet}](U, \nabla S) &= \mathrm{sSet}(\mathrm{Hom}_C(*, U), S) \\
&\simeq \mathrm{sSet}(\mathrm{Hom}_C(*, C(U)), S)\,, \\
&= [C^{\mathrm{op}}, \mathrm{sSet}](C(U), \nabla S)
\end{aligned}$$

where we use in the middle step that $\mathrm{sSet}_{\mathrm{Quillen}}$ is a simplicial model category so that homming the weak equivalence between cofibrant objects into the fibrant object S indeed yields a weak equivalence.

It remains to show that *pieces have points*, def. 3.4.5, in $\mathrm{Sh}_\infty(C)$. For the first statement we use the cofibrant replacement theorem from [Dugg01] for $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ which says that for X any simplicial presheaf, a functorial projective cofibrant replacement is given by the object

$$QX := \left(\cdots \overrightarrow{\overrightarrow{\longrightarrow}} \coprod_{U_0 \to U_1 \to X_1} U_0 \overrightarrow{\longrightarrow} \coprod_{U_0 \to X_0} U_0 \right) ,$$

where the coproducts are over the set of morphisms of presheaves from representables U_i as indicated. By the above discussion, the presentations of Γ and Π by left Quillen functors $\underset{\leftarrow}{\lim}$ and $\underset{\longrightarrow}{\lim}$ takes this to the morphism $\lim_{\leftarrow} QX \to \lim_{\rightarrow} QX$ induced in components by

$$\begin{array}{ccccc} \cdots \rightrightarrows & \coprod_{U_0 \to U_1 \to X_1} C(*.U_0) & \rightrightarrows & \coprod_{U_0 \to X_0} C(*, U_0) & . \\ & \downarrow & & \downarrow & \\ \cdots \rightrightarrows & \coprod_{U_0 \to U_1 \to X_1} * & \rightrightarrows & \coprod_{U_0 \to X_0} * & \end{array}$$

By assumption on C we have that all sets $C(*, U_0)$ are non-empty, so that this is componentwise an epimorphism and hence induces in particular an epimorphism on connected components.

Finally, for S a Kan complex we have by the above that Disc$S$ is the presheaf constant on S. Its homotopy sheaves are the presheaves constant on the homotopy groups of S. The inclusion of these into the homotopy sheaves of coDisc$S$ is over each $U \in C$ the diagonal injection

$$\pi_n(S, x) \hookrightarrow \pi_n(S, x)^{C(*,U)} .$$

Therefore also *discrete objects are concrete* in the ∞-topos over the ∞-cohesive site C. □

Below in 4 we discuss in detail the following examples.

Examples 3.4.10. The following sites are ∞-cohesive.

- The site $\mathrm{CartSp}_{\mathrm{top}}$ of Cartesian spaces, continuous maps between them and good open covers (prop. 4.3.2).
- The site $\mathrm{CartSp}_{\mathrm{smooth}}$ of Cartesian spaces, smooth maps between them and good open covers (prop. 4.4.6),
- The site $\mathrm{CartSp}_{\mathrm{SynthDiff}}$ of Cartesian spaces with infinitesimal thickening, smooth maps between the and good open covers that are the identity on the thickening (prop. 4.5.8).
- The site $\mathrm{CartSp}_{\mathrm{super}}$ of super-Cartesian spaces, morphisms of supermanifolds between them and good open covers (prop. 4.6.10).

We record a fact that is expected to hold quite generally for ∞-toposes, but for which we currently have a proof only over ∞-connected sites.

Theorem 3.4.11 (parameterized ∞-Grothendieck construction)**.** *Let* $\mathbf{H}$ *be an ∞-topos with an ∞-connected site of definition, def. 3.3.6, and let* $A \in \infty\mathrm{Grpd}$ *be any ∞-groupoid. Then there is an equivalence of ∞-categories*

$$\mathbf{H}/_{\mathrm{Disc}A} \simeq \mathbf{H}^A$$

between the slice ∞-topos of $\mathbf{H}$ *over the discrete cohesive ∞-groupoid on A and the ∞-category of ∞-functors* $A \to \mathbf{H}$.

Proof. For the case that the site of definition is terminal, hence that $\mathbf{H} \simeq \infty\mathrm{Grpd}$ this statement is the ∞-*Grothendieck construction* from section 2 of [LuHTT]. There the equivalence of ∞-categories

$$\infty\mathrm{Grpd}_{/A} \simeq \infty\mathrm{Grpd}^A$$

which takes a fibration to an ∞-functor that assigns its fibers is presented by a Quillen equivalence of model categories

$$\mathrm{sSet}^+/A \rightleftarrows [w(A)^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$$

between a model structure on *marked simplicial sets* sSet^+ over a Kan complex A and the global projective model structure on enriched presheaves on the simplically enriched category $w(A)$ corresponding to A by the discussion in section 1.1.5 of [LuHTT].

Now for C an ∞-connected site and $\mathbf{H} \simeq ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ$ we have by the proof of prop. 3.3.7 that with A a Kan complex, the constant simplicial presheaf $\mathrm{const}A : C^{\mathrm{op}} \to \mathrm{sSet}$ is a fibrant presentation in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ of $\mathrm{Disc}A$. Therefore the ∞-categorical slice $\mathbf{H}_{/\mathrm{Disc}A}$ is presented by the induced model structure on the 1-categorical slice category

$$\mathbf{H}_{/\mathrm{Disc}A} \simeq \left(([C^{\mathrm{op}}, \mathrm{sSet}]_{/\mathrm{const}A})_{\mathrm{proj,loc}/\mathrm{const}A}\right)^\circ .$$

We have an evident equivalence of 1-categories

$$[C^{\mathrm{op}}, \mathrm{sSet}]_{/\mathrm{const}A} \simeq [C^{\mathrm{op}}, \mathrm{sSet}_{/A}]$$

under which the above slice model structure is seen to become the model structure on presheaves with values in the slice model structure $(\mathrm{sSet}_{/A})_{\mathrm{Quillen}/A}$, hence

$$\mathbf{H}_{/\mathrm{Disc}A} \simeq \left([C^{\mathrm{op}}, (\mathrm{sSet}_{/A})_{\mathrm{Quillen}/A}]_{\mathrm{proj,loc}}\right)^\circ .$$

Since A is fibrant in the Quillen model structure, the slice model structure here presents the ∞-categorical slice of ∞-groupoids

$$\infty\mathrm{Grpd}_{/A} \simeq \left((\mathrm{sSet}_{/A})_{\mathrm{Quillen}/A}\right)^\circ .$$

By the above presentation of the ∞-Grothendieck construction by marked simplicial sets, this is equivalently

$$\cdots \simeq \left(\mathrm{sSet}^+/A\right)^\circ \simeq ([w(A)^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}})^\circ .$$

Since all model categories appearing here are combinatorial, it follows with prop. 4.2.4.4 in [LuHTT] that we have an equivalence of ∞-categories

$$\mathbf{H}_{/\mathrm{Disc}A} \simeq ([C^{\mathrm{op}}, [w(A)^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{proj,loc}})^\circ$$

and hence

$$\cdots \simeq ([w(A)^{\mathrm{op}}, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{proj}})^\circ \simeq \mathbf{H}^A .$$

□

3.4.2.2 Fibrancy over ∞-cohesive sites

The condition on an object $X \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ to be fibrant models the fact that X is an *∞-presheaf of ∞-groupoids*. The condition that X is also fibrant as an object in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ models the higher analog of the sheaf condition: it makes X an *∞-sheaf*. For generic sites C fibrancy in the local model structure is a property rather hard to check or establish concretely. But often a given site can be replaced by another site on which the condition is easier to control, without changing the corresponding ∞-topos, up to equivalence. Here we discuss for cohesive sites, def. 3.4.8 explicit conditions for a simplicial presheaf over them to be fibrant.

In order to discuss descent over C it is convenient to introduce the following notation for "cohomology over the site C". For the moment this is just an auxiliary technical notion. Later we will see how it relates to an intrinsically defined notion of cohomology.

Definition 3.4.12. For C an ∞-cohesive site, $A \in [C^{\mathrm{op}}, \mathrm{Set}]_{\mathrm{proj}}$ fibrant, and $\{U_i \to U\}$ a good cover in U, we write

$$H^n_C(\{U_i\}, A) := \pi_0 \mathrm{Maps}(C(\{U_i\}), A) .$$

Moreover, if A is equipped with (abelian) group structure we write

$$H^n_C(\{U_i\}, A) := \pi_0 \mathrm{Maps}(C(\{U_i\}), \overline{W}^n A) .$$

Definition 3.4.13. An object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is called *C-acyclic* if

1. it is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$;
2. for all $n \in \mathbb{N}$ the homotopy group presheaves π_n^{PSh} from def. 2.2.7 are already sheaves $\pi_n(A) \in \mathrm{Sh}(C)$;
3. for $n = 1$ and $k = 1$ as well as $n \geq 2$ and $k \geq 1$ we have $H_C^k(\{U_i\}, \pi_n(A)) \simeq *$ for all good covers $\{U_i \to U\}$.

Remark 3.4.14. This definition can be formulated and the following statements about it are true over any site whatsoever. However, on generic sites C the C-acyclic objects are not very interesting. On ∞-cohesive sites on the other hand they are of central importance.

Observation 3.4.15. If A is C-acyclic then for every point $x : * \to A$ also $\Omega_x A$ is C-acyclic (for any model of the loop space object in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$).

Proof. The standard statement in $\mathrm{sSet}_{\mathrm{Quillen}}$

$$\pi_n \Omega X \simeq \pi_{n+1} X$$

directly prolongs to $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. □

Theorem 3.4.16. *Let C be an ∞-cohesive site. Sufficient conditions for an object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]$ to be fibrant in the local model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ are*

- *A is 0-truncated and C-acyclic;*
- *A is connected and C-acyclic;*
- *A is a group object and C-acyclic.*

Here and in the following "truncated" and "connected" are as simplicial presheaves (not after sheafification of homotopy presheaves).

We demonstrate this statement in several stages.

Proposition 3.4.17. *A 0-truncated object is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ precisely if it is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and weakly equivalent to a sheaf: to an object in the image of the canonical inclusion*

$$\mathrm{Sh}_C \hookrightarrow [C^{\mathrm{op}}, \mathrm{Set}] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}] .$$

Proof. From general facts of left Bousfield localization we have that the fibrant objects in the local model structure are necessarily fibrant also in the global structure.

Since moreover $A \to \pi_0(A)$ is a weak equivalence in the global model structure by assumption, we have for every covering $\{U_i \to U\}$ in C a sequence of weak equivalences

$$\mathrm{Maps}(C(\{U_i\}), A) \xrightarrow{\simeq} \mathrm{Maps}(C(\{U_i\}), \pi_0(A)) \xrightarrow{\simeq} \mathrm{Maps}(\pi_0 C(\{U_i\}), \pi_0(A)) \xrightarrow{\simeq} \mathrm{Sh}_C(S(\{U_i\}), \pi_0(A)) ,$$

where $S(\{U_i\}) \hookrightarrow U$ is the sieve corresponding to the cover. Therefore the descent condition

$$\mathrm{Maps}(U, A) \xrightarrow{\simeq} \mathrm{Maps}(C(\{U_i\}), A)$$

is precisely the sheaf condition for $\pi_0(A)$. □

Proposition 3.4.18. *A connected fibrant object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ if for all objects $U \in C$*

1. *$H_C(U, A) \simeq *$;*

2. ΩA is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$,

where ΩA is any fibrant object in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ representing the looping of A.

Proof. For $\{U_i \to U\}$ a covering we need to show that the canonical morphism

$$\mathrm{Maps}(U, A) \to \mathrm{Maps}(C(\{U_i\}), A)$$

is a weak homotopy equivalence. This is equivalent to the two morphisms

1. $\pi_0 \mathrm{Maps}(U, A) \to \pi_0 \mathrm{Maps}(C(\{U_i\}), A)$
2. $\Omega \mathrm{Maps}(U, A) \to \Omega \mathrm{Maps}(C(\{U_i\}), A)$

being weak equivalences. Since A is connected the first of these says that there is a weak equivalence $* \stackrel{\simeq}{\to} H_C(U, A)$. The second condition is equivalent to $\mathrm{Maps}(U, \Omega A) \to \mathrm{Maps}(C(\{U_i\}), \Omega A)$, being a weak equivalence, hence to the descent of ΩA. □

Proposition 3.4.19. *An object A which is connected, 1-truncated and C-acyclic is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. Observe that for a connected and 1-truncated objects we have a weak equivalence $A \simeq \overline{W}\pi_1(A)$ in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. The first condition of prop. 3.4.18 is then implied by C-connectedness. The second condition there is that $\pi_1(A)$ satisfies descent. By C-acyclicity this is a sheaf and it is 0-truncated by assumption, therefore it satisfies descent by prop 3.4.17. □

Proposition 3.4.20. *Every connected and C-acyclic object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. We first show the statement for truncated A and afterwards for the general case.

The k-truncated case in turn we consider by induction over k. If A is 1-truncated the proposition holds by prop. 3.4.19. Assuming then that the statement has been shown for k-truncated A, we need to show it for $(k+1)$-truncated A.

This we do by decomposing A into its canonical Postnikov tower def. 3.6.25: For $n \in \mathbb{N}$ let

$$A(n) := A/_{\sim_n}$$

be the quotient simplicial presheaf where two cells

$$\alpha, \beta : \Delta^n \times U \to A$$

are identified, $\alpha \sim_n \beta$, precisely if they agree on their n-skeleton:

$$\mathrm{sk}_n \alpha = \mathrm{sk}_n \beta : \mathrm{sk}_n \Delta \hookrightarrow \Delta^n \to A(U)\,.$$

It is a standard fact (shown in [GoJa99], theorem VI 3.5 for simplicial sets, which generalizes immediately to the global model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$) that for all $n > 1$ we have sequences

$$K(n) \to A(n) \to A(n-1)\,,$$

where $A(n-1)$ is $(n-1)$-truncated with homotopy groups in degree $\leq n-1$ those of A, and where the right morphism is a Kan fibration and the left morphism is its kernel, such that

$$A = \varprojlim_n A(n)\,.$$

Moreover, there are canonical weak homotopy equivalences

$$K(n) \to \Xi((\pi_{n-1} A)[n])$$

to the Eilenberg-MacLane object on the nth homotopy group in degree n.

Since $A(n-1)$ is $(n-1)$-truncated and connected, the induction assumption implies that it is fibrant in the local model structure.

Moreover we see that $K(n)$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$: the first condition of 3.4.18 holds by the assumption that A is C-connected. The second condition is implied again by the induction hypothesis, since $\Omega K(n)$ is $(n-1)$-truncated, connected and still C-acyclic, by observation 3.4.15.

Therefore in the diagram (where $\mathrm{Maps}(-,-)$ denotes the simplicial hom complex)

$$\begin{array}{ccccc}
\mathrm{Maps}(U, K(n)) & \longrightarrow & \mathrm{Maps}(U, A(n)) & \longrightarrow & \mathrm{Maps}(U, A(n-1)) \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq \\
\mathrm{Maps}(C(\{U_i\}), K(n)) & \longrightarrow & \mathrm{Maps}(C(\{U_i\}), A(n)) & \longrightarrow & \mathrm{Maps}(C(\{U_i\}), A(n-1))
\end{array}$$

for $\{U_i \to U\}$ any good cover in C the top and bottom rows are fiber sequences (notice that all simplicial sets in the top row are connected because A is connected) and the left and right vertical morphisms are weak equivalences in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ (the right one since $A(n-1)$ is fibrant in the local model structure by induction hypothesis, as remarked before, and the left one by C-acyclicity of A). It follows that also the middle morphism is a weak equivalence. This shows that $A(n)$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. By completing the induction the same then follows for the object A itself.

This establishes the claim for truncated A. To demonstrate the claim for general A notice that the limit over a sequence of fibrations between fibrant objects is a homotopy limit (by example 2.3.16). Therefore we have

$$\begin{array}{ccc}
\mathrm{Maps}(U, A) & \simeq & \varprojlim_n \mathrm{Maps}(U, A(n)) \\
\downarrow & & \downarrow \simeq \\
\mathrm{Maps}(C(\{U_i\}), A) & \simeq & \varprojlim_n \mathrm{Maps}(C(\{U_i\}), A(n))
\end{array},$$

where the right vertical morphism is a morphism between homotopy limits in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ induced by a weak equivalence of diagrams, hence is itself a weak equivalence. Therefore A is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. □

Lemma 3.4.21. *For $G \in [C^{\mathrm{op}}, \mathrm{sSet}]$ a group object, the canonical sequence*

$$G_0 \to G \to G/G_0$$

is a homotopy fiber sequence in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. Since homotopy pullbacks of presheaves are computed objectwise, it is sufficient to show this for $C = *$, hence in $\mathrm{sSet}_{\mathrm{Quillen}}$. One checks that generally, for X a Kan complex and G a simplicial group acting on X, the quotient morphism $X \to X/G$ is a Kan fibration. Therefore the homotopy fiber of $G \to G/G_0$ is presented by the ordinary fiber in sSet. Since the action of G_0 on G is free, this is indeed $G_0 \to G$. □

Proposition 3.4.22. *Every C-acyclic group object $G \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ for which G_0 is a sheaf is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. By lemma 3.4.21 we have a fibration sequence

$$G_0 \to G \to G/G_0\,.$$

Since G_0 is assumed to be a sheaf it is fibrant in the local model structure by prop. 3.4.17. Since G/G_0 is evidently connected and C-acyclic it is fibrant in the local model structure by prop. 3.4.20. As before in the proof there this implies that also G is fibrant in the local model structure. □

We discuss some examples.

Proposition 3.4.23. *Let* $(\delta : G_1 \to G_0)$ *be a crossed module, def. 1.2.45, of sheaves over an* ∞*-cohesive site* C*. Then the simplicial delooping* $\bar{W}(G_1 \to G_0)$ *is fibrant in* $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *if the image factorization of* $G_0 \times G_1 \to G_0 \times G_0$ *has sections over each* $U \in C$ *and if the presheaf* $\ker\delta$ *is a sheaf.*

Proof. The existence of the lift ensures that the homotopy presheaf $\pi_1^{\mathrm{PSh}} \bar{W}G$ is a sheaf. Notice that $\pi_2^{\mathrm{PSh}} \bar{W}G = \ker(\delta)$. Since moreover $\bar{W}G$ is manifestly connected, the claim follows with theorem 3.4.16. □

3.5 Differential cohesive ∞-toposes

We discuss extra structure on a cohesive ∞-topos that encodes a refinement of the corresponding notion of cohesion to a notion of what may be called *infinitesimal cohesion* or *differential cohesion*. With respect to such it makes sense to ask if an object in the topos has *infinitesimal extension*.

A basic class of examples of objects with infinitesimal extension are *infinitesimal intervals* $\mathbb{D}$ that arise, in the presence of infinitesimal cohesion, from *line objects* $\mathbb{A}$ as the subobjects $\mathbb{D} \hookrightarrow \mathbb{A}$ of elements that square to 0 (in the internal logic of the topos)

$$\mathbb{D} = \{x \in \mathbb{A} | x \cdot x = 0\} .$$

These objects co-represent tangent spaces, in that for any other object X the internal hom object $TX := [\mathbb{D}, X]$ plays the role of the *tangent bundle* of X.

A well-known proposal for an axiomatic characterization of infinitesimal objects in a 1-topos goes by the name *synthetic differential geometry* [Lawv97], where infinitesimal extension is characterized by algebraic properties of dual function algebras, as above. From the point of view and in the presence of cohesion in an ∞-topos, however, there is a more immediate geometric characterization: an object $\mathbb{D}$ in a cohesive ∞-topos $\mathbf{H}$ behaves like a possibly infinitesimally thickened point if

1. it is geometrically contractible, $\Pi(\mathbb{D}) \simeq *$;

2. it has a single global point, $\Gamma(\mathbb{D}) \simeq *$.

This axiomatization we discuss in the following. We show that it formalizes a modern refinement of infinitesimal calculus called *$\mathcal{D}$-geometry* [BeDr04] [Lurie09c].

More precisely, we consider geometric inclusions $\mathbf{H} \hookrightarrow \mathbf{H}_{\mathrm{th}}$ of cohesive ∞-toposes that exhibit the objects of $\mathbf{H}_{\mathrm{th}}$ as infinitesimal cohesive neighbourhoods of objects in $\mathbf{H}$. Equivalently, if the cohesive ∞-topos $\mathbf{H}$ is itself regarded as a fat point by prop. 3.4.4, then $\mathbf{H}_{\mathrm{th}}$ is an infinitesimal thickening of that fat point itself. Below in 3.10.9 we furthermore consider the cofiber $\mathbf{H}_{\mathrm{inf}}$ of this inclusion

$$\begin{array}{ccc} \mathbf{H} & \hookrightarrow & \mathbf{H}_{\mathrm{th}} \\ \downarrow & & \downarrow \\ \infty\mathrm{Grpd} & \hookrightarrow & \mathbf{H}_{\mathrm{inf}} \end{array} .$$

This cofiber is interpreted accordingly as the respective infinitesimal thickening of the absolute point. We observe in 4.5.1.4 that the sub-∞-category of globally trivial objects of $\mathbf{H}_{\mathrm{inf}}$ is equivalent to that of L_∞-algebras, by the theory of "formal moduli problems" of [LurieFormalGeometry]. Moreover, the reflection along

$$\mathrm{Grp}(\mathbf{H}_{\mathrm{th}}) \simeq (\mathbf{H}_{\mathrm{th}})^{*/}_{\geq 1} \to (\mathbf{H}_{\mathrm{inf}})^{*/}_{\geq 1}$$

is Lie differentiation, sending a cohesive ∞-group to the L_∞-algebra that approximates it infinitesimally.

Below in 3.10 we discuss a list of structures that are canonically present in infinitesimal cohesive neighbourhoods.

Further below in 4.5 we discuss a model for these axioms which is an ∞-categorical generalization of a topos that is a model for *synthetic differential geometry.*

3.5.1 General abstract

Definition 3.5.1. Given a cohesive ∞-topos $\mathbf{H}$ we say that an *infinitesimal cohesive neighbourhood* of $\mathbf{H}$ is a geometric embedding $i : \mathbf{H} \hookrightarrow \mathbf{H}_{\mathrm{th}}$ into another cohesive ∞-topos $\mathbf{H}_{\mathrm{th}}$, such that there is an extra left adjoint $i_!$ (necessarily also full and faithful) and an extra right adjoint $i^!$

$$(i_! \dashv i^* \dashv i_* \dashv i^!) : \mathbf{H} \begin{array}{c} \xhookrightarrow{i_!} \\ \xleftarrow{i^*} \\ \xhookrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathbf{H}_{\mathrm{th}}$$

and such that $i_!$ preserves finite products.

When we think of this as exhibiting extra structure on $\mathbf{H}_{\mathrm{th}}$, we call $\mathbf{H}_{\mathrm{th}}$ equipped with this embedding a *differental cohesive ∞-topos* or *differential ∞-topos* for short.

Remark 3.5.2. This definition captures the characterization of infinitesimal objects as having a single global point surrounded by an infinitesimal neighbourhood: as we discuss in detail below in 3.10.1, the ∞-functor i^* may be thought of as contracting away any infinitesimal extension of an object. Thus X being an infinitesimal object amounts to $i^*X \simeq *$, and the ∞-adjunction $(i_! \dashv i^*)$ then implies that X has only a single global point, since

$$\begin{aligned} \mathbf{H}_{\mathrm{th}}(*, X) &\simeq \mathbf{H}_{\mathrm{th}}(i_!*, X) \\ &\simeq \mathbf{H}(*, i^*X) \\ &\simeq \mathbf{H}(*, *) \\ &\simeq * \end{aligned} .$$

Observation 3.5.3. The inclusion into the infinitesimal neighbourhood is necessarily a morphism of ∞-toposes over ∞Grpd.

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{(i^* \dashv i_*)} & \mathbf{H}_{\mathrm{th}} \\ & {\scriptstyle \Gamma_{\mathbf{H}}} \searrow \quad \swarrow {\scriptstyle \Gamma_{\mathbf{H}_{\mathrm{th}}}} & \\ & \infty\mathrm{Grpd} & \end{array}$$

as is the induced ∞-geometric morphism $(i_* \dashv i^!) : \mathbf{H}_{\mathrm{th}} \to \mathbf{H}$:

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{th}} & \xrightarrow{(i_* \dashv i^!)} & \mathbf{H} \\ & {\scriptstyle \Gamma_{\mathbf{H}_{\mathrm{th}}}} \searrow \quad \swarrow {\scriptstyle \Gamma_{\mathbf{H}}} & \\ & \infty\mathrm{Grpd} & \end{array} .$$

Proof. By essential uniqueness of the terminal global section geometric morphism, prop. 2.2.4. In both cases the direct image functor has as left adjoint that preserves the terminal object. Therefore we compute in the first case

$$\begin{aligned} \Gamma_{\mathbf{H}_{\mathrm{th}}}(i_*X) &\simeq \mathbf{H}_{\mathrm{th}}(*, i_*X) \\ &\simeq \mathbf{H}(i^**, X) \\ &\simeq \mathbf{H}(*, X) \\ &\simeq \Gamma_{\mathbf{H}}(X) \end{aligned}$$

and analogously in the second. □

Definition 3.5.4. For $(i_! \dashv i^* \dashv i_* \dashv i^!) : \mathbf{H} \to \mathbf{H}_{\mathrm{th}}$ an infinitesimal neighbourhood of a cohesive ∞-topos, we write

$$(\Pi_{\mathrm{inf}} \dashv \mathrm{Disc}_{\mathrm{inf}} \dashv \Gamma_{\mathrm{inf}}) := (i^* \dashv i_* \dashv i^!) ,$$

so that the locally connected terminal geometric morphism of $\mathbf{H}_{\mathrm{th}}$ factors as

$$(\Pi_{\mathbf{H}_{\mathrm{th}}} \dashv \mathrm{Disc}_{\mathbf{H}_{\mathrm{th}}} \dashv \flat_{\mathbf{H}_{\mathrm{th}}}) : \mathbf{H}_{\mathrm{th}} \underset{\underset{\Gamma_{\mathrm{inf}}}{\longrightarrow}}{\overset{\overset{\mathrm{Red}}{\hookleftarrow}}{\underset{\mathrm{Disc}_{\mathrm{inf}}}{\overset{\Pi_{\mathrm{inf}}}{\rightleftarrows}}}} \mathbf{H} \underset{\underset{\mathrm{coDisc}}{\hookleftarrow}}{\underset{\Gamma_{\mathbf{H}}}{\overset{\overset{\Pi_{\mathbf{H}}}{\longrightarrow}}{\hookleftarrow}}} \infty\mathrm{Grpd} \,.$$

The interrelation between overlapping adjoint triples here is discussed in more detail below in 3.10.1.

3.5.2 Presentations

We establish a presentation of differential ∞-toposes, def. 3.5.1, in terms of categories of simplicial presheaves over suitable neighbourhoods of ∞-cohesive sites.

Definition 3.5.5. Let C be an ∞-cohesive site, def. 3.4.8. We say a site C_{th}

- equipped with a coreflective embedding

$$(i \dashv p) : C \underset{p}{\overset{i}{\rightleftarrows}} C_{\mathrm{th}}$$

- such that

  1. i preserves finite products;
  2. i preserves pullbacks along morphisms in covering families;
  3. both i and p send covering families to covering families;
  4. for all $\mathbf{U} \in C_{\mathrm{th}}$ and for all covering families $\{U_i \to p(\mathbf{U})\}$ in C there is a lift through p to a covering family $\{\mathbf{U}_i \to \mathbf{U}\}$ in C_{th}

is an *infinitesimal neighbourhood site* of C.

Theorem 3.5.6. *Let C be an ∞-cohesive site and let $(i \dashv p) : C \underset{p}{\overset{i}{\rightleftarrows}} C_{\mathrm{th}}$ be an infinitesimal neighbourhood site.*

Then the ∞-category of ∞-sheaves on C_{th} is a cohesive ∞-topos and the restriction i^* along i exhibits it as an infinitesimal neighbourhood of the cohesive ∞-topos over C.

$$(i_! \dashv i^* \dashv i_* \dashv i^!) : \mathrm{Sh}_\infty(C) \to \mathrm{Sh}_\infty(C_{\mathrm{th}}) \,.$$

Moreover, $i_!$ restricts on representables to the ∞-Yoneda embedding factoring through i:

$$\begin{array}{ccc} C & \hookrightarrow & \mathrm{Sh}_\infty(C) \\ \downarrow^{i} & & \downarrow^{i_!} \\ C_{\mathrm{th}} & \hookrightarrow & \mathrm{Sh}_\infty(C_{\mathrm{th}}) \end{array} \,.$$

Proof. We demonstrate this in the model category presentation of $\mathrm{Sh}_\infty(C_{\mathrm{th}})$ as in the proof of prop. 3.4.9.

Consider the right Kan extension $\mathrm{Ran}_i : [C^{\mathrm{op}}, \mathrm{sSet}] \to [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}]$ of simplicial presheaves along the functor i. On an object $\mathbf{K} \in C_{\mathrm{th}}$ it is given by

$$\begin{aligned} \mathrm{Ran}_i F : \mathbf{K} \mapsto & \int_{U \in C} \mathrm{sSet}(C_{\mathrm{th}}(i(U), \mathbf{K}), F(U)) \\ \simeq & \int_{U \in C} \mathrm{sSet}(C(U, p(\mathbf{K})), F(U)) \\ \simeq & F(p(\mathbf{K}) \end{aligned} ,$$

where in the last step we use the Yoneda reduction-form of the Yoneda lemma.

This shows that the right adjoint to $(-)\circ i$ is itself given by precomposition with a functor, and hence has itself a further right adjoint, which gives us a total of four adjoint functors

$$[C^{\mathrm{op}}, \mathrm{sSet}] \; \begin{array}{c} \xrightarrow{\mathrm{Lan}_i} \\ \xleftarrow{(-)\circ i} \\ \xrightarrow{(-)\circ p} \\ \xleftarrow{\mathrm{Ran}_p} \end{array} \; [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}] \;\; .$$

From this are induced the corresponding simplicial Quillen adjunctions on the global projective and injective model structure on simplicial presheaves

$$(\mathrm{Lan}_i \dashv (-)\circ i) : \; [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \underset{(-)\circ i}{\overset{\mathrm{Lan}_i}{\rightleftarrows}} [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \;\; ;$$

$$((-)\circ i \dashv (-)\circ p) : \; [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \underset{(-)\circ p}{\overset{(-)\circ i}{\leftrightarrows}} [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \;\; ;$$

$$((-)\circ p \dashv \mathrm{Ran}_p) : \; [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}} \underset{\mathrm{Ran}_p}{\overset{(-)\circ p}{\rightleftarrows}} [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}} \;\; .$$

By prop. 2.1.34, for these Quillen adjunctions to descend to the Čech-local model structure on simplicial presheaves it suffices that the right adjoints preserve locally fibrant objects.

We first check that $(-)\circ i$ sends locally fibrant objects to locally fibrant objects. To that end, let $\{U_i \to U\}$ be a covering family in C. Write $\int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} (j(U_{i_0}) \times_{j(U)} j(U_{i_1}) \times_{j(U)} \cdots \times_{j(U)} j(U_k))$ for its Čech nerve, where j denotes the Yoneda embedding. Recall by the definition of the ∞-cohesive site C that all the fiber products of representable presheaves here are again themselves representable, hence $\cdots = \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} (j(U_{i_0} \times_U U_{i_1} \times_U \cdots \times_U U_k))$. Using that the left adjoint Lan_i preserves the coend and tensoring, that it restricts on representables to i and by the assumption that i preserves pullbacks along covers we have that

$$\begin{aligned} \mathrm{Lan}_i C(\{U_i \to U\}) &\simeq \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} \mathrm{Lan}_i (j(U_{i_0} \times_U U_{i_1} \times_U \cdots \times_U U_k)) \\ &\simeq \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} j(i(U_{i_0} \times_U U_{i_1} \times_U \cdots \times_U U_k)) \\ &\simeq \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} j(i(U_{i_0}) \times_{i(U)} i(U_{i_1}) \times_{i(U)} \cdots \times_{i(U)} i(U_k)) \end{aligned} \;\; .$$

By the assumption that i preserves covers, this is the Čech nerve of a covering family in C_{th}. Therefore for $F \in [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ fibrant we have for all coverings $\{U_i \to U\}$ in C that the descent morphism

$$i^* F(U) = F(i(U)) \xrightarrow{\simeq} [C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}](C(\{i(U_i)\}), F) = [C^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), i^* F)$$

is a weak equivalence.

To see that $(-)\circ p$ preserves locally fibrant objects, we apply the analogous reasoning after observing that its left adjoint $(-)\circ i$ preserves all limits and colimits of simplicial presheaves (as these are computed objectwise) and by observing that for $\{\mathbf{U}_I \xrightarrow{p_i} \mathbf{U}\}$ a covering family in C_{th} we have that its image under $(-)\circ i$ is its image under p, by the Yoneda lemma:

$$\begin{aligned} [C^{\mathrm{op}}, \mathrm{sSet}](K, ((-)\circ i)(\mathbf{U})) &\simeq C_{\mathrm{th}}(i(K), \mathbf{U}) \\ &\simeq C(K, p(\mathbf{U})) \end{aligned}$$

and using that p preserves covers by assumption.

Therefore $(-) \circ i$ is a left and right local Quillen functor with left local Quillen adjoint Lan_i and right local Quillen adjoint $(-) \circ p$.

Finally to see by the above reasoning that also Ran_p preserves locally fibant objects notice that for every covering family $\{U_i \to U\}$ in C and every morphism $\mathbf{K} \to p^* U$ in C_{th} we may find a covering $\{\mathbf{K}_j \to \mathbf{K}\}$ such that we have commuting diagrams as on the left of

$$\begin{array}{ccc} \mathbf{K}_j & \longrightarrow & p^* U_{i(j)} \\ \downarrow & & \downarrow \\ \mathbf{K} & \longrightarrow & p^* U \end{array} \quad \leftrightarrow \quad \begin{array}{ccccc} p(\mathbf{K}_j) & = & i^*(\mathbf{K}_j) & \longrightarrow & U_{i(j)} \\ \downarrow & & \downarrow & & \downarrow \\ p(\mathbf{K}) & = & i^*(\mathbf{K}) & \longrightarrow & U \end{array} ,$$

because by the $(i^* \dashv p^*)$ adjunction established above these correspond to the diagrams as indicated on the right, which exist by definition of coverage and the fact that, by definition, in C_{th} covers lift through p.

This implies that $\{p^* U_i \to p^* U\}$ is a *generalized cover* in the terminology of [DuHoIs04], which by the discussion there implies that the corresponding Čech nerve projection $C(\{p^* U_i\}) \to p^* U$ is a weak equivalence in $[C_{\mathrm{th}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

This establishes the quadruple of adjoint ∞-functors as claimed.

To see that Lan_i preserves products, use that, by the local formula for the left Kan extension, it is sufficient that for each $K \in C_{\mathrm{th}}$ the functor

$$X \mapsto \lim_{\to} (p^{\mathrm{op}}/K \to C^{\mathrm{op}} \xrightarrow{X} \mathrm{sSet})$$

preserves finite products. By a standard fact this is the case precisely if the slice category p^{op}/K is sifted. A sufficient condition for this is that it has coproducts. This is equivalent to K/p having products, and this is finally true due to the assumption that p preserves products.

It remains to see that $i_!$ is a full and faithful ∞-functor. For that notice the general fact that left Kan extension along a full and faithful functor i satisfies $\mathrm{Lan}_i \circ i \simeq \mathrm{id}$. It only remains to observe that since $(-) \circ i$ is not only right but also left Quillen by the above, we have that $i^* \circ \mathrm{Lan}_i$ applied to a cofibrant object is already the derived functor of the composite. □

3.6 Structures in an ∞-topos

We discuss here a list of fundamental homotopical and cohomological structures that exist in every ∞-topos but are particularly expressive in a *local* ∞-topos, def. 3.2.1, or rather: over a base ∞-topos that is local. As we discuss below in 3.6.6, every local ∞-topos has the *homotopy dimension* of the point and hence *all gerbes are delooped groups*. This means that group objects in a local ∞-topos, discussed in 3.6.8 below, behave as *absolute structured groups* rather than as ∞-sheaves of groups that vary over a fixed nontrivial space. This is the first central property of the *gros* toposes $\mathbf{H}$ that we are interested in here. For every object $X \in \mathbf{H}$ the slice ∞-topos $\mathbf{H}_{/X} \to \mathbf{H}$ is an ∞-topos relative to its local base $\mathbf{H}$, but is itself in general not local. Group objects in the slice are groups parameterized over X and pointed connected objects in the slice are the ∞-*gerbes* over X. This we discuss below in 3.6.15.

Structures entirely specific to local ∞-toposes we discuss below in 3.7. Additional structures that are present if we assume that $\mathbf{H}$ is locally ∞-connected are discussed below in 3.8, and those in an actual cohesive ∞-topos below in 3.9.

- 3.6.1 – Bundles
- 3.6.2 – Truncated objects and Postnikov towers
- 3.6.3 – Epi-/mono-morphisms, images and relative Postnikov systems
- 3.6.4 – Compact objects
- 3.6.5 – Homotopy
- 3.6.6 – Connected objects
- 3.6.7 – Groupoids
- 3.6.8 – Groups
- 3.6.9 – Cohomology
- 3.6.10 – Principal bundles
- 3.6.11 – Associated fiber bundles
- 3.6.12 – Sections and twisted cohomology
- 3.6.13 – Representations and group cohomology
- 3.6.14 – Extensions and twisted bundles
- 3.6.15 – Gerbes
- 3.6.16 – Relative cohomology

3.6.1 Bundles

We discuss the general notion of *bundles* or *objects in a slice* in an ∞-topos. In the following sections this general notion is specialized to *principal bundles*, 3.6.10, and *associated fiber bundles*, 3.6.11.

3.6.1.1 General abstract

For $X \in \mathbf{H}$ an object, a *bundle* over X is, in full generality, nothing but a morphism

$$\begin{array}{c} T \\ \downarrow^{p} \\ X \end{array}$$

in $\mathbf{H}$ with codomain X, and a *homomorphism of bundles* over X is a diagram of the form

$$\begin{array}{ccc} T_1 & \longrightarrow & T_2 \\ & \swarrow\!\!\!\!\!\!\Leftarrow\ \ \searrow & \\ & X & \end{array}$$

in $\mathbf{H}$. The full ∞-category of bundles over X in $\mathbf{H}$ is also called the *slice* of $\mathbf{H}$ over X:

Definition 3.6.1. For $\mathbf{H}$ an ∞-category and for $X \in \mathbf{H}$ an object, the *slice ∞-category* $\mathbf{H}_{/X}$ is the ∞-pullback

$$\mathbf{H}_{/X} := \mathbf{H}^{\Delta[1]} \underset{\mathbf{H}}{\times} \{X\}$$

in the diagram of ∞-categories

$$\begin{array}{ccccc} & & \overset{\sum_X}{\frown} & & \\ \mathbf{H}_{/X} & \longrightarrow & \mathbf{H}^{\Delta[1]} & \xrightarrow{\mathrm{dom}} & \mathbf{H} \\ \downarrow & & \downarrow^{\mathrm{cod}} & & \\ * & \xrightarrow{\vdash X} & \mathbf{H} & & \end{array} .$$

Proposition 3.6.2. *For $\mathbf{H}$ an ∞-topos and $X \in \mathbf{H}$, also the slice $\mathbf{H}_{/X}$, def. 3.6.1, is an ∞-topos. Moreover, the forgetful ∞-functor $\sum_X$ in def. 3.6.1 is the extra left adjoint in an essential geometric morphism of ∞-toposes*

$$\left(\sum_X \dashv X^* \dashv \prod_X\right) : \ \mathbf{H}_{/X} \underset{\overset{\longleftarrow X\times(-) \longrightarrow}{\prod_X}}{\overset{\sum_X}{\longrightarrow}} \mathbf{H}$$

called the étale geometric morphism *of $\mathbf{H}_{/X}$.*

Here $\prod_X$ is also called the dependent product *over X and $\sum_X$ is also called the* dependent sum *over X.*

Finally , $X \times (-)$ is a cartesian closed ∞-functor, which equivalently means that it satisfies Frobenius reciprocity*: for $U \in \mathbf{H}$ and $E \in \mathbf{H}_{/X}$ there is a natural equivalence*

$$\sum_X (E \times_X (X \times U)) \xrightarrow{\simeq} \left(\sum_X E\right) \times U$$

exhibited by the canonical morphism.

This is prop. 6.3.5.1 in [LuHTT].

Example 3.6.3. The terminal object of the slice $\mathbf{H}_{/X}$ is given by the identity morphism on X in $\mathbf{H}$.

Remark 3.6.4. The interpretation of these base change functors is as follows: an object in the slice $\mathbf{H}_{/X}$ corresponds to a morphism into X in $\mathbf{H}$. The functor $\sum_X$ picks out the domain of these morphisms: it forms the "sum (union) of all the fibers". Therefore an object $E \in \mathbf{H}_{/X}$ in the slice corresponds to a morphism of the form

$$\begin{array}{c} \sum_A E \\ \downarrow \\ X \end{array}$$

in $\mathbf{H}$. More generally, a morphism $f : E_1 \to E_2$ in the slice corresponds to a diagram of the form

$$\begin{array}{ccc} \sum_A E_1 & \xrightarrow{\sum_A f} & \sum_A E_2 \\ & \swArrow_{f} & \\ \searrow & & \swarrow \\ & X & \end{array}$$

in $\mathbf{H}$.

On the other hand, the right adjoint $\prod_A$ forms internal spaces of *sections* of these morphisms. With $E \in \mathbf{H}_{/X}$ as above we have

$$\prod_X E \;\simeq\; [X, \sum_X E] \underset{[X,X]}{\times} \{\mathrm{id}\}\,,$$

which says that $\prod_X E$ is the homotopy fiber of the projection $[X, \sum_X E] \to [X, X]$ from the internal hom space of maps from the base X to the domain $\sum_A E$, picking those morphisms in there which go to the identity on X, up to homotopy, when postcomposed with E, regarded as a morphism in $\mathbf{H}$.

This kind of relation also holds externally:

Proposition 3.6.5. *For $E_1, E_2 \in \mathbf{H}_{/X}$ two objects in a slice ∞-topos over $X \in \mathbf{H}$, the hom ∞-groupoid $\mathbf{H}_{/X}(E_1, E_2)$ between them is characterized as the homotopy fiber product*

$$\mathbf{H}_{/X}(E_1, E_2) \simeq \mathbf{H}\left(\sum_X E_1, \sum_X E_2\right) \underset{\mathbf{H}(\sum_X E_1, X)}{\times} \{E_1\}$$

of hom-∞-groupoids in $\mathbf{H}$, sitting in the ∞-pullback diagram

$$\begin{array}{ccc} \mathbf{H}_{/X}(E_1, E_2) & \longrightarrow & * \\ \downarrow & & \downarrow^{\vdash E_1} \\ \mathbf{H}(\sum_X E_1, \sum_X E_2) & \xrightarrow{E_2 \circ (-)} & \mathbf{H}(\sum_X E_1, X) \end{array} .$$

This appears as prop. 5.5.5.12 in [LuHTT].

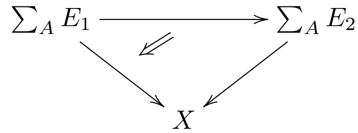
Therefore the slice ∞-topos $\mathbf{H}_{/X}$ may be regarded not only as living over the canonical base ∞-topos $\infty\mathrm{Grpd}$, but also as living over $\mathbf{H}$. As such its $\mathbf{H}$-valued hom is the dependent product of its interal hom.

Definition 3.6.6. For $X \in \mathbf{H}$ and $E_1, E_2 \in \mathbf{H}_{/X}$ we write

$$[E_1, E_2]_{\mathbf{H}} := \prod_X [E_1, E_2]$$

and speak of the $\mathbf{H}$-*valued hom* between E_1 and E_2 in the slice.

Remark 3.6.7. A global element of $\prod_X [E_1, E_2]$ corresponds again to a diagram of the form



in $\mathbf{H}$. The morphism of prop. 3.6.9 sends such a global element to the top horizontal morphism $\sum_A E_1 \to \sum_A E_2$, regarded as a global element of $[\sum_A E_1, \sum_A E_2]$.

Proposition 3.6.8. *The ∞-groupoid of global points of $[E_1, E_2]_{\mathbf{H}}$ is the slice hom $\mathbf{H}_{/X}(E_1, E_2)$:*

$$\mathbf{H}_{/X}(E_1, E_2) \simeq \Gamma\left([E_1, E_2]_{\mathbf{H}}\right) \simeq \mathbf{H}\left(*, [E_1, E_2]_{\mathbf{H}}\right) .$$

Proof. We compute

$$\begin{aligned} \mathbf{H}\left(*, [E_1, E_2]_{\mathbf{H}}\right) &\simeq \mathbf{H}_{/X}\left((* \times X), [E_1, E_2]\right) \\ &\simeq \mathbf{H}_{/X}\left(X \times_X E_1, E_2\right) \\ &\simeq \mathbf{H}_{/X}\left(E_1, E_2\right) \end{aligned} .$$

Here the first equivalence is that of the defining $\left((-) \times_X E_1 \dashv \prod\limits_X\right)$-adjunction of the dependent product, def. 3.6.2, the second is that of the $((-) \times_X E_1 \dashv [E_1, -])$-adjunction and the last one finally uses that X is the terminal object in $\mathbf{H}_{/X}$. □

We may compare the internal hom in the slice with that in the base by the following comparison morphism.

Proposition 3.6.9. *For $X \in \mathbf{H}$ and $E_1, E_2 \in \mathbf{H}_{/X}$, there is a natural morphism*

$$p_X : \prod_X [E_1, E_2] \to \left[\sum_X E_1, \sum_X E_2\right] .$$

Proof. Let $U \in \mathbf{H}$ be any object. Consider then the morphism of ∞-groupoids given by the composite

$$\begin{aligned} \mathbf{H}\left(U, \prod_X [E_1, E_2]\right) &\simeq \mathbf{H}_{/X}\left(X^* U, [E_1, E_2]\right) \\ &\simeq \mathbf{H}_{/X}\left(X^* U \times E_1, E_2\right) \\ &\to \mathbf{H}\left(\sum_X \left(f^* U \times E_1\right), \sum_X E_2\right) \\ &\simeq \mathbf{H}\left(U \times \sum_X E_1, \sum_X E_2\right) \\ &\simeq \mathbf{H}\left(U, \left[\sum_X E_1, \sum_X E_2\right]\right) \end{aligned} .$$

Here the first and last equivalences are the adjunction properties, the morphism in the middle is the relevant component of the ∞-functor $\sum_X : \mathbf{H}_{/X} \to \mathbf{H}$ and the step after that uses the *Frobenius reciprocity* property of the dependent sum (reflecting that $X*$ is a cartesian closed morphism). Since this morphism of ∞-groupoids is natural in U, the ∞-Yoneda lemma asserts that it is given by homming U into a morphism $\prod_X [E_1, E_2] \to [\sum_X E_1, \sum_X E_2]$ in $\mathbf{H}$. □

Proposition 3.6.10. *For $E_1, E_2 \in \mathbf{H}_{/X}$, there is an ∞-pullback diagram in $\mathbf{H}$ of the form*

$$
\begin{array}{ccc}
[E_1, E_2]_{\mathbf{H}} & \longrightarrow & * \\
\downarrow^{p_X} & & \downarrow^{\vdash E_1} \\
\left[\sum_X E_1, \sum_X E_2\right] & \xrightarrow{E_2 \circ (-)} & \left[\sum_X E_1, X\right]
\end{array}
,
$$

where the left vertical projections is the morphism of prop. 3.6.9.

Proof. We may check this on a set $U \in \mathbf{H}$ of generators of $\mathbf{H}$ (for instance the objects in a small ∞-site of definition). Since $\mathbf{H}(U, -)$ preserves ∞-limits (and detects them as U ranges over the set of generators), applying it to the above diagram (and using the definition $[E_1, E_2]_{\mathbf{H}} := \prod_X [E_1, E_2]$) yields the diagram

$$
\begin{array}{ccc}
\mathbf{H}_{/X}(U \times X, [E_1, E_2]) & \longrightarrow & * \\
\downarrow & & \downarrow^{\vdash (U \times X) \times_X E_1} \\
\mathbf{H}\left(U \times \sum_X E_1, \sum_X E_1\right) & \xrightarrow{\mathbf{H}(U \times \sum_X, E_1)} & \mathbf{H}\left(U \times \sum_X E_1, X\right)
\end{array}
.
$$

Here in the top left we can apply the $((-) \times_X E_1 \dashv [E_1, -])$-adjunction equivalence

$$\mathbf{H}_{/X}(U \times X, [E_1, E_2]) \simeq \mathbf{H}_{/X}((U \times X) \times_X E_1, E_2)\,,$$

and moreover by Frobenius reciprocity

$$\sum_X ((U \times X) \times_X E_1) \simeq U \times \sum_X E_1\,.$$

Therefore the above diagrams are ∞-pullbacks by prop. 3.6.5. □

Accordingly there is a Grp($\mathbf{H}$)-valued automorphism group construction:

Definition 3.6.11. For $X \in \mathbf{H}$ and $E \in \mathbf{H}_{/X}$ we say that the $\mathbf{H}$-*valued automorphism group* of E is the dependent product, def 3.6.2,

$$\mathbf{Aut}_{\mathbf{H}}(E) := \prod_X \mathbf{Aut}(E)$$

of the automorphism group of E in $\mathbf{H}_{/X}$, def. 3.6.212.

Proposition 3.6.12. *For $E \in \mathbf{H}_{/X}$ the object $\mathbf{Aut}_{\mathbf{H}}(E) \in \mathbf{H}$ of def. 3.6.11 sits in an ∞-pullback diagram of the form*

$$
\begin{array}{ccc}
\mathbf{Aut}_{\mathbf{H}}(E) & \longrightarrow & * \\
\downarrow & & \downarrow^{\vdash E} \\
\mathbf{Aut}(X) & \xrightarrow{E \circ (-)} & \left[\sum_X E, X\right]
\end{array}
.
$$

Proof. By prop. 3.6.10. □

More generally we have the following.

Proposition 3.6.13. *For* $\mathbf{H}$ *an* ∞*-topos and for* $f : X \to Y$ *a morphism in* $\mathbf{H}$*, the functor*

$$\sum_f := f \circ (-) : \mathbf{H}_{/X} \to \mathbf{H}_{/Y}$$

between the slices over the domain and codomain given by postcomposition with f *is the extra left adjoint in an essential geometric morphism*

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}_{/X} \underset{\prod_f}{\overset{\sum_f}{\underset{\longleftarrow f^* \longrightarrow}{\longrightarrow}}} \mathbf{H} \ ,$$

called the base change *geometric morphism. Here* f^* *is given by forming the* ∞*-pullback in* $\mathbf{H}$ *along* f*. As before* $\sum_f$ *is called the* dependent sum *along* f *and* $\prod_f$ *the* dependent product *along* f*.*

This is prop. 6.3.5.1, remark 6.3.5.10 of [LuHTT].

Proposition 3.6.14. *For* $\mathbf{H}$ *an* ∞*-topos, the* ∞*-functor*

$$\mathbf{H}_{/(-)} : \mathbf{H} \to \infty\mathrm{Topos}^{\mathrm{et}}/_{\mathbf{H}}$$

given by prop. 3.6.13, constitutes an equivalence of ∞*-categories between* $\mathbf{H}$ *and the full sub-*∞*-category of the slice of* ∞*-toposes and geometric morphisms over* $\mathbf{H}$ *on the étale geometric morphisms.*

This is [LuHTT], remark 6.3.5.10.

The internal hom in the slice is closely related to the dependent product:

Proposition 3.6.15. *For* $\mathbf{H}$ *an* ∞*-topos and* $X \in \mathbf{H}$ *an object, let* $E_1, E_2 \in \mathbf{H}_{/X}$ *be two object in the slice, corresponding to morphisms* $E_i : \sum_X E_i \to X$ *in* $\mathbf{H}$*. Then there is a natural equivalence*

$$[E_1, E_2] \simeq \prod_{f_1} f_1^* E_2 \, .$$

Proof. The product in the slice $\mathbf{H}_{/X}$ is given by the fiber product in $\mathbf{H}$ over X. Hence for $E \in \mathbf{H}_{/X}$ the product functor is

$$(-) \times E \simeq \sum_f f^* \, .$$

Since the internal hom is right adjoint to this functor, the statement follows by the defining adjoint tripe $(\prod_f \dashv f^* \dashv \sum_f)$. □

Proposition 3.6.16. *For* $\mathbf{H}$ *an* ∞*-topos,* $X \in \mathbf{H}$ *an object and* $E \in \mathbf{H}_{/X}$ *a slice, the* ∞*-fiber of the morphism* p_X *from def. 3.6.9 over the identity* $* \xrightarrow{\vdash \mathrm{id}_{\sum_X E}} [\sum_X E, \sum_X E]$ *is* $\Omega_E[\sum_X E, X]$*: there is a fiber sequence of the form*

$$\Omega_E[\textstyle\sum_X E, X] \hookrightarrow \prod_X [E, E] \xrightarrow{p_X} [\sum_X E, \sum_X E] \ .$$

Proof. This follows directly with prop. 3.6.10 and the pasting law, prop. 2.3.2.

More explicitly, by the proof of prop. 3.6.9 the morphism p_X is for any $U \in \mathbf{H}$ characterized, up to equivalence, as being the forgetful morphism

$$\mathbf{H}(U,p) : \ \mathbf{H}_{/X}(U \times E, E) \longrightarrow \mathbf{H}(U \times X, X)$$

that sends a morphism in the slice over X to the morphism obtained by forgetting the maps to X. Since $\mathbf{H}(U,-)$ preserves ∞-limits, it is sufficient to show that the homotopy fiber of this morphism (in ∞Grpd) is $\mathbf{H}(U, \Omega_E[\sum_X E, X])$, naturally for each U. To that end, notice that $\mathbf{H}(U, p_X)$ is the middle vertical morphism in the following diagram, where the right square is the ∞-pullback diagram that exhibits the hom space in the slice by prop. 3.6.5:

$$\begin{array}{ccccc}
\mathbf{H}(U, \Omega_E[\sum_X E, X]) & \longrightarrow & \mathbf{H}_{/X}(U \times E, E) & \longrightarrow & * \\
\downarrow & & \downarrow^{\mathbf{H}(U,p_X)} & & \downarrow^{\vdash U \times E} \\
* & \longrightarrow & \mathbf{H}(U \times \sum_X E, \sum_X E) & \xrightarrow{E \circ (-)} & \mathbf{H}(U \times \sum_X E, X)
\end{array} .$$

With the left square now denoting the ∞-pullback in question, we obtain the fiber in the top left by the pasting law for ∞-pullbacks, which says that also the total rectangle here is an ∞-pullback. But this total pullback rectangle is by example 3.6.114 the one that characterizes the loop space object and hence identifies the top left item in the above diagram as claimed. □

3.6.1.2 Presentations

We discuss presentations of slice ∞-categories, def. 3.6.1, by simplicial model categories, remark 2.1.29.

Proposition 3.6.17. *For C a model category and $X \in C$ an object, the slice category (overcategory) $C_{/X}$ as well as the co-slice category (undercategory) $C^{X/}$ inherit model category structures whose fibrations, cofibrations and weak equivalences are precisely those of C under the canonical forgetful functors $C_{/X} \to C$ and $C^{X/} \to C$, respectively.*

Proposition 3.6.18. *If the model category C is*

- *cofibrantly generated;*
- *or proper;*
- *or cellular*

then so are the (co)-slice model structures of prop. 3.6.17, for every object $X \in C$.

This is shown in [H].

Proposition 3.6.19. *If the model category C is combinatorial, then so is the slice model structure $C_{/X}$, for every object $X \in C$.*

Proof. With prop. 3.6.18 this follows form the fact that the slice of a locally presentable category is again locally presentable, (e.g. remark 3 in [CRV]). □

Proposition 3.6.20. *If C is a simplicial model category, then so is its slice $C_{/X}$, for every object $X \in C$.*

Proposition 3.6.21. *Let C be a simplicial model category and write $\mathcal{C}$ for the ∞-category that it presents. If X is fibrant in C, then the slice model structure $C_{/X}$ is a presentation of the ∞-categorical slicing $\mathcal{C}_{/X}$. If X is cofibrant in C, then the co-slice model structure $C^{X/}$ is a presentation of the ∞-categorical co-slicing $\mathcal{C}^{X/}$.*

Proof. We discuss the first case. The other one is dual. We need to check that the derived hom-spaces are the correct ∞-categorical hom-spaces. Let $A \xrightarrow{a} X$ and $B \xrightarrow{b} X$ be two objects of $\mathcal{C}_{/X}$. By prop. 3.6.5 the hom $\mathcal{C}_{/X}(a, b)$ is the ∞-pullback

$$\mathcal{C}_{/X}(a, b) \simeq \mathcal{C}(A, B) \times_{\mathcal{C}(A,X)} \{a\}$$

in ∞Grpd. Now write a for a cofibrant representative of this object in $C_{/X}$ and b for a fibrant representative. The sSet-hom object in $C_{/X}$ is the ordinary pullback

$$C_{/X}(a, b) \simeq C(A, B) \times_{C(A,X)} \{a\}$$

in sSet. One finds that a being cofibrant in $C_{/X}$ means that A is cofibrant in C and b being fibrant in $C_{/X}$ means that it is a fibration in C. Since by assumption X is fibrant in C, it follows that also B is fibrant in C. By the fact that $\mathrm{sSet}_{\mathrm{Quillen}}$ is itself a simplicial model category, it follows with prop. 2.1.32 that the simplicial hom-objects appearing in the above pullback are the correct hom-spaces, and that the pullback is along a fibration. Together this means by prop. 2.3.8 that the ordinary pullback is indeed a model for the above ∞-pullback. □

3.6.2 Truncated objects and Postnikov towers

We discuss general notions and presentations of truncated objects and Postnikov towers in an ∞-topos.

3.6.2.1 General abstract

Definition 3.6.22. For $n \in \mathbb{N}$ an ∞-groupoid $X \in \infty\mathrm{Grpd}$ is called *n-truncated* or a *homotopy n-type* if all its homotopy groups in degree $> n$ are trivial. It is called (-1)-*truncated* if it is either empty or contractible. It is called (-2)-*truncated* if it is non-empty and contractible.

For $\mathbf{H}$ an ∞-topos, and object $A \in \mathbf{H}$ is called *n-truncated* for $-2 \leq n \leq \infty$ if for all $X \in \mathbf{H}$ the hom ∞-groupoid $\mathbf{H}(X, A)$ is n-truncated.

An ∞-functor between ∞-groupoids is called *k-truncated* for $-2 \leq k \leq \infty$ if all its homotopy fibers are k-truncated. A morphism $f : A \to B$ in an ∞-topos $\mathbf{H}$ is k-truncated if for all objects $X \in \mathbf{H}$ the induced ∞-functor $\mathbf{H}(X, f) : \mathbf{H}(X, A) \to \mathbf{H}(X, B)$ is k-truncated.

This appears as [Re05] 7.1 and [LuHTT] def. 5.5.6.8.

Remark 3.6.23.

- A morphism is (-2)-truncated precisely if it is an equivalence.
- A morphism between ∞-groupoids that is (-1)-truncated is a *full and faithful ∞-functor*. A general morphism that is (-1)-truncated is an *∞-monomorphism*.

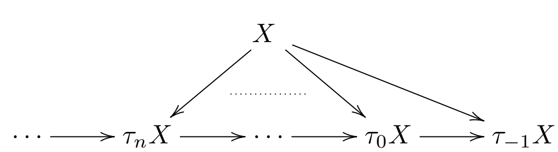
Proposition 3.6.24. *For all $(-2) \leq n \leq \infty$ the full sub-∞-category $\mathbf{H}_{\leq n}$ of $\mathbf{H}$ on the n-truncated objects is reflective in $\mathbf{H}$ in that the inclusion functor has a left adjoint ∞-functor τ_n*

$$\mathbf{H}_{\leq n} \underset{\hookrightarrow}{\overset{\tau_n}{\leftarrow}} \mathbf{H} \,.$$

Moreover, τ_n preserves finite products

This is [LuHTT] prop. 5.5.6.18, lemma 6.5.1.2.

Definition 3.6.25. For an object $X \in \mathbf{H}$ in an ∞-topos, we say that the canonical sequence



induced from the reflectors of prop. 3.6.24 is the *Postnikov tower* of X.

We say that the Postnikov tower *converges* if the above diagram exhibits X as the ∞-limit over its Postnikov tower

$$X \simeq \lim_{\longleftarrow n} \tau_n X .$$

This is def. 5.5.6.23 in [LuHTT].

Remark 3.6.26. Postnikov towers are a special cases of towers of higher *images*. This we discuss further below in 3.6.3.

3.6.2.2 Presentations

Proposition 3.6.27. *Let C be a small site of definition of an ∞-topos $\mathbf{H}$, so that*

$$\mathbf{H} \simeq L_W[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

according to theorem 2.2.15. Let $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\leq \mathrm{n}}$ be the left Bousfield localization of the local projective model structure on simplicial presheaves at the set of morphisms

$$\{\partial\Delta[k+1] \hookrightarrow U \to \Delta[k+1] \cdot U \mid U \in C; k > n\} .$$

This is a presentation of the sub-∞-category of n-truncated objects

$$\mathbf{H}_{\leq n} \simeq ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\leq n})^\circ$$

and the canonical Quillen adjunction

$$[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\leq n}$$

presents the reflection, $\tau_n \simeq \mathbb{L}\mathrm{id}$.

This appears in the proof of [Re05], prop. 7.5.

We now discuss an explicit presentation for n-truncation and Postnikov decompositions, def. 3.6.25, in terms of the projective model structure on simplicial presheaves. First recall the following classical notions, reviewed for instance in [GoJa99].

Definition 3.6.28. Let $\iota_{n+1} : \Delta_{\leq n+1} \hookrightarrow \Delta$ be the full subcatgeory of the simplex category on the objects $[k]$ for $k \leq n+1$. Write $\mathrm{sSet}_{\leq n+1} := \mathrm{Func}(\Delta^{\mathrm{op}}_{\leq n+1}, \mathrm{Set})$ for the category of $(n+1)$-stage simplicial sets.

Finally, write

$$\mathbf{cosk}_{n+1} : \mathrm{sSet} \xrightarrow{\iota^*_{n+1}} \mathrm{sSet}_{\leq n+1} \xhookrightarrow{\mathrm{cosk}_{n+1}} \mathrm{sSet}$$

for the composite of the pullback along ι_{n+1} with its *right adjoint* cosk_{n+1}.

For $X \in \mathrm{sSet}$ we say that $\mathbf{cosk}_{n+1} X$ is it *$(n+1)$-coskeleton*.

All of these constructions prolong to simplicial presheaves.

Theorem 3.6.29. *For $X \in \mathrm{sSet}$ a Kan complex, the tower of* **cosk**-*units*

$$\cdots \to \mathbf{cosk}_3 X \to \mathbf{cosk}_2 X \to \mathbf{cosk}_1 X$$

presents the Postnikov decomposition of X in $\infty\mathrm{Grpd}$.

This is a classical result due to [DwKa84b].

Proposition 3.6.30. *For C the site of definition of a hypercomplete ∞-topos, let $X \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ be a fibrant simplicial presheaf. Then the tower of* **cosk***-units*

$$\cdots \to \mathbf{cosk}_3 X \to \mathbf{cosk}_2 X \to \mathbf{cosk}_1 X$$

presents the Postnikov decomposition of X in $\mathrm{Sh}_\infty(X)$.

Proof. It is sufficient to show that $X \to \mathbf{cosk}_{n+1}X$ presents the n-truncation $X \to \tau_n X$ in $\mathrm{Sh}_\infty(X)$. For this, in turn, it is sufficient to observe that this morphism exhibits a fibrant resolution in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\leq n}$. By standard facts about left Bousfield localizations, $\mathbf{cosk}_{n+1}X$ is indeed fibrant in that model structure, since it is fibrant in the original structure by assumption and is local with respect to higher sphere inclusions by the nature of the coskeleton construction.

So it remains to see that the morphism $X \to \mathbf{cosk}_{n+1}X$ is a weak equivalence in the localized model structure. We notice that by assumption of hypercompleteness, the homotopy category is also computed by the derived hom in the truncation-localization of the Jardine model structure [Jard87]. By the nature of **cosk**, the morphism induces an isomorphism on all homotopy sheaves in degree $\leq n$ (since the homotopy presheaves of X and $\mathbf{cosk}_{n+1}X$ in these degrees are manifestly equal and $X \to \mathbf{cosk}_{n+1}$ is the identity on cells in these degrees). Since by prop. 3.6.27 also the localized Jardine structure presents the full sub-∞-category on n-truncated objects, the morphisms which are isos on homotpy groups in degree $\leq n$ are already equivalences here. □

3.6.3 Epi-/mono-morphisms, images and relative Postnikov systems

In an ∞-topos there is an infinite tower of notions of epimorphisms and monomorphisms: the $(n-2)$-connected and $(n-2)$-truncated morphisms for all $n \in \mathbb{N}$ [Re05, LuHTT]. Accordingly, factorization through these induces a notion of n-images of morphisms in an ∞-topos, for each $n \in \mathbb{N}$. The case when $n = -1$ is in some sense the most direct generalization of the 1-categorical notion.

3.6.3.1 General abstract

Definition 3.6.31. For $f : X \to Y$ a morphism in an ∞-topos $\mathbf{H}$ and for $n \in \mathbb{N}$, the *$(n-2)$-connected/$(n-2)$-truncated factorization* of f is the $(n-2)$-truncation of f, def. 3.6.22, as an object in the slice $\mathbf{H}_{/Y}$, def. 3.6.1:

$$\begin{array}{ccc} X & \longrightarrow & \sum_Y \tau_{n-2} f \\ & {\scriptstyle f}\searrow & \swarrow {\scriptstyle \tau_{n-2} f} \\ & Y & \end{array} .$$

We write

$$\mathrm{im}_n(f) := \sum_Y \tau_{n-2} f$$

and call this the *n-image* of f. We also say that

$$\mathrm{im}_\infty(f) := X$$

is the *1-image* of f.

Definition 3.6.32. A morphism $f : X \to Y$ is called

- an *n-epimorphism* if its n-image injection $\mathrm{im}_n(f) \to Y$ is an equivalence;
- an *n-monomorphism* if its n-image projection $X \to \mathrm{im}_n(f)$ is an equivalence.

Proposition 3.6.33. *For all n, the classes $(\mathrm{Epi}_n(\mathbf{H}), \mathrm{Mono}_n(\mathbf{H}))$ constitute an orthogonal factorization system.*

This is Proposition 8.5 in [Re05] and Example 5.2.8.16 in [LuHTT].
Moreover:

Proposition 3.6.34. *The factorization systems of prop. 3.6.33 are* stable*: for all n, the class of n-monomorphisms is preserved by ∞-pullback.*

This is [LuHTT], prop. 6.1.5.16(6).

Remark 3.6.35. By prop. 2.3.5 also 1-epimorphisms are preserved by ∞-pullback (as are 0-epimorphisms = equivalences), but the class of n-epimorphisms for $n > 1$ is in general not preserved by ∞-pullback.

Proposition 3.6.36. *A morphism $f : X \to Y$ is an n-monomorphism, precisely if its diagonal $X \to X \underset{Y}{\times} X$ is an $(n-1)$-monomorphism.*

This is [LuHTT], lemma 5.5.6.15.
Of particular interest are 1-epimorphisms/1-monomorphisms.

Definition 3.6.37. For $f : X \to Y$ a morphism in $\mathbf{H}$, we write its 1-epi/1-mono factorization given by Proposition 3.6.33 as

$$f : \; X \twoheadrightarrow \mathrm{im}_1(f) \hookrightarrow Y$$

and we call $\mathrm{im}_1(f) \hookrightarrow Y$ the *1-image* (or just *image*, for short) of f.

Equivalently the 1-image is the (-1)-truncation of $f : X \to Y$ regarded as an object in the slice ∞-topos.

Definition 3.6.38. Let $\mathbf{H}$ be an ∞-topos. For $X \to Y$ any morphism in $\mathbf{H}$, there is a simplicial object $\check{C}(X \to Y)$ in $\mathbf{H}$ (the *Čech nerve* of $f : X \to Y$) which in degree n is the $(n+1)$-fold ∞-fiber product of X over Y with itself

$$\check{C}(X \to Y) : [n] \mapsto X^{\times_Y^{n+1}}$$

A morphism $f : X \to Y$ in $\mathbf{H}$ is an *effective epimorphism* if it is the colimiting cocone under its own Čech nerve:

$$f : X \to \varinjlim \check{C}(X \to Y) \,.$$

Write $\mathrm{Epi}(\mathbf{H}) \subset \mathbf{H}^I$ for the collection of effective epimorphisms.

Proposition 3.6.39. *A morphism $f : X \to Y$ in the ∞-topos $\mathbf{H}$ is an effective epimorphism precisely if its 0-truncation $\tau_0 f : \tau_0 X \to \tau_0 Y$ is an epimorphism (necessarily effective) in the 1-topos $\tau_{\leq 0}\mathbf{H}$.*

This is Proposition 7.2.1.14 in [LuHTT].

Proposition 3.6.40. *The effective epimorphisms of def. 3.6.38 are equivalently the 1-epimorphisms of def. 3.6.31. In partiicular, for $f : X \to Y$ any morphism, its 1-image, def. 3.6.37, is given by the ∞-colimit over its Čech nerve, def. 3.6.38:*

$$\mathrm{im}_1(f) \simeq \varinjlim_n \left(X^{\times_Y^{n+1}} \right) \,.$$

Proof. Let $f : \; X \twoheadrightarrow \mathrm{im}_1(f) \longrightarrow Y$ be the essentially unique 1-image factorization. Then by prop. 3.6.36 the diagram exhibiting the ∞-fiber product of this morphism with itself decomposes into a pasting diagram of ∞-pullbacks of the form

$$
\begin{array}{ccccccc}
X \underset{Y}{\times} X & \simeq & X \underset{\mathrm{im}_1(f)}{\times} X & \longrightarrow & X & \xrightarrow{\simeq} & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & X & \longrightarrow\!\!\!\!\rightarrow & \mathrm{im}_1(f) & \xrightarrow{\simeq} & \mathrm{im}_1(f) \\
 & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow \\
 & & X & \longrightarrow & \mathrm{im}_1(f) & \hookrightarrow & Y
\end{array}
$$

with the composite $X \to Y$ along the bottom labeled f and the composite $X \to Y$ along the right labeled f.

By the pasting law, prop. 2.3.2 this identifis the ∞-fiber product of f with itself over Y with its product over $\mathrm{im}_1(f)$, as indicated, and hence the Čech nerve of f is equivalently that of its image projection $X \longrightarrow\!\!\!\!\rightarrow \mathrm{im}_1(f)$. Finally be the Giraud-Rezk-Lurie axiom, def. 2.2.2, satisfied by the ambient ∞-topos, the ∞-colimit over the Čech nerve of $X \longrightarrow\!\!\!\!\rightarrow \mathrm{im}_1(f)$ is that morphism itself. □

The following is a simple consequence which we will need.

Proposition 3.6.41. *For*

$$\iota : A \hookrightarrow B$$

a 1-monomorphism in $\mathbf{H}$ *and for* $X \in \mathbf{H}$ *any object, the image of* ϕ *under the internal hom* $[X, -] : \mathbf{H} \to \mathbf{H}$ *is again a 1-monomorphism.*

$$[X, \iota] : [X, A] \hookrightarrow [X, B]$$

Proof. By prop. 3.6.36 a morphism is a 1-monomorphism precisely if the ∞-fiber product with itself reproduces its domain. Since $[X, -]$ preserves ∞-limits, this implies the claim. □

Proposition 3.6.42. *For* $\iota : X \hookrightarrow *$ *a 1-monomorphism (exhibiting* X *as a* subterminal object*), and for* $E_1, E_2 \in \mathbf{H}_{/X}$ *two objects in the slice, the canonical map*

$$p_X : \prod_X [E_1, E_2] \to \left[\sum_X E_1, \sum_X E_2\right]$$

of prop. 3.6.9 is an equivalence.

Proof. By the proof of prop. 3.6.9 it suffices to show that the analogous statement holds for the external hom, hence that we have that the canonical map

$$\mathbf{H}_{/X}(E_1, E_2) \longrightarrow \mathbf{H}(\sum_X E_1, \sum_X E_1)$$

of prop. 3.6.5 is an equivalence. That morphism sits in the ∞-pullback on the left of the diagram

$$
\begin{array}{ccccc}
\mathbf{H}_{/X}(E_1, E_2) & \longrightarrow & * & & \\
\downarrow & & \downarrow{\scriptstyle \vdash E_1} & \searrow & \\
\mathbf{H}(\sum_X E_1, \sum_X E_1) & \xrightarrow{E_2 \circ (-)} & \mathbf{H}(\sum_X E_1, X) & \xhookrightarrow[\mathbf{H}(X,\iota)]{} & *
\end{array}
$$

in ∞Grpd. Here $\mathbf{H}(\sum_X E_1, X)$ is subterminal and inhabited, hence is terminal. Therefore the right vertical morphism is an equivalence and hence so is the left vertical morphism. □

By taking $\mathbf{H}$ in prop. 3.6.42 itself to be a slice of another ∞-topos, the statement implies the following seemingly more general statement:

Proposition 3.6.43. *for $f: X \hookrightarrow Y$ a 1-monomorphism in an ∞-topos $\mathbf{H}$ and for $E_1, E_2 \in \mathbf{H}_{/X}$ two objects in the slice over X, the canonical morphism*

$$\prod_Y p_f : [E_1, E_2]_{\mathbf{H}} \to \left[\sum_f E_1, \sum_f E_2\right]_{\mathbf{H}}$$

between the $\mathbf{H}$-valued slice homs of def. 3.6.6 is an equivalence.

The following is another simple fact that we will need.

Proposition 3.6.44. *For $f : X \to Y$ any morphism in $\mathbf{H}$ its homotopy fiber over any global point of Y in the image of f is equivalent to the homotopy fiber over the corresponding point in $\mathrm{im}_1(f)$.*

Proof. By the pasting law, prop. 2.3.2 the homotopy fiber sits in a pasting diagram of ∞-pullbacks.

$$\begin{array}{ccc} \mathrm{fib}_y(f) & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \xrightarrow{y} & \mathrm{im}_1(f) \\ \downarrow & & \downarrow \\ * & \xrightarrow{y} & Y \end{array} .$$

That y is in the image of f precisely says that we have the bottom square and the fact that that the bottom right morphism is a 1-monomorphism says that the bottom square is an ∞-pullback. This identifies the middle row of the digram as indicated. (For instance one can check this by applying $\mathbf{H}(U, -)$ to the diagram where U ranges over a set of generators and then using that the only suobobjects in ∞Grpd of $* \simeq \mathbf{H}(U, *)$ are $\emptyset$ and $*$ itself). □

Now we turn to discusson of the *towers* of n-image factorizations as n ranges, which are the *relative Postnikov towers* in an ∞-topos.

Remark 3.6.45. The n-images for all n form a tower

$$X \xrightarrow{=} \mathrm{im}_\infty(f) \longrightarrow \cdots \longrightarrow \mathrm{im}_2(f) \longrightarrow \mathrm{im}_1(f) \longrightarrow \mathrm{im}_0(f) \xrightarrow{\simeq} Y \ ,$$

, also called the *relative Postnikov tower* of f. For $Y \simeq *$ the terminal object this is the (absolute) *Postnikov tower* of the object X. For $X \simeq *$ the terminal object, this is the *Whitehead tower* of Y. Conversely, the relative Postnikov tower of f in $\mathbf{H}$ is equivalently the absolute Postnikov tower of f regarded as an object of the slice $\mathbf{H}_{/Y}$.

Remark 3.6.46. For $f : X \to *$ a terminal morphism, the n-image coincides with the $(n-2)$-truncation of X:

$$\tau_{n-2} X \simeq \mathrm{im}_n(X \to *) .$$

Proposition 3.6.47. *Let $f : X \to Y$ be a morphism in an ∞-topos $\mathbf{H}$ and let $x : * \to X$ be a base point. Then for all $n \in \mathbb{N}$, forming n-images commutes with forming loop space objects up to a shift in image-degree, in that there is a natural equivalence*

$$\Omega\left(\mathrm{im}_n(f)\right) \simeq \mathrm{im}_{n-1}(\Omega f) .$$

Proof. The corresponding statement in homotopy type theory is shown in [RiSp]. The above statement is the categorical semantics of that. □

3.6.3.2 Presentations We discuss presentations of n-images in ∞-toposes by constructions on simplicial presheaves.

In $\mathbf{H} = \infty\mathrm{Grpd}$, the general notion of relative Postnikov towers, remark 3.6.45, reproduces the traditional one.

Definition 3.6.48. For $X, Y \in \mathrm{sSet}$ two simplicial sets, let $f : X \to Y$ be a Kan fibration. For $n \in \mathbb{N}$ define an equivalence relation $\sim_n$ on $X_\bullet$ by declaring that two k-simplices $\sigma_1, \sigma_2 : \Delta^k \to X$ of X are equivalent if

1. they have the same n-skeleton $\mathrm{sk}_n \Delta^k \longrightarrow \Delta^k \xrightarrow{\sigma_1, \sigma_2} X$
2. and $f(\sigma_1) = f(\sigma_2)$.

Write then

$$\mathrm{im}_{n+1}(f) := X / \sim_n$$

for the quotient simplicial set. This comes equipped with canonical morphisms of simplicial sets

$$X \longrightarrow \mathrm{im}_{n+1}(f) \longrightarrow Y \,, \qquad X \longrightarrow Y .$$

This appears for instance as def. VI 2.9 in [GoJa99].

Proposition 3.6.49. *Under the equivalence* $\infty\mathrm{Grpd} \simeq L_{\mathrm{whe}}\mathrm{sSet}$*, the construction of def. 3.6.48 is a presentation of the relative Postnikov tower, remark 3.6.45, in* $\mathbf{H} = \infty\mathrm{Grpd}$.

This is essentially the statement of theorem VI 2.11 in [GoJa99].

For maps between low truncated objects, we have the following simple identification of their n-images.

Proposition 3.6.50. *A 1-functor between 1-groupoids is n-truncated as a morphism of ∞-groupoids precisely if*

- *for $n = -2$ it is an equivalence of categories;*
- *for $n = -1$ it is a full and faithful functor;*
- *for $n = 0$ it is a faithful functor.*

Proof. We consider the case $n = 0$. A functor $f : X \to Y$ between groupoids being faithful is equivalent to the induced morphisms on first homotopy groups being monomorphisms. Therefore for $F \to X \to Y$ the homotopy fiber over any point of Y, the long exact sequence of homotopy groups yields

$$\cdots \to \pi_1(F) \to \pi_1(X) \overset{f_*}{\hookrightarrow} \pi_1(Y) \to \cdots$$

and hence realizes $\pi_1(F)$ as the kernel of an injective map. Therefore $\pi(F) \simeq *$ and hence F is 0-truncated for every basepoint. This is the defining condition for f being 0-truncated. □

Proposition 3.6.51. *Let C be a site and let $f : X \to Y$ be a morphism of presheaves of groupoids on C which, under the nerve, are fibrant objects in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. If f is objectwise a) an equivalence, b) full and faithful or c) faithful, then the morphism presented by f in $\mathbf{H} := \mathrm{Sh}_\infty(X)$ is a) -2-truncated, b) (-1)-truncated, c) 0-truncated, respectively.*

Proof. We need to compute for every $A \in \mathbf{H}$ the homotopy fibers of $\mathbf{H}(A, f)$. Since by assumption X and Y are fibrant presentations, we may pick any cofibrant presentation of A and obtain this morphism as $[C^{\mathrm{op}}, \mathrm{sSet}](A, f)$. This is the nerve of a functor of groupoids which is a) an equivalence, b) full and faithful or c) faithful, respectively. The statement then follows with observation 3.6.50. □

Proposition 3.6.52. *Let $f : X \to Y$ be a morphism between presheaves of groupoids that are fibrant as objects of $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, and such that f is objectwise an essentially surjective and full functor.*

Then f presents a 0-connected morphism in $\mathrm{Sh}_\infty(C)$.

Proof. One checks that functors between 1-groupoids are 0-connected as morphisms in $\infty\mathrm{Grpd}$ precisely if they are essentially surjective and faithful.

The direction (eso+full) $\Rightarrow$ 0-connected of this argument goes through objectwise. □

More generally, we obtain a similarly simple and concrete presentation of n-image factorization of morphisms in the case that they are presented by homomorphisms of *strict* ∞-groupoids, def. 2.2.32.

Proposition 3.6.53. *Let $f : X \to Y$ be a morphism in $\infty\mathrm{Grpd}$ which is in the essential image of the inclusion*

$$\mathrm{Str}\infty\mathrm{Grpd} \hookrightarrow \mathrm{KanCplx} \to L_{\mathrm{whe}}\mathrm{sSet} \simeq \infty\mathrm{Grpd}$$

of a morphism strict ∞-groupoids, given by an underlying morphism of globular sets $f_\bullet : X_\bullet, Y_\bullet$. Then for $n \in \mathbb{N}$ the n-image factorization def. 3.6.31 of f is presented under this inclusion by the strict ∞-groupoid $\mathrm{im}_n(f)$ whose underlying globular set is

$$(\mathrm{im}_n(f))_k := \left\{ \begin{array}{cc} X_k & \forall k < n-1 \\ \mathrm{im}(X_{n-1}) \subset Y_{n-1} & \forall k = n-1 \\ Y_k & \forall k \geq n \end{array} \right.$$

equipped with the evident composition operations induced from those on $X_\bullet$ and $Y_\bullet$, and with the evident morphisms

$$X_\bullet \longrightarrow \mathrm{im}_n(f)_\bullet \longrightarrow Y_\bullet \ ,$$

the left one being the identity in degree $k < n-1$, the quotent projection in degree $n-1$ and f in degree $k \geq n$, and the right one being f in degree $k < n-1$, the image inclusion in degree $n-1$ and the identity in degree $k \geq n$.

Proof. The homotopy groups of a strict globular ∞-groupoid in any degree k are simply given by the groups of k-automorphisms of the identity $(k-1)$-morphism on a given baspoint modulo $(k+1)$-morphisms (hence the homology of the corresponding crossed complex, def. 1.2.60 in that degree). Therefore it is clear from the construction of $im_n(f)$ above that $X \to im_n(f)$ is surjective on π_0 and an isomorphism on $\pi_{k<n-1}$, and that $im_n(f)$ is a monomorphism on π_{n-1} and an isomorphism on $\pi_{k \geq n}$. □

Remark 3.6.54. For the case $Y = *$ the content of prop. 3.6.53 is discussed in [BFGM].

3.6.4 Compact objects

Traditionally there are two notions referred to as *compactness* of a space, which are closely related but subtly different.

1. On the one hand a space is called compact if regarded as an object of a certain *site* each of its covering families has a finite subfamily that is still covering.
2. On the other hand, an object in a category with colimits is called compact if the hom-functor out of that object commutes with all filtered colimits. Or more generally in the ∞-category context: if the hom-∞-functor out of the objects commutes with all filtered ∞-colimits (section 5.3 of [LuHTT]).

For instance in the site of topological spaces or of smooth manifolds, equipped with the usual open-cover coverage, the first definition reproduces the the traditional definition of *compact topological space* and of *compact smooth manifold*, respectively. But the notion of compact object in the category of topological spaces in the sense of the second definition is not quite equivalent. For instance the two-element set equipped with the indiscrete topology is compact in the first sense, but not in the second.

The cause of this mismatch, as we will discuss in detail below, becomes clearer once we generalize beyond 1-category theory to ∞-topos theory: in that context it is familiar that locality of morphisms out of an object X into an n-truncated object A (an n-stack) is no longer controled by just the notion of *covers* of X, but by the notion of *hypercover of height* n, which reduces to the ordinary notion of cover for $n = 0$. Accordingly it is clear that the ordinary condition on a compact topological space to admit finite refinement of any cover is just the first step in a tower of conditions: we may say an object is *compact of height* n if every hypercover of height n over the object is refined by a "finite hypercover" in a suitable sense.

Indeed, the condition on a *compact object* in a 1-category to distribute over filtered colimits turns out to be a compactness condition of *height 1*, which conceptually explains why it is stronger than the existence of finite refinements of covers. This state of affairs in the first two height levels has been known, in different terms, in topos theory, where one distinguishes between a topos being *compact* and being *strongly compact* [MoVe00]:

Definition 3.6.55. A 1-topos $(\Delta \dashv \Gamma) : \mathcal{X} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathrm{Set}$ is called

1. a *compact topos* if the global section functor Γ preserves filtered colimits of subterminal objects (= (-1)-truncated objects);

2. a *strongly compact topos* if Γ preserves all filtered colimits (hence of all 0-truncated objects).

Clearly these are the first two stages in a tower of notions which continues as follows.

Definition 3.6.56. For $(-1) \leq n \leq \infty$, an ∞-topos $(\Delta \dashv \Gamma) : \mathcal{X} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \infty\mathrm{Grpd}$ is called *compact of height* n if Γ preserves filtered ∞-colimits of n-truncated objects.

Since therefore the traditional terminology concerning "compactness" is not quite consistent across fields, with the category-theoretic "compact object" corresponding, as shown below, to the topos theoretic "strongly compact", we introduce for definiteness the following terminology.

Definition 3.6.57. For C a subcanonical site, call an object $X \in C \hookrightarrow \mathrm{Sh}(C) \hookrightarrow \mathrm{Sh}_\infty(C)$ *representably compact* if every covering family $\{U_\alpha \to X\}_{i \in I}$ has a finite subfamily $\{U_j \to X\}_{j \in J \subset I}$ which is still covering.

The relation to the traditional notion of compact spaces and compact objects is given by the following

Proposition 3.6.58. *Let* $\mathbf{H}$ *be a 1-topos and* $X \in \mathbf{H}$ *an object. Then*

1. *if* X *is representably compact, def. 3.6.57, with respect to the canonical topology, then the slice topos* $\mathbf{H}_{/X}$*, def. 3.6.1 is a compact topos;*

2. *the slice topos* $\mathbf{H}_{/X}$ *is strongly compact precisely if* X *is a compact object.*

Proof. Use that the global section functor Γ on the slice topos is given by

$$\Gamma([E \to X]) = \mathbf{H}(X, E) \times_{\mathbf{H}(X,X)} \{\mathrm{id}_X\}$$

and that colimits in the slice are computed as colimits in $\mathbf{H}$:

$$\varinjlim_i [E_i \to X] \simeq [(\varinjlim_i E_i) \to X] \,.$$

For the first statement, observe that the subterminal objects of $\mathbf{H}_{/X}$ are the monomorphisms in $\mathbf{H}$. Therefore Γ sends all subterminals to the empty set except the terminal object itself, which is sent to the singleton set. Accordingly, if $U_\bullet : I \to \mathbf{H}_{/X}$ is a filtered colimit of subterminals then

- either the $\{U_\alpha\}$ do not cover, hence in particular none of the U_α is X itself, and hence both $\Gamma(\varinjlim_i U_\alpha)$ as well as $\varinjlim_i \Gamma(U_\alpha)$ are the empty set;

- or the $\{U_\alpha\}_{i\in I}$ do cover. Then by assumption on X there is a finite subcover $J \subset I$, and then by assumption that $U_\bullet$ is filtered the cover contains the finite union $\varinjlim_{i\in J} U_\alpha = X$ and hence both $\Gamma(\varinjlim_i U_\alpha)$ as well as $\varinjlim_i \Gamma(U_\alpha)$ are the singleton set.

For the second statement, assume first that X is a compact object. Then using that colimits in a topos are preserved by pullbacks, it follows for all filtered diagrams $[E_\bullet \to X]$ in $\mathbf{H}_{/X}$ that

$$\begin{aligned}\Gamma(\varinjlim_i [E_i \to X]) &\simeq \mathbf{H}(X, \varinjlim_i E_i) \times_{\mathbf{H}(X,X)} \{\mathrm{id}\} \\ &\simeq (\varinjlim_i \mathbf{H}(X, E_i)) \times_{\mathbf{H}(X,X)} \{\mathrm{id}\} \\ &\simeq \varinjlim_i (\mathbf{H}(X, E_i) \times_{\mathbf{H}(X,X)} \{\mathrm{id}\}) \\ &\simeq \varinjlim_i \Gamma[E_i \to X]\end{aligned},$$

and hence $\mathbf{H}_{/X}$ is strongly compact.

Conversely, assume that $\mathbf{H}_{/X}$ is strongly compact. Observe that for every object $F \in \mathbf{H}$ we have a natural isomorphism $\mathbf{H}(X, F) \simeq \Gamma([X \times F \to X])$. Using this, we obtain for every filtered diagram $F_\bullet$ in $\mathbf{H}$ that

$$\begin{aligned}\mathbf{H}(X, \varinjlim_i F_i) &\simeq \Gamma([X \times (\varinjlim_i F_i) \to X]) \\ &\simeq \Gamma(\varinjlim_i [X \times F_i \to X]) \\ &\simeq \varinjlim_i \Gamma([X \times F_i \to X]) \\ &\simeq \varinjlim_i \mathbf{H}(X, F_i)\end{aligned}$$

and hence X is a compact object. □

Notice that a diagram of subterminal objects necessarily consists only of monomorphisms. We show now that a representably compact object generally distributes over such *monofiltered colimits*.

Definition 3.6.59. Call a filtered diagram $A : I \to D$ in a category D *mono-filtered* if for all morphisms $i_1 \to i_2$ in the diagram category I the morphism $A(i_1 \to i_2)$ is a monomorphism in D.

Lemma 3.6.60. *For C a site and $A : I \to \mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C)$ a monofiltered diagram of sheaves, its colimit $\varinjlim_i A_i \in \mathrm{PSh}(C)$ is a separated presheaf.*

Proof. For $\{U_\alpha \to X\}$ any covering family in C with $S(\{U_\alpha\}) \in \mathrm{PSh}(C)$ the corresponding sieve, we need to show that

$$\varinjlim_i A_i(X) \to \mathrm{PSh}_C(S(\{U_\alpha\}), \varinjlim_i A_i)$$

is a monomorphism. An element on the left is represented by a pair $(i \in I, a \in A_i(X))$. Given any other such element, we may assume by filteredness that they are both represented over the same index i. So let (i, a) and (i, a') be two such elements. Under the above function, (i, a) is mapped to the collection $\{i, a|_{U_\alpha}\}_\alpha$ and (i, a') to $\{i, a'|_{U_\alpha}\}_\alpha$. If a is different from a', then these families differ at stage i, hence at least one pair $a|_{U_\alpha}$, $a'|_{U_\alpha}$ is different at stage i. Then by mono-filteredness, this pair differs also at all later stages, hence the corresponding families $\{U_\alpha \to \varinjlim_i A_i\}_\alpha$ differ. □

Proposition 3.6.61. *For $X \in \mathrm{C} \hookrightarrow \mathrm{Sh}(C)$ a representably compact object, def. 3.6.57,* $\mathrm{Hom}_{\mathrm{Sh}(C)}(X, -)$ *commutes with all mono-filtered colimits.*

Proof. Let $A : I \to \mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C)$ be a mono-filtered diagram of sheaves, regarded as a diagram of presheaves. Write $\varinjlim_i A_i$ for its colimit. So with $L : \mathrm{PSh}(C) \to \mathrm{Sh}(C)$ denoting sheafification, $L \varinjlim_i A_i$ is the colimit of sheaves in question. By the Yoneda lemma and since colimits of presheaves are computed objectwise, it is sufficient to show that for X a representably compact object, the value of the sheafified colimit is the colimit of the values of the sheaves on X

$$(L \varinjlim_i A_i)(X) \simeq (\varinjlim_i A_i)(X) = \varinjlim_i A_i(X)\,.$$

To see this, we evaluate the sheafification by the plus construction. By lemma 3.6.60, the presheaf $\varinjlim_i A_i$ is already separated, so we obtain its sheafification by applying the plus-construction just *once*.

We observe now that *over a representably compact object* X the single plus-construction acts as the identity on the presheaf $\varinjlim_i A_i$. Namely the single plus-construction over X takes the colimit of the value of the presheaf on sieves

$$S(\{U_\alpha\}) := \varinjlim(\coprod_{\alpha,\beta} U_{\alpha,\beta} \rightrightarrows \coprod_\alpha U_\alpha)$$

over the opposite of the category of covers $\{U_\alpha \to X\}$ of X. By the very definition of compactness, the inclusion of (the opposite category of) the category of finite covers of X into that of all covers is a final functor. Therefore we may compute the plus-construction over X by the colimit over just the collection of finite covers. On a finite cover we have

$$\begin{aligned} \mathrm{PSh}(S(\{U_\alpha\}), \varinjlim_i A_i) &:= \mathrm{PSh}(\varinjlim(\coprod_{\alpha,\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha), \varinjlim_i A_i) \\ &\simeq \varprojlim(\prod_\alpha \varinjlim_i A_i(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \varinjlim_i A_i(U_{\alpha,\beta})) \\ &\simeq \varinjlim_i \varprojlim(\prod_\alpha A_i(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} A_i(U_{\alpha,\beta})) \\ &\simeq \varinjlim_i A_i(X) \end{aligned},$$

where in the second but last step we used that filtered colimits commute with finite limits, and in the last step we used that each A_i is a sheaf.

So in conclusion, for X a representably compact object and $A : I \to \mathrm{Sh}(C)$ a monofiltered diagram, we have found that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(C)}(X, L \varinjlim_i A_i) &\simeq (\varinjlim_i A_i)^+(X) \\ &\simeq \varinjlim_i A_i(X) \\ &\simeq \varinjlim_i \mathrm{Hom}_{\mathrm{Sh}(C)}(X, A_i) \end{aligned}$$

□

The discussion so far suggests that there should be conditions for "representantably higher compactness" on objects in a site that imply that the Yoneda-embedding of these objects into the ∞-topos over the site distribute over larger classes of filtered ∞-colimits.

Definition 3.6.62. For C a site, say that an object $X \in C$ is *representably paracompact* if each bounded hypercover over X can be refined by the Čech nerve of an ordinary cover.

The motivating example is

Proposition 3.6.63. *Over a paracompact topological space, every bounded hypercover is refined by the Čech nerve of an ordinary open cover.*

Proof. Let $Y \to X$ be a bounded hypercover. By lemma 7.2.3.5 in [LuHTT] we may find for each $k \in \mathbb{N}$ a refinement of the cover given by Y_0 such that the non-trivial $(k+1)$-fold intersections of this cover factor through Y_{k+1}. Let then $n \in \mathbb{N}$ be a bound for the height of Y and form the intersection of the covers obtained by this lemma for $0 \leq k \leq n$. Then the resulting Čech nerve projection factors through $Y \to X$. □

Proposition 3.6.64. *Let $X \in C \hookrightarrow \mathrm{Sh}_\infty(C) =: \mathbf{H}$ be an object which is*

1. *representably paracompact, def. 3.6.62;*
2. *representably compact, def. 3.6.57*

then it distributes over sequential ∞-colimits $A_\bullet : I \to \mathrm{Sh}_\infty(C)$ of n-truncated objects for every $n \in \mathbb{N}$.

Proof. Let $A_\bullet : I \to [C^{\mathrm{op}}, \mathrm{sSet}]$ be a presentation of a given sequential diagram in $\mathrm{Sh}_\infty(\mathrm{Mfd})$, such that it is fibrant and cofibrant in $[I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{proj}}$. Note for later use that this implies in particular that

- The ordinary colimit $\varinjlim_i A_i \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is a homotopy colimit.
- Every A_i is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ and hence also in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.
- Every morphism $A_i \to A_j$ is (by example 2.3.16) a cofibration in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, hence in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, hence in particular in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}$, hence is over each $U \in C$ a monomorphism.

Observe that $\varinjlim_i A_i$ is still fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$: since the colimit is taken in presheaves, it is computed objectwise, and since it is filtered, we may find the lift against horn inclusions (which are inclusions of degreewise finite simplicial sets) at some stage in the colimit, where it exists by assumption that $A_\bullet$ is projectively fibrant, so that each A_i is projectively fibrant in the local and hence in particular in the global model structure.

Since X, being representable, is cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, it also follows by this reasoning that the diagram

$$\mathbf{H}(X, A_\bullet) : I \to \infty\mathrm{Grpd}$$

is presented by

$$A_\bullet(X) : I \to \mathrm{sSet}\,.$$

Since the functors

$$[I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{proj}} \xrightarrow{\mathrm{id}} [I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{proj}} \xrightarrow{\mathrm{id}} [I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{proj}} \xrightarrow{\mathrm{id}} [I, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

all preserve cofibrant objects, it follows that $A_\bullet(X)$ is cofibrant in $[I, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$. Therefore also its ordinary colimit presents the corresponding ∞-colimit.

This means that the equivalence which we have to establish can be written in the form

$$\mathbb{R}\mathrm{Hom}(X, \varinjlim_i A_i) \simeq \varinjlim_i A_i(X)\,.$$

If here $\varinjlim_i A_i$ were fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, then the derived hom on the left would be given by the simplicial mapping space and the equivalence would hold trivially. So the remaining issue is now to deal with the fibrant replacement: the ∞-sheafification of $\varinjlim_i A_i$.

We want to appeal to theorem 7.6 c) in [DuHoIs04] to compute the derived hom into this ∞-stackification by a colimit over hypercovers of the ordinary simplicial homs out of these hypercovers into $\varinjlim_i A_i$ itself. To do so, we now argue that by the assumptions on X, we may in fact replace the hypercovers here with finite Čech covers.

So consider the colimit

$$\lim_{\{U_\alpha \to X\}_{\mathrm{finite}}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), \varinjlim_i A_i)$$

over all finite covers of X. Since by representable compactness of X these are cofinal in all covers of X, this is isomorphic to the colimit over all Čech covers

$$\cdots = \lim_{\{U_\alpha \to X\}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), \varinjlim_i A_i)\,.$$

Next, by representable paracomopactness of X, the Čech covers in turn are cofinal in all bounded hypercovers $Y \to X$, so that, furthermore, this is isomorphic to the colimit over all bounded hypercovers

$$\cdots = \lim_{\substack{\longrightarrow \\ Y \to X}} [C^{\mathrm{op}}, \mathrm{sSet}](Y, \lim_{\longrightarrow_i} A_i)\,.$$

Finally, by the assumption that the A_i are n-truncated, the colimit here may equivalently be taken over all hypercovers.

We now claim that the canonical morphism

$$\lim_{\substack{\longrightarrow \\ \{U_\alpha \to X\}_{\mathrm{finite}}}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), \lim_{\longrightarrow_i} A_i) \to \mathbb{R}\mathrm{Hom}(X, \lim_{\longrightarrow_i} A_i)$$

is a weak equivalence. Since the category of covers is filtered, we may first compute homotopy groups and then take the colimit. With the above isomorphisms, the statement is then given by theorem 7.6 c) in [DuHoIs04].

Now to conclude: since maps out of the finite Cech nerves pass through the filtered colimit, we have

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}(X, \lim_{\longrightarrow_i} A_i) &\simeq \lim_{\substack{\longrightarrow \\ \{U_\alpha \to X\}_{\mathrm{finite}}}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), \lim_{\longrightarrow_i} A_i) \\
&\simeq \lim_{\substack{\longrightarrow \\ \{U_\alpha \to X\}_{\mathrm{finite}}}} \lim_{\longrightarrow_i} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), A_i) \\
&\simeq \lim_{\longrightarrow_i} \lim_{\substack{\longrightarrow \\ \{U_\alpha \to X\}_{\mathrm{finite}}}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), A_i) \\
&\simeq \lim_{\longrightarrow_i} A_i(X)
\end{aligned}\,.$$

Here in the last step we used that each single A_i is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, so that for each $i \in I$

$$[C^{\mathrm{op}}, \mathrm{sSet}](X, A_i) \to [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), A_i)$$

is a weak equivalence. Moreover, the diagram $[C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), A_\bullet)$ in sSet is still projectively cofibrant, by example 2.3.16, since all morphisms are cofibrations in $\mathrm{sSet}_{\mathrm{Quillen}}$, and so the colimit in the second but last line is still a homotopy colimit and thus preserves these weak equivalences. □

3.6.5 Homotopy

3.6.5.1 General abstract

Definition 3.6.65. Let $\mathbf{H}$ an ∞-topos and $X \in \mathbf{H}$ an object. For $n \in \mathbb{N}$ write

$$(X^{(* \to \partial \Delta[n+1])} : X^{\Delta[n]} \to X) \in \mathbf{H}_{/X}$$

for the cotensoring of X by the point inclusion into the simplicial n-sphere, regarded as an object in the slice of $\mathbf{H}$ over X, def. 3.6.1. The *nth homotopy group* of X is the image of this under 0-truncation, prop. 3.6.24

$$\pi_n(X) := \tau_0(X^{* \to \partial \Delta[n+1]}) \in \tau_0(\mathbf{H}_{/X})\,.$$

This appears as def. 6.5.1.1 in [LuHTT].

Remark 3.6.66. Since truncation preserves finite products by prop. 3.6.24 we have that $\pi_n(X)$ is indeed a group object in the 1-topos $\tau_0()$ for $n \geq 1$ and is an abelian group object for $n \geq 2$.

Remark 3.6.67. For $\mathbf{H} = \infty\mathrm{Grpd} \simeq \mathrm{Top}$ and $x : * \to X \in \infty\mathrm{Grpd}$ a pointed object, we have for all $n \in \mathbb{N}$ that

$$\pi_n(X, x) := x^* \pi_n(X) \in \tau_0 \infty\mathrm{Grpd}_{/*} \simeq \mathrm{Set}$$

is the nth homotopy group of X at x as traditionally defined.

In [LuHTT] this is remark 6.5.1.6.

3.6.5.2 Presentations (...)

3.6.6 Connected objects

We discuss objects in an ∞-topos which are connected or higher connected in that their first non-trivial homotopy group, 3.6.5, is in some positive degree.

In a local ∞-topos and hence in particular in a cohesive ∞-topos, these are precisely the *deloopings* of *group objects*, discussed below in 3.6.8. In a more general ∞-topos, such as a slice of a cohesive ∞-topos, these are the (nonabelian/Giraud-)*gerbes*, discussed below in 3.6.15.

3.6.6.1 General abstract

Definition 3.6.68. Let $n \in \mathbb{Z}$, with $-1 \leq n$. An object $X \in \mathbf{H}$ is called *n-connected* if

1. the terminal morphism $X \to *$ is an effective epimorphism, def. 2.3.3;
2. all categorical homotopy groups $\pi_k(X)$, def. 3.6.65, for $k \leq n$ are trivial.

One also says

- *inhabited* or *well-supported* for (-1)-connected;
- *connected* for 0-connected;
- *simply connected* for 1-connected;
- $(n+1)$*-connective* for n-connected.

A morphism $f : X \to Y$ in $\mathbf{H}$ is called n-connected if it is n-connected regarded as an object of $\mathbf{H}_{/Y}$.

This is def. 6.5.1.10 in [LuHTT].

Example 3.6.69. An object $X \in \infty\mathrm{Grpd} \simeq \mathrm{Top}$ is n-connected precisely if it is n-connected in the traditional sense of higher connectedness of topological spaces. (A morphism in ∞Grpd is effective epi precisely if it induces an epimorphism on sets of connected components.)

Example 3.6.70. For C an ∞-site, a connected object in $\mathrm{Sh}_\infty(C)$ may also be called an ("nonabelian" or "Giraud"-) ∞*-gerbe* over C. This we discuss below in 3.6.15.

Definition 3.6.71. An ∞-topos $\mathbf{H}$ has *homotopy dimension* $n \in \mathbb{N}$ if n is the smallest number such that every $(n-1)$-connected object $X \in \mathbf{H}$ admits a morphism $* \to X$ from the terminal object

Remark 3.6.72. A morphism $* \to X$ is a *section* of the terminal geometric morphism. So in an ∞-topos of homotopy dimension n every $(n-1)$-connected object X has a section. For such X the terminal geometric morphism is therefore in fact a *split epimorphism.*

Example 3.6.73. The trivial ∞-topos $\mathbf{H} = *$ is, up to equivalence, the unique ∞-topos of homotopy dimension 0.

This is example 7.2.1.2 in [LuHTT].

Proposition 3.6.74. *An ∞-topos $\mathbf{H}$ has homotopy dimension $\leq n$ precisely if the global section geometric morphism $\Gamma : \mathbf{H} \to \infty\mathrm{Grpd}$, def. 2.2.4, sends $(n-1)$-connected morphisms to (-1)-connected morphisms (effective epimorphisms).*

Proof. This is essentially lemma 7.2.1.7 in [LuHTT]. The proof there shows a bit more, even. □

Proposition 3.6.75. *A local ∞-topos, def. 3.2.1, has homotopy dimension 0.*

Proof. By prop. 3.6.74 it is sufficient to show that effective epimorphisms are sent to effective epimorphisms. Since for a local ∞-topos the global section functor is a left adjoint, it preserves not only the ∞-limits involved in the characterization of effective epimorphisms, def. 2.3.3, but also the ∞-colimits. □

Remark 3.6.76. In particular an ∞-presheaf ∞-topos over an ∞-site with a terminal object is local. For this special case the statement of prop. 3.6.75 is example. 7.2.1.2 in [LuHTT], the argument above being effectively the same as the one given there.

Corollary 3.6.77. *A cohesive ∞-topos, def. 3.4.1, has homotopy dimension 0.*

Proof. By definition, a cohesive ∞-topos is in particular a local ∞-topos. □

In an ordinary topos every morphism has a unique factorization into an epimorphism followed by a monomorphism, the *image factorization*.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & A \\ & {}_{\text{epi}}\searrow \quad \nearrow_{\text{mono}} & \\ & \operatorname{im}(f) & \end{array} .$$

In an ∞-topos this notion generalizes to a tower of factorizations.

Proposition 3.6.78. *In an ∞-topos* $\mathbf{H}$ *for any $-2 \leq k \leq \infty$, every morphism $f : X \to Y$ admits a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & A \\ & \searrow \quad \nearrow & \\ & \operatorname{im}_{k+1}(f) & \end{array}$$

into a k-connected morphism, def. 3.6.68 followed by a k-truncated morphism, def. 3.6.22, and the space of choices of such factorizations is contractible.

This is [LuHTT], example 5.2.8.18.

Remark 3.6.79. For $k = -1$ this is the immediate generalization of the (epi,mono) factorization system in ordinary toposes. In particular, the 0-image factorization of a morphism between 0-truncated objects is the ordinary image factorization.

For $k = 1$ this is the generalization of the (essentially surhective and full, faithful) factorization system for functors between groupoids.

3.6.6.2 Presentations

We discuss presentations of connected and *pointed* connected objects in an ∞-topos by presheaves of pointed or reduced simplicial sets.

Observation 3.6.80. Under the presentation $\infty\mathrm{Grpd} \simeq (\mathrm{sSet}_{\mathrm{Quillen}})^\circ$, a Kan complex $X \in \mathrm{sSet}$ presents an n-connected ∞-groupoid precisely if

1. X is inhabited (not empty);
2. all simplicial homotopy groups of X in degree $k \leq n$ are trivial.

Definition 3.6.81. For $n \in \mathbb{N}$ a simplicial set $X \in \mathrm{sSet}$ is *n-reduced* if it has a single k-simplex for all $k \leq n$, hence if its n-skeleton is the point

$$\mathrm{sk}_n X = * .$$

For *0-reduced* we also just say *reduced*. Write

$$\mathrm{sSet}_n \hookrightarrow \mathrm{sSet}$$

for the full subcategory of n-reduced simplicial sets.

Proposition 3.6.82. *The n-reduced simplicial sets form a reflective subcategory*

$$\mathrm{sSet}_n \underset{\hookrightarrow}{\overset{\mathrm{red}_n}{\longleftarrow}} \mathrm{sSet}$$

of that of simplicial sets, where the reflector red_n *identifies all the n-vertices of a given simplicial set, in other words* $\mathrm{red}_n(X) = X/\mathrm{sk}_n X$ *for X a simplicial set.*

The inclusion $\mathrm{sSet}_n \hookrightarrow \mathrm{sSet}$ *uniquely factors through the forgetful functor* $\mathrm{sSet}^{*/} \to \mathrm{sSet}$ *from pointed simplicial sets, and that factorization is co-reflective*

$$\mathrm{sSet}_n \underset{E_{n+1}}{\overset{\hookrightarrow}{\longleftarrow}} \mathrm{sSet}^{*/} \,.$$

*Here the coreflector E_{n+1} sends a pointed simplicial set $* \stackrel{x}{\to} X$ to the sub-object $E_{n+1}(X,x)$ – the $(n+1)$-*Eilenberg subcomplex *(e.g. def. 8.3 in [May67]) – of cells whose n-faces coincide with the base point, hence to the fiber*

$$\begin{array}{ccc} E_{n+1}(X,x) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{cosk}_n X \end{array}$$

of the projection to the n-coskeleton.

For $(* \to X) \in \mathrm{sSet}^{*/}$ *such that* $X \in \mathrm{sSet}$ *is Kan fibrant and n-connected, the counit $E_{n+1}(X,*) \to X$ is a homotopy equivalence.*

The last statement appears for instance as part of theorem 8.4 in [May67].

Proposition 3.6.83. *Let C be a site with a terminal object and let $\mathbf{H} := \mathrm{Sh}_\infty(C)$. Then under the presentation $\mathbf{H} \simeq ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ$ every pointed n-connected object in $\mathbf{H}$ is presented by a presheaf of n-reduced simplicial sets, under the canonical inclusion $[C^{\mathrm{op}}, \mathrm{sSet}_n] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$.*

Proof. Let $X \in [C^{\mathrm{op}}, \mathrm{sSet}]$ be a simplicial presheaf presenting the given object. Then its objectwise Kan fibrant replacement $\mathrm{Ex}^\infty X$ is still a presentation, fibrant in the global projective model structure. Since the terminal object in $\mathbf{H}$ is presented by the terminal simplicial presheaf and since by assumption on C this is representable and hence cofibrant in the projective model structure, the point inclusion is presented by a morphism of simplicial presheaves $* \to \mathrm{Ex}^\infty X$, hence by a presheaf of pointed simplicial sets $(* \to \mathrm{Ex}^\infty X) \in [C^{\mathrm{op}}, \mathrm{sSet}^{*/}]$. So with observation 3.6.82 we obtain the presheaf of n-reduced simplicial sets

$$E_{n+1}(\mathrm{Ex}^\infty X, *) \in [C^{\mathrm{op}}, \mathrm{sSet}_n] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$$

and the inclusion $E_{n+1}(\mathrm{Ex}^\infty X, *) \to \mathrm{Ex}^\infty X$ is a global weak equivalence, hence a local weak equivalence, hence exhibits $E_{n+1}(\mathrm{Ex}^\infty X, *)$ as another presentation of the object in question. □

Proposition 3.6.84. *The category* sSet_0 *of reduced simplicial sets carries a left proper combinatorial model category structure whose weak equivalences and cofibrations are those in* $\mathrm{sSet}_{\mathrm{Quillen}}$ *under the inclusion* $\mathrm{sSet}_0 \hookrightarrow \mathrm{sSet}$.

Proof. The existence of the model structure itself is prop. V.6.2 in [GoJa99]. That this is left proper combinatorial follows for instance from prop. A.2.6.13 in [LuHTT], taking the set C_0 there to be

$$C_0 := \{\mathrm{red}(\Lambda^k[n] \to \Delta[n])\}_{n\in\mathbb{N}, 0\leq k\leq n}\,,$$

the image under of the horn inclusions (the generating cofibrations in $\mathrm{sSet}_{\mathrm{Quillen}}$) under the left adjoint, from observation 3.6.82, to the inclusion functor. □

Lemma 3.6.85. *Under the inclusion* $\mathrm{sSet}_0 \to \mathrm{sSet}$ *a fibration with respect to the model structure from prop. 3.6.84 maps to a fibration in* $\mathrm{sSet}_{\mathrm{Quillen}}$ *precisely if it has the right lifting property against the morphism* $(* \to S^1) := \mathrm{red}(\Delta[0] \to \Delta[1])$.

In particular it maps fibrant objects to fibrant objects.

The first statement appears as lemma 6.6. in [GoJa99]. The second (an immediate consequence) as corollary 6.8.

Proposition 3.6.86. *The adjunction*

$$\mathrm{sSet}_0 \underset{E_1}{\overset{i}{\rightleftarrows}} \mathrm{sSet}^{*/}_{\mathrm{Quillen}}$$

from observation 3.6.6.2 is a Quillen adjunction between the model structure form prop. 3.6.84 and the co-slice model structure, prop. 3.6.17, of $\mathrm{sSet}_{\mathrm{Quillen}}$ *under the point. This presents the full inclusion*

$$\infty\mathrm{Grpd}^{*/}_{\geq 1} \hookrightarrow \infty\mathrm{Grpd}^{*/}$$

of connected pointed ∞*-groupoids into all pointed* ∞*-groupoids.*

Proof. It is clear that the inclusion preserves cofibrations and acyclic cofibrations, in fact all weak equivalences. Since the point is necessarily cofibrant in $\mathrm{sSet}_{\mathrm{Quillen}}$, the model structure on the right is by prop. 3.6.21 indeed a presentation of $\infty\mathrm{Grpd}^{*/}$.

We claim now that the derived ∞-adjunction of this Quillen adjunction presents a homotopy full and faithful inclusion whose essential image consists of the connected pointed objects. For homotopy full- and faithfulness it is sufficient to show that for the derived functors there is a natural weak equivalence

$$\mathrm{id} \simeq \mathbb{R}E_1 \circ \mathbb{L}i\,.$$

This is the case, because by prop. 3.6.85 the composite derived functors are computed by the composite ordinary functors precomposed with a fibrant replacement functor P, so that we have a natural morphism

$$X \stackrel{\simeq}{\to} PX = E_1 \circ i(PX) \simeq (\mathbb{R}E_1) \circ (\mathbb{L}i)(X)\,.$$

Hence $\mathbb{L}i$ is homotopy full-and faithful and by prop. 3.6.83 its essential image consists of the connected pointed objects. □

3.6.7 Groupoids

In any ∞-topos $\mathbf{H}$ we may consider groupoids *internal* to $\mathbf{H}$, in the sense of internal category theory (as exposed for instance in the introduction of [Lurie09b]).

Such a *groupoid object* $\mathcal{G}$ in $\mathbf{H}$ is an $\mathbf{H}$-object $\mathcal{G}_0$ "of $\mathcal{G}$-objects" together with an $\mathbf{H}$-object $\mathcal{G}_1$ "of $\mathcal{G}$-morphisms" equipped with source and target assigning morphisms $s, t : \mathcal{G}_1 \to \mathcal{G}_0$, an identity-assigning morphism $i : \mathcal{G}_0 \to \mathcal{G}_1$ and a composition morphism $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ that all satisfy the axioms of a groupoid (unitalness, associativity, existence of inverses) up to coherent homotopy in $\mathbf{H}$. One way to formalize what it means for these axioms to hold up to coherent homotopy is the following.

One notes that ordinary groupoids, i.e. groupoid objects internal to Set, are characterized by the fact that their nerves are simplicial objects $\mathcal{G}_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Set}$ in Set such that all groupoidal Segal maps (see def. 3.6.88 below) are isomorphisms. This turns out to be a characterization that makes sense generally internal to higher categories: a groupoid object in $\mathbf{H}$ is an ∞-functor $\mathcal{G} : \Delta^{\mathrm{op}} \to \mathbf{H}$ such that all groupoidal Segal morphisms are equivalences in $\mathbf{H}$. This defines an ∞-category $\mathrm{Grpd}(\mathbf{H})$ of groupoid objects in $\mathbf{H}$.

Here a subtlety arises that is the source of a lot of interesting structure in higher topos theory: by the discussion in 2.2 the very objects of $\mathbf{H}$ are already to be regarded as "structured ∞-groupoids" themselves. Indeed, there is a full embedding $\mathrm{const} : \mathbf{H} \hookrightarrow \mathrm{Grpd}(\mathbf{H})$ that forms constant simplicial objects and thus regards every object $X \in \mathbf{H}$ as a groupoid object which, even though it has a trivial object of morphisms, already has a structured ∞-groupoid of objects. This embedding is in fact reflective, with the reflector given by forming the ∞-colimit over a simplicial diagram

$$\mathbf{H} \underset{\mathrm{const}}{\overset{\varinjlim}{\hookleftarrow\!\!\!\longrightarrow}} \mathrm{Grpd}(\mathbf{H}) \ .$$

For $\mathcal{G}$ a groupoid object in $\mathbf{H}$, the object $\varinjlim \mathcal{G}_\bullet$ in $\mathbf{H}$ may be thought of as the ∞-groupoid obtained from "gluing together the object of objects of $\mathcal{G}$ along the object of morphisms of $\mathcal{G}$". This idea that groupoid objects in an ∞-topos are like structured ∞-groupoids together with gluing information is formalized by the theorem that groupoid objects in $\mathbf{H}$ are equivalent to the *effective epimorphisms* $Y \twoheadrightarrow X$ in $\mathbf{H}$, the intrinsic notion of *cover* (of X by Y) in $\mathbf{H}$. The effective epimorphism / cover corresponding to a groupoid object $\mathcal{G}$ is the colimiting cocone $\mathcal{G}_0 \twoheadrightarrow \varinjlim \mathcal{G}_\bullet$. This state of affairs is a fundamental property of ∞-toposes, and as such part of the ∞-*Giraud axioms* def. 2.2.2.

The following statement refines the third ∞-Giraud axiom, Definition 2.2.2.

Theorem 3.6.87. *There is a natural equivalence of ∞-categories*

$$\mathrm{Grpd}(\mathbf{H}) \simeq (\mathbf{H}^{\Delta[1]})_{\mathrm{eff}} \, ,$$

where $(\mathbf{H}^{\Delta[1]})_{\mathrm{eff}}$ is the full sub-∞-category of the arrow category $\mathbf{H}^{\Delta[1]}$ of $\mathbf{H}$ on the effective epimorphisms, Definition 3.6.38.

This appears below Corollary 6.2.3.5 in [LuHTT].

3.6.7.1 General abstract We briefly recall the notion of *groupoid objects* in an ∞-topos from [LuHTT] with a note on how this notion axiomatizes that of ∞-groupoids with geometric structure and *equipped with an atlas* (a choice of *object of objects*) in 3.6.7.1.1. Then we discuss the notion of the ∞-group of *bisections* associated to such a choice of atlas in 3.6.7.1.2 and how these arrange to *Lie-Rinehart pairs* in 3.6.7.1.3 describing ∞-groupoids with atlases. Finally, by the 1-image factorization every morphism in an ∞-topos induces an atlas on its 1-image ∞-groupoid. This universal construction we identify as a generalization of the traditional notion of Atiyah groupoids, which we discuss in 3.6.7.1.4.

- 3.6.7.1.1 – Atlases;
- 3.6.7.1.2 – Group of bisections;
- 3.6.7.1.3 – Lie-Rinehart pairs;
- 3.6.7.1.4 – Atiyah groupoids.

3.6.7.1.1 Atlases On the one hand, *every* object in an ∞-topos $\mathbf{H}$ may be thought of as being an ∞-groupoid equipped with certain structure, notably with geometric or cohesive structure. On the other hand, traditional notions of geometric groupoids, such as *Lie groupoids* (discussed in detail in 4.4.3 below), typically involve (often implicitly) more data: the additional choice of an *atlas*, def. 2.3.4. An extreme example is the *pair groupoid* on some space X, which we discuss as example 3.6.93 below. As just an object of $\mathbf{H}$ every pair groupoid is trivial: it is equivalent to the point; but what traditonal literature really means (often implicitly) by the pair groupoid is the groupoid-with-atlas $X \to *$ with X regarded as an atlas of the point.

Abstractly, an atlas on an ∞-groupoid in $\mathbf{H}$ is just a 1-epimorphism in $\mathbf{H}$. Here we discuss this notion of *∞-groupoids with atlas.* This gives us occasion to put one of the Giraud-Rezk-Lurie axioms, def. 2.2.2, into a higher geometric context and to establish some perspetive on ∞-groupoids which is crucial in the succeeding discussion.

Definition 3.6.88. A *groupoid object* in an ∞-topos $\mathbf{H}$ is a simplicial object

$$\mathcal{G} : \Delta^{\mathrm{op}} \to \mathbf{H}$$

such that all its groupoidal Segal maps are equivalences: for every $n \in \mathbb{N}$ and every partition $[k] \cup [k'] \to [n]$ into two subsets with exactly one joint element $\{*\} = [k] \cap [k']$, the canonical diagram

$$\begin{array}{ccc} \mathcal{G}[n] & \longrightarrow & \mathcal{G}[k] \\ \downarrow & & \downarrow \\ \mathcal{G}[k'] & \longrightarrow & \mathcal{G}[*] \end{array}$$

is an ∞-pullback diagram.

Write

$$\mathrm{Grpd}(\mathbf{H}) \subset \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{H})$$

for the full subcategory of the ∞-category of simplicial objects in $\mathbf{H}$ on the groupoid objects.

This is def. 6.1.2.7 of [LuHTT], using prop. 6.1.2.6.

Example 3.6.89. For $Y \to X$ any morphism in $\mathbf{H}$, there is a groupoid object $\check{C}(Y \to X)$ which in degree n is the $(n+1)$-fold ∞-fiber product of Y over X with itself

$$\check{C}(Y \to X) : [n] \mapsto Y^{\times_X^{n+1}}$$

This appears in [LuHTT] as prop. 6.1.2.11. The following statement strengthens the third ∞-Giraud axiom of def. 2.2.2.

Theorem 3.6.90. *In an ∞-topos $\mathbf{H}$ we have*

1. *Every groupoid object in $\mathbf{H}$ is* effective*: the canonical morphism $\mathcal{G}_0 \to \varinjlim \mathcal{G}_\bullet$ is an effective epimorphism, and $\mathcal{G}$ is equivalent to the Čech nerve of this effective epimorphism.*

   Moreover, this extends to a natural equivalence of ∞-categories

   $$\mathrm{Grpd}(\mathbf{H}) \simeq (\mathbf{H}^{\Delta[1]})_{\mathrm{eff}} ,$$

   where on the right we have the full sub-∞-category of the arrow category of $\mathbf{H}$ on the effective epimorphisms.

2. *The ∞-pullback along any morphism preserves ∞-colimits*

$$\begin{array}{ccccc} \lim\limits_{\to i} f^* P_i & \simeq & f^* \lim\limits_{\to i} P_i & \longrightarrow & \lim\limits_{\to i} P_i \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow{f} & X \end{array}$$

This are two of the *Giraud-Rezk-Lurie axioms*, def. 2.2.2, that characterize ∞-toposes. (The equivalence of ∞-categories in the first point follows with the remark below corollary 6.2.3.5 of [LuHTT].)

Remark 3.6.91. If geometric structure is understood (as in a cohesive ∞-topos), there is a slight ambiguity in the word *groupoid* as usually used: in one sense every object of an ∞-topos itself is already a *parameterized ∞-groupoid* (an ∞-sheaf of ∞-groupoids, def. 2.2.1). However, for instance the literature on *Lie groupoid* theory often (and often implicitly) takes a choice of *object of objects* as part of the data of a Lie groupoid. For instance the notion of *group of bisection* of a Lie groupoid X or of its associated *Lie algebroid* both require that the inclusion of a manifold of objects is specified, a morphism $X_0 \to X$. This choice is genuine extra structure on X, as it is not in general preserved by equivalences on X. The main technical requirement on this choice is that it indeed captures "all objects" of the groupoid, up to equivalence. One often says that the inclusion has to be an *atlas* of X. In the general abstract terms of ∞-topos theory this means simply that $X_0 \to X$ is a *1-epimorphism*, remark 2.3.4.

In view of this we interpret theorem 3.6.90: if we follow remark 2.3.4 and call a 1-epimorphism in an ∞-topos an *atlas* of its codomain parameterized ∞-groupoid, then the *groupoid objects* of def. 3.6.88 are really the "parameterized ∞-groupoids equipped with a choice of atlas". (In traditional geometric groupoid theory the atlas (the domain object) is usually required to be 0-truncated, and this is often the choice of interest, also in applications of higher geometry, but in general every 1-epimorphism qualifies as an *atlas* in this sense.)

With this understood, the following definitions axiomatize and generalize standard constructions in traditional geometric groupoid theory. That they indeed reduce to these traditional notions is shown below in 4.4.3.

Example 3.6.92. For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, 3.6.8, its delooping $\mathbf{B}G$ is essentially uniquely pointed, and this point inclusion $* \twoheadrightarrow \mathbf{B}G$ is a 1-epimorphism (for instance by prop. 3.6.39). Hence this is the canonical incarnation of the delooping of G as an ∞-groupoid with atlas. In terms of this we may read theorem 3.6.116 as saying that *∞-groups are equivalent to their delooping ∞-groupoids with canonical atlases.*

Example 3.6.93. By def. 3.6.68 an object $X \in \mathbf{H}$ is called *inhabited* if the canonical morphism to the terminal object is a 1-epimorphism. Therefore for X inhabited the map $X \twoheadrightarrow *$ may be regarded as an ∞-groupoid with atlas. To see what this means consider its Čech nerve, which is of course of the form

$$\left(\cdots \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} X \times X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \right) \in \mathbf{H}^{\Delta^{\mathrm{op}}} .$$

This is a groupoid object whose objects are the points of X, whose morphisms are ordered pairs of points in X, and where composition is given in the evident way. This is what in the literature is known as the *pair groupoid* of X.

$$\mathrm{Pair}(X) := \left(X \twoheadrightarrow * \right) \in (\mathbf{H}^{\Delta^1})_{\mathrm{eff}} \simeq \mathrm{Grpd}(\mathbf{H}) .$$

Almost trivial as it may seem, the pair groupoid plays an important role for instance in the theory of Atiyah groupoids, discussed below in 3.6.7.1.4.

As these examples show, often it is more convenient to work with the atlas than with the groupoid object that it equivalently corresponds to. The following propositions shows how to compute ∞-limits in this perspective.

Proposition 3.6.94. *An ∞-limit of a diagram in in $(\mathbf{H}^{\Delta^1})_{\mathrm{eff}}$ is given by the (-1)-truncation projection of the ∞-limit of the underlying diagram in $\mathbf{H}^{\Delta^1}$. Hence if $A : J \to (\mathbf{H}^{\Delta^1})_{\mathrm{eff}}$ is a diagram with underlying diagrams $X := \partial_1 \circ A$ and $Y := \partial_2 \circ A$ in $\mathbf{H}$, then*

$$\varprojlim_j A_j \simeq \left(\varprojlim_j X_j \to \mathrm{im}_1 \left(\varprojlim_j X_j \longrightarrow \varprojlim_j Y_j \right) \right) .$$

Proof. One checks the defining universal property by the orthogonal factorization system of prop. 3.6.33. □

3.6.7.1.2 Group of Bisections We discuss here the description of ∞-groupoids $X \in \mathbf{H}$ equipped with *atlases* $X_0 \longrightarrow\!\!\!\!\rightarrow X$ in terms of their ∞-groups $\mathbf{Aut}_X(X_0)$ of autoequivalences of X_0 over X. In the case that $\mathbf{H}$ is the ∞-topos of smooth cohesion described below in 4.4 and for the example that X is presented by a traditional *Lie groupoid* this is the group which is traditionally known as the *group of bisections* of X, this we discuss in 4.4.3.1 below. Since this is a good descriptive term also in the general case, we here generally speak of $\mathbf{Aut}_X(X_0)$ as the ∞*-group of bisections*.

Due to their special construction, groups of bisections have special properties. In the traditional literature these are best known after Lie differentiation: again for X a Lie groupoid, the pair $(C^\infty(X_0), \mathrm{Lie}(\mathbf{Aut}_X(X_0)))$ consisting of the associative algebra of smooth functions on X_0 and the Lie algebra of the group of bisections is known as the *Lie-Rinehart algebra pair* associated with the groupoid. It enjoys the special property that each of the two algebras is equipped with an action of the other algebra in a compatible way. This is an equivalent way of encoding the *Lie algebroid* associated with the Lie groupoid X. Below in **??** we discuss the generalization of this perspective to ∞-groupoids.

Definition 3.6.95. For $X_\bullet \in \mathbf{H}^{\Delta^{\mathrm{op}}}$ a groupoid object in an ∞-topos, def. 3.6.88, with $\phi_X : X_0 \longrightarrow\!\!\!\!\rightarrow X$ the corresponding 1-epimorphism by theorem 3.6.90 (the *atlas* by remark 3.6.91), we say that the *group of bisections* $\mathbf{BisSect}(\phi_X) \in \mathrm{Grp}(\mathbf{H})$ of $X_\bullet$ (also written $\mathbf{BiSect}_X(X_0)$ if the morphism p_X is understood) is the relative automorphism group, def. 3.6.11, of X_0 over X:

$$\mathbf{BiSect}_X(X_0) := \mathbf{Aut}_{\mathbf{H}}(p_X) := \prod_X \mathbf{Aut}(p_X) .$$

Remark 3.6.96. We discuss how this general abstract notion reduces to that of the group of bisections of a Lie groupoid as traditionally defined below in prop. 4.4.23.

Definition 3.6.97. The *atlas automorphisms* $\mathbf{AtlasAut}_X(X_0)$ of the atlas $\phi_X : X_0 \longrightarrow\!\!\!\!\rightarrow X$ is the 1-image of the morphism p_X of def. 3.6.9, hence the factorization of p_X as

$$\mathbf{BiSect}_X(X_0) \xrightarrow{p} \mathbf{AtlasAut}_X(X_0) \hookrightarrow \mathbf{Aut}(X) .$$

Proposition 3.6.98. *For $X_\bullet \in \mathbf{H}^{\Delta^{\mathrm{op}}}$ a groupoid object in an ∞-topos, def. 3.6.88, with $\phi_X : X_0 \longrightarrow\!\!\!\!\rightarrow X$ the corresponding 1-epimorphism by theorem 3.6.90, we have a fiber sequence*

$$\Omega_{\phi_X}[X_0, X] \hookrightarrow \mathbf{BiSect}_X(X_0) \xrightarrow{p} \mathbf{AtlasAut}_X(X_0)$$

in $\mathrm{Grp}(\mathbf{H})$ *which exhibits* $\mathbf{BiSect}_X(X_0)$ *as an ∞-group extension of* $\mathbf{AtlasAut}_X(X_0)$ *by the automorphism ∞-group of the atlas X_0 inside X.*

Proof. Since $\mathbf{AtlasAut}_X(X_0)$ is by definition the 1-image of the morphism $p : \mathbf{BiSect}_X(X_0) \to \mathbf{Aut}(X)$ the statement is equivalent to the diagram

$$\Omega_\nabla[X,X] \hookrightarrow \mathbf{BisSect}_X(X_0) \xrightarrow{p} \mathbf{Aut}(X)$$

being a fiber sequence, since, by the pasting law, with the bottom square in the following diagram being an ∞-pullback, the top square is precisely so if the outer rectangle is.

$$\begin{array}{ccc}
\Omega_\nabla[X_0,X] & \longrightarrow & \mathbf{BiSect}_X(X_0) \\
\downarrow & & \downarrow^{p} \\
* & \longrightarrow & \mathbf{AtlasAut}_X(X_0) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{Aut}(X)
\end{array}$$

That the outer rectangle is an ∞-pullback is the statement of prop. 3.6.16. □

Remark 3.6.99. The sequence of prop. 3.6.98 is actually the sequence of bisection groups induced by a fiber sequence of ∞-groupoids with atlases: the generalized *Atiyah sequence*. This we discuss below in 3.6.7.1.4.

Example 3.6.100. For $X \in \mathbf{H}$ inhabited, the group of bisections of the pair groupoid $\mathrm{Pair}(X)$, example 3.6.93, is canonically equivalent to $\mathbf{Aut}(X)$:

$$\mathbf{BiSect}(\mathrm{Pair}(X)) \simeq \mathbf{Aut}(X)\,.$$

Example 3.6.101. For $X \in \mathbf{H} \overset{\mathrm{const}}{\hookrightarrow} \mathrm{Grpd}(\mathbf{H})$ the constant groupoid object on X, its group of bisections is the trivial group

$$\mathbf{BiSect}(\mathrm{const}X) \simeq *\,.$$

Proof. By example 3.6.3 the identity morphism on X is the terminal object in the slice ∞-topos $\mathbf{H}_{/X}$. □

3.6.7.1.3 Lie-Rinehart pairs (...)

3.6.7.1.4 Atiyah groupoids By the 1-image factorization, def. 3.6.31, every morphism in an ∞-topos induces an atlas for an ∞-groupoid, in the sense discussed above in 3.6.7.1.1. If the codomain is a pointed connected object, hence of the form $\mathbf{B}G$ for some ∞-group G, then we may equivalently think of this ∞-groupoid with atlas as associated to the corresponding G-principal ∞-bundle over the domain, discussed below in 3.6.10. One finds that this construction generalizes the traditional notion of the Lie groupoid which Lie integrates the *Atiyah Lie algebroid* of a smooth principal bundle (this traditional example we discuss in 4.4.3.2 below). Therefore we generally speak of *Atiyah ∞-groupoids*.

A special case this construction relevant for codomains that are moduli ∞-stacks specifically for *differential* cocycles are *Courant groupoids* which we discuss below in 3.9.13.8.

Note. This section partly refers to definitions and results in the theory of principal ∞-bundles which we discuss only below in 3.6.10. We nevertheless group the discussion of Atiyah groupoids here since one of the key aspects of their general definition in ∞-toposes is that they apply much more generally than just to principal ∞-bundles.

A fundamental construction in the traditional theory of G-principal bundles $P \to X$ is that of the corresponding *Atiyah Lie algebroid* and that of the Lie groupoid which integrates it, which we will call the *Atiyah groupoid* $\mathrm{At}(P)$. In words this is the Lie groupoid whose manifold of objects is X, and whose morphisms between two points are the G-equivariant maps between the fibers of P over these points. Observing that a G-equivariant map between two G-torsors over the point is fixed by its image on any one point, this groupoid is usually written as on the left of

$$\begin{array}{ccc} \mathrm{At}(P) & \to & \mathrm{Pair}(X) \\ = & & = \\ \left(\begin{array}{c} (P \times P)/_{\mathrm{diag}} G \\ \downarrow\uparrow\downarrow \\ X \end{array} \right) & & \left(\begin{array}{c} X \times X \\ \downarrow\uparrow\downarrow \\ X \end{array} \right) \end{array} .$$

There is a conceptual simplification to this construction when expressed in terms of the smooth moduli stack $\mathbf{B}G$ of G-principal bundles (in the smooth model for cohesion, discussed below in 4.4): if $\nabla^0 : X \to \mathbf{B}G$ is the map which modulates $P \to X$, then

Proposition 3.6.102. *The space of morphisms of* $\mathrm{At}(P)$ *is naturally identified with the* homotopy fiber product *of* ∇^0 *with itself:*

$$(P \times P)/_{\mathrm{diag}} G \simeq X \underset{\mathbf{B}G}{\times} X .$$

Moreover, the canonical atlas *of the Atiyah groupoid, given by the canonical inclusion* $p_{\mathrm{At}(P)} : X \longrightarrow\!\!\!\!\!\rightarrow \mathrm{At}(P)$ *, is equivalently the homotopy-colimiting cocone under the full Čech nerve of the classifying map* ∇^0*:*

$$\cdots\cdots X \underset{\mathbf{B}G}{\times} X \underset{\mathbf{B}G}{\times} X \begin{array}{c}\longrightarrow\\[-4pt]\longleftarrow\\[-4pt]\longrightarrow\\[-4pt]\longleftarrow\\[-4pt]\longrightarrow\end{array} X \underset{\mathbf{B}G}{\times} X \begin{array}{c}\longrightarrow\\[-4pt]\longleftarrow\\[-4pt]\longrightarrow\end{array} X \xrightarrow{p_{\mathrm{At}(P)}} \left(\varinjlim_n X^{\times^{n+1}_{\mathbf{B}G}} \right) \simeq \mathrm{At}(P) .$$

This is by direct verification, the details of this example are discussed below in 4.4.3.2. In terms of groups of bisections the above proposition 3.6.102 becomes:

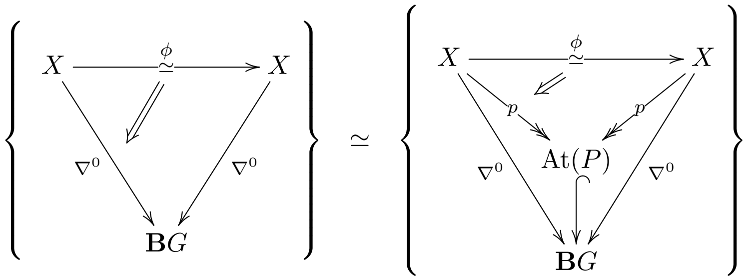
Proposition 3.6.103. *The Atiyah groupoid* $\mathrm{At}(P)$ *of a smooth G-principal bundle $P \to X$ is the Lie groupoid which is universal with the property that its group of bisections is naturally equivalent to the group of automorphisms of the modulating map* ∇^0 *of $P \to X$ in the slice:*

$$\begin{array}{ccc} \mathbf{BiSect}(\mathrm{At}(P)) & \simeq & \mathbf{Aut}_{\mathbf{H}}(\nabla^0) \\ = & & = \\ \left\{ \begin{array}{ccc} X & \xrightarrow[\simeq]{\phi} & X \\ & \swarrow\!\!\!\swarrow & \\ {\scriptstyle p_{\mathrm{At}(P)}} \searrow & & \swarrow {\scriptstyle p_{\mathrm{At}(P)}} \\ & \mathrm{At}(P) & \end{array} \right\} & & \left\{ \begin{array}{ccc} X & \xrightarrow[\simeq]{\phi} & X \\ & \swarrow\!\!\!\swarrow & \\ {\scriptstyle \nabla^0} \searrow & & \swarrow {\scriptstyle \nabla^0} \\ & \mathbf{B}G & \end{array} \right\} \end{array} .$$

In terms of 1-image factorizations we may naturally understand proposition 3.6.102 as saying that (the atlas of) the Atiyah groupoid provides the essentially unique factorization

$$\nabla^0 : X \xrightarrow{p_{\mathrm{At}(P)}} \mathrm{At}(P) \hookrightarrow \mathbf{B}G$$

of the modulating map ∇^0 of $P \to X$ by a 1-epimorphism of stacks followed by a 1-monomorphism, namely the *first relative Postnikov stage* of ∇^0, in the context of smooth stacks. As for traditional relative Postnikov theory in traditional homotopy theory, this characterizes $\mathrm{At}(P)$ uniquely as receiving an epimorphism on smooth connected components from X (the atlas $p_{\mathrm{At}(P)}$), while at the same time having a *fully faithful embedding* into $\mathbf{B}G$. This being fully faithful directly implies that the components of any natural transformation from ∇^0 to itself necessarily factor through this fully faithful inclusion:



This relation translates to a proof of prop. 3.6.103.

This discussion of Atiyah groupoids of traditional G-principal bundles generalizes directly now to bundles in an ∞-topos.

Definition 3.6.104. Let $\phi : X \to \mathbf{F}$ a morphism in $\mathbf{H}$. We say that its 1-image projection, def. 3.6.31,

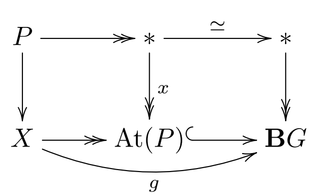
$$X \twoheadrightarrow \mathrm{im}_1(\phi) \ ,$$

regarded as an ∞-groupoid $\mathrm{im}_1(\phi)$ with atlas X by remark 2.3.4, is the *Atiyah groupoid* $At(\phi) \in \mathrm{Epi}_1(\mathbf{H})$ of ϕ.

Here for the direct generalization of the traditional notion of Atiyah groupoids we set $\mathbf{F} = \mathbf{B}G$ the delooping of some ∞-group. But the definition and many of its uses does not depend on this restriction. An exception os the following fact, which generalizes a standard theorem about Atiyah groupoids known from textbooks on differential geometry.

Proposition 3.6.105. *For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, every G-principal ∞-bundle $P \to X$ in $\mathbf{H}$, def. 3.6.155, over an inhabited object X, def. 3.6.68, is equivalently the source-fiber of a transitive higher groupoid $\mathcal{G} \in \mathrm{Grpd}(\mathbf{H})$ with vertex ∞-group G. Here in particular we can set $\mathcal{G} = \mathrm{At}(P)$.*

Proof. For $P \to X$ a G-principal ∞-bundle, write $g : X \to \mathbf{B}G$ for the map that modulates it by theorem 3.6.170. Then the outer rectangle of



is an ∞-pullback by that theorem 3.6.170. Also the right sub-square is an ∞-pullback (for any global point $x \in X$) because by ∞-pullback stability of 1-epimorphisms (prop. 2.3.5) and 1-monomorphisms (prop. 3.6.34), the top right morphism is a 1-monomorphism from an inhabited object to the terminal object, hence is not just a 1-mono but also a 1-epi and hence is an equivalence. Now by the pasting law for ∞-pullbacks, prop. 2.3.2, also the left sub-square is an ∞-pullback and this exhibits P as the source fiber of $\mathrm{At}(P)$ over $x \in X$. □

Proposition 3.6.106. *For $\phi : X \to \mathbf{F}$ a morphism, there is a canonical equivalence*

$$\mathbf{BiSect}(\mathrm{At}(\phi)) \simeq \mathbf{Aut}_{\mathbf{H}}(\phi)$$

between the ∞-group of bisections, def. 3.6.95, of the higher Atiyah groupoid of ϕ, def. 3.6.104, and the $\mathbf{H}$-valued automorphism ∞-group of ϕ

Moreover, the ∞-group of bisections of the higher Atiyah ∞-groupoid sits in a homotopy fiber sequence of ∞-groups of the form

$$\begin{array}{ccccc} \Omega_\phi[X,\mathbf{F}] & \longrightarrow & \mathbf{BiSect}(\mathbf{At}(\phi)) & \longrightarrow & \mathbf{Aut}(X) \;, \\ & & \simeq & & \\ & & \mathbf{Aut}_{\mathbf{H}}(\phi) & & \end{array}$$

where on the right we have the canonical forgetful map.

Proof. This is the restriction of the statement of prop. 3.6.43 to those endomorphisms that are equivalences. □

Definition 3.6.107 (Atiyah sequence)**.** For $\phi : X \to \mathbf{B}G$ a cocycle, write

$$\mathrm{At}(\phi) \xrightarrow{\;p\;} \mathrm{Pair}(X)$$

for the morphism of groupoid objects to the *pair groupoid* of X, example 3.6.93, given by the canonical map of atlases

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow & & \downarrow \\ \mathrm{im}_1(\phi) & \longrightarrow & * \end{array} \;.$$

We say that the ∞-fiber sequence of this morphism over X

$$\mathrm{ad}(\phi) \longrightarrow \mathrm{At}(\phi) \longrightarrow \mathrm{Pair}(X) \;,$$

is the *Atiyah sequence* of ϕ, hence the sequence given by the ∞-pullback diagram

$$\begin{array}{ccc} \mathrm{ad}(\phi) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{At}(\phi) & \xrightarrow{\;p\;} & \mathrm{Pair}(X) \end{array} \;.$$

Proposition 3.6.108. *Given $\phi : X \to \mathbf{B}G$, the induced sequence of groups of bisections, def. 3.6.95, is the sequence of prop. 3.6.98.*

Proof. By prop. 3.6.100 and prop. 3.6.106 the morphism of groupoid objects $\mathrm{At}(\phi) \to \mathrm{Pair}(X)$ induces the morphism of groups of bisections $\mathbf{Aut}(\phi) \to \mathbf{Aut}(X)$. Therefore it remains to show that $\mathrm{ad}(\phi) \to \mathrm{At}(\phi)$ is as claimed.

By prop. 3.6.94 we obtain $\mathrm{ad}(\phi)$ as the 1-image factorization of the limit in $\mathbf{H}^{\Delta^1}$ over

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{id}} & X & \longleftarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{im}_1(\phi) & \longrightarrow & * & \longleftarrow & X \end{array} \;,$$

hence the 1-image factorization of the diagonal $X \twoheadrightarrow X \times \mathrm{im}_1(\phi)$. Moreover by prop. 3.6.106 the group of bisections of this image factorization is equivalently that of the morphism itself. Now a bisection of the diagonal, hence a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\simeq\quad} & X \\ & \searrow \;\; \Swarrow \;\; \swarrow & \\ & X \times \mathrm{im}_1(\phi) & \end{array}$$

is equivalently a pair of a diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow[\simeq]{f} & X \\ & {}^{\mathrm{id}}\searrow \;\; \Swarrow \;\; \swarrow{}^{\mathrm{id}} & \\ & X & \end{array} \quad , \quad \begin{array}{ccc} X & \xrightarrow[\simeq]{f} & X \\ & \searrow \;\; \Swarrow \;\; \swarrow & \\ & \mathrm{im}_1(\phi) & \end{array}$$

that share the top horizontal morphism, as indicated. By example 3.6.101 the ∞-groupoid of diagrams as on the left is contractible, hence up to essentially unique equivalence we have $f = \mathrm{id}$. This reduces the diagram on the right to an automorphism of ϕ, as claimed. □

The Atiyah groupoid acts on sections of the corresponding bundle and its associated bundles:

Definition 3.6.109. For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, for $P \to X$ a G-principal ∞-bundle modulated by a map $g : X \to \mathbf{B}G$, and for $\rho : V//G \to \mathbf{B}G$ an action of G on some $V \in \mathbf{H}$, write

$$(P \times_G V)//\mathrm{At}(P) \to \mathrm{At}(P)$$

for the ∞-pullback of ρ along the defining 1-monomorphism from the Atiyah groupoid of P. Then by the pasting law, prop. 2.3.2, and by the characterization of the universal ρ-associated bundle, prop. 3.6.209, we have an ∞-pullback square as on the left of the following diagram:

$$\begin{array}{ccccc} P \times_G V & \longrightarrow & (P \times_G V)//\mathrm{At}(P) & \longrightarrow & V//G \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho} \\ X & \twoheadrightarrow & \mathrm{At}(P) & \hookrightarrow & \mathbf{B}G \end{array} \quad .$$

By def. ?? this exhibits $(P \times_G V)//\mathrm{At}(P)$ as a groupoid action of $\mathrm{At}(P)$ on the associated V-fiber bundle $P \times_G V \to X$. This we call the *canonical Atiyah-groupoid action on sections.*

3.6.7.2 Presentations

For $\mathbf{H} = \mathrm{Sh}_\infty(C)$ the ∞-topos over a site C, we discuss a presentation of the ∞-category $\mathrm{Grpd}(\mathbf{H})$ of groupoid object in $\mathbf{H}$, def. 3.6.88, in terms of a model category structure on the category of I-simplicial objects in a model category of simplicial presheaves, where an I-simplicial object is a simplicial object equipped with extra structure that encodes equivalences under reversion of the order of vertices.

Definition 3.6.110. Write $I\Delta$ for the category (...).

[Ber08a]

Definition 3.6.111. Write

$$[I\Delta^{\mathrm{op}}, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}]_{\mathrm{proj,Segal}}$$

for the left Bousfield localization of $[I\Delta^{\mathrm{op}}, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}]_{\mathrm{proj}}$ at the set of simplex spine inclusions. (...)

Example 3.6.112. For $C = *$ this gives the model structur of *invertible Segal spaces* discussed in section 3 of [Ber08b].

(...)

3.6.8 Groups

Every ∞-topos $\mathbf{H}$ comes with a notion of ∞-*group objects* that generalizes the ordinary notion of group objects in a topos as well as that of grouplike A_∞-spaces in Top $\simeq \infty$Grpd [Sta63b]. Operations of *looping* and *delooping* identify ∞-group objects with pointed connected objects. If moreover $\mathbf{H}$ is cohesive then it follows that every connected object is canonically pointed, and hence every connected object uniquely corresponds to an ∞-group object.

This section to a large extent collects and reviews general facts about ∞-group objects in ∞-toposes from [LuHTT] and [Lurie11]. We add some observations that we need later on.

3.6.8.1 General abstract

Definition 3.6.113. Write

- $\mathbf{H}^{*/}$ for the ∞-category of pointed objects in $\mathbf{H}$;
- $\mathbf{H}_{\geq 1}$ for for the full sub-∞-category of $\mathbf{H}$ on the connected objects;
- $\mathbf{H}^{*/}_{\geq 1}$ for the full sub-∞-category of the pointed and connected objects.

Definition 3.6.114. Write

$$\Omega : \mathbf{H}^{*/} \to \mathbf{H}$$

for the ∞-functor that sends a pointed object $* \to X$ to its *loop space object*: the ∞-pullback

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array} .$$

Definition 3.6.115. An ∞-*group* in $\mathbf{H}$ is an A_∞-algebra G in $\mathbf{H}$ such that $\pi_0(G)$ is a group object. Write $\mathrm{Grp}(\mathbf{H})$ for the ∞-category of ∞-groups in $\mathbf{H}$.

This is def. 5.1.3.2 in [Lurie11], together with remark 5.1.3.3.

Theorem 3.6.116. *Every loop space object canonically has the structure of an ∞-group, and this construction extends to an ∞-functor*

$$\Omega : \mathbf{H}^{*/} \to \mathrm{Grp}(\mathbf{H}) .$$

This constitutes an equivalence of ∞-categories

$$(\Omega \dashv \mathbf{B}) : \ \mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\Omega}{\underset{\simeq}{\leftrightarrows}}} \mathbf{H}^{*/}_{\geq 1}$$

of ∞-groups with connected pointed objects in $\mathbf{H}$.

This is lemma 7.2.2.1 in [LuHTT]. (See also theorem 5.1.3.6 of [Lurie11] where this is the equivalence denoted ϕ_0 in the proof.)

Definition 3.6.117. We call the inverse $\mathbf{B} : \mathrm{Grp}(\mathbf{H}) \to \mathbf{H}^{*/}_{\geq 1}$ the *delooping* functor of $\mathbf{H}$. By convenient abuse of notation we write $\mathbf{B}$ also for the composite $\mathbf{B} : \infty\mathrm{Grpd}(\mathbf{H}) \to \mathbf{H}^{*/}_{\geq 1} \to \mathbf{H}$ with the functor that forgets the basepoint and the connectedness.

Remark 3.6.118. While by prop. 3.4.4 every connected object in a cohesive ∞-topos has a unique point, nevertheless the homotopy type of the full hom-∞-groupoid $\mathbf{H}^{*/}(\mathbf{B}G, \mathbf{B}H)$ of pointed objects in general differs from the hom ∞-groupoid $\mathbf{H}(\mathbf{B}G, \mathbf{B}H)$ of the underlying unpointed objects.

For instance let $\mathbf{H} := \infty\mathrm{Grpd}$ and let G be an ordinary group, regarded as a group object in ∞Grpd. Then $\mathbf{H}^{*/}(\mathbf{B}G, \mathbf{B}G) \simeq \mathrm{Aut}(G)$ is the ordinary automorphism group of G, but $\mathbf{H}(\mathbf{B}G, \mathbf{B}G) = \mathrm{AUT}(G)$ is the automorphism 2-group, example 1.2.51.

The more deloopings an ∞-group admits, the "more abelian" it is:

Definition 3.6.119. A *braided* ∞-group in $\mathbf{H}$ is an ∞-group $G \in \mathrm{Grp}(\mathbf{H})$ equipped with the following equivalent additional structures:

1. a lift of the groupal $A_\infty \simeq E_1$-algebra structure to an E_2-algebra structure;
2. the structure of an ∞-group on the delooping $\mathbf{B}G$;
3. a choice of double delooping $\mathbf{B}^2G$.

Definition 3.6.120. An *abelian* ∞-group in $\mathbf{H}$ is an ∞-group $G \in \mathrm{Grp}(\mathbf{H})$ equipped with the following equivalent additional structures:

1. a lift of the groupal $A_\infty \simeq E_1$-algebra structure to an E_∞-algebra structure;
2. coinductively: a choice of abelian ∞-group structure on its delooping $\mathbf{B}G$.

Proposition 3.6.121. *∞-groups G in $\mathbf{H}$ are equivalently those groupoid objects, def. 3.6.88, $\mathcal{G}$ in $\mathbf{H}$ for which $\mathcal{G}_0 \simeq *$.*

This is the statement of the compound equivalence $\phi_3\phi_2\phi_1$ in the proof of theorem 5.1.3.6 in [Lurie11].

Remark 3.6.122. This means that for G an ∞-group object the Čech nerve extension of its delooping fiber sequence $G \to * \to \mathbf{B}G$ is the simplicial object

$$\cdots \ \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \ G \times G \ \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \ G \ \substack{\longrightarrow\\[-2pt]\longrightarrow} \ * \longrightarrow \mathbf{B}G$$

that exhibits G as a groupoid object over $*$. In particular it means that for G an ∞-group, the essentially unique morphism $* \to \mathbf{B}G$ is an effective epimorphism.

Definition 3.6.123. For $f : Y \to Z$ any morphism in $\mathbf{H}$ and $z : * \to Z$ a point, the *∞-fiber* or *homotopy fiber* of f over this point is the ∞-pullback $X := * \times_Z Y$

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & Z \end{array} .$$

Observation 3.6.124. Suppose that also Y is pointed and f is a morphism of pointed objects. Then the ∞-fiber of an ∞-fiber is the loop object of the base.

This means that we have a diagram

$$\begin{array}{ccccc} \Omega_z Z & \longrightarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & Y & \xrightarrow{f} & Z \end{array} .$$

where the outer rectangle is an ∞-pullback if the left square is an ∞-pullback. This follows from the pasting law prop. 2.3.2.

3.6.8.2 Presentations We discuss presentations of the notion of ∞-groups, 3.6.8.1, by simplicial groups in a category with weak equivalences.

Definition 3.6.125. One writes $\overline{W}$ for the composite functor from simplicial groups to simplicial sets given by

$$\overline{W} : [\Delta^{\mathrm{op}}, \mathrm{Grpd}] \xrightarrow{[\Delta^{\mathrm{op}}, \mathbf{B}]} [\Delta^{\mathrm{op}}, \mathrm{Grpd}] \xrightarrow{[\Delta^{\mathrm{op}}, N]} [\Delta^{\mathrm{op}}, \mathrm{sSet}] \xrightarrow{T} \mathrm{sSet} \;,$$

where

- $[\Delta^{\mathrm{op}}, \mathbf{B}] : [\Delta^{\mathrm{op}}, \mathrm{Grp}] \to [\Delta^{\mathrm{op}}, \mathrm{Grpd}]$ is the functor from simplicial groups to simplicial groupoids that sends degreewise a group to the corresponding one-object groupoid;
- $T : [\Delta^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{sSet}$ is the total simplicial set functor, def. 2.3.23.

This simplicial delooping $\overline{W}$ was originally introduced in components in [EiML], now a classical construction. The above formulation is due to [Dus75], see lemma 15 in [St11].

Remark 3.6.126. This functor takes values in *reduced* simplicial sets $\mathrm{sSet}_{\geq 1} \hookrightarrow \mathrm{sSet}$, those with precisely one vertex.

Remark 3.6.127. For G a simplicial group, the simplicial set $\overline{W}G$ is, by corollary 2.3.27, the homotopy colimit over a simplicial diagram in simplicial sets. Below in 3.6.10.4 we see that this simplicial diagram is that presenting the groupoid object $*//G$ which is the action groupoid of G acting trivially on the point.

Proposition 3.6.128. *The category* sGrpd *of simplicial groups carries a cofibrantly generated model structure for which the fibrations and the weak equivalences are those of* $\mathrm{sSet}_{\mathrm{Quillen}}$ *under the forgetful functor* $\mathrm{sGrpd} \to \mathrm{sSet}$*.*

Proof. This is theorem 2.3 in [GoJa99]. Since model structure is therefore transferred along the forgetful functor, it inherits generating (acyclic) cofibrations from those of $\mathrm{sSet}_{\mathrm{Quillen}}$. □

Theorem 3.6.129. *The functor* $\overline{W}$ *is the right adjoint of a Quillen equivalence*

$$(L \dashv \overline{W}) : \mathrm{sGrp} \underset{\overline{W}}{\overset{L}{\rightleftarrows}} \mathrm{sSet}_{\geq 1} \;,$$

with respect to the model structures of prop. 3.6.128 and prop. 3.6.84. In particular

- *the adjunction unit is a weak equivalence*

  $$Y \xrightarrow{\simeq} \overline{W}LY$$

  for every $Y \in \mathrm{sSet}_0 \hookrightarrow \mathrm{sSet}_{\mathrm{Quillen}}$
- $\overline{W}LY$ *is always a Kan complex.*

This is discussed for instance in chapter V of [GoJa99]. A new proof is given in [St11].

Definition 3.6.130. For G a simplicial group, write

$$WG \to \overline{W}G$$

for the décalage, def. 2.3.31, on $\overline{W}G$.

This characterization by décalage of the object going by the classical name WG is made fairly explicit on p. 85 of [Dus75]. The fully explicit statement is in [RoSt12].

Proposition 3.6.131. *The morphism $WG \to \overline{W}G$ is a Kan fibration resolution of the point inclusion $* \to \overline{W}G$.*

Proof. This follows directly from the characterization of $WG \to \overline{W}G$ by décalage. $\square$

Pieces of this statement appear in [May67]: lemma 18.2 there gives the fibration property, prop. 21.5 the contractibility of WG.

Corollary 3.6.132. *For G a simplicial group, the sequence of simplicial sets*

$$G \longrightarrow WG \longrightarrow\!\!\!\!\!\rightarrow \overline{W}G$$

is a presentation in $\mathrm{sSet}_{\mathrm{Quillen}}$ *by a pullback of a Kan fibration of the looping fiber sequence, theorem. 3.6.116,*

$$G \to * \to \mathbf{B}G$$

in ∞Grpd.

Proof. One finds that G is the 1-categorical fiber of $WG \to \overline{W}G$. The statement then follows using prop. 3.6.131 in prop. 2.3.8. $\square$

The explicit statement that the sequence $G \to WG \to \overline{W}G$ is a model for the looping fiber sequence appears on p. 239 of [Por]. The universality of $WG \to \overline{W}G$ for G-principal simplicial bundles is the topic of section 21 in [May67], where however it is not made explicit that the "twisted cartesian products" considered there are precisely the models for the pullbacks as above. This is made explicit for instance on page 148 of [Por].

Corollary 3.6.133. *The Quillen equivalence $(L \dashv \overline{W})$ from theorem 3.6.129 is a presentation of the looping/delooping equivalence, theorem 3.6.116.*

We now lift all these statements from simplicial sets to simplicial presheaves.

Proposition 3.6.134. *If the cohesive ∞-topos $\mathbf{H}$ has site of definition C with a terminal object, then*

- *every ∞-group object has a presentation by a presheaf of simplicial groups*

  $$G \in [C^{\mathrm{op}}, \mathrm{sGrp}] \xrightarrow{U} [C^{\mathrm{op}}, \mathrm{sSet}]$$

  which is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$;

- *the corresponding delooping object is presented by the presheaf*

  $$\overline{W}G \in [C^{\mathrm{op}}, \mathrm{sSet}_0] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$$

  which is given over each $U \in C$ by $\overline{W}(G(U))$.

Proof. By theorem 3.6.116 every ∞-group is the loop space object of a pointed connected object. By prop. 3.6.83 every such is presented by a presheaf X of reduced simplicial sets. By the simplicial looping/delooping Quillen equivalence, theorem 3.6.129, the presheaf

$$\overline{W}LX \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$$

is weakly equivalent to the simplicial presheaf X. From this the statement follows with corollary 3.6.132, combined with prop. 2.3.13, which together say that the presheaf LX of simplicial groups presents the given ∞-group. $\square$

Remark 3.6.135. We may read this as saying that every ∞-group may be *strictified.*

Example 3.6.136. Every 2-group in $\mathbf{H}$ (1-truncated group object) has a presentation by a crossed module, def. 1.2.45, in simplicial presheaves.

3.6.9 Cohomology

There is an intrinsic notion of *cohomology* in every ∞-topos. It is the joint generalization of the definition of cohomology in Top in terms of maps into classifying spaces and of *sheaf cohomology* over any site of definition of the ∞-topos.

For the case of abelian coefficients, as disucssed in 2.2.6, this perspective of (sheaf) cohomology as the cohomology intrinsic to an ∞-topos is essentially made explicit already in [Br73]. In more modern language analogous discussion is in section 7.2.2 of [LuHTT].

Here we review central concepts and discuss further aspects that will be needed later on.

3.6.9.1 General abstract

Definition 3.6.137. For $X, A \in \mathbf{H}$ two objects, we say that

$$H(X,A) := \pi_0 \mathbf{H}(X,A)$$

is the *cohomology set* of X with coefficients in A. If $A = G$ is an ∞-group we write

$$H^1(X,G) := \pi_0 \mathbf{H}(X, \mathbf{B}G)$$

for cohomology with coefficients in its delooping. Generally, if $K \in \mathbf{H}$ has a p-fold delooping for some $p \in \mathbb{N}$, we write

$$H^p(X,K) := \pi_0 \mathbf{H}(X, \mathbf{B}^p K)\,.$$

In the context of cohomology on X wth coefficients in A we we say that

- the hom-space $\mathbf{H}(X,A)$ is the *cocycle ∞-groupoid*;
- a morphism $g : X \to A$ is a *cocycle*;
- a 2-morphism : $g \Rightarrow h$ is a *coboundary* between cocycles.
- a morphism $c : A \to B$ represents the *characteristic class*

$$[c] : H(-,A) \to H(-,B)\,.$$

Remark 3.6.138. Traditionally attention is often concentrated on the case that $K \in \tau_0 \mathrm{Grp}(\mathbf{H})$ is an abelian 0-truncated group object and $A := \mathbf{B}^p K$ is the Eilenber-MacLane object with K in degree p. The corresponding cohomology $H^p(-,K) \simeq \pi_0 \mathbf{H}(-, \mathbf{B}^p K)$ is sometimes called *ordinary cohomology* with coefficients in K, to distinguish it from the generalizations obtained by allowing more general K, which traditionally go by the term *hypercohomology* (if K is not necessarily concentrated in a single degree but is still an abelian ∞-group, def. 3.6.120) and more generally *nobabelian cohomology* (if A is allowed to be any homotopy type).

Below in 3.6.10 we discuss the notion of an *∞-group G acting on a space X* and the corresponding (homotopy) quotient $X//G$. Then we say

Definition 3.6.139. The cohomology of $X//G$ is the *G-equivariant cohomology* of X with respect to the given action. .

Remark 3.6.140. There is also a notion of cohomology in the *petit* ∞-topos of $X \in \mathbf{H}$, the slice of $\mathbf{H}$ over X

$$\mathcal{X} := \mathbf{H}_{/X}\,.$$

This is canonically equipped with the étale geometric morphism, prop. 3.6.13

$$(X_! \dashv X^* \dashv X_*) : \mathbf{H}/X \underset{X_*}{\overset{X_!}{\underset{\longleftarrow X^* -}{\rightrightarrows}}} \mathbf{H}\,,$$

where $X_!$ simply forgets the morphism to X and where $X^* = X \times (-)$ forms the product with X. Accordingly $X^*(*_{\mathbf{H}}) \simeq *_{\mathcal{X}} =: X$ and $X_!(*_{\mathcal{X}}) = X \in \mathbf{H}$. Therefore cohomology over X with coefficients of the form X^*A is equivalently the cohomology in $\mathbf{H}$ of X with coefficients in A:

$$\mathcal{X}(X, X^*A) \simeq \mathbf{H}(X, A)\,.$$

For a general coeffcient object $A \in \mathcal{X}$ the A-cohomology over X in $\mathcal{X}$ is a *twisted* cohomology of X in $\mathbf{H}$, discussed below in 3.6.12.

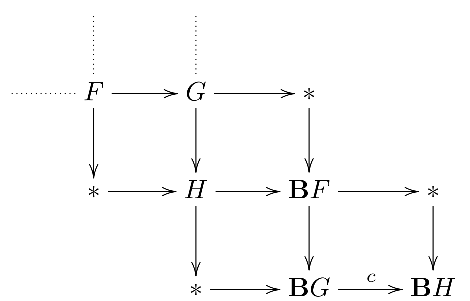
Typically one thinks of a morphism $A \to B$ in $\mathbf{H}$ as presenting a *characteristic class* of A if B is "simpler" than A, notably if B is an Eilenberg-MacLane object $B = \mathbf{B}^n K$ for K a 0-truncated abelian group in $\mathbf{H}$. In this case the characteristic class may be regarded as being in the degree-n K-cohomology of A

$$[c] \in H^n(A, K)\,.$$

Definition 3.6.141. For every morphism $c : \mathbf{B}G \to \mathbf{B}H \in \mathbf{H}$ define the *long fiber sequence to the left*

$$\cdots \to \Omega G \to \Omega H \to F \to G \to H \to \mathbf{B}F \to \mathbf{B}G \xrightarrow{c} \mathbf{B}H$$

to be given by the consecutive pasting diagrams of ∞-pullbacks



Proposition 3.6.142. *This is well-defined, in that the objects in the fiber sequence are indeed as indicated.*

Proof. Repeatedly apply the pasting law 2.3.2 and definition 3.6.114. □

Proposition 3.6.143. *1. The long fiber sequence to the left of $c : \mathbf{B}G \to \mathbf{B}H$ becomes constant on the point after n iterations if H is n-truncated.*

2. For every object $X \in \mathbf{H}$ we have a long exact sequence of pointed cohomology sets

$$\cdots \to H^0(X, G) \to H^0(X, H) \to H^1(X, F) \to H^1(X, G) \to H^1(X, H)\,.$$

Proof. The first statement follows from the observation that a loop space object $\Omega_x A$ is a fiber of the free loop space object $\mathcal{L}A$ and that this may equivalently be computed by the ∞-powering A^{S^1}, where $S^1 \in \mathrm{Top} \simeq \infty\mathrm{Grpd}$ is the circle.

The second statement follows by observing that the ∞-hom-functor $\mathbf{H}(X, -)$ preserves all ∞-limits, so that we have ∞-pullbacks

$$\begin{array}{ccc} \mathbf{H}(X, F) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{H}(X, G) & \longrightarrow & \mathbf{H}(X, H) \end{array}$$

etc. in ∞Grpd at each stage of the fiber sequence. The statement then follows with the familiar long exact sequence for homotopy groups in $\mathrm{Top} \simeq \infty\mathrm{Grpd}$. □

Remark 3.6.144. To every cocycle $g : X \to \mathbf{B}G$ is canonically associated its homotopy fiber $P \to X$, the ∞-pullback

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} .$$

We discuss below in 3.6.10 that such P canonically has the structure of a *G-principal ∞-bundle* and that $\mathbf{B}G$ is the fine moduli space – the moduli ∞-stack – for G-principal ∞-bundles.

Proposition 3.6.145 (Mayer-Vietoris fiber sequence)**.** *Let $\mathbf{H}$ be an ∞-topos with a 1-site of definition (for instance an ∞-cohesive site as in def. 3.4.8) and let B be an ∞-group object in $\mathbf{H}$. Then for any two morphisms $f : X \to B$ and $g : Y \to B$ the ∞-pullback $X \times_B Y$ is equivalently the ∞-pullback*

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ X \times Y & \xrightarrow{f \cdot g^{-1}} & B \end{array} ,$$

where the bottom morphism is the composite

$$f \cdot g^{-1} : \; X \times Y \xrightarrow{(f,g)} B \times B \xrightarrow{(\mathrm{id}, (-)^{-1})} B \times B \xrightarrow{\cdot} B$$

of the pair (f, g) with the morphism that inverts the second factor and the morphism that exhibits the group product on B.

We have then a fiber sequence that starts out as

$$\cdots \longrightarrow \Omega B \longrightarrow X \times_B Y \longrightarrow X \times Y \xrightarrow{f \cdot g^{-1}} B \; .$$

Proof. By prop 3.6.134 there is a presheaf of simplicial groups presenting B over the site C, which we shall denote by the same symbol, $B \in [C^{\mathrm{op}}, \mathrm{sGrp}] \to [C^{\mathrm{op}}, \mathrm{sSet}]$. In terms of this the morphism $- : B \times B \to B$ is, objectwise over $U \in C$, given by the simplicial morphism $-_U : B(U) \times B(U) \to B(U)$ that sends k-cells $(a, b) : \Delta[k] \to B(U) \times B(U)$ to $a \cdot b^{-1}$, using the degreewise group structure.

We observe first that this morphism is objectwise a Kan fibration and hence a fibration in $[S^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. To see this, let

$$\begin{array}{ccc} \Lambda[k]_i & \xrightarrow{(ha,hb)} & B(U) \times B(U) \\ \downarrow^{j} & & \downarrow^{-} \\ \Delta[k] & \xrightarrow{\sigma} & B(U) \end{array}$$

be a lifting problem. Since $B(U)$, being the simplicial set underlying a simplicial group, is a Kan complex, there is a filler $b : \Delta[k] \to B(U)$ of the horn hb. Define then a k-cell

$$a := \sigma \cdot b \, .$$

This is a filler of ha, since the face maps are group homomorphisms:

$$\begin{aligned} \delta_l a &= \delta_l(\sigma \cdot b) \\ &= \delta_l(\sigma) \cdot \delta_l(b) \\ &= \delta_l(\sigma) \cdot (hb)_l \\ &= (ha)_l \end{aligned} .$$

So we have a filler

$$\begin{array}{ccc} \Lambda[k]_i & \xrightarrow{(ha,hb)} & B(U) \times B(U) \\ \downarrow^{j} & \nearrow^{(a,b)} & \downarrow^{-} \\ \Delta[k] & \xrightarrow{\sigma} & B(U) \end{array} .$$

Observe then that there is a pullback diagram of simplicial presheaves

$$\begin{array}{ccc} B & \longrightarrow & * \\ \downarrow^{\Delta_B} & & \downarrow^{e} \\ B \times B & \xrightarrow{-} & B \end{array} ,$$

where the left morphism is the diagonal on B and where the right morphism picks the neutral element in B. Since, by the above, the bottom morphism is a fibration, this presents a homotopy pullback.

Next, by the *factorization lemma*, lemma 2.3.9, and using prop. 2.3.13, the homotopy pullback of f along g is presented by the ordinary pullback of simplicial presheaves

$$\begin{array}{ccc} Q & \longrightarrow & B^{\Delta[0]} \\ \downarrow & & \downarrow \\ X \times Y & \xrightarrow{(f,g)} & B \times B \end{array} ,$$

where the right morphism is endpoint evaluation out of the canonical path object of B, which is a fibration replacement of the diagonal Δ_B. Therefore this presents an ∞-pullback

$$\begin{array}{ccc} Q & \longrightarrow & B \\ \downarrow & & \downarrow^{\Delta_B} \\ X \times Y & \xrightarrow{(f,g)} & B \times B \end{array} .$$

Now by the pasting law, prop. 2.3.2, Q is also an ∞-pullback for the total outer diagram in

$$\begin{array}{ccccc} Q & \longrightarrow & B & \longrightarrow & * \\ \downarrow & & \downarrow^{\Delta_B} & & \downarrow^{e} \\ X \times Y & \xrightarrow{(f,g)} & B \times B & \xrightarrow{-} & B \end{array} , \qquad X \times Y \xrightarrow{f \cdot g^{-1}} B .$$

□

3.6.9.2 Presentations

We discuss explicit presentations of cocycles, cohomology classes and fiber sequences in an ∞-topos.

3.6.9.2.1 Cocycle ∞-groupoids and cohomology classes

We discuss a useful presentation of cocycle ∞-groupoids and of cohomology classes by a construction that exists when the ambient ∞-topos is presented by a category with weak equivalences that is equipped with the structure of a *category of fibrant objects* [Br73].

Definition 3.6.146 (Brown)**.** A *category of fibrant objects* is a category equipped with two distinguished classes of morphisms, called *fibrations* and *weak equivalences*, such that

1. the category has a terminal object $*$ and finite products;
2. fibrations and weak equivalences form subcategories that contain all isomorphisms; weak equivalences moreover satisfy the 2-out-of-3 property;
3. for any object B the map $B \to *$ is a fibration;
4. the classes of fibrations and of *acyclic fibrations* (the fibration that are also weak equivalences) are stable under pullback. That means: given a diagram $A \xrightarrow{g} C \xleftarrow{f} B$ where f is a (acyclic) fibration then the pullback $A \times_C B$ exists and the morphism $A \times_C B \to A$ is again a (acyclic) fibration.
5. For every object B there is a path object B^I, i.e. a factorization of the diagonal $\Delta \colon B \to B \times B$ into

$$B \xrightarrow{\simeq} B^I \twoheadrightarrow B \times B$$

such that left map is weak equivalence and the right map a fibration. We assume here moreover for simplicity that this B^I can be chosen functorial in B.

Given a category of fibrant objects, we will denote the class of weak equivalence by W and the class of fibrations by F.

Examples 3.6.147. We have the following well known examples of categories of fibrant objects.

- For any model category (with functorial factorization) the full subcategory of fibrant objects is a category of fibrant objects.
- The category of stalkwise Kan simplicial presheaves on any site with enough points. In this case the fibrations are the stalkwise fibrations and the weak equivalences are the stalkwise weak equivalences.

Remark 3.6.148. Notice that (over a non-trivial site) the second example above is *not* a special case of the first: while there are model structures on categories of simplicial presheaves whose weak equivalences are the stalkwise weak equivalences, their fibrations (even between fibrant objects) are much more restricted than just being stalkwise fibrations.

Theorem 3.6.149. *Let the ∞-category $\mathbf{H}$ be presented by a category with weak equivalences $(\mathcal{C}, W)$ that carries a compatible structure of a category of fibrant objects, def. 3.6.146.*

Then for X, A and two objects in $\mathcal{C}$, presenting two objects in $\mathbf{H}$, the ∞-groupoid $\mathbf{H}(X, A)$ is presented in $\mathrm{sSet}_{\mathrm{Quillen}}$ by the nerve of the category whose

- *objects are spans (cocycles / ∞-anafunctors)*

$$X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} A$$

in $\mathcal{C}$;

- *morphisms $f : (\hat{X}, g) \to (\hat{X}', g')$ are given by morphisms $f : \hat{X} \to \hat{X}'$ in $\mathcal{C}$ such that the diagram*

$$\begin{array}{ccccc} & & \hat{X} & & \\ & \overset{\simeq}{\swarrow} & \downarrow{\scriptstyle f} & \overset{g}{\searrow} & \\ X & & & & A \\ & \underset{\simeq}{\nwarrow} & & \underset{g'}{\nearrow} & \\ & & \hat{X}' & & \end{array}$$

commutes.

This appears for instance as prop. 3.23 in [Cis10].

Example 3.6.150. By the discussion in 2.2.3, if $\mathbf{H}$ has a 1-site of definition C with enough 1-topos points, then it is presented by the category $\mathrm{Sh}(C)^{\Delta^{\mathrm{op}}}$ of simplicial sheaves on C with weak equivalences the stalkwise weak equivalences of simplicial sets, and equivalently by its full subcategory of stalkwise Kan fibrant simplicial sheaves. With the local fibrations, def. 2.2.13 as fibrations, this is a category of fibrant objects. So in this case the cocycle ∞-groupoid $\mathbf{H}(X, A)$ is presented by the Kan fibrant replacement of the category whose objects are spans

$$X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} A$$

for $\hat{X} \to X$ a stalkwise acylic Kan fibration, and whose morphisms are as above.

3.6.9.2.2 Fiber sequences

We discuss explicit presentations of certain fiber sequences, def. 3.6.141, in an ∞-topos.

Proposition 3.6.151. *Let $A \to \hat{G} \to G$ be a central extension of (ordinary) groups. Then there is a long fiber sequence in* $\infty\mathrm{Grpd}$ *of the form*

$$A \longrightarrow \hat{G} \longrightarrow G \xrightarrow{\Omega \mathbf{c}} \mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \longrightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^2 A \ ,$$

where the connecting homomorphism is presented by the correspondence of crossed modules, def. 1.2.45, given by

$$(1 \to G) \overset{\simeq}{\longleftarrow} (A \to \hat{G}) \longrightarrow (A \to 1) \ .$$

Here in the middle appears the crossed module defined by the central extension, def. 1.2.52.

3.6.10 Principal bundles

For G an ∞-group object in a cohesive ∞-topos $\mathbf{H}$ and $\mathbf{B}G$ its delooping in $\mathbf{H}$, as discussed in 3.6.8, the cohomology over an object X with coefficients in $\mathbf{B}G$, as in 3.6.9, classifies maps $P \to X$ that are equipped with a G-action that is *principal*. We discuss here these *G-principal ∞-bundles*.

3.6.10.1 Introduction and survey

We give an exposition of some central ideas and phenomena of higher principal bundles, discussed in detail below.

This section draws from [NSSa].

Let G be a topological group, or Lie group or some similar such object. The traditional definition of *G-principal bundle* is the following: there is a map

$$P \to X := P/G$$

which is the quotient projection induced by a *free* action

$$\rho : P \times G \to P$$

of G on a space (or manifold, depending on context) P, such that there is a cover $U \to X$ over which the quotient projection is isomorphic to the trivial one $U \times G \to U$.

In higher geometry, if G is a topological or smooth ∞-group, the quotient projection must be replaced by the ∞-quotient (homotopy quotient) projection

$$P \to X := P//G$$

for the action of G on a topological or smooth ∞-groupoid (or ∞-stack) P. It is a remarkable fact that this single condition on the map $P \to X$ already implies that G acts freely on P and that $P \to X$ is locally trivial, when the latter notions are understood in the context of higher geometry. We will therefore define a G-principal ∞-bundle to be such a map $P \to X$.

As motivation for this, notice that if a Lie group G acts properly, but not freely, then the quotient $P \to X := P/G$ differs from the homotopy quotient. Specifically, if precisely the subgroup $G_{\mathrm{stab}} \hookrightarrow G$ acts trivially, then the homotopy quotient is instead the *quotient stack* $X//G_{\mathrm{stab}}$ (sometimes written $[X//G_{\mathrm{stab}}]$, which is an orbifold if G_{stab} is finite). The ordinary quotient coincides with the homotopy quotient if and only if the stabilizer subgroup G_{stab} is trivial, and hence if and only if the action of G is free.

Conversely this means that in the context of higher geometry a non-free action may also be principal: with respect not to a base space, but with respect to a base groupoid/stack. In the example just discussed, we have that the projection $P \to X//G_{\mathrm{stab}}$ exhibits P as a G-principal bundle over the action groupoid $P//G \simeq X//G_{\mathrm{stab}}$. For instance if $P = V$ is a vector space equipped with a G-representation, then $V \to V//G$ is a G-principal bundle over a groupoid/stack. In other words, the traditional requirement of freeness in a principal action is not so much a characterization of principality as such, as rather a condition that ensures that the base of a principal action is a 0-truncated object in higher geometry.

Beyond this specific class of 0-truncated examples, this means that we have the following noteworthy general statement: in higher geometry *every* ∞-action is principal with respect to *some* base, namely with respect to its ∞-quotient. In this sense the notion of principal bundles is (even) more fundamental to higher geometry than it is to ordinary geometry. Also, several constructions in ordinary geometry that are traditionally thought of as conceptually different from the notion of principality turn out to be special cases of principality in higher geometry. For instance a central extension of groups $A \to \hat{G} \to G$ turns out to be equivalently a higher principal bundle, namely a $\mathbf{B}A$-principal 2-bundle of moduli stacks $\mathbf{B}\hat{G} \to \mathbf{B}G$. Following this through, one finds that the topics of principal ∞-bundles, of ∞-group extensions (3.6.14), of ∞-representations (3.6.13), and of ∞-group cohomology are all different aspects of just one single concept in higher geometry.

More is true: in the context of an ∞-topos every ∞-quotient projection of an ∞-group action is locally trivial, with respect to the canonical intrinsic notion of cover, hence of locality. Therefore also the condition of local triviality in the classical definition of principality becomes automatic. This is a direct consequence of the third ∞-Giraud axiom, Definition 2.2.2 that "all ∞-quotients are effective". This means that the projection map $P \to P//G$ is always a cover (an *effective epimorphism*) and so, since every G-principal ∞-bundle trivializes over itself, it exhibits a local trivialization of itself; even without explicitly requiring it to be locally trivial.

As before, this means that the local triviality clause appearing in the traditional definition of principal bundles is not so much a characteristic of principality as such, as rather a condition that ensures that a given quotient taken in a category of geometric spaces coincides with the "correct" quotient obtained when regarding the situation in the ambient ∞-topos.

Another direct consequence of the ∞-Giraud axioms is the equivalence of the definition of principal bundles as quotient maps, which we discussed so far, with the other main definition of principality: the condition that the "shear map" $(\mathrm{id}, \rho) : P \times G \to P \times_X P$ is an equivalence. It is immediate to verify in traditional 1-categorical contexts that this is equivalent to the action being properly free and exhibiting X as its quotient (we discuss this in detail in [NSSc]). Simple as this is, one may observe, in view of the above discussion, that the shear map being an equivalence is much more fundamental even: notice that $P \times G$ is the first stage of the *action groupoid object* $P//G$, and that $P \times_X P$ is the first stage of the *Čech nerve groupoid object* $\check{C}(P \to X)$ of the corresponding quotient map. Accordingly, the shear map equivalence is the first stage in the equivalence of groupoid objects in the ∞-topos

$$P//G \simeq \check{C}(P \to X)\,.$$

This equivalence is just the explicit statement of the fact mentioned before: the groupoid object $P//G$ is effective – as is any groupoid object in an ∞-topos – and, equivalently, its principal ∞-bundle map $P \to X$ is an effective epimorphism.

Fairly directly from this fact, finally, springs the classification theorem of principal ∞-bundles. For we have a canonical morphism of groupoid objects $P//G \to *//G$ induced by the terminal map $P \to *$. By the ∞-Giraud theorem the ∞-colimit over this sequence of morphisms of groupoid objects is a G-cocycle on X

(Definition 3.6.137) canonically induced by P:

$$\varinjlim \left(\check{C}(P \to X)_\bullet \simeq (P//G)_\bullet \to (*//G)_\bullet\right) = (X \to \mathbf{B}G) \quad \in \mathbf{H}(X, \mathbf{B}G)\,.$$

Conversely, from any such G-cocycle one finds that one obtains a G-principal ∞-bundle simply by forming its ∞-fiber: the ∞-pullback of the point inclusion $* \to \mathbf{B}G$. We show in [NSSb] that in presentations of the ∞-topos theory by 1-categorical tools, the computation of this homotopy fiber is *presented* by the ordinary pullback of a big resolution of the point, which turns out to be nothing but the universal G-principal bundle. This appearance of the universal ∞-bundle as just a resolution of the point inclusion may be understood in light of the above discussion as follows. The classical characterization of the universal G-principal bundle $\mathbf{E}G$ is as a space that is homotopy equivalent to the point and equipped with a *free* G-action. But by the above, freeness of the action is an artefact of 0-truncation and not a characteristic of principality in higher geometry. Accordingly, in higher geometry the universal G-principal ∞-bundle for any ∞-group G may be taken to *be* the point, equipped with the trivial (maximally non-free) G-action. As such, it is a bundle not over the classifying *space* BG of G, but over the full moduli ∞-stack $\mathbf{B}G$.

This way we have natural assignments of G-principal ∞-bundles to cocycles in G-nonabelian cohomology, and vice versa. We find (see Theorem 3.6.170 below) that precisely the second ∞-Giraud axiom of Definition 2.2.2, namely the fact that in an ∞-topos ∞-colimits are preserved by ∞-pullback, implies that these constructions constitute an equivalence of ∞-groupoids, hence that G-principal ∞-bundles are classified by G-cohomology.

The following table summarizes the relation between ∞-bundle theory and the ∞-Giraud axioms as indicated above, and as proven in the following section.

∞-Giraud axioms	**principal ∞-bundle theory**
quotients are effective	every ∞-quotient $P \to X := P//G$ is principal
colimits are preserved by pullback	G-principal ∞-bundles are classified by $\mathbf{H}(X, \mathbf{B}G)$

3.6.10.2 Definition and classification We discuss the general definition and the central classification theorem of principal ∞-bundles.

This section draws from [NSSa].

Definition 3.6.152. For $G \in \mathrm{Grp}(\mathbf{H})$ a group object, we say a *G-action* on an object $P \in \mathbf{H}$ is a groupoid object $P//G$ (Definition 3.6.88) of the form

$$\cdots \; P \times G \times G \;\substack{\to\\\to\\\to}\; P \times G \;\underset{d_1}{\overset{\rho := d_0}{\rightrightarrows}}\; P$$

such that $d_1 : P \times G \to P$ is the projection, and such that the degreewise projections $P \times G^n \to G^n$ constitute a morphism of groupoid objects

$$\begin{array}{ccccccc} \cdots & P \times G \times G & \substack{\to\\\to\\\to} & P \times G & \rightrightarrows & P \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots & G \times G & \substack{\to\\\to\\\to} & G & \rightrightarrows & * \end{array}$$

where the lower simplicial object exhibits G as a groupoid object over $*$.

With convenient abuse of notation we also write

$$P//G := \varinjlim (P \times G^{\times^\bullet}) \quad \in \mathbf{H}$$

for the corresponding ∞-colimit object, the ∞-*quotient* of this action.

Write

$$G\mathrm{Action}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H})_{/(*//G)}$$

for the full sub-∞-category of groupoid objects over $*//G$ on those that are G-actions.

Remark 3.6.153. The remaining face map d_0

$$\rho := d_0 : P \times G \to P$$

is the action itself.

Remark 3.6.154. Using this notation in Proposition 3.6.121 we have

$$\mathbf{B}G \simeq *//G\,.$$

We list examples of ∞-actions below as Example 3.6.216. This is most conveniently done after establishing the theory of principal ∞-actions, to which we now turn.

Definition 3.6.155. Let $G \in \infty\mathrm{Grp}(\mathbf{H})$ be an ∞-group and let X be an object of $\mathbf{H}$. A *G-principal ∞-bundle* over X (or *G-torsor over X*) is

1. a morphism $P \to X$ in $\mathbf{H}$;
2. together with a G-action on P;

such that $P \to X$ is the colimiting cocone exhibiting the quotient map $X \simeq P//G$ (Definition 3.6.152).

A *morphism* of G-principal ∞-bundles over X is a morphism of G-actions that fixes X; the ∞-category of G-principal ∞-bundles over X is the homotopy fiber of ∞-categories

$$G\mathrm{Bund}(X) := G\mathrm{Action}(\mathbf{H}) \times_{\mathbf{H}} \{X\}$$

over X of the quotient map

$$G\mathrm{Action}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H})_{/(*//G)} \longrightarrow \mathrm{Grpd}(\mathbf{H}) \xrightarrow{\varinjlim} \mathbf{H}\ .$$

Remark 3.6.156. By the third ∞-Giraud axiom (Definition 2.2.2) this means in particular that a G-principal ∞-bundle $P \to X$ is an effective epimorphism in $\mathbf{H}$.

Remark 3.6.157. Even though $G\mathrm{Bund}(X)$ is by definition a priori an ∞-category, Proposition 3.6.169 below says that in fact it happens to be ∞-groupoid: all its morphisms are invertible.

Proposition 3.6.158. *A G-principal ∞-bundle $P \to X$ satisfies the* principality condition*: the canonical morphism*

$$(\rho, p_1) : \ P \times G \xrightarrow{\simeq} P \times_X P$$

is an equivalence, where ρ is the G-action.

Proof. By the third ∞-Giraud axiom (Definition 2.2.2) the groupoid object $P//G$ is effective, which means that it is equivalent to the Čech nerve of $P \to X$. In first degree this implies a canonical equivalence $P \times G \to P \times_X P$. Since the two face maps $d_0, d_1 : P \times_X P \to P$ in the Čech nerve are simply the projections out of the fiber product, it follows that the two components of this canonical equivalence are the two face maps $d_0, d_1 : P \times G \to P$ of $P//G$. By definition, these are the projection onto the first factor and the action itself. □

Proposition 3.6.159. *For $g : X \to \mathbf{B}G$ any morphism, its homotopy fiber $P \to X$ canonically carries the structure of a G-principal ∞-bundle over X.*

Proof. That $P \to X$ is the fiber of $g : X \to \mathbf{B}G$ means that we have an ∞-pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}G. \end{array}$$

By the pasting law for ∞-pullbacks, Proposition 2.3.2, this induces a compound diagram

$$\begin{array}{ccccccccc} \cdots \substack{\to\\\to\\\to\\\to} & P \times G \times G & \substack{\to\\\to\\\to} & P \times G & \rightrightarrows & P & \twoheadrightarrow & X \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ \cdots \substack{\to\\\to\\\to\\\to} & G \times G & \substack{\to\\\to\\\to} & G & \rightrightarrows & * & \twoheadrightarrow & \mathbf{B}G \end{array}$$

where each square and each composite rectangle is an ∞-pullback. This exhibits the G-action on P. Since $* \to \mathbf{B}G$ is an effective epimorphism, so is its ∞-pullback $P \to X$. Since, by the ∞-Giraud theorem, ∞-colimits are preserved by ∞-pullbacks we have that $P \to X$ exhibits the ∞-colimit $X \simeq P//G$. □

Lemma 3.6.160. *For $P \to X$ a G-principal ∞-bundle obtained as in Proposition 3.6.159, and for $x : * \to X$ any point of X, we have a canonical equivalence*

$$x^*P \xrightarrow{\simeq} G$$

between the fiber x^*P and the ∞-group object G.

Proof. This follows from the pasting law for ∞-pullbacks, which gives the diagram

$$\begin{array}{ccccc} G & \longrightarrow & P & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{x} & X & \xrightarrow{g} & \mathbf{B}G \end{array}$$

in which both squares as well as the total rectangle are ∞-pullbacks. □

Definition 3.6.161. The *trivial* G-principal ∞-bundle $(P \to X) \simeq (X \times G \to X)$ is, up to equivalence, the one obtained via Proposition 3.6.159 from the morphism $X \to * \to \mathbf{B}G$.

Observation 3.6.162. For $P \to X$ a G-principal ∞-bundle and $Y \to X$ any morphism, the ∞-pullback $Y \times_X P$ naturally inherits the structure of a G-principal ∞-bundle.

Proof. This uses the same kind of argument as in Proposition 3.6.159 (which is the special case of the pullback of what we will see is the universal G-principal ∞-bundle $* \to \mathbf{B}G$ below in Proposition 3.6.166). □

Definition 3.6.163. A G-principal ∞-bundle $P \to X$ is called *locally trivial* if there exists an effective epimorphism $U \twoheadrightarrow X$ and an equivalence of G-principal ∞-bundles

$$U \times_X P \simeq U \times G$$

from the pullback of P (Observation 3.6.162) to the trivial G-principal ∞-bundle over U (Definition 3.6.161).

Proposition 3.6.164. *Every G-principal ∞-bundle is locally trivial.*

Proof. For $P \to X$ a G-principal ∞-bundle, it is, by Remark 3.6.156, itself an effective epimorphism. The pullback of the G-bundle to its own total space along this morphism is trivial, by the principality condition (Proposition 3.6.158). Hence setting $U := P$ proves the claim. □

Remark 3.6.165. This means that every G-principal ∞-bundle is in particular a G-fiber ∞-bundle (in the evident sense of Definition 3.6.204 below). But not every G-fiber bundle is G-principal, since the local trivialization of a fiber bundle need not respect the G-action.

Proposition 3.6.166. *For every G-principal ∞-bundle $P \to X$ the square*

$$\begin{array}{ccccccc} & & P & \longrightarrow & * & & \\ & & \downarrow & & \downarrow & & \\ X & \simeq & \varinjlim_n (P \times G^{\times_n}) & \longrightarrow & \varinjlim_n G^{\times_n} & \simeq & \mathbf{B}G \end{array}$$

is an ∞-pullback diagram.

Proof. Let $U \to X$ be an effective epimorphism such that $P \to X$ pulled back to U becomes the trivial G-principal ∞-bundle. By Proposition 3.6.164 this exists. By definition of morphism of G-actions and by functoriality of the ∞-colimit, this induces a morphism in $\mathbf{H}^{\Delta[1]}{}_{/(* \to \mathbf{B}G)}$ corresponding to the diagram

$$\begin{array}{ccccc} U \times G & \twoheadrightarrow & P & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow^{\mathrm{pt}} \\ U & \twoheadrightarrow & X & \longrightarrow & \mathbf{B}G \end{array} \quad \simeq \quad \begin{array}{ccccc} U \times G & & \longrightarrow & & * \\ \downarrow & & & & \downarrow^{\mathrm{pt}} \\ U & \longrightarrow & * & \xrightarrow{\mathrm{pt}} & \mathbf{B}G \end{array}$$

in $\mathbf{H}$. By assumption, in this diagram the outer rectangles and the square on the very left are ∞-pullbacks. We need to show that the right square on the left is also an ∞-pullback.

Since $U \to X$ is an effective epimorphism by assumption, and since these are stable under ∞-pullback, $U \times G \to P$ is also an effective epimorphism, as indicated. This means that

$$P \simeq \varinjlim_n (U \times G)^{\times_P^{n+1}} .$$

We claim that for all $n \in \mathbb{N}$ the fiber products in the colimit on the right are naturally equivalent to $(U^{\times_X^{n+1}}) \times G$. For $n = 0$ this is clearly true. Assume then by induction that it holds for some $n \in \mathbb{N}$. Then with the pasting law (Proposition 2.3.2) we find an ∞-pullback diagram of the form

$$\begin{array}{ccccccc} (U^{\times_X^{n+1}}) \times G & \simeq & (U \times G)^{\times_P^{n+1}} & \longrightarrow & (U \times G)^{\times_P^{n}} & \simeq & (U^{\times_X^{n}}) \times G \\ & & \downarrow & & \downarrow & & \\ & & U \times G & \longrightarrow & P & & \\ & & \downarrow & & \downarrow & & \\ & & U & \longrightarrow & X. & & \end{array}$$

This completes the induction. With this the above expression for P becomes

$$\begin{aligned} P &\simeq \varinjlim_n (U^{\times_X^{n+1}}) \times G \\ &\simeq \varinjlim_n \mathrm{pt}^* (U^{\times_X^{n+1}}) \\ &\simeq \mathrm{pt}^* \varinjlim_n (U^{\times_X^{n+1}}) \\ &\simeq \mathrm{pt}^* X, \end{aligned}$$

where we have used that by the second ∞-Giraud axiom (Definition 2.2.2) we may take the ∞-pullback out of the ∞-colimit and where in the last step we used again the assumption that $U \to X$ is an effective epimorphism. $\square$

Example 3.6.167. The fiber sequence

$$\begin{array}{ccc} G & \longrightarrow & * \\ & & \downarrow \\ & & \mathbf{B}G \end{array}$$

which exhibits the delooping $\mathbf{B}G$ of G according to Theorem 3.6.116 is a G-principal ∞-bundle over $\mathbf{B}G$, with *trivial* G-action on its total space $*$. Proposition 3.6.166 says that this is the *universal G-principal ∞-bundle* in that every other one arises as an ∞-pullback of this one. In particular, $\mathbf{B}G$ is a classifying object for G-principal ∞-bundles.

Below in Theorem 3.6.255 this relation is strengthened: every *automorphism* of a G-principal ∞-bundle, and in fact its full automorphism ∞-group arises from pullback of the above universal G-principal ∞-bundle: $\mathbf{B}G$ is the fine *moduli ∞-stack* of G-principal ∞-bundles.

The traditional definition of universal G-principal bundles in terms of contractible objects equipped with a free G-action has no intrinsic meaning in higher topos theory. Instead this appears in *presentations* of the general theory in model categories (or categories of fibrant objects) as *fibrant representatives* $\mathbf{E}G \to \mathbf{B}G$ of the above point inclusion. This we discuss in [NSSb].

The main classification Theorem 3.6.170 below implies in particular that every morphism in $G\mathrm{Bund}(X)$ is an equivalence. For emphasis we note how this also follows directly:

Lemma 3.6.168. *Let $\mathbf{H}$ be an ∞-topos and let X be an object of $\mathbf{H}$. A morphism $f\colon A \to B$ in $\mathbf{H}_{/X}$ is an equivalence if and only if p^*f is an equivalence in $\mathbf{H}_{/Y}$ for any effective epimorphism $p\colon Y \to X$ in $\mathbf{H}$.*

Proof. It is clear, by functoriality, that p^*f is a weak equivalence if f is. Conversely, assume that p^*f is a weak equivalence. Since effective epimorphisms as well as equivalences are preserved by pullback we get a simplicial diagram of the form

$$\begin{array}{ccccccc} \cdots \rightrightarrows & p^*A \times_A p^*A & \rightrightarrows & p^*A & \twoheadrightarrow & A \\ & \downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{f} \\ \cdots \rightrightarrows & p^*B \times_B p^*B & \rightrightarrows & p^*B & \twoheadrightarrow & B \end{array}$$

where the rightmost horizontal morphisms are effective epimorphisms, as indicated. By definition of effective epimorphisms this exhibits f as an ∞-colimit over equivalences, hence as an equivalence. $\square$

Proposition 3.6.169. *Every morphism between G-actions over X that are G-principal ∞-bundles over X is an equivalence.*

Proof. Since a morphism of G-principal bundles $P_1 \to P_2$ is a morphism of Čech nerves that fixes their ∞-colimit X, up to equivalence, and since $* \to \mathbf{B}G$ is an effective epimorphism, we are, by Proposition 3.6.166, in the situation of Lemma 3.6.168. $\square$

Theorem 3.6.170. *For all $X, \mathbf{B}G \in \mathbf{H}$ there is a natural equivalence of ∞-groupoids*

$$G\mathrm{Bund}(X) \simeq \mathbf{H}(X, \mathbf{B}G)$$

which on vertices is the construction of Definition 3.6.159: a bundle $P \to X$ is mapped to a morphism $X \to \mathbf{B}G$ such that $P \to X \to \mathbf{B}G$ is a fiber sequence.

We therefore say

- $\mathbf{B}G$ is the *classifying object* or *moduli ∞-stack* for G-principal ∞-bundles;
- a morphism $c : X \to \mathbf{B}G$ is a *cocycle* for the corresponding G-principal ∞-bundle and its class $[c] \in \mathrm{H}^1(X, G)$ is its *characteristic class.*

Proof. By Definitions 3.6.152 and 3.6.155 and using the refined statement of the third ∞-Giraud axiom (Theorem 3.6.87), the ∞-groupoid of G-principal ∞-bundles over X is equivalent to the fiber over X of the sub-∞-category of the slice of the arrow ∞-topos on those squares

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{B}G \end{array}$$

that exhibit $P \to X$ as a G-principal ∞-bundle. By Proposition 3.6.159 and Proposition 3.6.166 these are the ∞-pullback squares $\mathrm{Cart}({\mathbf{H}^{\Delta[1]}}_{/(*\to\mathbf{B}G)}) \hookrightarrow {\mathbf{H}^{\Delta[1]}}_{/(*\to\mathbf{B}G)}$, hence

$$G\mathrm{Bund}(X) \simeq \mathrm{Cart}({\mathbf{H}^{\Delta[1]}}_{/(*\to\mathbf{B}G)}) \times_{\mathbf{H}} \{X\}\,.$$

By the universality of the ∞-pullback the morphisms between these are fully determined by their value on X, so that the above is equivalent to

$$\mathbf{H}_{/\mathbf{B}G} \times_{\mathbf{H}} \{X\}\,.$$

(For instance in terms of model categories: choose a model structure for $\mathbf{H}$ in which all objects are cofibrant, choose a fibrant representative for $\mathbf{B}G$ and a fibration resolution $\mathbf{E}G \to \mathbf{B}G$ of the universal G-bundle. Then the slice model structure of the arrow model structure over this presents the slice in question and the statement follows from the analogous 1-categorical statement.) This finally is equivalent to

$$\mathbf{H}(X, \mathbf{B}G)\,.$$

(For instance in terms of quasi-categories: the projection $\mathbf{H}_{/\mathbf{B}G} \to \mathbf{H}$ is a fibration by Proposition 2.1.2.1 and 4.2.1.6 in [LuHTT], hence the homotopy fiber $\mathbf{H}_{/\mathbf{B}G} \times_{\mathbf{X}} \{X\}$ is the ordinary fiber of quasi-categories. This is manifestly the $\mathrm{Hom}^R_{\mathbf{H}}(X, \mathbf{B}G)$ from Proposition 1.2.2.3 of [LuHTT]. Finally, by Proposition 2.2.4.1 there, this is equivalent to $\mathbf{H}(X, \mathbf{B}G)$.) □

Corollary 3.6.171. *Equivalence classes of G-principal ∞-bundles over X are in natural bijection with the degree-1 G-cohomology of X:*

$$G\mathrm{Bund}(X)_{/\sim} \simeq H^1(X, G)\,.$$

Proof. By Definition 3.6.137 this is the restriction of the equivalence $G\mathrm{Bund}(X) \simeq \mathbf{H}(X, \mathbf{B}G)$ to connected components. □

3.6.10.3 Universal principal ∞-bundles and the Borel construction By prop. 3.6.134 every ∞-group in an ∞-topos over an ∞-cohesive site is presented by a (pre-)sheaf of simplicial groups, hence by a strict group object G in a 1-category of simplicial (pre-)sheaves. We have seen in 3.6.8.2 that for such a presentation the delooping $\mathbf{B}G$ is presented by $\bar{W}G$. By the above discussion in 3.6.10.2 the theory of G-principal ∞-bundles is essentially that of homotopy fibers of morphisms into $\mathbf{B}G$, hence into $\bar{W}G$. By prop. 2.3.8 such homotopy fibers are computed as ordinary pullbacks of fibration resolutions of the point inclusion into $\bar{W}G$. Here we discuss these fibration resolutions. They turn out to be the classical *universal simplicial principal bundles* $WG \to \bar{W}G$.

This section draws from [NSSb].

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Let C be some site. We consider group objects in the category of simplicial presheaves $[C^{\mathrm{op}}, \mathrm{sSet}]$. Since sheafification preserves finite limits, all of the following statements hold verbatim also in the category $\mathrm{Sh}(C)^{\Delta^{\mathrm{op}}}$ of simplicial sheaves over C.

Definition 3.6.172. For G be a group object in $[C^{\mathrm{op}}, \mathrm{sSet}]$ and for $\rho : P \times G \to P$ a G-action, its *action groupoid object* is the simplicial object

$$P//G \in [\Delta^{\mathrm{op}}, [C^{\mathrm{op}}, \mathrm{sSet}$$

whose value in degree n is

$$(P//G)_n := P \times G^{\times^n} \quad \in [C^{\mathrm{op}}, \mathrm{sSet}]\,,$$

whose face maps are given by

$$d_i(p, g_1, \ldots, g_n) = \begin{cases} (pg_1, g_2, \ldots, g_n) & \text{if } i = 0, \\ (p, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 1 \le i \le n-1, \\ (p, g_1, \ldots, g_{n-1}) & \text{if } i = n, \end{cases}$$

and whose degeneracy maps are given by

$$s_i(p, g_1, \ldots, g_n) = (p, g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n)\,.$$

Definition 3.6.173. For $\rho : P \times G \to P$ an action, write

$$P/_h G := T(P//G) \in [C^{\mathrm{op}}, \mathrm{sSet}]$$

for the corresponding total simplicial object, def. 2.3.23.

Remark 3.6.174. According to corollary 2.3.27 the object $P/_h G$ presents the homotopy colimit over the simplicial object $P//G$. We say that $P/_h G$ is the *homotopy quotient* of P by the action of G.

Example 3.6.175. The unique trivial action of a group object G on the terminal object $*$ gives rise to a canonical action groupoid $*//G$. According to def. 3.6.125 we have

$$*/_h G = \overline{W}G\,.$$

The multiplication morphism $\cdot : G \times G \to G$ regarded as an action of G on itself gives rise to a canonical action groupoid $G//G$. The terminal morphism $G \to *$ induces a morphism of simplicial objects

$$G//G \to *//G\,.$$

Defined this way $G//G$ carries a *left* G-action relative to this morphism. To stay with our convention that actions on bundles are from the right, we consider in the following instead the right action of G on itself given by

$$G \times G \xrightarrow{\sigma} G \times G \xrightarrow{((-)^{-1}, \mathrm{id})} G \times G \xrightarrow{\cdot} G\ ,$$

where σ exchanges the two cartesian factors

$$(h, g) \mapsto g^{-1} h\,.$$

With respect to this action, the action groupoid object $G//G$ is canonically equipped with the right G-action by multiplication from the right. Whenever in the following we write

$$G//G \to *//G$$

we are referring to this latter definition.

Definition 3.6.176. Given a group object in $[C^{\mathrm{op}}, \mathrm{sSet}]$, write

$$(WG \to \bar{W}G) := (G/_hG \to */_hG) \quad \in [C^{\mathrm{op}}, \mathrm{sSet}]$$

for the morphism induced on homotopy quotients, def. 3.6.173, by the morphism of canonical action groupoid objects of example 3.6.175.

We will call this the *universal weakly G-principal bundle.*

This term will be justified by prop. 3.6.181, remark 3.6.182 and theorem 3.6.201 below. We now discuss some basic properties of this morphism.

Definition 3.6.177. For $\rho : P \times G \to P$ a G-action in $[C^{\mathrm{op}}, \mathrm{sSet}]$, we write

$$P \times_G WG := (P \times WG)/G \in [C^{\mathrm{op}}, \mathrm{sSet}]$$

for the quotient by the diagonal G-action with respect to the given right G action on P and the canonical right G-action on WG from prop. 3.6.181. We call this quotient the *Borel construction* of the G-action on P.

Proposition 3.6.178. *For $P \times G \to P$ an action in $[C^{\mathrm{op}}, \mathrm{sSet}]$, there is an isomorphism*

$$P/_hG \simeq P \times_G WG\,,$$

between the homotopy quotient, def. 3.6.173, and the Borel construction. In particular, for all $n \in \mathbb{N}$ there are ismorphisms

$$(P/_hG)_n \simeq P_n \times G_{n-1} \times \cdots \times G_0\,.$$

Proof. This follows by a straightforward computation.

Lemma 3.6.179. *Let P be a Kan complex, G a simplicial group and $\rho : P \times G \to P$ an action. The following holds.*

1. *The qotient map $P \to P/G$ is a Kan fibration.*
2. *If the action is free, then P/G is a Kan complex.*

The second statement is for instance lemma V3.7 in [GoJa99].

Lemma 3.6.180. *For P a Kan complex and $P \times G \to P$ an action by a group object, the homotopy quotient $P/_hG$, def. 3.6.173, is itself a Kan complex.*

Proof. By prop. 3.6.178 the homotopy quotient is isomorphic to the Borel construction. Since G acts freely on WG it acts freely on $P \times WG$. The statement then follows with lemma 3.6.179. □

Proposition 3.6.181. *For G a group object in $[C^{\mathrm{op}}, \mathrm{sSet}]$, the morphism $WG \to \overline{W}G$ from def. 3.6.176 has the following properties.*

1. *It is isomorphic to the traditional morphism denoted by these symbols, e.g. [May67].*
2. *It is isomorphic to the décalage morphism $\mathrm{Dec}_0\overline{W}G \to \overline{W}G$, def. 2.3.31.*

3. *It is canonically equipped with a right G-action over $\overline{W}G$ that makes it a weakly G-princial bundle (in fact the shear map is an isomorphism).*

4. *It is an objectwise Kan fibration replacement of the point inclusion $* \to \bar{W}G$.*

This is lemma 10 in [RoSt12].

Remark 3.6.182. Let $\hat{X} \to \bar{W}G$ be a morphism in $[C^{\mathrm{op}}, \mathrm{sSet}]$, presenting, by prop. 3.6.134, a morphism $X \to \mathbf{B}G$ in the ∞-topos $\mathbf{H} = \mathrm{Sh}_\infty(C)$. By prop. 3.6.166 every G-principal ∞-bundle over X arises as the homotopy fiber of such a morphism. By using prop. 3.6.181 in prop. 2.3.8 it follows that the principal ∞-bundle classified by $\hat{X} \to \bar{W}G$ is presented by the ordinary pullback of $WG \to \bar{W}G$. This is the defining property of the universal principal bundle.

In 3.6.10.4 below we show how this observation leads to a complete presentation of the theory of principal ∞-bundles by simplical weakly principal bundles.

3.6.10.4 Presentation in locally fibrant simplicial sheaves We discuss a presentation of the general notion of principal ∞-bundles, 3.6.10.2 by weakly principal bundles in a 1-category of simplicial sheaves.

Let $\mathbf{H}$ be a hypercomplete ∞-topos (for instance a cohesive ∞-topos), such that it admits a 1-site C with enough points.

Observation 3.6.183. By prop. 2.2.12 a category with weak equivalences that presents $\mathbf{H}$ under simplicial localization, def. 2.1.19, is the category of simplicial 1-sheaves on C, $\mathrm{sSh}(C)$, with the weak equivalences $W \subset \mathrm{Mor}(\mathrm{sSh}(C))$ being the stalkwise weak equivalences:

$$\mathbf{H} \simeq L_W \mathrm{sSh}(C) \,.$$

Also the full subcategory

$$\mathrm{sSh}(C)_{\mathrm{lfib}} \hookrightarrow \mathrm{sSh}(C)$$

on the locally fibrant objects is a presentation.

Corollary 3.6.184. *Regard $\mathrm{sSh}(C)_{\mathrm{lfib}}$ as a category of fibrant objects, def. 3.6.146, with weak equivalences and fibrations the stalkwise weak equivalences and firations in $\mathrm{sSet}_{\mathrm{Quillen}}$, respectively, as in example 3.6.147.*

Then for any two objects $X, A \in \mathbf{H}$ there are simplicial sheaves, to be denoted by the same symbols, such that the hom ∞-groupoid in $\mathbf{H}$ from X to A is presented in $\mathrm{sSet}_{\mathrm{Quillen}}$ by the Kan complex of cocycles 3.6.9.2.

Proof. By theorem 3.6.149. □

We now discuss for the general theory of principal ∞-bundles in $\mathbf{H}$ from 3.6.10.2 a corresponding realization in the presentation for $\mathbf{H}$ given by $(\mathrm{sSh}(C), W)$.

By prop. 3.6.134 every ∞-group in $\mathbf{H}$ is presented by an ordinary group in $\mathrm{sSh}(C)$. It is too much to ask that also every G-principal ∞-bundle is presented by a principal bundle in $\mathrm{sSh}(C)$. But something close is true: every principal ∞-bundle is presented by a *weakly principal* bundle in $\mathrm{sSh}(C)$.

Definition 3.6.185. Let $X \in \mathrm{sSh}(C)$ be any object, and let $G \in \mathrm{sSh}(C)$ be equipped with the structure of a group object. A *weakly G-principal bundle* is

- an object $P \in \mathrm{sSh}(C)$ (the *total space*);
- a local fibration $\pi \colon P \to X$ (the *bundle projection*);
- a right action

$$\begin{array}{ccc} P \times G & \xrightarrow{\rho} & P \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

of G on P over X

such that

- the action of G is *weakly principal* in that the *shear map*

$$(p_1, \rho) : P \times G \to P \times_X P \qquad (p, g) \mapsto (p, pg)$$

  is a local weak equivalence.

Remark 3.6.186. We do not ask the G-action to be degreewise free as in [JaLu04], where a similar notion is considered. However we show in Corollary 3.6.203 below that each weakly G-principal bundle is equivalent to one with free G-action.

Definition 3.6.187. A morphism of weakly G-principal bundles $(\pi, \rho) \to (\pi', \rho')$ over X is a morphism $f : P \to P'$ in $\mathrm{sSh}(C)$ that is G-equivariant and commutes with the bundle projections, hence such that it makes this diagram commute:

$$\begin{array}{ccc} P \times G & \xrightarrow{(f,\mathrm{id})} & P' \times G \\ \downarrow^{\rho} & & \downarrow^{\rho'} \\ P & \xrightarrow{f} & P' \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & X & \end{array} .$$

Write

$$\mathrm{w}G\mathrm{Bund}(X) \in \mathrm{sSet}_{\mathrm{Quillen}}$$

for the nerve of the category of weakly G-principal bundles and morphisms as above. The ∞-groupoid that this presents under $\infty\mathrm{Grpd} \simeq (\mathrm{sSet}_{\mathrm{Quillen}})^\circ$ we call the *∞-groupoid of weakly G-principal bundles over X*.

Lemma 3.6.188. *Let $\pi : P \to X$ be a weakly G-principal bundle. Then the following statements are true:*

1. *For any point $p : * \to P$ the action of G induces a weak equivalence*

$$G \longrightarrow P_x$$

   where $x = \pi p$ and where P_x is the fiber of $P \to X$ over x.

2. *For all $n \in \mathbb{N}$, the multi-shear maps*

$$P \times G^n \to P^{\times_X^{n+1}} \qquad (p, g_1, ..., g_n) \mapsto (p, pg_1, ..., pg_n)$$

   are weak equivalences.

Proof. We consider the first statement. Regard the weak equivalence $P \times G \xrightarrow{\simeq} P \times_X P$ as a morphism over P where in both cases the map to P is given by projection onto the first factor. By basic properties of categories of fibrant objects, both of these morphisms are fibrations. Therefore, by prop. 2.3.12 the pullback of the shear map along p is still a weak equivalence. But this pullback is just the map $G \to P_x$, which proves the claim.

For the second statement, we use induction on n. Suppose that $P \times G^n \to P^{\times_X^{n+1}}$ is a weak equivalence. By prop. 2.3.12, the pullback $P^{\times_X^n} \times_X (P \times G) \to P^{\times_X^{n+2}}$ of the shear map itself along $P^{\times_X^n} \to X$ is again a weak equivalence, as is the product $P \times G^n \times G \to P^{\times_X^{n+1}} \times G$ of the n-fold shear map with G. The composite of these two weak equivalences is the multi-shear map $P \times G^{n+1} \to P^{\times_X^{n+2}}$, which is hence a also weak equivalence.

Proposition 3.6.189. *Let $P \to X$ be a weakly G-principal bundle and let $f : Y \to X$ be an arbitrary morphism. Then the pullback $f^*P \to Y$ exists and is also canonically a weakly G-principal bundle. This operation extends to define a pullback morphism*

$$f^* : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{w}G\mathrm{Bund}(Y) .$$

Proof. By basic properties of a category of fibrant objects:

The pullback f^*P exists and the morphism $f^*P \to Y$ is again a local fibration. Thus it only remains to show that f^*P is weakly principal, i.e. that the morphism $f^*P \times G \to f^*P \times_Y f^*P$ is a weak equivalence. This follows from prop. 2.3.12.

Remark 3.6.190. The functor f^* associated to the map $f\colon Y \to X$ above is the restriction of a functor $f^*\colon \mathrm{sSh}(C)/X \to \mathrm{sSh}(C)/Y$ mapping from simplicial sheaves over X to simplicial sheaves over Y. This functor f^* has a left adjoint $f_!\colon \mathrm{sSh}(C)/Y \to \mathrm{Sh}^{\Delta^{\mathrm{op}}}/X$ given by composition along f, in other words

$$f_!(E \to Y) = E \to Y \xrightarrow{f} X.$$

Note that the functor $f_!$ does not usually restrict to a functor $f_!\colon \mathrm{w}G\mathrm{Bund}(Y) \to \mathrm{w}G\mathrm{Bund}(X)$. But when it does, we say that principal ∞-bundles *satisfy descent along* f. In this situation, if P is a weakly G-principal bundle on Y, then P is weakly equivalent to the pulled-back principal ∞-bundle $f^*f_!P$ on Y, in other words P 'descends' to $f_!P$.

The next result says that weakly G-principal bundles satisfy descent along local acyclic fibrations (hypercovers).

Proposition 3.6.191. *Let $p : Y \to X$ be a local acyclic fibration in* $\mathrm{sSh}(C)$*. Then the functor $p_!$ defined above restricts to a functor $p_!\colon \mathrm{w}G\mathrm{Bund}(Y) \to \mathrm{w}G\mathrm{Bund}(X)$, left adjoint to $p^*\colon \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{w}G\mathrm{Bund}(Y)$, hence to a homotopy equivalence in* $\mathrm{sSet}_{\mathrm{Quillen}}$.

Proof. Given a weakly G-principal bundle $P \to Y$, the first thing we have to check is that the map $P \times G \to P \times_X P$ is a weak equivalence. This map can be factored as $P \times G \to P \times_Y P \to P \times_X P$. Hence it suffices to show that the map $P \times_Y P \to P \times_X P$ is a weak equivalence. But this follows by prop. 2.3.12, since both pullbacks are along local fibrations and $Y \to X$ is a local weak equivalence by assumption.

This establishes the existence of the functor $p_!$. It is easy to see that it is left adjoint to p^*. This implies that it induces a homotopy equivalence in $\mathrm{sSet}_{\mathrm{Quillen}}$.

Corollary 3.6.192. *For $f : Y \to X$ a local weak equivalence, the induced functor $f^* : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{w}G\mathrm{Bund}(Y)$ is a homotopy equivalence.*

Proof. By lemma 2.3.9 we can factor the weak equivalence f into a composite of a local acyclic fibration and a left inverse to a local acyclic fibration. Therefore, by prop. 3.6.191, f^* may be factored as the composite of two homotopy equivalences, hence is itself a homotopy equivalence.

We discuss now how weakly G-principal bundles arise from the universal G-principal bundle, def. 3.6.176 by pullback, and how this establishes their equivalence with G-ccoycles.

Proposition 3.6.193. *For G a group object in* $\mathrm{sSh}(C)$*, the map $WG \to \overline{W}G$ from def. 3.6.176 equipped with the G-action of prop. 3.6.181 is a weakly G-principal bundle.*

Indeed, it is a strictly G-principal bundle. This is a classical fact, for instance around lemma V4.1 in [GoJa99].In terms of the total simplicial set functor it is observed in section 4 of [RoSt12].

Proof. By inspection one finds that

$$\begin{array}{ccc} (G//G) \times G & \longrightarrow & G//G \\ \downarrow & & \downarrow \\ G//G & \longrightarrow & *//G \end{array}$$

is a pullback diagram in $[\Delta^{\mathrm{op}}, \mathrm{sSh}(C)]$. Since the total simplicial object functor T of def. 2.3.23 is right adjoint it preserves this pullback. This shows the principality of the shear map.

Definition 3.6.194. For $Y \to X$ a morphism in $\mathrm{sSh}(C)$, write

$$\check{C}(Y) \in [\Delta^{\mathrm{op}}, \mathrm{sSh}(C)]$$

for its *Čech nerve*, given in degree n by the n-fold fiber product of Y over X

$$\check{C}(Y)_n := Y^{\times_X^{n+1}} .$$

Observation 3.6.195. The canonical morphism of simplicial objects $\check{C}(Y) \to X$, with X regarded as a constant simplicial object induces under totalization, def. 2.3.23, and by prop. 2.3.26 a canonical morphism

$$T\check{C}(Y) \to X \quad \in \mathrm{sSh}(C) .$$

Lemma 3.6.196. *For $p : Y \to X$ a local acyclic fibration, the morphism $T\check{C}(Y) \to X$ from observation 3.6.195 is a local weak equivalence.*

Proof. By pullback stability of local acylic fibrations, for each $n \in \mathbb{N}$ the morphism $Y^{\times_X^n} \to X$ is a local weak equivalence. By remark. 2.3.25 and prop. 2.3.26 this degreewise local weak equivalence is preserved by the functor T.

The main statement now is the following.

Theorem 3.6.197. *For $P \to X$ a weakly G-principal bundle in* $\mathrm{sSh}(C)$*, the canonical morphism*

$$P/_hG \longrightarrow X$$

is a local acyclic fibration.

Proof. To see that the morphism is a local weak equivalence, factor $P//G \to X$ in $[\Delta^{\mathrm{op}}, \mathrm{sSh}(C)]$ via the multi-shear maps from lemma 3.6.188 through the Čech nerve, def. 3.6.194, as

$$P//G \to \check{C}(P) \to X .$$

Applying to this the total simplicial object functor T, def. 2.3.23, yields a factorization

$$P/_hG \to T\check{C}(P) \to X .$$

The left morphism is a weak equivalence because, by lemma 3.6.188, the multi-shear maps are weak equivalences and by corollary 2.3.27 T preserves sends degreewise weak equivalences to weak equivalences. The right map is a weak equivalence by lemma 3.6.196.

We now prove that $P/_hG \to X$ is a local fibration. We need to show that for each topos point p of $\mathrm{Sh}(C)$ the morphism of stalks $p(P/_hG) \to p(X)$ is a Kan fibration of simplicial sets. By prop. 3.6.178 this means equivalently that the morphism

$$p(P \times_G WG) \to p(X)$$

is a Kan fibration. By definition of topos point, p commutes with all the finite products and colimits involved here. Therefore equivalently we need to show that

$$p(P) \times_{p(G)} W p(G) \to p(X)$$

is a Kan fibration for all topos points p.

Observe that this morphism factors the projection $p(P) \times W(p(G)) \to p(X)$ as

$$p(P) \times W(p(G)) \to p(P) \times_{p(G)} W(p(G)) \to p(X)$$

in sSet. Here the first morphism is a Kan fibration by lemma 3.6.179, which in particular is also surjective on vertices. Also the total composite morphism is a Kan fibration, since $W(p(G))$ is Kan fibrant. From this the desired result follows with the next lemma 3.6.198.

Lemma 3.6.198. *Suppose that $X \xrightarrow{p} Y \xrightarrow{q} Z$ is a diagram of simplicial sets such that p is a Kan fibration surjective on vertices and qp is a Kan fibration. Then q is also a Kan fibration.*

Proof. Consider a lifting problem of the form

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle q} \\ \Delta[n] & \longrightarrow & Z. \end{array}$$

Choose a 0-simplex of X which projects to the 0-simplex of Y corresponding to the image of the vertex 0 under the map $\Lambda^k[n] \to Y$. Since $\Delta[0] \to \Lambda^k[n]$ is an acyclic cofibration, we may choose a map $\Lambda^k[n] \to X$ such that the diagram

$$\begin{array}{ccc} \Delta[0] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Lambda^k[n] & \longrightarrow & Y \end{array}$$

commutes. This map gives rise to a commutative diagram

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle qp} \\ \Delta[n] & \longrightarrow & Z \end{array}$$

and any diagonal filler in this diagram gives a solution of the original lifting problem.

We now discuss the equivalence between weakly G-principal bundles and G-cocycles. For $X, A \in \mathrm{sSh}(C)$, write $\mathrm{Cocycle}(X, A)$ for the category of cocycles from X to A, according to 3.6.9.2.

Definition 3.6.199. Let $X, G \in \mathrm{sSh}(C)$ with G equipped with the structure of a group object (hence necessarily locally fibrant) and also with X being locally fibrant.

Define a functor

$$\mathrm{Extr} : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{Cocycle}(X, \overline{W}G)$$

("extracting" a cocycle) on objects by sending a weakly G-principal bundle $P \to X$ to the cocycle

$$X \overset{\sim}{\twoheadleftarrow} P/_h G \longrightarrow \overline{W}G \;,$$

where the left morphism is the local acyclic fibration from theorem 3.6.197, and where the right morphism is the image under the total simplicial object functor, def. 2.3.23, of the canonical morphism $P//G \to *//G$ of simplicial objects.

Define also a functor

$$\mathrm{Rec} : \mathrm{Cocycle}(X, \overline{W}G) \to \mathrm{w}G\mathrm{Bund}(X)$$

("reconstruction" of the bundle) which on objects takes a cocycle $X \xleftarrow{\pi} Y \xrightarrow{g} \overline{W}G$ to the weakly G-principal bundle

$$g^* WG \to Y \xrightarrow{\pi} X \,,$$

which is the pullback of the universal G-principal bundle, def. 3.6.176, along g, and which on morphisms takes a coboundary to the morphism between pullbacks induced from the corresponding morphism of pullback diagrams.

Observation 3.6.200. The functor Extr sends the universal G-principal bundle $WG \to \overline{W}G$ to the cocycle

$$\overline{W}G \simeq * \times_G WG \overset{\simeq}{\leftarrow} WG \times_G WG \overset{\simeq}{\rightarrow} WG \times_G * \simeq \overline{W}G\,.$$

Write

$$q : \mathrm{Cocycle}(X, \overline{W}G) \to \mathrm{Cocycle}(X, \overline{W}G)$$

for the functor given by postcomposition with this universal cocycle. This has an evident left and right adjoint $\bar{q}$. Therefore under the simplicial nerve these functors induce homotopy equivalences in $\mathrm{sSet}_{\mathrm{Quillen}}$.

Theorem 3.6.201. *The functors* Extr *and* Rec *from def. 3.6.199 induce weak equivalences*

$$N\mathrm{w}G\mathrm{Bund}(X) \simeq N\mathrm{Cocycle}(X, \overline{W}G) \quad \in \mathrm{sSet}_{\mathrm{Quillen}}$$

between the simplicial nerves of the category of weakly G-principal bundles and of cocycles, respectively.

Proof. We construct natural transformations

$$\mathrm{Extr} \circ \mathrm{Rec} \Rightarrow q$$

and

$$\mathrm{Rec} \circ \mathrm{Extr} \Rightarrow \mathrm{id}\,,$$

where q is the homotopy equivalence from observation 3.6.200.

For

$$X \overset{\pi}{\leftarrow} Y \overset{f}{\rightarrow} \overline{W}G.$$

a cocycle, its image under $\mathrm{Extr} \circ \mathrm{Rec}$ is

$$X \leftarrow (f^* WG)/_h G \to \overline{W}G\,.$$

The morphism $(f^*WG)/_hG$ factors through Y by construction, so that the left triangle in the diagram

$$\begin{array}{ccccc} & & (f^*WG)/_hG & & \\ & \overset{\sim}{\swarrow} & \downarrow & \searrow & \\ X & & & & \overline{W}G \\ & \underset{\sim}{\nwarrow} & \downarrow & \overset{q(f)}{\nearrow} & \\ & & Y & & \end{array}$$

commutes. The top right morphism is by definition the image under the total simplicial set functor, def. 2.3.23, of $(f^*WG)//G \to *//G$. This factors the top horizontal morphism in

$$\begin{array}{ccccc} (f^*WG)//G & \longrightarrow & (WG)//G & \longrightarrow & *//G \\ \downarrow & & \downarrow & & \\ Y & \xrightarrow{f} & \overline{W}G & & \end{array} .$$

Applying the total simplicial object functor to this diagram gives the above commuting triangle on the right. Clearly this construction is natural and hence provides a natural transformation Extr Rec $\Rightarrow q$.

For the other natural trasformation, let now $P \to X$ be a weakly G-principal bundle. This induces the following commutative diagram of simplicial objects (with P and X regarded as constant simplicial objects)

$$\begin{array}{ccccccc} P & \longleftarrow & P \times_X (P//G) & \underset{\sim}{\overset{\phi}{\longleftarrow}} & (P \times G)//G & \longrightarrow & G//G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & P//G & = & P//G & \longrightarrow & *//G \end{array} ,$$

where the left and the right square are pullbacks, and where the top horizontal morphism ϕ is the degreewise local weak equivalence which is degreewise induced by the shear map, composed with exchange of the two factors.

Explicitly, in degree 0 the morphism ϕ is given on generalized elements by

$$(p', g) \xleftarrow{\phi_0} (p'g, p')$$

and in degree 1 by

$$\begin{array}{ccc} (p'g, (p', h)) & \xleftarrow{\phi_1} & ((p', g), h) \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\ (p'g, p'h) & \xleftarrow{\phi_0} & ((p'h, h^{-1}g) \end{array} ,$$

etc. Here the top horizontal morphisms also respect the right G-actions ρ induced from the weakly G-principal bundle structure on $P \to X$ and on $G//G \to *//G$. For instance the respect of the right G-action of ϕ in degree 0 is on elements verified by

$$\begin{array}{ccc} ((p'g, p'), k) & \xleftarrow{\phi_0} & ((p', g), k) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ (p'gk, p') & \xleftarrow{\phi_0} & ((p', gk) \end{array} .$$

The image of the above diagram under the total simplicial object functor, which preserves all the pullbacks and weak equivalences involved, is

$$\begin{array}{ccccccc} P & \overset{\sim}{\twoheadleftarrow} & P \times_X P/_h G & \overset{\sim}{\leftarrow} & (P \times G)/_h G & \longrightarrow & WG \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \overset{\sim}{\twoheadleftarrow} & P/_h G & = & P/_h G & \longrightarrow & \overline{W}G \end{array} .$$

Here the total bottom span is the cocycle $\mathrm{Extr}(P)$, and so the object $(P \times G)/_h G$ over X is $\mathrm{Rec}(\mathrm{Extr}(P))$. Therefore this exhibits a natural morphism $\mathrm{Rec}\,\mathrm{Extr}P \to P$.

Remark 3.6.202. By theorem 3.6.149 the simplicial set $N\mathrm{Cocycle}(X, \overline{W}G)$ is a presentation of the intrinsic cocycle ∞-groupoid $\mathbf{H}(X, \mathbf{B}G)$ of the hypercomplete ∞-topos $\mathbf{H} = \mathrm{Sh}_\infty^{\mathrm{hc}}(C)$. Therefore the equivalence of theorem 3.6.201 is a presentation of that of theorem 3.6.170,

$$G\mathrm{Bund}_\infty(X) \simeq \mathbf{H}(X, \mathbf{B}G)$$

between the ∞-groupoid of G-principal ∞-bundles in $\mathbf{H}$ and the intrinsic cocycle ∞-groupoid of $\mathbf{H}$.

Corollary 3.6.203. *For each weakly G-principal bundle $P \to X$ there is a weakly G-principal bundle P^f with a levelwise free G-action and a weak equivalence $P^f \xrightarrow{\sim} P$ of weakly G-principal bundles over X. In fact, the assignment $P \mapsto P^f$ is an homotopy inverse to the full inclusion of weakly G-principal bundles with free action into all weakly G-principal bundles.*

Proof. Note that the universal bundle $WG \to \overline{W}G$ carries a free G-action, in the sense that the levelwise action of G_n on $(WG)_n$ is free. This means that the functor Rec from the proof of theorem 3.6.201 indeed takes values in weakly G-principal budles with free action. Hence we can set

$$P^f := \mathrm{Rec}(\mathrm{Extr}(P)) = (P \times G)/_h G\,.$$

By the discussion there we have a natural morphism $P^f \to P$ and one checks that this exhibits the homomotopy inverse.

3.6.11 Associated fiber bundles

We discuss the notion of representations/actions/modules of ∞-groups in an ∞-topos and the structures directly induced by this: the corresponding twisted cohomology is cohomology with coefficients in *modules* (the generalization of group cohomology with coefficients in a module) and the corresponding notion of *associated ∞-bundles*.

3.6.11.1 General abstract

This section draws from [NSSa].

Let $\mathbf{H}$ be an ∞-topos, $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group. Fix an action $\rho : V \times G \to V$ (Definition 3.6.152) on an object $V \in \mathbf{H}$. We discuss the induced notion of *ρ-associated V-fiber ∞-bundles*. We show that there is a *universal* ρ-associated V-fiber bundle over $\mathbf{B}G$ and observe that under Theorem 3.6.170 this is effectively identified with the action itself. Accordingly, we also further discuss ∞-actions as such.

Definition 3.6.204. For $V, X \in \mathbf{H}$ any two objects, a *V-fiber ∞-bundle* over X is a morphism $E \to X$, such that there is an effective epimorphism $U \twoheadrightarrow X$ and an ∞-pullback of the form

$$\begin{array}{ccc} U \times V & \longrightarrow & E \\ \downarrow & & \downarrow \\ U & \twoheadrightarrow & X \,. \end{array}$$

We say that $E \to X$ locally trivializes with respect to U. As usual, we often say *V-bundle* for short.

Definition 3.6.205. For $P \to X$ a G-principal ∞-bundle, we write

$$P \times_G V := (P \times V)//G$$

for the ∞-quotient of the diagonal ∞-action of G on $P \times V$. Equipped with the canonical morphism $P \times_G V \to X$ we call this the ∞-bundle *ρ-associated* to P.

Remark 3.6.206. The diagonal G-action on $P \times V$ is the product in $G\mathrm{Action}(\mathbf{H})$ of the given actions on P and on V. Since $G\mathrm{Action}(\mathbf{H})$ is a full sub-∞-category of a slice category of a functor category, the product is given by a degreewise pullback in $\mathbf{H}$:

$$\begin{array}{ccc} P \times V \times G^{\times n} & \longrightarrow & V \times G^{\times n} \\ \downarrow & & \downarrow \\ P \times G^{\times n} & \longrightarrow & G^{\times n} \,. \end{array}$$

and so

$$P \times_G V \simeq \varinjlim_n (P \times V \times G^{\times n}) \,.$$

The canonical bundle morphism of the corresponding ρ-associated ∞-bundle is the realization of the left morphism of this diagram:

$$\begin{array}{ccc} P \times_G V & := & \varinjlim_n (P \times V \times G^{\times n}) \\ \downarrow & & \downarrow \\ X & \simeq & \varinjlim_n (P \times G^{\times n}) \,. \end{array}$$

Example 3.6.207. By Theorem 3.6.170 every ∞-group action $\rho : V \times G \to V$ has a classifying morphism $\mathbf{c}$ defined on its homotopy quotient, which fits into a fiber sequence of the form

$$
\begin{array}{ccc}
V & \longrightarrow & V//G \\
 & & \downarrow^{\mathbf{c}} \\
 & & \mathbf{B}G\,.
\end{array}
$$

Regarded as an ∞-bundle, this is ρ-associated to the universal G-principal ∞-bundle $* \longrightarrow \mathbf{B}G$ from Example 3.6.167:

$$
V//G \simeq * \times_G V\,.
$$

Lemma 3.6.208. *The realization functor* $\varinjlim : \mathrm{Grpd}(\mathbf{H}) \to \mathbf{H}$ *preserves the ∞-pullback of Remark 3.6.206:*

$$
P \times_G V \simeq \varinjlim_n (P \times V \times G^{\times_n}) \simeq (\varinjlim_n P \times G^{\times_n}) \times_{(\varinjlim_n G^{\times_n})} (\varinjlim_n V \times G^{\times_n})\,.
$$

Proof. Generally, let $X \to Y \leftarrow Z \in \mathrm{Grpd}(\mathbf{H})$ be a diagram of groupoid objects, such that in the induced diagram

$$
\begin{array}{ccccc}
X_0 & \longrightarrow & Y_0 & \longleftarrow & Z_0 \\
\downarrow & & \downarrow & & \downarrow \\
\varinjlim_n X_n & \longrightarrow & \varinjlim_n Y_n & \longleftarrow & \varinjlim_n Z_n
\end{array}
$$

the left square is an ∞-pullback. By the third ∞-Giraud axiom (Definition 2.2.2) the vertical morphisms are effective epi, as indicated. By assumption we have a pasting of ∞-pullbacks as shown on the left of the following diagram, and by the pasting law (Proposition 2.3.2) this is equivalent to the pasting shown on the right:

$$
\begin{array}{ccc}
X_0 \times_{Y_0} Z_0 & \longrightarrow & Z_0 \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & Y_0 \\
\downarrow & & \downarrow \\
\varinjlim_n X_n & \longrightarrow & \varinjlim_n Y_n
\end{array}
\quad \simeq \quad
\begin{array}{ccc}
X_0 \times_{Y_0} Z_0 & \longrightarrow & Z_0 \\
\downarrow & & \downarrow \\
(\varinjlim_n X_n) \times_{(\varinjlim_n Y_n)} (\varinjlim_n Z_n) & \longrightarrow & \varinjlim_n Z_n \\
\downarrow & & \downarrow \\
\varinjlim_n X_n & \longrightarrow & \varinjlim_n Y_n.
\end{array}
$$

Since effective epimorphisms are stable under ∞-pullback, this identifies the canonical morphism

$$
X_0 \times_{Y_0} Z_0 \to (\varinjlim_n X_n) \times_{(\varinjlim_n Y_n)} (\varinjlim_n Z_n)
$$

as an effective epimorphism, as indicated.

Since ∞-limits commute over each other, the Čech nerve of this morphism is the groupoid object $[n] \mapsto X_n \times_{Y_n} Z_n$. Therefore the third ∞-Giraud axiom now says that $\varinjlim$ preserves the ∞-pullback of groupoid objects:

$$
\varinjlim (X \times_Y Z) \simeq \varinjlim_n (X_n \times_{Y_n} Z_n) \simeq (\varinjlim_n X_n) \times_{(\varinjlim_n Y_n)} (\varinjlim_n Z_n)\,.
$$

Consider this now in the special case that $X \to Y \leftarrow Z$ is $(P \times G^{\times \bullet}) \to G^{\times \bullet} \leftarrow (V \times G^{\times \bullet})$. Theorem 3.6.170 implies that the initial assumption above is met, in that $P \simeq (P//G) \times_{*//G} * \simeq X \times_{\mathbf{B}G} *$, and so the claim follows. $\square$

Proposition 3.6.209. *For $g_X : X \to \mathbf{B}G$ a morphism and $P \to X$ the corresponding G-principal ∞-bundle according to Theorem 3.6.170, there is a natural equivalence*

$$g_X^*(V//G) \simeq P \times_G V$$

over X, between the pullback of the ρ-associated ∞-bundle $V//G \xrightarrow{\mathbf{c}} \mathbf{B}G$ of Example 3.6.207 and the ∞-bundle ρ-associated to P by Definition 3.6.205.

Proof. By Remark 3.6.206 the product action is given by the pullback

$$\begin{array}{ccc} P \times V \times G^{\times \bullet} & \longrightarrow & V \times G^{\times \bullet} \\ \downarrow & & \downarrow \\ P \times G^{\times \bullet} & \longrightarrow & G^{\times \bullet} \end{array}$$

in $\mathbf{H}^{\Delta^{\mathrm{op}}}$. By Lemma 3.6.208 the realization functor preserves this ∞-pullback. By Remark 3.6.206 it sends the left morphism to the associated bundle, and by Theorem 3.6.170 it sends the bottom morphism to g_X. Therefore it produces an ∞-pullback diagram of the form

$$\begin{array}{ccc} V \times_G P & \longrightarrow & V//G \\ \downarrow & & \downarrow^{\mathbf{c}} \\ X & \xrightarrow{g_X} & \mathbf{B}G\,. \end{array}$$

□

Remark 3.6.210. This says that $V//G \xrightarrow{\mathbf{c}} \mathbf{B}G$ is both, the V-fiber ∞-bundle ρ-associated to the universal G-principal ∞-bundle, Observation 3.6.207, as well as the universal ∞-bundle for ρ-associated ∞-bundles.

Proposition 3.6.211. *Every ρ-associated ∞-bundle is a V-fiber ∞-bundle, Definition 3.6.204.*

Proof. Let $P \times_G V \to X$ be a ρ-associated ∞-bundle. By the previous Proposition 3.6.209 it is the pullback $g_X^*(V//G)$ of the universal ρ-associated bundle. By Proposition 3.6.164 there exists an effective epimorphism $U \longrightarrow\!\!\!\!\to X$ over which P trivializes, hence such that $g_X|_U$ factors through the point, up to equivalence. In summary and by the pasting law, Proposition 2.3.2, this gives a pasting of ∞-pullbacks of the form

$$\begin{array}{ccccc} U \times V & \longrightarrow & P \times_G V & \longrightarrow & V//G \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow\!\!\!\!\to & X & \xrightarrow{g_X} & \mathbf{B}G \\ & \searrow & & \nearrow & \\ & & * & & \end{array}$$

which exhibits $P \times_G V \to X$ as a V-fiber bundle by a local trivialization over U. □

So far this shows that every ρ-associated ∞-bundle is a V-fiber bundle. We want to show that, conversely, every V-fiber bundle is associated to a principal ∞-bundle.

Definition 3.6.212. Let $V \in \mathbf{H}$ be a κ-compact object, for some regular cardinal κ. By the characterization of Definition 2.2.3, there exists an ∞-pullback square in $\mathbf{H}$ of the form

$$\begin{array}{ccc} V & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow \\ * & \xrightarrow{\vdash V} & \mathrm{Obj}_\kappa \end{array}$$

Write

$$\mathbf{BAut}(V) := \mathrm{im}(\vdash V)$$

for the 1-image, Definition 3.6.37, of the classifying morphism $\vdash V$ of V. By definition this comes with an effective epimorphism

$$* \twoheadrightarrow \mathbf{BAut}(V) \hookrightarrow \mathrm{Obj}_\kappa \ ,$$

and hence, by Proposition 3.6.121, it is the delooping of an ∞-group

$$\mathbf{Aut}(V) \in \mathrm{Grp}(\mathbf{H})$$

as indicated. We call this the *internal automorphism ∞-group* of V.

By the pasting law, Proposition 2.3.2, the image factorization gives a pasting of ∞-pullback diagrams of the form

$$\begin{array}{ccccc} V & \longrightarrow & V/\!/\mathbf{Aut}(V) & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow^{\mathbf{c}_V} & & \downarrow \\ * & \overset{\vdash V}{\twoheadrightarrow} & \mathbf{BAut}(V) & \hookrightarrow & \mathrm{Obj}_\kappa \end{array}$$

By Theorem 3.6.170 this defines a canonical ∞-action

$$\rho_{\mathbf{Aut}(V)} : V \times \mathbf{Aut}(V) \to V$$

of $\mathbf{Aut}(V)$ on V with homotopy quotient $V/\!/\mathbf{Aut}(V)$ as indicated.

Proposition 3.6.213. *Every V-fiber ∞-bundle is $\rho_{\mathbf{Aut}(V)}$-associated to an $\mathbf{Aut}(V)$-principal ∞-bundle.*

Proof. Let $E \to V$ be a V-fiber ∞-bundle. By Definition 3.6.204 there exists an effective epimorphism $U \twoheadrightarrow X$ along which the bundle trivializes locally. It follows by the second Axiom in Definition 2.2.3 that on U the morphism $X \xrightarrow{\vdash E} \mathrm{Obj}_\kappa$ which classifies $E \to X$ factors through the point

$$\begin{array}{ccccc} U \times V & \longrightarrow & E & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow & & \downarrow \\ U & \twoheadrightarrow & X & \overset{\vdash E}{\longrightarrow} & \mathrm{Obj}_\kappa . \\ & \searrow & & \nearrow_{\vdash V} & \\ & & * & & \end{array}$$

Since the point inclusion, in turn, factors through its 1-image $\mathbf{BAut}(V)$, Definition 3.6.212, this yields the outer commuting diagram of the following form

$$\begin{array}{ccccc} U & \longrightarrow & * & \longrightarrow & \mathbf{BAut}(V) \\ \downarrow & & & \overset{g}{\nearrow} & \downarrow \\ X & & \underset{\vdash E}{\longrightarrow} & & \mathrm{Obj}_\kappa \end{array}$$

By the epi/mono factorization system of Proposition 3.6.33 there is a diagonal lift g as indicated. Using again the pasting law and by Definition 3.6.212 this factorization induces a pasting of ∞-pullbacks of the form

$$\begin{array}{ccccc} E & \longrightarrow & V/\!/\mathbf{Aut}(V) & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow^{\mathbf{c}_V} & & \downarrow \\ X & \overset{g}{\longrightarrow} & \mathbf{BAut}(V) & \hookrightarrow & \mathrm{Obj}_\kappa \end{array}$$

Finally, by Proposition 3.6.209, this exhibits $E \to X$ as being $\rho_{\mathbf{Aut}(V)}$-associated to the $\mathbf{Aut}(V)$-principal ∞-bundle with class $[g] \in H^1(X, G)$. □

Theorem 3.6.214. *V-fiber ∞-bundles over $X \in \mathbf{H}$ are classified by $H^1(X, \mathbf{Aut}(V))$.*

Under this classification, the V-fiber ∞-bundle corresponding to $[g] \in H^1(X, \mathbf{Aut}(V))$ is identified, up to equivalence, with the $\rho_{\mathbf{Aut}(V)}$-associated ∞-bundle (Definition 3.6.205) to the $\mathbf{Aut}(V)$-principal ∞-bundle corresponding to $[g]$ by Theorem 3.6.170.

Proof. By Proposition 3.6.213 every morphism $X \xrightarrow{\vdash E} \mathrm{Obj}_\kappa$ that classifies a small ∞-bundle $E \to X$ which happens to be a V-fiber ∞-bundle factors via some g through the moduli for $\mathbf{Aut}(V)$-principal ∞-bundles

$$X \xrightarrow{g} \mathbf{BAut}(V) \hookrightarrow \mathrm{Obj}_\kappa \, , \qquad (\vdash E : X \to \mathrm{Obj}_\kappa)$$

Therefore it only remains to show that also every homotopy $(\vdash E_1) \Rightarrow (\vdash E_2)$ factors through a homotopy $g_1 \Rightarrow g_2$. This follows by applying the epi/mono lifting property of Proposition 3.6.33 to the diagram

$$\begin{array}{ccc} X \coprod X & \xrightarrow{(g_1, g_2)} & \mathbf{BAut}(V) \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & \mathrm{Obj}_\kappa \end{array}$$

The outer diagram exhibits the original homotopy. The left morphism is an effective epi (for instance immediately by Proposition 3.6.39), the right morphism is a monomorphism by construction. Therefore the dashed lift exists as indicated and so the top left triangular diagram exhibits the desired factorizing homotopy. □

Remark 3.6.215. In the special case that $\mathbf{H} = \mathrm{Grpd}_\infty$, the classification Theorem 3.6.214 is classical [St63a, May67], traditionally stated in (what in modern terminology is) the presentation of Grpd_∞ by simplicial sets or by topological spaces. Recent discussions include [BC]. For $\mathbf{H}$ a general 1-localic ∞-topos (meaning: with a 1-site of definition), the statement of Theorem 3.6.214 appears in [We11], formulated there in terms of the presentation of $\mathbf{H}$ by simplicial presheaves. (We discuss the relation of these presentations to the above general abstract result in [NSSb].) Finally, one finds that the classification of *G-gerbes* [Gir71] and *G-2-gerbes* in [Br94] is the special case of the general statement, for $V = \mathbf{B}G$ and G a 1-truncated ∞-group. This we discuss below in Section 3.6.15.

We close this section with a list of some fundamental classes of examples of ∞-actions, or equivalently, by Remark 3.6.210, of universal associated ∞-bundles. For doing so we use again that, by Theorem 3.6.170, to give an ∞-action of G on V is equivalent to giving a fiber sequence of the form $V \to V//G \to \mathbf{B}G$.

Example 3.6.216. The following is a list of examples for ∞-actions of ∞-groups $G \in \mathrm{Grp}(\mathbf{H})$ on objects in $\mathbf{H}$.

We display the universal associated ∞-bundles, remark 3.6.210, over the moduli $\mathbf{B}G$ of G-principal ∞-bundles, that characterize these ∞-actions according to theorem 3.6.170, as discussed in 3.6.11.

So an ∞-action of some ∞-group G on an object V is displayed as

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow \\ & & \mathbf{B}G \end{array} \quad \leftrightarrow \quad \begin{array}{ccc} \text{Representation space} & \longrightarrow & \begin{array}{l}\text{Quotient space/} \\ \text{total space of} \\ \text{universal associated } V\text{-bundle}\end{array} \\ & & \downarrow \\ & & \text{Moduli of } G\text{-principal bundles} \end{array}$$

The examples are listed roughly ordered by generality. The first are classes of examples that exist in every ∞-topos. The more axioms on the ambient ∞-topos are needed, the further down the list the example appears.

1. For every $V \in \mathbf{H}$, the fiber sequence

$$\begin{array}{ccc} V & \xrightarrow{(\mathrm{id}_V, \mathrm{pt}_{\mathbf{B}G})} & V \times \mathbf{B}G \\ & & \downarrow^{p_2} \\ & & \mathbf{B}G \end{array}$$

is the *trivial ∞-action* of G on V.

2. For every $G \in \mathrm{Grp}(\mathbf{H})$, the fiber sequence

$$\begin{array}{ccc} G & \longrightarrow & * \\ & & \downarrow \\ & & \mathbf{B}G \end{array}$$

which defines $\mathbf{B}G$ by Theorem 3.6.116 induces the *right action of G on itself*

$$* \simeq G//G\,.$$

At the same time this sequence, but now regarded as a bundle over $\mathbf{B}G$, is the universal G-principal ∞-bundle, Remark 3.6.167.

3. For every object $X \in \mathbf{H}$ write

$$\mathbf{L}X := X \times_{X \times X} X$$

for its *free loop space* object, the ∞-fiber product of the diagonal on X along itself

$$\begin{array}{ccc} \mathbf{L}X & \longrightarrow & X \\ {}^{\mathrm{ev}_*}\downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array} .$$

For every $G \in \mathrm{Grp}(\mathbf{H})$ there is a fiber sequence

$$\begin{array}{ccc} G & \longrightarrow & \mathbf{L}\mathbf{B}G \\ & & \downarrow^{\mathrm{ev}_*} \\ & & \mathbf{B}G \end{array} .$$

This exhibits the *adjoint action of G on itself*

$$\mathbf{L}\mathbf{B}G \simeq G//_{\mathrm{ad}}G\,.$$

4. For every $V \in \mathbf{H}$ there is the canonical *∞-action by automorphisms* of the *automorphism ∞-group* $\mathbf{Aut}(V)$, def. 3.6.212, on V, exhibited by the fiber sequence on the left of the pasting diagram of ∞-pullbacks

$$\begin{array}{ccccc} V & \longrightarrow & V//\mathbf{Aut}(V) & \longrightarrow & \widehat{\mathrm{Obj}} \\ \downarrow & & \downarrow & & \downarrow \\ * & \twoheadrightarrow & \mathbf{B}\mathbf{Aut}(V) & \hookrightarrow & \mathrm{Obj} \end{array} ,$$

with the composite bottom morphism $* \to \mathrm{Obj}$ labeled $\vdash V$.

5. For $\rho_1, \rho_2 \in \mathbf{H}_{/\mathbf{B}G}$ two G-∞-actions on objects $V_1, V_2 \in \mathbf{H}$, respectively, their internal hom $[\rho_1, \rho_2] \in \mathbf{H}_{/\mathbf{B}G}$ in the slice over $\mathbf{B}G$ is a G-∞-action on the internal hom $[V_1, V_2] \in \mathbf{H}$:

$$\begin{array}{ccc} [V_1, V_2] \longrightarrow [V_1, V_2]//G & \simeq & \sum_{\mathbf{B}G} [\rho_1, \rho_2] \;, \\ \downarrow & & \\ \mathbf{B}G & & \end{array}$$

hence $[V_1, V_2]//G \simeq \sum_{\mathbf{B}G} [\rho_1, \rho_2]$ (this follows by the fact that the inverse image of base change along $\mathrm{pt}_{\mathbf{B}G} : * \to \mathbf{B}G$ is a cartesian closed ∞-functor and hence preserves internal homs[8]) This is the *conjugation ∞-action* of G on morphisms $V_1 \to V_2$ by pre- and postcomposition with the action of G on V_1 and V_2, respectively.

6. The *precomposition action* of the automorphism ∞-group $\mathbf{Aut}(V)$ on a mapping space $[V, A]$ is given by

$$\begin{array}{c} [V, A] \longrightarrow \sum_{\mathbf{B}G} [\rho_{\mathrm{aut}}, \mathbf{B}G^* A] \\ \downarrow \\ \mathbf{BAut}(V) \end{array} .$$

7. Let now $\mathbf{H}$ be a differential cohesive ∞-topos, 3. Let $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ be a braided ∞-group, def. 3.6.119, and write $\Omega^2_{\mathrm{cl}}(-, \mathbb{G}) \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ for the corresponding moduli of $\mathbb{G}$-differential cocycles, 3.9.6.2.

   Let furthermore

$$\begin{array}{ccc} & & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \\ & \overset{\nabla}{\nearrow} & \downarrow{\scriptstyle F_{(-)}} \\ X & \xrightarrow{\omega} & \mathbf{B}\mathbb{G} \end{array}$$

   a presymplectic structure ω with prequantization ∇, 3.9.13 and let ρ be an action of $\mathbb{G}$ on some V. Then the (higher Heisenberg group-) *∞-action of higher prequantum operators on the space $\mathbf{\Gamma}_X(E)$ of higher prequantum states* is

$$\begin{array}{c} \mathbf{\Gamma}_X(E) \longrightarrow \prod_{\mathbf{B}\mathbb{G}} \left(\left[\sum_U \nabla, \rho \right] // \prod_U \mathbf{Aut}(\nabla) \right) \;, \\ \downarrow \\ \mathbf{Aut}_{\mathbf{H}}(\nabla) \end{array}$$

   where E is the ρ-associated V-bundle to $\sum_U \nabla$.

8. Let specifically $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$, 4.4.

   There we have

   (a) the ∞-action of the moduli of circle principal bundles (the circle 2-group) $\mathbf{B}U(1)$ on the moduli of unitary bundles, 5.4.3,

$$\begin{array}{ccc} \mathbf{B}U(n) & \longrightarrow & \mathbf{B}\mathrm{PU}(n) \\ & & \downarrow{\scriptstyle \mathbf{dd}_n} \\ & & \mathbf{B}^2 U(1) \end{array}$$

[8] U.S. thanks Mike Shulman for discussion of this point.

(b) the ∞-action of the moduli of circle principal 2-bundles (the circle 3-group) $\mathbf{B}^2U(1)$ on the moduli for String-principal 2-bundles, 5.1.4,

$$\begin{array}{ccc} \mathbf{B}\mathrm{String} & \longrightarrow & \mathbf{B}\mathrm{Spin} \\ & & \downarrow{\scriptstyle \frac{1}{2}\mathbf{p}_1} \\ & & \mathbf{B}^3U(1) \end{array}$$

(c) the ∞-action of the moduli of circle principal 6-bundles (the circle 7-group) $\mathbf{B}^6U(1)$ on the moduli for Fivebrane-principal 6-bundles, 5.1.5,

$$\begin{array}{ccc} \mathbf{B}\mathrm{Fivebrane} & \longrightarrow & \mathbf{B}\mathrm{String} \\ & & \downarrow{\scriptstyle \frac{1}{6}\mathbf{p}_2} \\ & & \mathbf{B}^7U(1) \end{array}$$

For more examples along these lines see 5.4.1.

3.6.11.2 Presentation in locally fibrant simplicial sheaves We discuss associated ∞-bundles in an ∞-topos $\mathbf{H} = \mathrm{Sh}_\infty(C)$ in terms of the presentation of $\mathbf{H}$ by locally fibrant simplicial sheaves, corresponding to the respective presentation of principal ∞-bundles from 3.6.10.4.

This section draws from [NSSb].

Let C be a site with terminal object.

By prop. 3.6.134 every ∞-group over C has a presentation by a sheaf of simplicial groups $G \in \mathrm{Grp}(\mathrm{sSh}(C)_{\mathrm{lfib}})$. Moreover, by theorem 3.6.201 every ∞-action of G on an object V, def. 3.6.152, is exhibited by a weakly principal simplicial bundle

$$\begin{array}{ccc} V & \longrightarrow & V/_hG \\ & & \downarrow{\scriptstyle \rho} \\ & & \overline{W}G \end{array} .$$

By example 3.6.209 this is a presentation for the *universal ρ-associated V-bundle.*

We now spell out what this means in the presentation.

Lemma 3.6.217. *The morphism $V/_hG \to \overline{W}G$ is a local fibration.*

Proof. By the same argument as in the proof of theorem 3.6.197. □

Proposition 3.6.218. *Let $P \to X$ in $\mathrm{sSh}(C)_{\mathrm{lfib}}$ be a weakly G-principal bundle with classifying cocycle $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \overset{g}{\to} \overline{W}G$. Then the corresponding ρ-associated ∞-bundle, def. 3.6.209, is presented by the ordinary V-associated bundle $P \times_G V$ formed in $\mathrm{sSh}(C)_{\mathrm{lfib}}$.*

Proof. By def. 3.6.209 the associated ∞-bundle is the ∞-pullback of $V//G \to \mathbf{B}G$ along g. Using lemma 3.6.217 in prop. 2.3.12 we find that this is presented already by the ordinary pullback of $V/_hG \to \overline{W}G$ along g. By prop. 3.6.178 this in turn is isomorphic to the pullback of $V \times_G WG \to \overline{W}G$. Since $\mathrm{sSh}(C)$ is a 1-topos, pullbacks preserve quotients, and so this pullback finally is

$$g^*(WG \times_G V) \simeq (g^*WG) \times_G V \simeq P \times_G WG\,.$$

□

3.6.12 Sections and twisted cohomology

We discuss here how the general notion of cohomology in an ∞-topos considered above in 3.6.9, already subsumes the notion of *twisted cohomology* and we discuss the corresponding geometric structure classified by twisted cohomology: *twisted ∞-bundles*.

Where ordinary cohomology is given by a derived hom-∞-groupoid, twisted cohomology is given by the ∞-groupoid of *sections of a local coefficient bundle* in an ∞-topos. This is a geometric and unstable variant of the picture of twisted cohomology developed in [ABG10] [MaSi07]. It is fairly immediate that given a *universal coefficient bundle*, the induced twisted cohomology is equivalently the ordinary cohomology in the corresponding slice ∞-topos. This identification provides a clean formulation of the contravariance of twisted cocycles. Finally, we observe that twisted cohomology in an ∞-topos equivalently classifies extensions of structure groups of principal ∞-bundles.

This section draws from [NSSa] and [NSSb].

3.6.12.1 General abstract

Definition 3.6.219. Let $p : E \to X$ be any morphism in $\mathbf{H}$, to be regarded as an ∞-bundle over X. A *section* of E is a diagram

$$\begin{array}{ccc} & & E \\ & {}^{\sigma}\nearrow & \downarrow{}^{p} \\ X & \xrightarrow[\mathrm{id}]{\Downarrow\simeq} & X \end{array}$$

(where for emphasis we display the presence of the homotopy filling the diagram). The *∞-groupoid of sections* of $E \stackrel{p}{\to} X$ is the homotopy fiber

$$\Gamma_X(E) := \mathbf{H}(X,E) \times_{\mathbf{H}(X,X)} \{\mathrm{id}_X\}$$

of the space of all morphisms $X \to E$ on those that cover the identity on X.

We record two elementary but important observations about spaces of sections.

Observation 3.6.220. There is a canonical identification

$$\Gamma_X(E) \simeq \mathbf{H}_{/X}(\mathrm{id}_X, p)$$

of the space of sections of $E \to X$ with the hom-∞-groupoid in the slice ∞-topos $\mathbf{H}_{/X}$ between the identity on X and the bundle map p.

Proof. By prop. 3.6.5. □

Lemma 3.6.221. *Let*

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow{}^{p_1} & & \downarrow{}^{p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

be an ∞-pullback diagram in $\mathbf{H}$ and let $X \xrightarrow{g_X} B_1$ be any morphism. Then post-composition with f induces a natural equivalence of hom-∞-groupoids

$$\mathbf{H}_{/B_1}(g_X, p_1) \simeq \mathbf{H}_{/B_2}(f \circ g_X, p_2)\,.$$

Proof. By Proposition 3.6.5, the left hand side is given by the homotopy pullback

$$\begin{array}{ccc} \mathbf{H}_{/B_1}(g_X, p_1) & \longrightarrow & \mathbf{H}(X, E_1) \\ \downarrow & & \downarrow^{\mathbf{H}(X,p_1)} \\ \{g_X\} & \longrightarrow & \mathbf{H}(X, B_1)\,. \end{array}$$

Since the hom-∞-functor $\mathbf{H}(X, -) : \mathbf{H} \to \mathrm{Grpd}_\infty$ preserves the ∞-pullback $E_1 \simeq f^* E_2$, this extends to a pasting of ∞-pullbacks, which by the pasting law (Proposition 2.3.2) is

$$\begin{array}{ccccc} \mathbf{H}_{/B_1}(g_X, p_1) & \longrightarrow & \mathbf{H}(X, E_1) & \longrightarrow & \mathbf{H}(X, E_2) \\ \downarrow & & \downarrow^{\mathbf{H}(X,p_1)} & & \downarrow^{\mathbf{H}(X,p_2)} \\ \{g_X\} & \longrightarrow & \mathbf{H}(X, B_1) & \underset{\mathbf{H}(X,f)}{\longrightarrow} & \mathbf{H}(X, B_2) \end{array} \quad \simeq \quad \begin{array}{ccc} \mathbf{H}_{/B_2}(f \circ g_X, p_2) & \longrightarrow & \mathbf{H}(X, E_2) \\ \downarrow & & \downarrow^{\mathbf{H}(X,p_2)} \\ \{f \circ g_X\} & \longrightarrow & \mathbf{H}(X, B_2). \end{array}$$

□

Fix now an ∞-group $G \in \mathrm{Grp}(\mathbf{H})$ and an ∞-action $\rho : V \times G \to V$. Write

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow^{\mathbf{c}} \\ & & \mathbf{B}G \end{array}$$

for the corresponding *universal ρ-associated ∞-bundle* as discussed in Section 3.6.13.

Proposition 3.6.222. *For $g_X : X \to \mathbf{B}G$ a cocycle and $P \to X$ the corresponding G-principal ∞-bundle according to Theorem 3.6.170, there is a natural equivalence*

$$\Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$$

between the space of sections of the corresponding ρ-associated V-bundle (Definition 3.6.205) and the hom-∞-groupoid of the slice ∞-topos of $\mathbf{H}$ over $\mathbf{B}G$, between g_X and $\mathbf{c}$. Schematically:

$$\left\{ \begin{array}{ccc} & & E \\ & {}^{\sigma}\nearrow \;\; \Downarrow\simeq & \downarrow^{p} \\ X & \underset{\mathrm{id}}{\longrightarrow} & X \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & & V//G \\ & {}^{\sigma}\nearrow \;\; \Downarrow\simeq & \downarrow^{\mathbf{c}} \\ X & \underset{\mathrm{g_X}}{\longrightarrow} & \mathbf{B}G \end{array} \right\}$$

Proof. By Observation 3.6.220 and Lemma 3.6.221. □

Observation 3.6.223. If in the above the cocycle g_X is trivializable, in the sense that it factors through the point $* \to \mathbf{B}G$ (equivalently if its class $[g_X] \in H^1(X, G)$ is trivial) then there is an equivalence

$$\mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}) \simeq \mathbf{H}(X, V)\,.$$

Proof. In this case the homotopy pullback on the right in the proof of Proposition 3.6.222 is

$$\begin{array}{ccccc} \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}) & \simeq & \mathbf{H}(X, V) & \longrightarrow & \mathbf{H}(X, V//G) \\ & & \downarrow & & \downarrow^{\mathbf{H}(X,\mathbf{c})} \\ \{g_X\} & \simeq & \mathbf{H}(X, *) & \longrightarrow & \mathbf{H}(X, \mathbf{B}G) \end{array}$$

using that $V \to V//G \stackrel{\mathbf{c}}{\to} \mathbf{B}G$ is a fiber sequence by definition, and that $\mathbf{H}(X, -)$ preserves this fiber sequence. □

Remark 3.6.224. Since by Proposition 3.6.164 every cocycle g_X trivializes locally over some cover $U \longrightarrow\!\!\!\!\to X$ and equivalently, by Proposition 3.6.211, every ∞-bundle $P \times_G V$ trivializes locally, Observation 3.6.223 says that elements $\sigma \in \Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$ *locally* are morphisms $\sigma|_U : U \to V$ with values in V. They fail to be so *globally* to the extent that $[g_X] \in H^1(X, G)$ is non-trivial, hence to the extent that $P \times_G V \to X$ is non-trivial.

This motivates the following definition.

Definition 3.6.225. We say that the ∞-groupoid $\Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$ from Proposition 3.6.222 is the ∞-groupoid of $[g_X]$-*twisted cocycles* with values in V, with respect to the *local coefficient ∞-bundle* $V//G \stackrel{\mathbf{c}}{\to} \mathbf{B}G$.

Accordingly, its set of connected components we call the $[g_X]$-*twisted V-cohomology* with respect to the local coefficient bundle $\mathbf{c}$ and write:

$$H^{[g_X]}(X, V) := \pi_0 \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}) .$$

Remark 3.6.226. The perspective that twisted cohomology is the theory of sections of associated bundles whose fibers are classifying spaces is maybe most famous for the case of twisted K-theory, where it was described in this form in [Ros]. But already the old theory of *ordinary cohomology with local coefficients* is of this form, as is made manifest in [BFG] (we discuss this in detail in [NSSc]).

A proposal for a comprehensive theory in terms of bundles of topological spaces is in [MaSi07] and a systematic formulation in ∞-category theory and for the case of multiplicative generalized cohomology theories is in [ABG10]. The formulation above refines this, unstably, to geometric cohomology theories/(nonabelian) sheaf hypercohomology, hence from bundles of classifying spaces to ∞-bundles of moduli ∞-stacks.

A wealth of examples and applications of such geometric nonabelian twisted cohomology of relevance in quantum field theory and in string theory is discussed in 3.9.8.

Remark 3.6.227. Of special interest is the case where V is pointed connected, hence (by Theorem 3.6.116) of the form $V = \mathbf{B}A$ for some ∞-group A, and so (by Definition 3.6.137) the coefficient for degree-1 A-cohomology, and hence itself (by Theorem 3.6.170) the moduli ∞-stack for A-principal ∞-bundles. In this case $H^{[g_X]}(X, \mathbf{B}A)$ is *degree-1 twisted A-cohomology*. Generally, if $V = \mathbf{B}^n A$ it is *degree-n twisted A-cohomology*. In analogy with Definition 3.6.137 this is sometimes written

$$H^{n+[g_X]}(X, A) := H^{[g_X]}(X, \mathbf{B}^n A) .$$

Moreover, in this case $V//G$ is itself pointed connected, hence of the form $\mathbf{B}\hat{G}$ for some ∞-group $\hat{G}$, and so the universal local coefficient bundle

$$\begin{array}{ccc} \mathbf{B}A & \longrightarrow & \mathbf{B}\hat{G} \\ & & \downarrow^{\mathbf{c}} \\ & & \mathbf{B}G \end{array}$$

exhibits $\hat{G}$ as an *extension of ∞-groups* of G by A. This case we discuss below in Section 3.6.14.

In this notation the local coefficient bundle $\mathbf{c}$ is left implicit. This convenient abuse of notation is justifed to some extent by the fact that there is a *universal local coefficient bundle*:

Example 3.6.228. The classifying morphism of the $\mathbf{Aut}(V)$-action on some $V \in \mathbf{H}$ from Definition 3.6.212 according to Theorem 3.6.170 yields a local coefficient ∞-bundle of the form

$$\begin{array}{ccc} V & \longrightarrow & V//\mathbf{Aut}(V) \\ & & \downarrow \\ & & \mathbf{BAut}(V) \end{array}$$

which we may call the *universal local V-coefficient bundle*. In the case that V is pointed connected and hence of the form $V = \mathbf{B}G$

$$\begin{array}{ccc} \mathbf{B}G & \longrightarrow & (\mathbf{B}G)//\mathbf{Aut}(\mathbf{B}G) \\ & & \downarrow \\ & & \mathbf{BAut}(\mathbf{B}G) \end{array}$$

the universal twists of the corresponding twisted G-cohomology are the *G-∞-gerbes*. These we discuss below in section 3.6.15.

We now internalize the formulation of spaces of sections, to obtain objects of sections in the ambient ∞-topos.

Definition 3.6.229. For $p : E \to X$ a ρ-associated V-fiber bundle, its object of sections is is the dependent product, def. 3.6.2:

$$\mathbf{\Gamma}_X(E) \simeq \prod_X p\,.$$

Proposition 3.6.230. *For $p : E \to X$ a ρ-associated V-fiber bundle, its object of sections is equivalently given by*

$$\mathbf{\Gamma}_X(E) \simeq \prod_{\mathbf{B}G} [g, \rho]\,,$$

where $g : X \to \mathbf{B}G$ is the modulus of the G-principal bundle to which E is associated.

Proof. By functoriality we have

$$\begin{aligned} \prod_X g^*\rho &\simeq \prod_{\mathbf{B}G}\prod_g g^*\rho \\ &\simeq \prod_{\mathbf{B}G}[g, \rho] \end{aligned}\,,$$

where the second step is prop. 3.6.15. □

3.6.12.2 Presentations

Remark 3.6.231. When the ∞-topos $\mathbf{H}$ is presented by a model structure on simplicial presheaves as in 2.2.3 and presentations for X and C have been chosen, then the cocycle ∞-groupoid $\mathbf{H}(X, C)$ is presented by an explicit simplicial set $\mathbf{H}(X, C)_{\mathrm{simp}} \in \mathrm{sSet}$. Once these choices are made, there is therefore the inclusion of simplicial presheaves

$$\mathrm{const}(\mathbf{H}(X, C)_{\mathrm{simp}})_0 \to \mathbf{H}(X, C)_{\mathrm{simp}}\,,$$

where on the left we have the simplicially constant object on the vertices of $\mathbf{H}(X, C)_{\mathrm{simp}}$. This morphism, in turn, presents a morphism in $\infty\mathrm{Grpd}$ that in general contains a multitude of copies of the components of any $H(X,C) \to \mathbf{H}(X,C)$, a multitude of representatives of twists for each cohomology class of twists. Since the twisted cohomology does not depend, up to equivalence, on the choice of representative of the twist, the corresponding ∞-pullback yields in general a larger coproduct of ∞-groupoids as the corresponding twisted cohomology. This however just contains copies of the homotopy types already present in $\mathbf{H}_{\mathrm{tw}}(X,A)$ as defined above and therefore constitutes no additional information.

However, the choice of effective epimorphism $H(X,C) \to \mathbf{H}(X,C)$, while unique up to equivalence, can usually not be made functorially in X. Therefore twisted cohomology can have a *representing object* only if one does consider multiple twist representatives in a suitable way. An example of this situation appears in the discussion of differential cohomology below in 3.9.6.

3.6.13 Representations and group cohomology

We further discuss the notion of representations/actions/modules of ∞-groups in an ∞-topos and the related notions of quotients, invariants and group cohomology

3.6.13.1 General abstract. Let $G \in \mathrm{Grp}(\mathbf{H})$ be a group object. By the discussion in 3.6.11 we may identify the slice ∞-topos over its delooping with the ∞-category of G-actions:

Proposition 3.6.232. *We have an equivalence of ∞-categories*

$$G\mathrm{Act} \simeq \mathbf{H}_{/\mathbf{B}G}\,,$$

under which an action of G on some $V \in \mathbf{H}$ is identified with a morphism $V//G \to \mathbf{B}G$, regarded as an object in $\mathbf{H}_{/\mathbf{B}G}$, whose ∞-fiber is V:

$$V \longrightarrow V//G \longrightarrow \mathbf{B}G\;.$$

It is useful to identify the structure seen here more formally: write

$$\mathbf{H}_{/\mathbf{B}G} \underset{\prod_{\mathbf{B}G}}{\overset{\sum_{\mathbf{B}G}}{\underset{\xleftarrow{(\mathbf{B}G)\times(-)}}{\longrightarrow}}} \mathbf{H}$$

for the induced étale geometric morphism, prop. 3.6.13. We introduce some basic terminology on G-actions and analyze some properties.

Definition 3.6.233. For $\rho \in \mathbf{H}_{/\mathbf{B}G}$ a G-action on some $V \in \mathbf{H}$, we say that

1. its dependent sum $\sum_{\mathbf{B}G} \rho \in \mathbf{H}$ is the *quotient object* of the action;
2. its dependent product $\prod_{\mathbf{B}G} \rho \in \mathbf{H}$ is the *object of invariants* of the action.

Moreover, for $V \in \mathbf{H}$ any object, we say that $(\mathbf{B}G)^* V \in \mathbf{H}_{/\mathbf{B}G}$ is the *trivial action* of G on V. .

Proposition 3.6.234. *1. The quotient object in the sense of def. 3.6.233 coincides with the quotient in the sense of def. 3.6.205:*

$$\sum_{\mathbf{B}G} \rho \simeq V//G\,.$$

2. The object of invariants coincides with the object of sections of the universal V-associated bundle, def. 3.6.222:

$$\prod_{\mathbf{B}G} \rho \simeq \mathbf{\Gamma}_{\mathbf{B}G}(V//G)\,.$$

Definition 3.6.235. For $\rho_1, \rho_2 \in \mathbf{H}_{/\mathbf{B}G}$ two G-actions on objects $V_1, V_2 \in \mathbf{H}$, respectively, write $[\rho_1, \rho_2] \in \mathbf{H}_{/\mathbf{B}G}$ for their internal hom in the slice. This we call the *conjugation action* of G on morphisms $V_1 \to V_2$. We say its direct image under the above étale geometric morphism is the object of *action homomorphisms* and write

$$\mathbf{Hom}_G(\rho_1, \rho_2) := \prod_{\mathbf{B}G} [\rho_1, \rho_2] \in \mathbf{H}\,.$$

Remark 3.6.236. In words this says that a G-action homomorphism is a morphism $V_1 \to V_2$ which is an invariant (up to homotopy) of the conjugation action of G.

Proposition 3.6.237. *The conjugation action $[\rho_1, \rho_2]$, def. 3.6.235, is a G-action on the internal hom object $[V_1, V_2] \in \mathbf{H}$.*

Proof. By def. 3.6.205 we need to show that the internal hom $[\rho_1, \rho_2]$ in the slice sits in a fiber sequence in $\mathbf{H}$ of the form

$$\begin{array}{ccc} [V_1, V_2] & \longrightarrow & \sum_{\mathbf{B}G} [\rho_1, \rho_2] \\ & & \downarrow \\ & & \mathbf{B}G \end{array} .$$

Observe that forming the homotopy fiber is applying the inverse image of base change along the point inclusion $\mathrm{pt}_{\mathbf{B}G} : * \to \mathbf{B}G$ and that base change inverse images are cartesian closed functors[9], hence preserve fibers. Using this we compute

$$\begin{aligned} (\mathrm{pt}_{\mathbf{B}G})^* [\rho_1, \rho_2] &\simeq [(\mathrm{pt}_{\mathbf{B}G})^*(V_1, \rho_1), (\mathrm{pt}_{\mathbf{B}G})^*(V_2, \rho_2)] \\ &\simeq [V_1, V_2] \end{aligned} .$$

□

Definition 3.6.238. For $G_1, G_2 \in \mathrm{Grp}(\mathbf{H})$ two groups and $f : G_1 \to G_2$ a group homomorphism, hence $\mathbf{B}f : \mathbf{B}G_1 \to \mathbf{B}G_2$ a morphism in $\mathbf{H}$ we say that

1. the base change
$$(\mathbf{B}f)^* : \ \mathrm{Act}(G_2) \simeq \mathbf{H}_{/\mathbf{B}G_2} \longrightarrow \mathbf{H}_{/\mathbf{B}G_1} \simeq \mathrm{Act}(G_1)$$
is the *pullback representation* functor (or *restricted representation* functor if f is a monomorphism);

2. the dependent sum
$$\sum_{\mathbf{B}f} : \ \mathrm{Act}(G_1) \simeq \mathbf{H}_{/\mathbf{B}G_1} \longrightarrow \mathbf{H}_{/\mathbf{B}G_2} \simeq \mathrm{Act}(G_2)$$
is the *induced representation* functor.

3. the dependent product
$$\prod_{\mathbf{B}f} : \ \mathrm{Act}(G_1) \simeq \mathbf{H}_{/\mathbf{B}G_1} \longrightarrow \mathbf{H}_{/\mathbf{B}G_2} \simeq \mathrm{Act}(G_2)$$
is the *coinduced representation* functor.

Remark 3.6.239. For the case of permutation representations of discrete groups, this identification of dependent sum/dependent product along contexts of pointed connected discrete groupoids has been mentioned on p. 14 of [Lawv06].

[9] Thanks to Mike Shulman for discussion of this point.

Example 3.6.240. For $X \in \mathbf{H}$ any object, the automorphism group $\mathbf{Aut}(X)$ of def. 3.6.212 has a canonical action $\rho_{\mathrm{aut}(X)}$ on X, given by the pasting of ∞-pullback diagrams

$$\begin{array}{ccccc} X & \longrightarrow & V//\mathbf{Aut}(X) & \longrightarrow & \widehat{\mathrm{Obj}} \\ \downarrow & & \downarrow{\scriptstyle \rho_{\mathrm{aut}(X)}} & & \downarrow \\ * & \twoheadrightarrow & \mathbf{BAut}(X) & \hookrightarrow & \mathrm{Obj} \end{array} ,$$

with the bottom composite $* \xrightarrow{\vdash X} \mathrm{Obj}$,

where the morphism on the right is the universal small object bundle.

Example 3.6.241. For $X, Y \in \mathbf{H}$ two objects, the automorphism group $\mathbf{Aut}(X)$ of X, def. 3.6.212 has a canonical action ρ_{prec} by *precomposition* on the internal hom $[X, Y] \in \mathbf{H}$, given itself by the internal hom

$$\rho_{\mathrm{prec}} := \left[\rho_{\mathrm{aut}(X)}, \mathbf{BAut}(X)^* Y\right]$$

in $\mathrm{Act}(\mathbf{Aut}(X))$, hence by the congugation action on morphisms from X to Y with Y regarded as equipped with the trivial $\mathbf{Aut}(X)$-action; we have a fiber sequence

$$\begin{array}{ccc} [X,Y] & \longrightarrow & [X,Y]//\mathbf{Aut}(X) \\ & & \downarrow{\scriptstyle \rho_{\mathrm{prec}}} \\ & & \mathbf{BAut}(X) \end{array}$$

in $\mathbf{H}$.

Definition 3.6.242. For $*$ the point equipped with the (necessarily) trivial G-action, and for $(V, \rho) \in \mathbf{H}_{/\mathbf{B}G}$ we say that

$$\mathbf{Hom}_G(*, V) \in \mathbf{H}$$

is the *cocycle ∞-groupoid* of G-group cohomology with coefficients in V. We say that

$$H_{\mathrm{Grp}}(G, V) := \pi_0 \mathbf{Hom}_G(*, V)$$

is the *group cohomology* of G with coefficients in V.

Remark 3.6.243. By remark 3.6.236 and since the action on $*$ is trivial, this says in words that group cohomology with coefficients in V is the collection of equivalence classes of invariants of V.

3.6.13.2 Presentations.

Remark 3.6.244. In the case that $V \in \mathbf{H}$ is presented by a chain complex under the Dold-Kan correspondence, def. 2.2.31 and that $G \in \mathrm{Grp}(\mathbf{H})$ is a 0-truncated group, def. 3.6.242 of group cohomology of G with coefficients in V manifestly reduces to the traditional definition of group cohomology in homological algebra, given by the derived functor of the invariants functor of G-modules.

3.6.14 Extensions and twisted bundles

We discuss the notion of *extensions* of ∞-groups (see Section 3.6.8), generalizing the traditional notion of group extensions. This is in fact a special case of the notion of principal ∞-bundle, Definition 3.6.155, for base space objects that are themselves deloopings of ∞-groups. For every extension of ∞-groups, there is the corresponding notion of *lifts of structure ∞-groups* of principal ∞-bundles. These are classified equivalently by trivializations of an *obstruction class* and by the twisted cohomology with coefficients in the extension itself, regarded as a local coefficient ∞-bundle.

Moreover, we show that principal ∞-bundles with an extended structure ∞-group are equivalent to principal ∞-bundles with unextended structure ∞-group but carrying a principal ∞-bundle for the *extending* ∞-group on their total space, which on fibers restricts to the given ∞-group extension. We formalize these *twisted (principal) ∞-bundles* and observe that they are classified by twisted cohomology, Definition 3.6.225.

Definition 3.6.245. We say a sequence of ∞-groups,

$$A \to \hat{G} \to G$$

in $\mathrm{Grp}(\mathbf{H})$ *exhibits $\hat{G}$ as an extension of G by A* if the delooping

$$\mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G$$

is a fiber sequence in $\mathbf{H}$, def. 3.6.141.

Remark 3.6.246. By continuing the fiber sequence to the left

$$A \to \hat{G} \to G \to \mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G$$

this implies by theorem 3.6.170 that $\hat{G} \to G$ is an A-principal bundle and that

$$G \simeq \hat{G} \to A$$

is the quotient of the A-action.

Definition 3.6.247. For A a braided ∞-group, def. 3.6.119, a *central extension* $\hat{G}$ of G by A is an extension $A \to \hat{G} \to G$, such that the defining delooping extends one step further to the right:

$$\mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \xrightarrow{\mathbf{p}} \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^2 A\,.$$

We write

$$\mathrm{Ext}(G, A) := \mathbf{H}(\mathbf{B}G, \mathbf{B}^2 A) \simeq (\mathbf{B}A)\mathrm{Bund}(\mathbf{B}G)$$

for the *∞-groupoid of extensions* of G by A.

Definition 3.6.248. Given an ∞-group extension $A \longrightarrow \hat{G} \xrightarrow{\Omega\mathbf{c}} G$ and given a G-principal ∞-bundle $P \to X$ in $\mathbf{H}$, we say that a *lift* $\hat{P}$ of P to a $\hat{G}$-principal ∞-bundle is a lift $\hat{g}_X$ of its classifying cocycle $g_X : X \to \mathbf{B}G$, under the equivalence of Theorem 3.6.170, through the extension:

$$\begin{array}{ccc} & & \mathbf{B}\hat{G} \\ & {}^{\hat{g}_X}\nearrow & \downarrow^{\mathbf{p}} \\ X & \xrightarrow[g_X]{} & \mathbf{B}G. \end{array}$$

Accordingly, the *∞-groupoid of lifts* of P with respect to $\mathbf{p}$ is

$$\mathrm{Lift}(P, \mathbf{p}) := \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{p})\,.$$

Observation 3.6.249. By the universal property of the ∞-pullback, a lift exists precisely if the cohomology class

$$[\mathbf{c}(g_X)] := [\mathbf{c} \circ g_X] \in H^2(X, A)$$

is trivial.

This is implied by Theorem 3.6.251, to which we turn after introducing the following terminology.

Definition 3.6.250. In the above situation, we call $[\mathbf{c}(g_X)]$ the *obstruction class* to the extension; and we call $[\mathbf{c}] \in H^2(\mathbf{B}G, A)$ the *universal obstruction class* of extensions through $\mathbf{p}$.

We say that a *trivialization* of the obstruction cocycle $\mathbf{c}(g_X)$ is a morphism $\mathbf{c}(g_X) \to *_X$ in $\mathbf{H}(X, \mathbf{B}^2 A)$, where $*_X : X \to * \to \mathbf{B}^2 A$ is the trivial cocycle. Accordingly, the *∞-groupoid of trivializations of the obstruction* is

$$\mathrm{Triv}(\mathbf{c}(g_X)) := \mathbf{H}_{/\mathbf{B}^2 A}(\mathbf{c} \circ g_X, *_X)\,.$$

We give now three different characterizations of spaces of extensions of ∞-bundles. The first two, by spaces of twisted cocycles and by spaces of trivializations of the obstruction class, are immediate consequences of the previous discussion:

Theorem 3.6.251. *Let $P \to X$ be a G-principal ∞-bundle corresponding by Theorem 3.6.170 to a cocycle $g_X : X \to \mathbf{B}G$.*

1. *There is a natural equivalence*
$$\mathrm{Lift}(P, \mathbf{p}) \simeq \mathrm{Triv}(\mathbf{c}(g_X))$$
between the ∞-groupoid of lifts of P through $\mathbf{p}$, Definition 3.6.248, and the ∞-groupoid of trivializations of the obstruction class, Definition 3.6.250.

2. *There is a natural equivalence* $\mathrm{Lift}(P, \mathbf{p}) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{p})$ *between the ∞-groupoid of lifts and the ∞-groupoid of g_X-twisted cocycles relative to $\mathbf{p}$, Definition 3.6.225, hence a classification*
$$\pi_0 \mathrm{Lift}(P, \mathbf{P}) \simeq H^{1+[g_X]}(X, A)$$
of equivalence classs of lifts by the $[g_X]$-twisted A-cohomology of X relative to the local coefficient bundle
$$\begin{array}{ccc} \mathbf{B}A & \longrightarrow & \mathbf{B}\hat{G} \\ & & \downarrow{\scriptstyle \mathbf{p}} \\ & & \mathbf{B}G\,. \end{array}$$

Proof. The first statement is the special case of Lemma 3.6.221 where the ∞-pullback $E_1 \simeq f^* E_2$ in the notation there is identified with $\mathbf{B}\hat{G} \simeq \mathbf{c}^* *$. The second is evident after unwinding the definitions. □

Remark 3.6.252. For the special case that A is 0-truncated, we may, by the discussion in [NW11a, NSSc], identify $\mathbf{B}A$-principal ∞-bundles with *A-bundle gerbes*, [Mur]. Under this identification the ∞-bundle classified by the obstruction class $[\mathbf{c}(g_X)]$ above is what is called the *lifting bundle gerbe* of the lifting problem, see for instance [CBMMS02] for a review. In this case the first item of Theorem 3.6.251 reduces to Theorem 2.1 in [Wal09] and Theorem A (5.2.3) in [NW11b]. The reduction of this statement to connected components, hence the special case of Observation 3.6.249, was shown in [Br90].

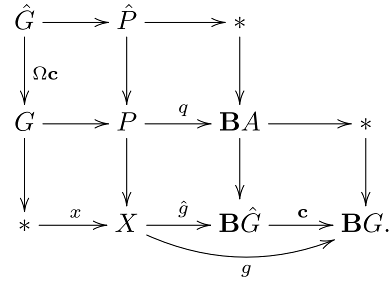
While, therefore, the discussion of extensions of ∞-groups and of lifts of structure ∞-groups is just a special case of the discussion in the previous sections, this special case admits geometric representatives of cocycles in the corresponding twisted cohomology by twisted principal ∞-bundles. This we turn to now.

Definition 3.6.253. Given an extension of ∞-groups $A \to \hat{G} \xrightarrow{\Omega \mathbf{c}} G$ and given a G-principal ∞-bundle $P \to X$, with class $[g_X] \in H^1(X, G)$, a *$[g_X]$-twisted A-principal ∞-bundle* on X is an A-principal ∞-bundle $\hat{P} \to P$ such that the cocycle $q : P \to \mathbf{B}A$ corresponding to it under Theorem 3.6.170 is a morphism of G-∞-actions.

The *∞-groupoid of $[g_X]$-twisted A-principal ∞-bundles on X* is

$$A\mathrm{Bund}^{[g_X]}(X) := G\mathrm{Action}(P, \mathbf{B}A) \subset \mathbf{H}(P, \mathbf{B}A)\,.$$

Observation 3.6.254. Given an ∞-group extension $A \to \hat{G} \overset{\Omega\mathbf{c}}{\to} G$, an extension of a G-principal ∞-bundle $P \to X$ to a $\hat{G}$-principal ∞-bundle, Definition 3.6.248, induces an A-principal ∞-bundle $\hat{P} \to P$ fitting into a pasting diagram of ∞-pullbacks of the form



In particular, it has the following properties:

1. $\hat{P} \to P$ is a $[g_X]$-twisted A-principal bundle, Definition 3.6.253;
2. for all points $x : * \to X$ the restriction of $\hat{P} \to P$ to the fiber P_x is equivalent to the ∞-group extension $\hat{G} \to G$.

Proof. This follows from repeated application of the pasting law for ∞-pullbacks, Proposition 2.3.2.

The bottom composite $g : X \to \mathbf{B}G$ is a cocycle for the given G-principal ∞-bundle $P \to X$ and it factors through $\hat{g} : X \to \mathbf{B}\hat{G}$ by assumption of the existence of the extension $\hat{P} \to P$.

Since also the bottom right square is an ∞-pullback by the given ∞-group extension, the pasting law asserts that the square over $\hat{g}$ is also an ∞-pullback, and then that so is the square over q. This exhibits $\hat{P}$ as an A-principal ∞-bundle over P classified by the cocycle q on P. By Proposition 3.6.255 this $\hat{P} \to P$ is twisted G-equivariant.

Now choose any point $x : * \to X$ of the base space as on the left of the diagram. Pulling this back upwards through the diagram and using the pasting law and the definition of loop space objects $G \simeq \Omega\mathbf{B}G \simeq * \times_{\mathbf{B}G} *$ the diagram completes by ∞-pullback squares on the left as indicated, which proves the claim. □

Theorem 3.6.255. *The construction of Observation 3.6.254 extends to an equivalence of ∞-groupoids*

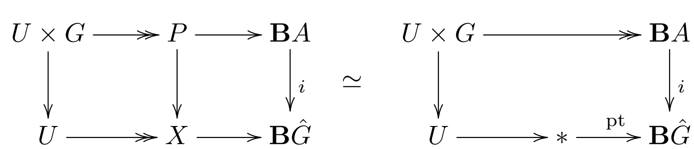
$$A\mathrm{Bund}^{[g_X]}(X) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$$

between that of $[g_X]$-twisted A-principal bundles on X, Definition 3.6.253, and the cocycle ∞-groupoid of degree-1 $[g_X]$-twisted A-cohomology, Definition 3.6.225.

In particular the classification of $[g_X]$-twisted A-principal bundles is

$$A\mathrm{Bund}^{[g_X]}(X)_{/\sim} \simeq H^{1+[g_X]}(X, A)\,.$$

Proof. For $G = *$ the trivial group, the statement reduces to Theorem 3.6.170. The general proof works along the same lines as the proof of that theorem. The key step is the generalization of the proof of Proposition 3.6.166. This proceeds verbatim as there, only with $\mathrm{pt} : * \to \mathbf{B}G$ generalized to $i : \mathbf{B}A \to \mathbf{B}\hat{G}$. The morphism of G-actions $P \to \mathbf{B}A$ and a choice of effective epimorphism $U \to X$ over which $P \to X$ trivializes gives rise to a morphism in $\mathbf{H}^{\Delta[1]}_{/(* \to \mathbf{B}G)}$ which involves the diagram



in $\mathbf{H}$. (We are using that for the 0-connected object $\mathbf{B}\hat{G}$ every morphism $* \to \mathbf{B}G$ factors through $\mathbf{B}\hat{G} \to \mathbf{B}G$.) Here the total rectangle and the left square on the left are ∞-pullbacks, and we need to show that the right square on the left is then also an ∞-pullback. Notice that by the pasting law the rectangle on the right is indeed equivalent to the pasting of ∞-pullbacks

$$\begin{array}{ccccc} U \times G & \longrightarrow & G & \longrightarrow & \mathbf{B}A \\ \downarrow & & \downarrow & & \downarrow^{i} \\ U & \longrightarrow & * & \xrightarrow{\mathrm{pt}} & \mathbf{B}\hat{G} \end{array}$$

so that the relation

$$U^{\times_X^{n+1}} \times G \simeq i^*(U^{\times_X^{n+1}})$$

holds. With this the proof finishes as in the proof of Proposition 3.6.166, with pt^* generalized to i^*. □

Remark 3.6.256. Aspects of special cases of this theorem can be identified in the literature. For the special case of ordinary extensions of ordinary Lie groups, the equivalence of the corresponding extensions of a principal bundle with certain equivariant structures on its total space is essentially the content of [Mac, And]. In particular the twisted unitary bundles or *gerbe modules* of twisted K-theory [CBMMS02] are equivalent to such structures.

For the case of $\mathbf{B}U(1)$-extensions of Lie groups, such as the String-2-group, the equivalence of the corresponding String-principal 2-bundles, by the above theorem, to certain bundle gerbes on the total spaces of principal bundles underlies constructions such as in [Redd06]. Similarly the bundle gerbes on double covers considered in [SSW05] are $\mathbf{B}U(1)$-principal 2-bundles on $\mathbb{Z}_2$-principal bundles arising by the above theorem from the extension $\mathbf{B}U(1) \to \mathbf{Aut}(\mathbf{B}U(1)) \to \mathbb{Z}_2$, a special case of the extensions that we consider in the next Section 3.6.15.

These and more examples we discuss in detail below.

3.6.15 Gerbes

We discuss the general notion of (nonabelian) *gerbes* and higher gerbes in an ∞-topos.

This section draws from [NSSa].

Remark 3.6.227 above indicates that of special relevance are those V-fiber ∞-bundles $E \to X$ in an ∞-topos $\mathbf{H}$ whose typical fiber V is pointed connected, and hence is the moduli ∞-stack $V = \mathbf{B}G$ of G-principal ∞-bundles for some ∞-group G. Due to their local triviality, when regarded as objects in the slice ∞-topos $\mathbf{H}_{/X}$, these $\mathbf{B}G$-fiber ∞-bundles are themselves *connected objects*. Generally, for $\mathcal{X}$ an ∞-topos regarded as an ∞-topos of ∞-stacks over a given space X, it makes sense to consider its connected objects as ∞-bundles over X. Here we discuss these ∞-*gerbes*.

In the following discussion it is useful to consider two ∞-toposes:

1. an "ambient" ∞-topos $\mathbf{H}$ as before, to be thought of as an ∞-topos "of all geometric homotopy types" for a given notion of geometry, in which ∞-bundles are given by *morphisms* and the terminal object plays the role of the geometric point $*$;

2. an ∞-topos $\mathcal{X}$, to be thought of as the topos-theoretic incarnation of a single geometric homotopy type (space) X, hence as an ∞-topos of "geometric homotopy types étale over X", in which an ∞-bundle over X is given by an *object* and the terminal object plays the role of the base space X.

   In practice, $\mathcal{X}$ is the slice $\mathbf{H}_{/X}$ of the previous ambient ∞-topos over $X \in \mathbf{H}$, or the smaller ∞-topos $\mathcal{X} = \mathrm{Sh}_\infty(X)$ of (internal) ∞-stacks over X.

In topos-theory literature the role of $\mathbf{H}$ above is sometimes referred to as that of a *gros* topos and then the role of $\mathcal{X}$ is referred to as that of a *petit* topos. The reader should beware that much of the classical literature on gerbes is written from the point of view of only the *petit* topos $\mathcal{X}$.

The original definition of a *gerbe* on X [Gir71] is: a stack E (i.e. a 1-truncated ∞-stack) over X that is 1. *locally non-empty* and 2. *locally connected*. In the more intrinsic language of higher topos theory, these two conditions simply say that E is a *connected object* (Definition 6.5.1.10 in [LuHTT]): 1. the terminal morphism $E \to *$ is an effective epimorphism and 2. the 0th homotopy sheaf is trivial, $\pi_0(E) \simeq *$. This reformulation is made explicit in the literature for instance in Section 5 of [JaLu04] and in Section 7.2.2 of [LuHTT]. Therefore:

Definition 3.6.257. For $\mathcal{X}$ an ∞-topos, a *gerbe* in $\mathcal{X}$ is an object $E \in \mathcal{X}$ which is

1. connected;
2. 1-truncated.

For $X \in \mathbf{H}$ an object, a *gerbe E over X* is a gerbe in the slice $\mathbf{H}_{/X}$. This is an object $E \in \mathbf{H}$ together with an effective epimorphism $E \to X$ such that $\pi_i(E) = X$ for all $i \neq 1$.

Remark 3.6.258. Notice that conceptually this is different from the notion of *bundle gerbe* introduced in [Mur] (see [NW11a] for a review). We discuss in [NSSc] that bundle gerbes are presentations of *principal* ∞-bundles (Definition 3.6.155). But gerbes – at least the *G-gerbes* considered in a moment in Definition 3.6.264 – are V-fiber ∞-bundles (Definition 3.6.204) hence *associated* to principal ∞-bundles (Proposition 3.6.213) with the special property of having pointed connected fibers. By Theorem 3.6.214 V-fiber ∞-bundles may be identified with their underlying $\mathbf{Aut}(V)$-principal ∞-bundles and so one may identify G-gerbes with nonabelian $\mathrm{Aut}(\mathbf{B}G)$-bundle gerbes (see also around Proposition 3.6.267 below), but considered generally, neither of these two notions is a special case of the other. Therefore the terminology is slightly unfortunate, but it is standard.

Definition 3.6.257 has various obvious generalizations. The following is considered in [LuHTT].

Definition 3.6.259. For $n \in \mathbb{N}$, an *EM n-gerbe* is an object $E \in \mathcal{X}$ which is

1. $(n-1)$-connected;
2. n-truncated.

Remark 3.6.260. This is almost the definition of an *Eilenberg-Mac Lane object* in $\mathcal{X}$, only that the condition requiring a global section $* \to E$ (hence $X \to E$) is missing. Indeed, the Eilenberg-Mac Lane objects of degree n in $\mathcal{X}$ are precisely the EM n-gerbes of *trivial class*, according to Proposition 3.6.267 below.

There is also an earlier established definition of *2-gerbes* in the literature [Br94], which is more general than EM 2-gerbes. Stated in the above fashion it reads as follows.

Definition 3.6.261 (Breen [Br94])**.** A *2-gerbe* in $\mathcal{X}$ is an object $E \in \mathcal{X}$ which is

1. connected;
2. 2-truncated.

This definition has an evident generalization to arbitrary degree, which we adopt here.

Definition 3.6.262. An *n-gerbe* in $\mathcal{X}$ is an object $E \in \mathcal{X}$ which is

1. connected;
2. n-truncated.

In particular an ∞-*gerbe* is a connected object.

The real interest is in those ∞-gerbes which have a prescribed typical fiber:

Remark 3.6.263. By the above, ∞-gerbes (and hence EM n-gerbes and 2-gerbes and hence gerbes) are much like deloopings of ∞-groups (Theorem 3.6.116) only that there is no requirement that there exists a global section. An ∞-gerbe for which there exists a global section $X \to E$ is called *trivializable*. By Theorem 3.6.116 trivializable ∞-gerbes are equivalent to ∞-group objects in $\mathcal{X}$ (and the ∞-groupoids of all of these are equivalent when transformations are required to preserve the canonical global section).

But *locally* every ∞-gerbe E is of this form. For let

$$(x^* \dashv x_*) : \mathrm{Grpd}_\infty \underset{x_*}{\overset{x^*}{\rightleftarrows}} \mathcal{X}$$

be a topos point. Then the stalk $x^* E \in \mathrm{Grpd}_\infty$ of the ∞-gerbe is connected: because inverse images preserve the finite ∞-limits involved in the definition of homotopy sheaves, and preserve the terminal object. Therefore

$$\pi_0\, x^* E \simeq x^* \pi_0 E \simeq x^* * \simeq * \,.$$

Hence for every point x we have a stalk ∞-group G_x and an equivalence

$$x^* E \simeq B G_x \,.$$

Therefore one is interested in the following notion.

Definition 3.6.264. For $G \in \mathrm{Grp}(\mathcal{X})$ an ∞-group object, a G-∞-*gerbe* is an ∞-gerbe E such that there exists

1. an effective epimorphism $U \longrightarrow\!\!\!\!\to X$;
2. an equivalence $E|_U \simeq \mathbf{B}G|_U$.

Equivalently: a G-∞-gerbe is a $\mathbf{B}G$-fiber ∞-bundle, according to Definition 3.6.204.

In words this says that a G-∞-gerbe is one that locally looks like the moduli ∞-stack of G-principal ∞-bundles.

Example 3.6.265. For X a topological space and $\mathcal{X} = \mathrm{Sh}_\infty(X)$ the ∞-topos of ∞-sheaves over it, these notions reduce to the following.

- a 0-group object $G \in \tau_0\mathrm{Grp}(\mathcal{X}) \subset \mathrm{Grp}(\mathcal{X})$ is a sheaf of groups on X (here $\tau_0\mathrm{Grp}(\mathcal{X})$ denotes the 0-truncation of $\mathrm{Grp}(\mathcal{X})$);
- for $\{U_i \to X\}$ any open cover, the canonical morphism $\coprod_i U_i \to X$ is an effective epimorphism to the terminal object;
- $(\mathbf{B}G)|_{U_i}$ is the stack of $G|_{U_i}$-principal bundles ($G|_{U_i}$-torsors).

It is clear that one way to construct a G-∞-gerbe should be to start with an $\mathbf{Aut}(\mathbf{B}G)$-principal ∞-bundle, Remark 3.6.228, and then canonically *associate* a fiber ∞-bundle to it.

Example 3.6.266. For $G \in \tau_0\mathrm{Grp}(\mathrm{Grpd}_\infty)$ an ordinary group, $\mathbf{Aut}(\mathbf{B}G)$ is usually called the *automorphism 2-group* of G. Its underlying groupoid is equivalent to

$$\mathbf{Aut}(G) \times G \rightrightarrows \mathbf{Aut}(G),$$

the action groupoid for the action of G on $\mathbf{Aut}(G)$ via the homomorphism $\mathrm{Ad}\colon G \to \mathbf{Aut}(G)$.

Corollary 3.6.267. *Let $\mathcal{X}$ be a 1-localic ∞-topos (i.e. one that has a 1-site of definition). Then for $G \in \mathrm{Grp}(\mathcal{X})$ any ∞-group object, G-∞-gerbes are classified by $\mathbf{Aut}(\mathbf{B}G)$-cohomology:*

$$\pi_0 G\mathrm{Gerbe} \simeq \pi_0 \mathcal{X}(X, \mathbf{BAut}(\mathbf{B}G)) =: H^1_{\mathcal{X}}(X, \mathbf{Aut}(\mathbf{B}G)) .$$

Proof. This is the special case of Theorem 3.6.214 for $V = \mathbf{B}G$. □
For the case that G is 0-truncated (an ordinary group object) this is the content of Theorem 23 in [JaLu04].

Example 3.6.268. For $G \in \mathrm{Grp}(\mathcal{X}) \subset \tau_{\leq 0}\mathrm{Grp}(\mathcal{X})$ an ordinary 1-group object, this reproduces the classical result of [Gir71], which originally motivated the whole subject: by Example 3.6.266 in this case $\mathbf{Aut}(\mathbf{B}G)$ is the traditional automorphism 2-group and $H^1_{\mathcal{X}}(X, \mathbf{Aut}(\mathbf{B}G))$ is Giraud's nonabelian G-cohomology that classifies G-gerbes (for arbitrary *band*, see Definition 3.6.274 below).

For $G \in \tau_{\leq 1}\mathrm{Grp}(\mathcal{X}) \subset \mathrm{Grp}(\mathcal{X})$ a 2-group, we recover the classification of 2-gerbes as in [Br94, Br06].

Remark 3.6.269. In Section 7.2.2 of [LuHTT] the special case that here we called *EM-n-gerbes* is considered. Beware that there are further differences: for instance the notion of morphisms between n-gerbes as defined in [LuHTT] is more restrictive than the notion considered here. For instance with our definition (and hence also that in [Br94]) each group automorphism of an abelian group object A induces an automorphism of the trivial A-2-gerbe $\mathbf{B}^2 A$. But, except for the identity, this is not admitted in [LuHTT] (manifestly so by the diagram above Lemma 7.2.2.24 there). Accordingly, the classification result in [LuHTT] is different: it involves the cohomology group $H^{n+1}_{\mathcal{X}}(X, A)$. Notice that there is a canonical morphism

$$H^{n+1}_{\mathcal{X}}(X, A) \to H^1_{\mathcal{X}}(X, \mathbf{Aut}(\mathbf{B}^n A))$$

induced from the morphism $\mathbf{B}^{n+1} A \to \mathbf{Aut}(\mathbf{B}^n A)$.

We now discuss how the ∞-group extensions, Definition 3.6.245, given by the Postnikov stages of $\mathbf{Aut}(\mathbf{B}G)$ induces the notion of *band* of a gerbe, and how the corresponding twisted cohomology, according to Remark 3.6.251, reproduces the original definition of nonabelian cohomology in [Gir71] and generalizes it to higher degree.

Definition 3.6.270. Fix $k \in \mathbb{N}$. For $G \in \infty\mathrm{Grp}(\mathcal{X})$ a k-truncated ∞-group object (a $(k+1)$-group), write

$$\mathbf{Out}(G) := \tau_k \mathbf{Aut}(\mathbf{B}G)$$

for the k-truncation of $\mathbf{Aut}(\mathbf{B}G)$. (Notice that this is still an ∞-group, since by Lemma 6.5.1.2 in [LuHTT] τ_n preserves all ∞-colimits and additionally all products.) We call this the *outer automorphism n-group* of G.

In other words, we write

$$\mathbf{c} : \mathbf{BAut}(\mathbf{B}G) \to \mathbf{BOut}(G)$$

for the top Postnikov stage of $\mathbf{BAut}(\mathbf{B}G)$.

Example 3.6.271. Let $G \in \tau_0\mathrm{Grp}(\mathrm{Grpd}_\infty)$ be a 0-truncated group object, an ordinary group,. Then by Example 3.6.266, $\mathbf{Out}(G) = \mathbf{Out}(G)$ is the coimage of $\mathrm{Ad} : G \to \mathrm{Aut}(G)$, which is the traditional group of outer automorphisms of G.

Definition 3.6.272. Write $\mathbf{B}^2\mathbf{Z}(G)$ for the ∞-fiber of the morphism $\mathbf{c}$ from Definition 3.6.270, fitting into a fiber sequence

$$\begin{array}{ccc} \mathbf{B}^2\mathbf{Z}(G) & \longrightarrow & \mathbf{BAut}(\mathbf{B}G) \\ & & \downarrow{\scriptstyle \mathbf{c}} \\ & & \mathbf{BOut}(G) \end{array} .$$

We call $\mathbf{Z}(G)$ the *center* of the ∞-group G.

Example 3.6.273. For G an ordinary group, so that $\mathbf{Aut}(\mathbf{B}G)$ is the automorphism 2-group from Example 3.6.266, $\mathbf{Z}(G)$ is the center of G in the traditional sense.

By theorem 3.6.267 there is an induced morphism

$$\mathrm{Band} : \pi_0 G\mathrm{Gerbe} \to H^1(X, \mathbf{Out}(G))\,.$$

Definition 3.6.274. For $E \in G\mathrm{Gerbe}$ we call $\mathrm{Band}(E)$ the *band* of E.

By using Definition 3.6.272 in Definition 3.6.225, given a band $[\phi_X] \in H^1(X, \mathbf{Out}(G))$, we may regard it as a twist for twisted $\mathbf{Z}(G)$-cohomology, classifying G-gerbes with this band:

$$\pi_0 G\mathrm{Gerbe}^{[\phi_X]}(X) \simeq H^{2+[\phi_X]}(X, \mathbf{Z}(G))\,.$$

Remark 3.6.275. The original definition of *gerbe with band* in [Gir71] is slightly more general than that of *G-gerbe* (with band) in [Br94]: in the former the local sheaf of groups whose delooping is locally equivalent to the gerbe need not descend to the base. These more general Giraud gerbes are 1-gerbes in the sense of Definition 3.6.262, but only the slightly more restrictive G-gerbes of Breen have the good property of being connected fiber ∞-bundles. From our perspective this is the decisive property of gerbes, and the notion of band is relevant only in this case.

Example 3.6.276. For G a 0-group this reduces to the notion of band as introduced in [Gir71], for the case of G-gerbes as in [Br94].

3.6.16 Relative cohomology

We discuss the notion of *relative cohomology* internal to any ∞-topos $\mathbf{H}$.

Definition 3.6.277. Let $i : Y \to X$ and $f : B \to A$ be two morphisms in $\mathbf{H}$. We say that the ∞-groupoid of *relative cocycles* on i with coefficients in f is the hom ∞-groupoid $\mathbf{H}^I(i, f)$, where $\mathbf{H}^I := \mathrm{Funct}(\Delta[1], \mathbf{H})$. The corresponding set of equivalence classes / homotopy classes we call the *relative cohomology*

$$H_Y^B(X, A) := \pi_0 \mathbf{H}^I(i, f)\,.$$

When A is understood to be a pointed object, $B = *$ is the terminal object and $f : B \simeq * \to A$ is the point inclusion, we speak for short of the *cohomology of X with coefficients in A relative to Y* and write

$$H_Y(X, A) := H_Y^*(X, A)\,.$$

Proposition 3.6.278. *The ∞-groupoid of relative cocycles fits into an ∞-pullback diagram of the form*

$$\begin{array}{ccc} \mathbf{H}^I(i,f) & \longrightarrow & \mathbf{H}(X,A) \\ \downarrow & & \downarrow^{i^*} \\ \mathbf{H}(Y,B) & \xrightarrow{f_*} & \mathbf{H}(Y,A) \end{array}\,.$$

Proof. Let C be an ∞-site of definition of $\mathbf{H}$ and

$$\mathbf{H} \simeq ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ$$

be a presentatin by simplicial presheaves as in 2.2.3. Then $\mathbf{H}^I$ is presented by the, say, Reedy model structure on simplicial functors from $\Delta[1]$ to simplicial presheaves

$$\mathbf{H}^I \simeq ([\Delta[1],\ [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{Reedy}})^\circ\,.$$

We may find for $i : Y \to X$ in $\mathbf{H}$ a presentation by a cofibration between cofibrant objects in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, and similarly for $f : B \to A$ a presentation by a fibration between fibrant objects. Let these same symbols now denote these presentations. Then i is also cofibrant in the above presentation for $\mathbf{H}^I$ and similarly f is fibrant there.

This implies that the ∞-categorical hom space in question is given by the hom-simplicial set

$$\mathbf{H}^I(i, f) \simeq [\Delta[1], [C^{\mathrm{op}}, \mathrm{sSet}(i, f) .$$

This in turn is computed as the 1-categorical pullback of simplicial sets

$$\begin{array}{ccc} [\Delta[1], [C^{\mathrm{op}}, \mathrm{sSet}(i, f) & \longrightarrow & [C^{\mathrm{op}}, \mathrm{sSet}](X, A) \\ \downarrow & & \downarrow^{i^*} \\ [C^{\mathrm{op}}, \mathrm{sSet}](Y, A) & \xrightarrow{f_*} & [C^{\mathrm{op}}, \mathrm{sSet}](Y, A) \end{array} .$$

Since $[C^{\mathrm{op}}, \mathrm{sSet}]$ is a simplicial model category, and by assumption on our presentations for i and f, here the bottom and the right morphism are Kan fibrations. Therefore by prop. 2.3.8 this presents a homotopy pullback diagram, which proves the claim. □

Remark 3.6.279. This says in words that a cocycle relative to $Y \to X$ with coefficients in $B \to A$ is an A-cocycle on X whose pullback to Y is equipped with a coboundary to a B-cocycle. In particular, in the case that $B \simeq *$ it is an A-cocycle on X equipped with a trivialization of its pullback to Y.

In the case that B is not trivial, this definition of relative cohomology is a generalization of the twisted cohomology discussed in 3.6.12.

Observation 3.6.280. Let $\mathbf{c} : X \to A$ be a fixed A-cocycle on X. Then the fiber of the ∞-groupoid of (i, f)-relative cocycles over $\mathbf{c}$ is equivalently the ∞-groupoid of $[i^*\mathbf{c}]$-twisted cohomology on Y, according to def. 3.6.225.

Proof. By the pasting law, prop. 2.3.2 the relative cocycles over $\mathbf{c}$ sitting in the top ∞-pullback square of

$$\begin{array}{ccc} \mathbf{H}^I(i, f)|_{\mathbf{c}} & \longrightarrow & * \\ \downarrow & & \downarrow^{\mathbf{c}} \\ \mathbf{H}^I(i, f) & \longrightarrow & \mathbf{H}(X, A) \\ \downarrow & & \downarrow^{i^*} \\ \mathbf{H}(Y, B) & \xrightarrow{f_*} & \mathbf{H}(Y, A) \end{array}$$

also form the ∞-pullback of the total rectangle, which is the ∞-groupoid of $[i^*\mathbf{c}]$-twisted cocycles on Y. □

Remark 3.6.281. In the special case that the coefficients B and A have a presentation by sheaves of chain complexes in the image of the Dold-Kan correspondence, prop. 2.2.31, the morphism $i^* : \mathbf{H}(X, A) \to \mathbf{H}(Y, A)$ has a presentation by a morphism of cochain complexes and the above ∞-pullback may be computed in terms of the dual mapping cone on this morphism. Specicially in the case that $B \simeq *$ the homotopy pullback is presented by that dual mapping cone itself, and hence the relative cohomology is the cochain cohomology of the mapping cone on i^*. In this form relative cohomology is traditionally defined in the literature.

3.7 Structures in a local ∞-topos

We discuss structures present in a *local ∞-topos*, def. 3.2.1.

- 3.7.1 – Codiscrete objects;
- 3.7.2 – Concrete objects.

3.7.1 Codiscrete objects

Observation 3.7.1. The cartesian internal hom $[-,-] : \mathbf{H}^{\mathrm{op}} \times \mathbf{H} \to \mathbf{H}$ is related to the external hom $\mathbf{H}(-,-) : \mathbf{H}^{\mathrm{op}} \times \mathbf{H} \to \infty\mathrm{Grpd}$ by

$$\mathbf{H}(-,-) \simeq \Gamma[-,-]\,..$$

Proof. The ∞-Yoneda lemma implies, by the same argument as for 1-categorical sheaf toposes, that the internal hom is the ∞-stack given on any test object U by

$$[X,A](U) \simeq \mathbf{H}(U,[X,A]) \simeq \mathbf{H}(X \times U, A).$$

By prop. 2.2.5 the global section functor Γ is given by evaluation on the point, so that

$$\Gamma([X,A]) \simeq \mathbf{H}(*,[X,A]) \simeq \mathbf{H}(X \times *, A) \simeq \mathbf{H}(X,A)\,.$$

□

Proposition 3.7.2. *The codiscrete objects in a local ∞-topos, hence in a cohesive ∞-topos,* $\mathbf{H}$ *are stable under internal exponentiation: for all $X \in \mathbf{H}$ and $A \in \infty\mathrm{Grpd}$ we have*

$$[X, \mathrm{coDisc}A] \in \mathbf{H}$$

is codiscrete. Specifically, the internal hom into a codiscrete object is the codiscretificartion of the external hom

$$[X, \mathrm{coDisc}A] \simeq \mathrm{coDisc}\mathbf{H}(X, \mathrm{coDisc}A)\,.$$

Proof. The internal hom is the ∞-stack given by the assignment

$$[X, \mathrm{coDisc}A] : U \mapsto \mathbf{H}(X \times U, \mathrm{coDisc}A)\,.$$

By the $(\Gamma \dashv \mathrm{Disc})$-adjunction the right hand is

$$\simeq \infty\mathrm{Grpd}(\Gamma(X \times U), A)\,.$$

Since Γ is also a right adjoint it preserves the product, so that

$$\cdots \simeq \infty\mathrm{Grpd}(\Gamma(X) \times \Gamma(U), A)\,.$$

Using the cartesian closure of ∞Grpd this is

$$\cdots \simeq \infty\mathrm{Grpd}(\Gamma(U), [\Gamma(X), A])\,.$$

Using again the $(\Gamma \dashv \mathrm{coDisc})$-adjunction this is

$$\cdots \simeq \mathbf{H}(U, \mathrm{coDisc}[\Gamma(X), A]).$$

Since all of these equivalence are natural, with the ∞-Yoneda lemma it finally follows that

$$[X, \mathrm{coDisc}A] \simeq \mathrm{coDisc}\infty\mathrm{Grpd}(\Gamma(X), A) \simeq \mathrm{coDisc}\mathbf{H}(X, \mathrm{coDisc}A)\,.$$

□

3.7.2 Concrete objects

The cohesive structure on an object in a cohesive ∞-topos need not be supported by points. We discuss a general abstract characterization of objects that do have an interpretation as bare n-groupoids equipped with cohesive structure. Further refinements of these constructions are discussed further below in 3.9.6.4 for objects that serve as moduli of differetial cocycles.

The content of this section is partly taken from [CarSch].

3.7.2.1 General abstract

Proposition 3.7.3. *On a cohesive ∞-topos $\mathbf{H}$ both* Disc *and* coDisc *are full and faithful ∞-functors and* coDisc *exhibits* ∞Grpd *as a sub-∞-topos of $\mathbf{H}$ by an ∞-geometric embedding*

$$\infty\mathrm{Grpd} \underset{\mathrm{coDisc}}{\overset{\Gamma}{\hookleftarrow\!\!\longrightarrow}} \mathbf{H} \,.$$

Proof. The full and faithfulness of Disc was shown in prop. 3.3.4 and that for coDisc follows from the same kind of argument. Since Γ is also a right adjoint it preserves in particular finite ∞-limits, so that $(\Gamma \dashv \mathrm{coDisc})$ is indeed an ∞-geometric morphism. □

Recall that we write

$$\sharp := \mathrm{coDisc} \circ \Gamma$$

Corollary 3.7.4. *The ∞-topos* ∞Grpd *is equivalent to the full sub-∞-category of $\mathbf{H}$ on those objects $X \in \mathbf{H}$ for which the canonical morphism $X \to \sharp X$ is an equivalence.*

Proof. This follows by general facts about reflective sub-∞-categories ([LuHTT], section 5.5.4). □

Proposition 3.7.5. *Let $\mathbf{H}$ be the ∞-topos over an ∞-cohesive site C. For a 0-truncated object X in $\mathbf{H}$ the morphism*

$$X \to \sharp X$$

is a monomorphism precisely if X is a concrete sheaf *in the traditional sense of [Dub79].*

Proof. Monomorphisms of sheaves are detected objectwise. So by the Yoneda lemma and using the $(\Gamma \dashv \mathrm{coDisc})$-adjunction we have that $X \to \mathrm{coDisc}\,\Gamma\, X$ is a monomorphism precisely if for all $U \in C$ the morphism

$$X(U) \simeq \mathbf{H}(U, X) \to \mathbf{H}(U, \mathrm{coDisc}\,\Gamma\, X) \simeq \mathbf{H}(\Gamma(U), \Gamma(X))$$

is a monomorphism. This is the traditional definition. □

Definition 3.7.6. For $X \in \mathbf{H}$, write

$$X =: \sharp_\infty X \longrightarrow \cdots \longrightarrow \sharp_2 X \longrightarrow \sharp_1 X \longrightarrow \sharp_0 X := \sharp X$$

for the tower of n-image factorizations, def. 3.6.31, of $X \to \sharp X$, hence with

$$\sharp_n X := \mathrm{im}_n(X \to \sharp X)$$

for all $n \in \mathbb{N}$.

Definition 3.7.7. For $n \in \mathbb{N}$ and $X \in \mathbf{H}$ an n-truncated object, we say that $X \to \sharp_{n+1} X$ is its *n-concretification*. If this is an equivalence we say that X is *n-concrete*.

3.7.2.2 Presentations

We discuss presentations of n-concrete objects for low n.

Proposition 3.7.8. *Let C be an ∞-cohesive site, 3.4.2.1, and let $A \in \mathrm{Sh}_\infty(C)$ be a 1-truncated object that has a presentation by a groupoid-valued presheaf on C which is fibrant as a simplicial presheaf. Then it is 1-concrete if in degree 1 this is a concrete sheaf. Moreover, its 1-concretification, def. 3.7.7, has a presentation by a presheaf of groupoids which in degree 1 is a concrete sheaf.*

Proof. Any functor $f : X \to Y$ between groupoids has a factorization $X \to \mathrm{im}_1 f \to Y$, where the groupoid $\mathrm{im}_1 f$ has the same objects as X and has as morphisms equivalence classes $[\xi]$ of morphisms ξ in X under the relation $[\xi_1] = [\xi_2]$ precisely if $f(\xi_1) = f(\xi)_2$. The evident functor $\mathrm{im}_1 f \to Y$ is manifestly faithful and this factorization is natural. Therefore if now f is a morphism of presheaves of groupoids, it, too, has a factorization wich is objectwise of this form.

By the discussion in 3.4.2.1, over an ∞-cohesive site the units $\eta_X : X \to \sharp X$ of the $(\Gamma \dashv \mathrm{coDisc})$-$\infty$-adjunction are presented for fibrant simplicial presheaf representatives X by morphisms of simplicial presheaves that object- and degreewise send the value set of a presheaf to the set of concrete values. By the previous paragraph and prop. 3.6.51 it follows that the 1-image factorization $X \to \mathrm{im}_1 \eta_X \to \sharp X$ is in the second morphism objectwise a faithful functor. This means that the hom-presheaf $(\mathrm{im}_1 \eta_X)_1$ is a concrete sheaf on C. □

3.8 Structures in a locally ∞-connected ∞-topos

We discuss here homotopical, cohomological and geometrical structures that are canonically present in a locally ∞-connected ∞-topos $\mathbf{H}$, 3.3.1. The existence of the extra left adjoint Π for a locally ∞-connected ∞-topos encodes an intrinsic notion of *geometric paths* in the objects of $\mathbf{H}$.

If $\mathbf{H}$ is in addition *cohesive*, then these Π-geometric structures combine with the cohomological structures of a local ∞-topos, discussed in 3.7 to produce differential geometry and differential cohomological structures. This we discuss below in 3.9.

- 3.8.1 – Geometric homotopy / Étale homotopy
- 3.8.2 – Concordance
- 3.8.3 – Paths and geometric Postnikov towers
- 3.8.4 – Universal coverings and geometric Whitehead towers
- 3.8.5 – Flat connections and local systems
- 3.8.6 – Galois theory

3.8.1 Geometric homotopy / Étale homotopy

We discuss internal realizations of the notions of *geometric realization*, and *geometric homotopy* in any cohesive ∞-topos $\mathbf{H}$.

Definition 3.8.1. For $\mathbf{H}$ a locally ∞-connected ∞-topos and $X \in \mathbf{H}$ an object, we call $\Pi(X) \in \infty\mathrm{Grpd}$ the *fundamental ∞-groupoid* of X.

The ordinary homotopy groups of $\Pi(X)$ we call the *geometric homotopy groups* of X

$$\pi_\bullet^{\mathrm{geom}}(X \in \mathbf{H}) := \pi_\bullet(\Pi(X \in \infty\mathrm{Grpd}))\,.$$

Definition 3.8.2. For $|-| : \infty\mathrm{Grpd} \stackrel{\simeq}{\to} \mathrm{Top}$ the canonical equivalence of ∞-toposes, we write

$$|X| := |\Pi X| \in \mathrm{Top}$$

and call this the *geometric realization* of X.

Remark 3.8.3. In presentations of $\mathbf{H}$ by simplicial presheaves, as in prop. 3.4.9, aspects of this abstract notion are more or less implicit in the literature. See for instance around remark 2.22 of [SiTe]. The key insight is already in [ArMa69], if somewhat implicitly. This we discuss in detail in 4.3.4.

In some applications we need the following characterization of geometric homotopies in a cohesive ∞-topos.

Definition 3.8.4. We say a *geometric homotopy* between two morphisms $f, g : X \to Y$ in $\mathbf{H}$ is a diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle (\mathrm{Id},i)}\downarrow & \searrow^{f} & \\ X \times I & \xrightarrow{\eta} & Y \\ {\scriptstyle (\mathrm{Id},o)}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

such that I is geometrically connected, $\pi_0^{geom}(I) = *$.

Proposition 3.8.5. *If two morphism $f, g : X \to Y$ in a cohesive ∞-topos $\mathbf{H}$ are geometrically homotopic then their images $\Pi(f), \Pi(g)$ are equivalent in ∞Grpd.*

Proof. By the condition that Π preserves products in a strongly ∞-connected ∞-topos we have that the image of the geometric homotopy in ∞Grpd is a diagram of the form

$$\begin{array}{ccc} \Pi(X) & & \\ {\scriptstyle (\mathrm{Id},\Pi(i))}\downarrow & \searrow^{\Pi(f)} & \\ \Pi(X) \times \Pi(I) & \xrightarrow{\Pi(\eta)} & \Pi(Y) \\ {\scriptstyle (\mathrm{Id},\Pi(o))}\uparrow & \nearrow_{\Pi(g)} & \\ \Pi(X) & & \end{array}$$

Since $\Pi(I)$ is connected by assumption, there is a diagram

$$\begin{array}{ccc} & & * \\ & \nearrow & \downarrow{\scriptstyle \Pi(i)} \\ * & \longrightarrow & \Pi(I) \\ & \searrow & \uparrow{\scriptstyle \Pi(o)} \\ & & * \end{array}$$

in ∞Grpd (filled with homotopies, which we do not display, as usual, that connect the three points in $\Pi(I)$). Taking the product of this diagram with $\Pi(X)$ and pasting the result to the above image $\Pi(\eta)$ of the geometric homotopy constructs the equivalence $\Pi(f) \Rightarrow \Pi(g)$ in ∞Grpd. □

We consider a refinement of these kinds of considerations below in 3.9.1.

Proposition 3.8.6. *For $\mathbf{H}$ a locally ∞-connected ∞-topos, also all its objects $X \in \mathbf{H}$ are locally ∞-connected, in the sense that their over-∞-toposes $\mathbf{H}/X$ are locally ∞-connected $(\Pi_X \dashv \Delta_X \dashv \Gamma_X) : \mathbf{H}/X \to \infty\mathrm{Grpd}$.*

The two notions of fundamental ∞-groupoids of any object X induced this way do agree, in that there is a natural equivalence

$$\Pi_X(X \in \mathbf{H}/X) \simeq \Pi(X \in \mathbf{H}) .$$

Proof. By the general properties of over-∞-toposes ([LuHTT], prop 6.3.5.1) we have a a composite essential ∞-geometric morphism

$$(\Pi_X \dashv \Delta_X \dashv \Gamma_X) : \ \mathbf{H}/X \underset{X_*}{\overset{X_!}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{X^*}}}}} \mathbf{H} \underset{\Gamma}{\overset{\Pi}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\Delta}}}}} \infty\mathrm{Grpd}$$

and $X_!$ is given by sending $(Y \to X) \in \mathbf{H}/X$ to $Y \in \mathbf{H}$. □

3.8.2 Concordance

We formulate the notion of *concordance* (of bundles or cocycles) abstractly internal to a cohesive ∞-topos.

Definition 3.8.7. For $\mathbf{H}$ a cohesive ∞-topos and $X, A \in \mathbf{H}$ two objects, we say that the ∞-groupoid of *concordances* from X to A is

$$\mathrm{Concord}(X, A) := \Pi[X, A] \,,$$

where $[-,-] : \mathbf{H}^{\mathrm{op}} \times \mathbf{H} \to \mathbf{H}$ is the internal hom.

Observation 3.8.8. For $X, A, B \in \mathbf{H}$ three objects, there is a canonical composition ∞-functor of concordances between them

$$\mathrm{Concord}(X, A) \times \mathrm{Concord}(A, B) \to \mathrm{Concord}(X, B) \,.$$

Using that, by the axioms of cohesion, Π preserves products, this is the image under Π of the composition on internal homs

$$[X, A] \times [A, B] \to [X, B] \,.$$

3.8.3 Paths and geometric Postnikov towers

The fundamental ∞-groupoid ΠX of objects X in $\mathbf{H}$ may be reflected back into $\mathbf{H}$, where it gives a notion of *geometric homotopy path n-groupoids* and a geometric notion of Postnikov towers of objects in $\mathbf{H}$.

Recall from def. 3.4.3 the pair of adjoint endofunctors

$$(\mathbf{\Pi} \dashv \flat) : \mathbf{H} \to \mathbf{H}$$

on any locally connected ∞-topos $\mathbf{H}$.

We say for any $X, A \in \mathbf{H}$

- $\mathbf{\Pi}(X)$ is the *path ∞-groupoid* of X – the reflection of the fundamental ∞-groupoid from 3.8.1 back into the cohesive context of $\mathbf{H}$;
- $\flat A$ (“flat A”) is the coefficient object for *flat differential A-cohomology* or for *A-local systems* (discussed below in 3.8.5).

Write

$$(\tau_n \dashv i_n) : \mathbf{H}_{\leq n} \underset{i}{\overset{\tau_n}{\hookleftarrow\!\!\rightarrow}} \mathbf{H}$$

for the reflective sub-∞-category of n-truncated objects ([LuHTT], section 5.5.6) and

$$\tau_n : \mathbf{H} \xrightarrow{\tau_n} \mathbf{H}_{\leq n} \hookrightarrow \mathbf{H}$$

for the localization funtor. We say

$$\mathbf{\Pi}_n : \mathbf{H} \xrightarrow{\mathbf{\Pi}_n} \mathbf{H} \xrightarrow{\tau_n} \mathbf{H}$$

is the *homotopy path n-groupoid* functor. The (truncated) components of the $(\Pi \dashv \mathrm{Disc})$-unit

$$X \to \mathbf{\Pi}_n(X)$$

we call the *constant path inclusion*. Dually we have canonical morphisms

$$\flat A \to A$$

natural in $A \in \mathbf{H}$.

Definition 3.8.9. For $X \in \mathbf{H}$ we say that the *geometric Postnikov tower* of X is the categorical Postnikov tower ([LuHTT] def. 5.5.6.23) of $\mathbf{\Pi}(X) \in \mathbf{H}$:

$$\mathbf{\Pi}(X) \to \cdots \to \mathbf{\Pi}_2(X) \to \mathbf{\Pi}_1(X) \to \mathbf{\Pi}_0(X)\,.$$

The main purpose of geometric Postnikov towers for us is the notion of *geometric Whitehead towers* that they induce, discussed in the next section.

3.8.4 Universal coverings and geometric Whitehead towers

We discuss an intrinsic notion of Whitehead towers in a locally ∞-connected ∞-topos $\mathbf{H}$.

Definition 3.8.10. For $X \in \mathbf{H}$ a pointed object, the *geometric Whitehead tower* of X is the sequence of objects

$$X^{(\infty)} \to \cdots \to X^{(\mathbf{2})} \to X^{(\mathbf{1})} \to X^{(\mathbf{0})} \simeq X$$

in $\mathbf{H}$, where for each $n \in \mathbb{N}$ the object $X^{(n+1)}$ is the homotopy fiber of the canonical morphism $X \to \mathbf{\Pi}_{n+1}X$ to the path $(n+1)$-groupoid of X (3.8.3). We call $X^{(\mathbf{n+1})}$ the $(n+1)$-fold *universal covering space* of X. We write $X^{(\infty)}$ for the homotopy fiber of the untruncated constant path inclusion.

$$X^{(\infty)} \to X \to \mathbf{\Pi}(X)\,.$$

Here the morphisms $X^{(\mathbf{n})} \to X^{\mathbf{n-1}}$ are those induced from this pasting diagram of ∞-pullbacks

$$\begin{array}{ccccc} X^{(\mathbf{n})} & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \\ X^{(\mathbf{n-1})} & \longrightarrow & \mathbf{B}^n\pi_n(X) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}_n(X) & \xrightarrow{\tau_{\leq(n-1)}} & \mathbf{\Pi}_{(n-1)}(X) \end{array} \quad ,$$

where the object $\mathbf{B}^n\pi_n(X)$ is defined as the homotopy fiber of the bottom right morphism.

Proposition 3.8.11. *Every object X in a cohesive ∞-topos $\mathbf{H}$ is covered by objects of the form $X^{(\infty)}$ for different choices of base points in X, in the sense that every X is the ∞-colimit over a diagram whose vertices are of this form.*

Proof. Consider the diagram

$$\begin{array}{ccc} \varinjlim_{s\in\Pi(X)}(i^* *_s) & \longrightarrow & \varinjlim_{s\in\Pi(X)} *_s \\ \downarrow\simeq & & \downarrow\simeq \\ X & \xrightarrow{i} & \mathbf{\Pi}(X) \end{array} \; .$$

The bottom morphism is the constant path inclusion, the $(\Pi \dashv \mathrm{Disc})$-unit. The right morphism is the equivalence that is the image under Disc of the decomposition $\lim_{\longrightarrow S} * \stackrel{\simeq}{\to} S$ of every ∞-groupoid as the ∞-colimit over itself of the ∞-functor constant on the point. The left morphism is the ∞-pullback along i of this equivalence, hence itself an equivalence. By universality of ∞-colimits in the ∞-topos $\mathbf{H}$, the top left object is the ∞-colimit over the single homotopy fibers $i^* *_s$ of the form $X^{(\infty)}$ as indicated. □

We would like claim that moreover each of the patches $i^* *$ of the object X in a cohesive ∞-topos is geometrically contractible, thus exhibiting a generic cover of any object by contractibles. The following states something slightly weaker.

Proposition 3.8.12. *The inclusion $\Pi(i^* *) \to \Pi(X)$ of the fundamental ∞-groupoid $\Pi(i^* *)$ of each of these patches into $\Pi(X)$ is homotopic to the point.*

Proof. We apply $\Pi(-)$ to the above diagram over a single vertex s and attach the $(\Pi \dashv \mathrm{Disc})$-counit to get

$$
\begin{array}{ccccc}
\Pi(i^* *) & \longrightarrow & & \longrightarrow & * \\
\downarrow & & & & \downarrow \\
\Pi(X) & \xrightarrow{\Pi(i)} & \Pi\,\mathrm{Disc}\,\Pi(X) & \longrightarrow & \Pi(X)
\end{array} .
$$

Then the bottom morphism is an equivalence by the $(\Pi \dashv \mathrm{Disc})$-zig-zag-identity. □

This implies that in a cohesive ∞-topos every principal

3.8.5 Flat connections and local systems

We describe for a locally ∞-connected ∞-topos $\mathbf{H}$ a canonical intrinsic notion of *flat connections on ∞-bundles*, *flat higher parallel transport* and *∞-local systems*.

Let $\mathbf{\Pi} : \mathbf{H} \to \mathbf{H}$ be the path ∞-groupoid functor from def. 3.4.3, discussed in 3.8.3.

Definition 3.8.13. For $X, A \in \mathbf{H}$ we write

$$\mathbf{H}_{\mathrm{flat}}(X, A) := \mathbf{H}(\mathbf{\Pi} X, A)$$

and call $H_{\mathrm{flat}}(X, A) := \pi_0 \mathbf{H}_{\mathrm{flat}}(X, A)$ the *flat (nonabelian) differential cohomology* of X with coefficients in A. We say a morphism $\nabla : \mathbf{\Pi}(X) \to A$ is a *flat ∞-connnection* on the principal ∞-bundle corresponding to $X \to \mathbf{\Pi}(X) \stackrel{\nabla}{\to} A$, or an *$A$-local system* on X.

The induced morphism

$$\mathbf{H}_{\mathrm{flat}}(X, A) \to \mathbf{H}(X, A)$$

we say is the forgetful functor that *forgets flat connections*.

The object $\mathbf{\Pi}(X)$ has the interpretation of the path ∞-groupoid of X: it is a cohesive ∞-groupoid whose k-morphisms may be thought of as generated from the k-morphisms in X and k-dimensional cohesive paths in X. Accordingly a mophism $\mathbf{\Pi}(X) \to A$ may be thought of as assigning

- to each point of X a fiber in A;
- to each path in X an equivalence between these fibers;
- to each disk in X a 2-equivalalence between these equivaleces associated to its boundary
- and so on.

This we think of as encoding a flat *higher parallel transport* on X, coming from some flat ∞-connection and *defining* this flat ∞-connection.

Observation 3.8.14. By the $(\mathbf{\Pi} \dashv \flat)$-adjunction we have a natural equivalence

$$\mathbf{H}_{\text{flat}}(X, A) \simeq \mathbf{H}(X, \flat A) .$$

A cocycle $g : X \to A$ for a principal ∞-bundle on X is in the image of

$$\mathbf{H}_{\text{flat}}(X, A) \to \mathbf{H}(X, A)$$

precisely if there is a lift ∇ in the diagram

$$\begin{array}{ccc} & & \flat A \\ & \overset{\nabla}{\nearrow} & \downarrow \\ X & \xrightarrow{g} & A \end{array} .$$

We call $\flat A$ the *coefficient object for flat A-connections.*

Proposition 3.8.15. *For* $G := \text{Disc}(G_0) \in \mathbf{H}$ *discrete* ∞*-group (3.6.8) the canonical morphism* $\mathbf{H}_{\text{flat}}(X, \mathbf{B}G) \to \mathbf{H}(X, \mathbf{B}G)$ *is an equivalence.*

Proof. This follows by definition 3.4.3 $\flat = \text{Disc}\,\Gamma$ and using that Disc is full and faithful. □

This says that for discrete structure ∞-groups G there is an essentially unique flat ∞-connection on any G-principal ∞-bundle. Moreover, the further equivalence

$$\mathbf{H}(\mathbf{\Pi}(X), \mathbf{B}G) \simeq \mathbf{H}_{\text{flat}}(X, \mathbf{B}G) \simeq \mathbf{H}(X, \mathbf{B}G)$$

may be read as saying that the G-principal ∞-bundle for discrete G is entirely characterized by the flat higher parallel transport of this unique ∞-connection.

Below in 3.8.6 we discuss in more detail the total spaces classified by ∞-local systems.

3.8.6 Galois theory

We discuss a canonical internal realization of *locally constant* ∞*-stacks* and their classification by *Galois theory* inside any cohesive ∞-topos.

Classical Galois theory is the classification of certain extensions of a field K. Viewing the formal dual $\text{Spec}(K)$ as a space, this generalizes to *Galois theory of schemes*, which classifies κ-compact étale morphisms $E \to X$ over a connected scheme X by functors

$$\Pi_1(X) \simeq \mathbf{B}\pi_1(X) \to \text{Set}_\kappa$$

from the classifying groupoid of the fundamental group of X (defined thereby) to the category of κ-small sets. See for instance [Len85] for an account.

From the point of view of topos theory over the étale site, κ-compact étale morphisms are equivalently sheaves (namely the sheaves of local sections of the étale morphism) that are locally constant on κ-small sets. The notion of locally constant sheaves of course exists over any site and in any topos whatsoever, and hence *topos theoretic Galois theory* more generally classifies locally constant sheaves. A general abstract category theoretic discussion of such generalized Galois theory is given by Janelidze, whose construction in the form of [CJKP97] we generalize below to locally connected ∞-toposes.

A generalization of Galois theory from topos theory to ∞-topos theory as a classification of *locally constant* ∞*-stacks* was envisioned by Grothendieck and, for the special case over topological spaces, first formalized in [Toën00], where it is shown that the homotopy type of a connected locally contractible topological space X is the automorphism ∞-group of the fiber functor on locally constant ∞-stacks over X. Similar disucssion appeared later in [PoWa05] and [Shu07].

We show below that this central statement of *higher Galois theory* holds generally in every ∞-connected ∞-topos.

For κ an uncountable regular cardinal, write

$$\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa \in \infty\mathrm{Grpd}$$

for the ∞-groupoid of κ-small ∞-groupoids, def. 4.1.19.

Definition 3.8.16. For $X \in \mathbf{H}$ write

$$\mathrm{LConst}(X) := \mathbf{H}(X, \mathrm{Disc}(\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa))$$

for the cocycle ∞-groupoid on X with coefficients in the discretely cohesive ∞-groupoid on the ∞-groupoid of κ-small ∞-groupoids. We call this the ∞-groupoid of *locally constant ∞-stacks* on X.

Observation 3.8.17. Since Disc is left adjoint and right adjoint, it commutes with coproducts and with delooping, def. 3.6.116, so that by remark 4.1.20 we have

$$\mathrm{Disc}(\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa) \simeq \coprod_i \mathbf{B}\,\mathrm{Disc}(\mathrm{Aut}(F_i))\,.$$

Therefore, by the discussion in 3.6.10, a locally constant ∞-stack $P \in \mathrm{LConst}(X)$ may be identified on each geometric connected component of X with the total space of a $\mathrm{Disc}\,\mathrm{Aut}(F_i)$-principal ∞-bundle $P \to X$.

Moreover, by the discussion in 3.6.13, to each such $\mathrm{Aut}(F_i)$-principal ∞-bundle is canonically associated a $\mathrm{Disc}(F_i)$-fiber ∞-bundle $E \to X$. This is the ∞-pullback

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Disc}(F_i)//\mathrm{Disc}(\mathrm{Aut}(F_i)) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{B}\mathrm{Disc}(\mathrm{Aut}(F_i)) \end{array} .$$

Since by corollary 4.1.25 every discrete ∞-bundle with κ-small fibers over connected X arises this way, essentially uniquely, we may canonically identify the morphism $E \to X$ with an object $E \in \mathbf{H}_{/X}$ in the little topos over X, which interprets as the ∞-topos of ∞-stacks over X, as discussed at the beginning of 3.6.15. This way the objects of $\mathrm{LConst}(X)$ are indeed identified with ∞-stacks over X.

The following proposition says that the central statement of Galois theory holds for the notion of locally constant ∞-stacks in a cohesive ∞-topos.

Proposition 3.8.18. *For $\mathbf{H}$ locally and globally ∞-connected, we have*

1. *a natural equivalence*
$$\mathrm{LConst}(X) \simeq \infty\mathrm{Grpd}(\Pi(X), \infty\mathrm{Grpd}_\kappa)$$
*of locally constant ∞-stacks on X with ∞-permutation representations of the fundamental ∞-groupoid of X (*local systems *on X);*

2. *for every point $x : * \to X$ a natural equivalence of the endomorphisms of the fiber functor*
$$x^* : \mathrm{LConst}(X) \to \infty\mathrm{Grpd}_\kappa$$
and the loop space of $\Pi(X)$ at x
$$\mathrm{End}(x^*) \simeq \Omega_x \Pi(X)\,.$$

Proof. The first statement is essentially the $(\Pi \dashv \mathrm{Disc})$-adjunction :

$$\begin{aligned} \mathrm{LConst}(X) &:= \mathbf{H}(X, \mathrm{Disc}(\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa)) \\ &\simeq \infty\mathrm{Grpd}(\Pi(X), \mathrm{Core}\,\infty\mathrm{Grpd}_\kappa)\,. \\ &\simeq \infty\mathrm{Grpd}(\Pi(X), \infty\mathrm{Grpd}_\kappa) \end{aligned}$$

Using this and that Π preserves the terminal object, so that the adjunct of $(* \to X \to \mathrm{Disc}\,\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa)$ is $(* \to \Pi(X) \to \infty\mathrm{Grpd}_\kappa)$, the second statement follows with an iterated application of the ∞-Yoneda lemma:

The fiber functor $x^* : \mathrm{Func}_\infty(\Pi(X), \infty\mathrm{Grpd}) \to \infty\mathrm{Grpd}$ evaluates an ∞-presheaf on $\Pi(X)^{\mathrm{op}}$ at $x \in \Pi(X)$. By the ∞-Yoneda lemma this is the same as homming out of $j(x)$, where $j : \Pi(X)^{\mathrm{op}} \to \mathrm{Func}(\Pi(X), \infty\mathrm{Grpd})$ is the ∞-Yoneda embedding:

$$x^* \simeq \mathrm{Hom}_{\mathrm{PSh}(\Pi(X)^{op})}(j(x), -)\,.$$

This means that x^* itself is a representable object in $\mathrm{PSh}_\infty(\mathrm{PSh}_\infty(\Pi(X)^{\mathrm{op}})^{\mathrm{op}})$. If we denote by $\tilde{j}$: $\mathrm{PSh}_\infty(\Pi(X)^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{PSh}_\infty(\mathrm{PSh}_\infty(\Pi(X)^{\mathrm{op}})^{\mathrm{op}})$ the corresponding Yoneda embedding, then

$$x^* \simeq \tilde{j}(j(x))\,.$$

With this, we compute the endomorphisms of x^* by applying the ∞-Yoneda lemma two more times:

$$\begin{aligned} \mathrm{End}(x^*) &\simeq \mathrm{End}_{\mathrm{PSh}(\mathrm{PSh}(\Pi(X)^{\mathrm{op}})^{\mathrm{op}})}(\tilde{j}(j(x))) \\ &\simeq \mathrm{End}(\mathrm{PSh}(\Pi(X))^{\mathrm{op}})(j(x)) \\ &\simeq \mathrm{End}_{\Pi(X)^{\mathrm{op}}}(x,x) \\ &\simeq \mathrm{Aut}_x \Pi(X) \\ &=: \Omega_x \Pi(X) \end{aligned} \quad .$$

□

Next we discuss how this intrinsic Galois theory in a cohesive ∞-topos is in line with the *categorical Galois theory* of Janelidze, as treated in [CJKP97]. This revolves around factorization systems associated with the path functor $\mathbf{\Pi}$ from 3.8.3.

Theorem 3.8.19. *If* $\mathbf{H}$ *has an* ∞*-cohesive site of definition, def. 3.4.8, the functor* $\Pi : \mathbf{H} \to \infty\mathrm{Grpd}$ *preserves* ∞*-pullbacks over discrete objects.*

This was pointed out by Mike Shulman.
Proof. By prop. 5.2.5.1 in [LuHTT] the $(\Pi \dashv \mathrm{Disc})$-adjunction passes for each $A \in \infty\mathrm{Grpd}$ to the slice as

$$(\Pi_{/\mathrm{Disc}A} \dashv \mathrm{Disc}_{/\mathrm{Disc}A}) : \mathbf{H}_{/\mathrm{Disc}A} \to \infty\mathrm{Grpd}_{/A}\,.$$

Under the parameterized ∞-Grothendieck construction, prop. 3.4.11, we have that $\Pi_{/\mathrm{Disc}A}$ becomes

$$\Pi^A : \mathbf{H}^A \to \infty\mathrm{Grpd}^A\,.$$

Since ∞-limits of functor ∞-categories are computed objectwise, and since Π preserves finite products by the axioms of cohesion, Π^A preserves finite products and hence so does $\Pi_{/\mathrm{Disc}A}$. Since a binary product in $\mathbf{H}_{/\mathrm{Disc}A}$ is an ∞-pullback over $\mathrm{Disc}A$ in $\mathbf{H}$, this completes the proof. □

Remark 3.8.20. We find below that over some ∞-cohesive sites of interest Π preserves further ∞-pullbacks. See prop. 4.3.47.

Definition 3.8.21. For $f : X \to Y$ a morphism in $\mathbf{H}$, write

$$c_{\mathbf{\Pi}} f := Y \times_{\mathbf{\Pi} Y} \mathbf{\Pi}(X) \to Y$$

for the ∞-pullback in

$$\begin{array}{ccc} c_{\mathbf{\Pi}} f & \longrightarrow & \mathbf{\Pi} X \\ \downarrow & & \downarrow^{\mathbf{\Pi} f} \\ Y & \longrightarrow & \mathbf{\Pi} Y \end{array} \;,$$

where the bottom morphism is the $(\Pi \dashv \mathrm{Disc})$-unit. We say that $c_{\mathbf{\Pi}} f$ is the $\mathbf{\Pi}$-*closure* of f, and that f is $\mathbf{\Pi}$-*closed* if $X \simeq c_{\mathbf{\Pi}} f$.

Remark 3.8.22. In the discussion of *differential* cohesion below in 3.5 we see that the *infinitesimal* analog of $\mathbf{\Pi}$-closesness is *formal étaleness*, see def. 3.10.19 below. There is a close conceptual relation: as we now discuss (prop. 3.8.30 below) morphisms $X \stackrel{f}{\to} Y$ that are $\mathbf{\Pi}$-closed may be identified with the total space projections of *locally constant ∞-stacks over Y*. Accordingly in a context of differential cohesion, $\mathbf{\Pi}_{\mathrm{inf}}$-closed such morphisms may be interpreted as projections out of total spaces of general ∞-stacks over Y.

Definition 3.8.23. Call a morphism $f : X \to Y$ in $\mathbf{H}$ a Π-*equivalence* if $\Pi(f)$ is an equivalence in $\infty\mathrm{Grpd}$.

Remark 3.8.24. Since $\mathrm{Disc} : \infty\mathrm{Grpd} \to \mathbf{H}$ is full and faithful, we may equivalently speak of $\mathbf{\Pi}$-*equivalences*.

Proposition 3.8.25. *If $\mathbf{H}$ has an ∞-connected site of definition, then every morphism $f : X \to Y$ in $\mathbf{H}$ factors as*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle f'}\searrow \quad \nearrow & \\ & c_{\mathbf{\Pi}} f & \end{array} ,$$

where f' is a $\mathbf{\Pi}$-equivalence.

Proof. The naturality of the adjunction unit together with the universality of the ∞-pullback that defines $c_{\mathbf{\Pi}} f$ gives the factorization

$$\begin{array}{ccccc} X & \stackrel{f'}{\dashrightarrow} & Y \times_{\mathbf{\Pi} Y} \mathbf{\Pi} X & \longrightarrow & \mathbf{\Pi} X \\ & {\scriptstyle f}\searrow & \downarrow & & \downarrow {\scriptstyle \mathbf{\Pi} f} \\ & & Y & \longrightarrow & \mathbf{\Pi} Y \end{array} .$$

By theorem 3.8.19 the functor Π preserves the above ∞-pullback. Since $\Pi(X \to \mathbf{\Pi} X)$ is an equivalence, it follows that $\mathbf{\Pi} X$ is also a pullback of the $\mathbf{\Pi}$-image of the diagram, and hence $\mathbf{\Pi}(f')$ is an equivalence. $\square$

Proposition 3.8.26. *For $\mathbf{H}$ with an ∞-cohesive site of definition, the pair of classes of morphisms*

$$(\mathbf{\Pi}\text{-}\textit{equivalences},\ \mathbf{\Pi}\text{-}\textit{closed morphisms}) \subset \mathrm{Mor}(\mathbf{H}) \times \mathrm{Mor}(\mathbf{H})$$

constitutes an orthogonal factorization system (5.2.8 in [LuHTT]).

Proof. The factorization is given by prop. 3.8.25. It remains to check orthogonality.

Let therefore

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

be any commuting diagram in $\mathbf{H}$, where the left morphism is a $\mathbf{\Pi}$-equivalence and the right morphism is $\mathbf{\Pi}$-closed. Then, by assumption, there exists a pullback diagram on the right in

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & \mathbf{\Pi} X \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & Y & \longrightarrow & \mathbf{\Pi} Y \end{array} .$$

By the naturality of the $(\Pi \dashv \mathrm{Disc})$-unit, the outer rectangle above is equivalent to the outer rectangle of

$$\begin{array}{ccccc} A & \longrightarrow & \mathbf{\Pi} A & \longrightarrow & \mathbf{\Pi} X \\ \downarrow & & \downarrow \simeq & & \downarrow \\ B & \longrightarrow & \mathbf{\Pi} B & \longrightarrow & \mathbf{\Pi} Y \end{array} ,$$

where now, again by assumption, the middle vertical morphism is an equivalence. Therefore there exists an essentially unique lift in the right square of this diagram. This induces a lift in the outer rectangle. By the universality of the adjunction unit, such lifts factor essentially uniquely through $\mathbf{\Pi}B$ and hence this lift, too, is essentially unique. Finally by the universal property of the pullback $X \simeq c_{\mathbf{\Pi}} f$, this gives the required essentially unique lift on the left of

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & \mathbf{\Pi}X \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ B & \longrightarrow & Y & \longrightarrow & \mathbf{\Pi}Y \end{array} .$$

□

We now identifiy the $\mathbf{\Pi}$-closed morphisms with covering spaces, hence with total spaces of locally constant ∞-stacks.

Observation 3.8.27. For $f : X \to Y$ a $\mathbf{\Pi}$-closed morphism, its fibers X_y over global points $y : * \to Y$ are discrete objects.

Proof. By assumption and using the pasting law, prop. 2.3.2, it follows that the fibers of f are the fibers of $\mathbf{\Pi}f$. Since the terminal object is discrete and since Disc preserves ∞-pullbacks, these are the images under Disc of fibers of Πf, and hence are discrete. □

Conversely we have:

Example 3.8.28. Let $X \in \mathbf{H}$ be any object, and let $A \in \infty\mathrm{Grpd}$ be any discrete ∞-groupoid. Then the projection morphism $p : X \times \mathrm{Disc}(A) \to X$ out of the product is $\mathbf{\Pi}$-closed.

Proof. Since $\mathbf{\Pi}$ preserves products, by the axioms of cohesion, and Disc preserves products as a right adjoint and is moreover full and faithful, we have that $\mathbf{\Pi}(p)$ is the projection

$$\mathbf{\Pi}(p) : \mathbf{\Pi}(X) \times \mathrm{Disc}(A) \to \mathbf{\Pi}(X) .$$

Since ∞-limits commute with ∞-limits, it follows that

$$\begin{array}{ccc} X \times \mathrm{Disc}(A) & \longrightarrow & \mathbf{\Pi}(X) \times \mathrm{Disc}(A) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}(X) \end{array}$$

is an ∞-pullback. □

Remark 3.8.29. Morphisms of the form $X \times \mathrm{Disc}(A) \to X$ fit into pasting diagrams of ∞-pullbacks of the form

$$\begin{array}{ccccc} X \times \mathrm{Disc}(A) & \longrightarrow & \mathrm{Disc}(A) & \longrightarrow & \mathrm{Disc}(A//\mathrm{Aut}(A)) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & * & \longrightarrow & \mathbf{B}\mathrm{Disc}(\mathrm{Aut}(A)) \end{array} ,$$

where the square on the right is the universal discrete A-bundle, by the discussion in 3.6.13. According to def. 3.8.16 the composite morphism on the bottom classifies the *trivial* locally constant ∞-stack with fiber A over X, hence the *constant* ∞-stack with fiber A over X. Therefore the above ∞-pullback exhibits $X \times \mathrm{Disc}(A) \to X$ as the total space incarnation of that constant ∞-stack on X.

The following proposition generalizes this statement to all locally constant ∞-stacks over X.

Proposition 3.8.30. *Let $\mathbf{H}$ have an ∞-cohesive site of definition, 3.4.2.1. Then for any $X \in \mathbf{H}$ the locally constant ∞-stacks $E \in \mathrm{LConst}(X)$, regarded as ∞-bundle morphisms $p : E \to X$ by observation 3.8.17, are precisely the $\mathbf{\Pi}$-closed morphisms into X.*

Proof. We may without restriction of generality assume that X has a single geometric connected component. Then $E \to X$ is given by an ∞-pullback of the form

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Disc}(F_i//\mathrm{Aut}(F_i)) \\ \downarrow^{p} & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}\mathrm{Disc}\mathrm{Aut}(F_i) \end{array} .$$

By theorem 3.8.19 the functor Π preserves this ∞-pullback, so that also

$$\begin{array}{ccc} \mathbf{\Pi}E & \longrightarrow & \mathrm{Disc}(F_i//\mathrm{Aut}(F_i)) \\ \downarrow & & \downarrow \\ \mathbf{\Pi}X & \xrightarrow{\mathbf{\Pi}g} & \mathbf{B}\mathrm{Disc}\,\mathrm{Aut}(F_i) \end{array}$$

is an ∞-pullback, where we used that, by the axioms of cohesion, $\mathbf{\Pi}$ sends discrete objects to themselves.

By def. 3.8.21 the factorization in question is given by forming the ∞-pullback on the left of

$$\begin{array}{ccccc} X \times_{\mathbf{\Pi}X} \mathbf{\Pi}E & \longrightarrow & \mathbf{\Pi}E & \longrightarrow & \mathrm{Disc}(F_i//\mathrm{Aut}(F_i)) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}X & \xrightarrow{\mathbf{\Pi}g} & \mathbf{B}\mathrm{Disc}\mathrm{Aut}(F_i) \end{array} .$$

By the universal property of the $(\Pi \dashv \mathrm{Disc})$-reflection, the bottom composite is again equivalent to g, hence by the pasting law, prop. 2.3.2, it follows that the pullback on the left is equivalent to $E \to X$.

Conversely, if the ∞-pullback diagram on the left is given, it follows with prop. 4.1.23 and using, by definition of cohesion, that Disc is full and faithful, that an ∞-pullback square as on the right exists. Again by the pasting law, this implies that the morphism on the left is the total space projection of a locally constant ∞-stack over X. □

Remark 3.8.31. In the "1-categorical Galois theory" of [CJKP97] only the trivial discrete ∞-bundles arise as pullbacks this way, and much of the theory deals with getting around this restriction. In our language, this is because in the context of 1-categorical cohesion, as in [Lawv07], the ∞-functor $\mathbf{\Pi}$ reduces to the 1-functor $\mathbf{\Pi}_0 \simeq \tau_0 \circ \mathbf{\Pi}$, discussed in 3.8.3, on a locally connected and connected 1-topos, which assigns only the set of connected components, instead of the full path ∞-groupoid.

Clearly, the pullback over an object of the form $\mathbf{\Pi}_0 K$ is indeed a locally constant ∞-stack that is trivial as a discretely fibered ∞-bundle. But this restriction is lifted by passing from cohesive 1-toposes to cohesive ∞-toposes.

We now characterize locally constant ∞-stacks over X as precisely the "relatively discrete" objects over X. To that end, recall, by prop. 3.8.6, that for $\mathbf{H}$ a locally ∞-connected ∞-topos also all the slice ∞-toposes $\mathcal{X} := \mathbf{H}_{/X}$ for all objects $X \in \mathbf{H}$ are locally ∞-connected.

Definition 3.8.32. For $X \in \mathbf{H}$ an object in a cohesive ∞-topos $\mathbf{H}$ and

$$\mathbf{H}_{/X} \underset{p_*}{\overset{\overset{p_!}{\longrightarrow}}{\underset{\longleftarrow p^* \longrightarrow}{}}} \infty\mathrm{Grpd}$$

the corresponding locally ∞-connected terminal geometric morphism, write

$$\mathbf{H}_{/X} \underset{\longleftarrow p^*/X \longrightarrow}{\overset{p_!/X}{\longrightarrow}} \infty\mathrm{Grpd}_{/\Pi(X)}$$

for the induced ∞-adjunction on the slices, by prop. 5.2.5.1 in [LuHTT], where the left adjoint $p_!/X$ sends $(E \to X)$ to $(\Pi(E) \to \Pi(X))$.

Proposition 3.8.33. *Let the cohesive ∞-topos H have an ∞-cohesive site of definition, def. 3.4.8 and let $X \in \mathbf{H}$ be any object.*

The full sub-∞-category of $\mathbf{H}_{/X}$ on the $\mathbf{\Pi}$-closed morphisms into X, def. 3.8.21, hence on the locally constant ∞-stacks over X, prop. 3.8.30, is equivalent to the image of the moprhism $p^*/X : \infty\mathrm{Grpd}_{/\Pi(X)} \to \mathbf{H}_{/X}$.

Proof. By prop 5.2.5.1 in [LuHTT], the ∞-functor p^*/X is the composite

$$p^*/X : \ \infty\mathrm{Grpd}_{/\Pi(X)} \xrightarrow{\mathrm{Disc}} \mathbf{H}_{/\mathbf{\Pi}} \xrightarrow{X \times_{\mathbf{\Pi}(X)} (-)} \mathbf{H}_{/X} \ .$$

This sends a morphism $Q \to \Pi(X)$ to the pullback on the left of the pullback square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Disc}(Q) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}(X) \end{array} .$$

Since Π preserves this ∞-pullback, by theorem 3.8.19, and sends $X \to \mathbf{\Pi}(X)$ to an equivalence, it follows that $\Pi(E \to X)$ is equivalent to $Q \to \Pi(X)$ and hence the above pullback diagram looks like

$$\begin{array}{ccc} E & \longrightarrow & \mathbf{\Pi}(E) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}(X) \end{array} .$$

The naturality of the $(\Pi \dashv \mathrm{Disc})$-unit and the universality of the pullback imply that the top horizontal morphism here is indeed the E-component of the $(\Pi \dashv \mathrm{Disc})$ unit.

This shows that, up to equivalence, precisely the $\mathbf{\Pi}$-closed morphism $E \to X$ arise this way. □

Remark 3.8.34. A definition of locally constant objects in general ∞-toposes is given in section A.1 of [Lurie11]. The above prop. 3.8.33 together with theorem A.1.15 in [LuHTT] shows that restricted to the slices $\mathbf{H}_{/X}$ it coincides with the definition discussed here.

3.9 Structures in a cohesive ∞-topos

We discuss differential geometric and differential cohomological structures that exist in any *cohesive ∞-topos*, def. 3.4.1. These are obtained from the Π-geometric structures of a locally ∞-connected ∞-topos, discussed in 3.8 by interpreting them in the *gros* cohomological context of a local ∞-topos, dscussed in 3.7.

- 3.9.1 – $\mathbb{A}^1$-Homotopy / The Continuum
- 3.9.2 – Manifolds
- 3.9.3 – de Rham cohomology
- 3.9.4 – Exponentiated Lie algebras
- 3.9.5 – Maurer-Cartan forms and curvature characteristic forms
- 3.9.6 – Differential cohomology
- 3.9.7 – Chern-Weil homomorphism

- 3.9.8 – Twisted differential structures
- 3.9.9 – Higher holonomy
- 3.9.10 – Transgression
- 3.9.11 – Chern-Simons functionals
- 3.9.12 – Wess-Zumino-Witten functionals
- 3.9.13 – Geometric prequantization

3.9.1 $\mathbb{A}^1$-Homotopy / The Continuum

We formalize in a cohesive ∞-topos $\mathbf{H}$ the notion of *the continuum* in the sense in which the standard real line $\mathbb{R}$ is traditionally called *the continuum*. Abstractly this is an object $\mathbb{A}^1 \in \mathbf{H}$ which, when regarded as a *line object*, *induces* the geometric homotopy in $\mathbf{H}$ as discussed in 3.8.1. Explicitly this means that $\mathbf{\Pi} : \mathbf{H} \xrightarrow{\Pi} \infty\mathrm{Grpd} \hookrightarrow \mathbf{H}$ exhibits the *localization* of $\mathbf{H}$ which inverts all those morphisms that are products of an object with the terminal morphism $\mathbb{A}^1 \to *$. Since by cohesion $\mathbf{\Pi}(*) \simeq *$, this means in particular that such an $\mathbb{A}^1$ is a geometrically contractible object in that $\mathbf{\Pi}(\mathbb{A}^1) \simeq *$. Together this are the characterizing property of the archetypical "continuum" $\mathbb{R}$. Below in 3.9.2 we discuss how a continuum line object induces a notion of *manifold* objects in $\mathbf{H}$.

Remark 3.9.1. The ∞-topos $\mathbf{H}$, being in particular a presentable ∞-category, admits a choice of a small set $\{c_i \in \mathbf{H}\}_i$ of generating objects, and every small set of morphisms in $\mathbf{H}$ induces a full reflective sub-∞-category of objects that are *local* with respect to these morphisms.

This is [LuHTT], section 5.

Definition 3.9.2. For $\mathbf{H}$ a cohesive ∞-topos, we say an object $I \in \mathbf{H}$ is an *continuum line object exhibiting the cohesion* of $\mathbf{H}$ if the reflective inclusion of the discrete objects

$$(\Pi \dashv \mathrm{Disc}) : \infty\mathrm{Grpd} \underset{\mathrm{Disc}}{\overset{\Pi}{\hookleftarrow\!\!\to}} \mathbf{H}$$

is induced by the localization at the set of morphisms

$$S := \{c_i \times (I \to *)\}_i, \, ,$$

for $\{c_i\}_i$ some small set of generators of $\mathbf{H}$.

Remark 3.9.3. In this situation, for $X \in \mathbf{H}$ we may think of $\Pi(X)$ also as the I*-localization* of X.

A class of examples of this situation is the following.

Proposition 3.9.4. *Let C be an ∞-cohesive site, def. 3.4.8, which moreover is the syntactic category of a Lawvere algebraic theory (see chapter 3, volume 2 of [Borc94]), in that it has finite products and there is an object*

$$\mathbb{A}^1 \in C$$

such that every other object is isomorphic to an n-fold cartesian product $\mathbb{A}^n = (\mathbb{A}^1)^n$.

Then $\mathbb{A}^1 \in C \hookrightarrow \mathrm{Sh}_\infty(C)$ is a geometric interval exhibiting the cohesion of the ∞-topos over C.

Proof. A set of generating objects of $\mathbf{H} = \mathrm{Sh}_\infty(C)$ is given by the set of isomorphism classes of objects of C, hence, by assumption, by $\{\mathbb{A}^n\}_{n\in\mathbb{N}}$. The set of localizing morphisms is therefore

$$S := \{\mathbb{A}^{n+1} \to \mathbb{A}^n \mid n \in \mathbb{N}\}\,.$$

By prop. 3.4.9, $\mathbf{H}$ is presented by the model category $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. By the proof of [LuHTT] cor. A.3.7.10 the localization of $\mathbf{H}$ as S is presented by the left Bousfield localization of this model category at S, given by a Quillen adjunction to be denoted

$$(L_{\mathbb{A}^1} \dashv R_{\mathbb{A}^1}) : [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\mathbb{A}1} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}} .$$

Observe that we also have a Quillen adjunction

$$(\mathrm{const} \dashv (-)_*) : [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\mathbb{A}^1} \underset{(-)_*}{\overset{\mathrm{const}}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}} ,$$

where the right adjoint evaluates at the terminal object $\mathbb{A}^0$, and where the left adjoint produces constant simplicial presheaves. This is because the two functors are clearly a Quillen adjunction before localization (on $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$) and so by [LuHTT] cor. A.3.7.2 it is sufficient to observe that on the local structure the right adjoint still preserves fibrant objects, which it does because the fibrant objects in the localization are in particular fibrant in the unlocalized structure.

Moreover, we claim that $(\mathrm{const} \dashv (-)_*)$ is in fact a Quillen equivalence, by observing that the derived adjunction unit and counit are equivalences. For the derived adjunction unit, notice that by the proof of prop. 3.4.9 a constant simplicial presheaf is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, and so it is clearly fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\mathbb{A}^1}$. Therefore the plain adjunction unit, which is the identity, is already the derived adjunction unit. For the derived counit, let $X \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc},\mathbb{A}1}$ be fibrant. Then also the adjunction counit

$$\eta : \mathrm{const}(X(\mathbb{A}^0)) \to X$$

is already the derived counit (since $X(\mathbb{A}^1) \in \mathrm{sSet}_{\mathrm{Quillen}}$ is necessarily cofibrant). At every $\mathbb{A}^n \in C$ it is isomorphic to the sequence of morphisms

$$\eta(\mathbb{A}^n); X(\mathbb{A}^0) \to X(\mathbb{A}^1) \to \cdots \to X(\mathbb{A}^n) ,$$

each of which is a weak equivalence by the $\mathbb{A}^1$-locality of X.

Now observe that we have an equivalence of ∞-functors

$$\mathrm{Disc} \simeq \mathbb{R}R_{\mathbb{A}^1} \circ \mathbb{L}\mathrm{const} : \infty\mathrm{Grpd} \to \mathbf{H} .$$

Because for $A \in \mathrm{sSet}$ fibrant, $\mathbb{L}\mathrm{const}(A) \simeq A$ is still fibrant, by the proof of prop. 3.4.9, and so $(\mathbb{R}R_{\mathbb{A}^1})((\mathbb{L}\mathrm{const})(A)) \simeq \mathrm{const}A$ is presented simply by the constant simplicial presheaf on A, which indeed is a presentation for $\mathrm{Disc}A$, again by the proof of prop. 3.4.9.

Finally, since by the above $\mathbb{L}\mathrm{const}$ is in fact an equivalence, by essential uniqueness of ∞-adjoints it follows now that $\mathbb{L}L_{\mathbb{A}^1}$ is left adjoint to the ∞-functor Disc, and this proves the claim. □

Remark 3.9.5. Below in 4.3.5 we show that in the models of Euclidean-topological cohesion and of smooth cohesion the standard real line is indeed the continuum line object in the above abstract sense.

3.9.2 Manifolds (unseparated)

We discuss a general abstract realization of the notion of *unseparated manifolds* internal to a cohesive ∞-topos. In order to formalize separated manifolds (Hausdorff manifolds) we need the extra axioms of differential cohesion. This is discussed below in 3.10.6.

Remark 3.9.6. The theory of principal ∞-bundles in 3.6.10 extensively used two of the three Giraud-Rezk-Lurie axioms characterizing ∞-toposes, def. 2.2.2 (universal coproducts and effective groupoid objects). Here we now use the third one, that *coproducts are disjoint*.

Proposition 3.9.7. *If $A \in \mathbf{H}$ is 0-truncated, def. 3.6.22 is geometrically connected in that $\Pi(A) \in \infty\mathrm{Grpd}$ is connected, then morphisms $A \to X \coprod Y$ into a coproduct of 0-truncated objects in $\mathbf{H}$ factor through one of the two inclusions $X \hookrightarrow X \coprod Y$ or $Y \hookrightarrow X \coprod Y$.*

Proof. The 1-topos $\tau_{\leq 0}\mathbf{H}$ of 0-truncated objects of a locally ∞-connected ∞-topos is a locally connected 1-topos by prop. 3.3.3. Under this identification, $A \in \tau_0\mathbf{H}$ as above is a connected object, and hence is in particular not a coproduct of two non-initial objects. Since moreover coproducts in $\mathbf{H}$ and in $\tau_{\leq 0}\mathbf{H}$ are disjoint and since truncation (being a left adjoint) preserves them, the statement reduces to a standard fact in topos theory (for instance [John03], p. 34). □

Let now $\mathbb{A}^1 \in \mathbf{H}$ be a continuum line object that *exhibits the cohesion of* $\mathbf{H}$ in the sense of def. 3.9.2. For $n \in \mathbb{N}$, write

$$\mathbb{A}^n := \underbrace{\mathbb{A}^1 \times \cdots \times \mathbb{A}^1}_{n \text{ factors}} .$$

Proposition 3.9.8. *For all $n \in \mathbb{N}$ the objects $\mathbb{A}^n \in \mathbf{H}$ are geometrically connected.*

Proof. By cohesion, $\Pi : \mathbf{H} \to \infty\mathrm{Grpd}$ preserves finite products and so the statement reduces to the fact that the product of two connected ∞-groupoids is itself a connected ∞-groupoid. □

Definition 3.9.9. Given an object $\mathbb{A}^1 \in \mathbf{H}$ exhibiting the cohesion of the cohesive topos $\mathbf{H}$, an object $X \in \mathbf{H}$ is an *unseparated* $\mathbb{A}$*-manifold* of *dimension* $n \in \mathbb{N}$ if there exists a small set of monomorphisms of the form

$$\{\mathbb{A}^n \overset{\phi_j}{\hookrightarrow} X\}_j$$

such that for the corresponding

$$\phi : \ \coprod_j \mathbb{A}^n \xrightarrow{(\phi_j)_j} X$$

we have

1. ϕ is an effective epimorphism, def. 2.3.3;
2. the nerve simplicial object $C_\bullet(\phi)$ of ϕ is degreewise a coproduct of copies of $\mathbb{A}^n$.

Remark 3.9.10. Since monomorphisms are stable under pullback and since by the Giraud-Rezk-Lurie axioms coproducts are preserved under pullback, it follows that the simplicial object in def. 3.9.9 is such that all components $\mathbb{A}^n \to \mathbb{A}^n$ of all face maps (given by prop. 3.9.7 and prop. 3.9.8) are monomorphisms.

Remark 3.9.11. Below in 4.3.6 and 4.4.11 is discussed that in the standard model of Euclidean-topological and of smooth cohesion this abstract definition reproduces the traditional definition of topological and of smooth manifolds, respectively.

3.9.3 de Rham cohomology

We discuss how in every locally ∞-connected ∞-topos $\mathbf{H}$ there is an intrinsic notion of *nonabelian de Rham cohomology*.

We have already seen the notions of *Principal bundles*, 3.6.10, and of flat ∞-connections on principal ∞-bundles, 3.8.5, in any locally ∞-connected ∞-topos. In traditional differential geometry, flat connection on the *trivial* principal bundle may be canonically identified with flat differential 1-forms on the base space. In the following we take this idea to be the *definition* of flat ∞-group/∞-Lie algebra valued forms: flat ∞-connections on trivial principal ∞-bundles.

Definition 3.9.12. Let $\mathbf{H}$ be a locally ∞-connected ∞-topos. For $X \in \mathbf{H}$ an object, write $\mathbf{\Pi}_{\mathrm{dR}}X := * \coprod_X \mathbf{\Pi}X$ for the ∞-pushout

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{\Pi}(X) & \longrightarrow & \mathbf{\Pi}_{\mathrm{dR}}X \end{array} .$$

We call this the *cohesive de Rham homotopy type* of X (see remark 3.9.19 below).

For $\mathrm{pt}_A : * \to A$ any pointed object in $\mathbf{H}$, write $\flat_{\mathrm{dR}}A := * \prod_A \flat A$ for the ∞-pullback

$$\begin{array}{ccc} \flat_{\mathrm{dR}}A & \longrightarrow & \flat A \\ \downarrow & & \downarrow \\ * & \longrightarrow & A \end{array} .$$

We call this the *de Rham coefficient object* of $\mathrm{pt}_A : * \to A$.

Proposition 3.9.13. *This construction yields a pair of adjoint ∞-functors*

$$(\mathbf{\Pi}_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}) : \; */\mathbf{H} \underset{\flat_{\mathrm{dR}}}{\overset{\mathbf{\Pi}_{\mathrm{dR}}}{\leftrightarrows}} \mathbf{H} \; .$$

Proof. We check the defining natural hom-equivalence

$$*/\mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) \simeq \mathbf{H}(X, \flat_{\mathrm{dR}}A) \, .$$

The hom-space in the under-∞-category $*/\mathbf{H}$ is computed by prop. 3.6.5 as the ∞-pullback

$$\begin{array}{ccc} */\mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) & \longrightarrow & \mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\mathrm{pt}_A} & \mathbf{H}(*, A) \end{array} .$$

By the fact that the hom-functor $\mathbf{H}(-,-) : \mathbf{H}^{\mathrm{op}} \times \mathbf{H} \to \infty\mathrm{Grpd}$ preserves ∞-limits in both arguments we have a natural equivalence

$$\begin{aligned} \mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) &:= \mathbf{H}(* \coprod_X \mathbf{\Pi}(X), A) \\ &\simeq \mathbf{H}(*, A) \prod_{\mathbf{H}(X,A)} \mathbf{H}(\mathbf{\Pi}(X), A) \end{aligned} .$$

We paste this pullback to the above pullback diagram to obtain

$$\begin{array}{ccccc} */\mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) & \longrightarrow & \mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) & \longrightarrow & \mathbf{H}(\mathbf{\Pi}(X), A) \\ \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{\mathrm{pt}_A} & \mathbf{H}(*, A) & \longrightarrow & \mathbf{H}(X, A) \end{array} .$$

By the pasting law for ∞-pullbacks, prop. 2.3.2, the outer diagram is still a pullback. We may evidently rewrite the bottom composite as in

$$\begin{array}{ccccc} */\mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}X, A) & & \longrightarrow & & \mathbf{H}(\mathbf{\Pi}(X), A) \\ \downarrow & & & & \downarrow \\ * & \xrightarrow{\simeq} & \mathbf{H}(X, *) & \xrightarrow{(\mathrm{pt}_A)_*} & \mathbf{H}(X, A) \end{array} .$$

This exhibits the hom-space as the pullback

$$*/\mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}}(X), A) \simeq \mathbf{H}(X, *) \prod_{\mathbf{H}(X,A)} \mathbf{H}(X, \flat A)\,,$$

where we used the $(\mathbf{\Pi} \dashv \flat)$-adjunction. Now using again that $\mathbf{H}(X, -)$ preserves pullbacks, this is

$$\cdots \simeq \mathbf{H}(X, * \prod_{A} \flat A) \simeq \mathbf{H}(X, \flat_{\mathrm{dR}} A)\,.$$

$\square$

Observation 3.9.14. If $\mathbf{H}$ is also local, then there is a further right adjoint $\mathbf{\Gamma}_{\mathrm{dR}}$

$$(\mathbf{\Pi}_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}} \dashv \mathbf{\Gamma}_{\mathrm{dR}}) : \mathbf{H} \underset{\mathbf{\Gamma}_{\mathrm{dR}}}{\overset{\overset{\mathbf{\Pi}_{\mathrm{dR}}}{\longrightarrow}}{\underset{\flat_{\mathrm{dR}}}{\longleftarrow}}} */\mathbf{H}$$

given by

$$\mathbf{\Gamma}_{\mathrm{dR}} X := * \coprod_{X} \mathbf{\Gamma}(X)\,.$$

Definition 3.9.15. For $X, A \in \mathbf{H}$ we write

$$\mathbf{H}_{\mathrm{dR}}(X, A) := \mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}} X, A) \simeq \mathbf{H}(X, \flat_{\mathrm{dR}} A)\,.$$

A cocycle $\omega : X \to \flat_{\mathrm{dR}} A$ we call a *flat A-valued differential form* on X.

We say that $H_{\mathrm{dR}}(X, A) := \pi_0 \mathbf{H}_{\mathrm{dR}}(X, A)$ is the *de Rham cohomology* of X with coefficients in A.

Observation 3.9.16. A cocycle in de Rham cohomology

$$\omega : \mathbf{\Pi}_{\mathrm{dR}} X \to A$$

is precisely a flat ∞-connection on a *trivializable* A-principal ∞-bundle. More precisely, $\mathbf{H}_{\mathrm{dR}}(X, A)$ is the homotopy fiber of the forgetful functor from ∞-bundles with flat ∞-connection to ∞-bundles: we have an ∞-pullback diagram

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{dR}}(X, A) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{H}_{\mathrm{flat}}(X, A) & \longrightarrow & \mathbf{H}(X, A) \end{array}\,.$$

Proof. This follows by the fact that the hom-functor $\mathbf{H}(X, -)$ preserves the defining ∞-pullback for $\flat_{\mathrm{dR}} A$. $\square$

Just for emphasis, notice the dual description of this situation: by the universal property of the ∞-colimit that defines $\mathbf{\Pi}_{\mathrm{dR}} X$ we have that ω corresponds to a diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{\Pi}(X) & \xrightarrow{\omega} & A \end{array}\,.$$

The bottom horizontal morphism is a flat connection on the ∞-bundle which in turn is given by the composite cocycle $X \to \mathbf{\Pi}(X) \stackrel{\omega}{\to} A$. The diagram says that this is equivalent to the trivial bundle given by the trivial cocycle $X \to * \to A$.

Proposition 3.9.17. *The de Rham cohomology with coefficients in discrete objects is trivial: for all* $S \in \infty\mathrm{Grpd}$ *we have*

$$\flat_{\mathrm{dR}}\mathrm{Disc}S \simeq * \,.$$

Proof. Using that in a ∞-connected ∞-topos the functor Disc is a full and faithful ∞-functor so that unit $\mathrm{Id} \to \Gamma\mathrm{Disc}$ is an equivalence and using that by the zig-zag identity the counit component $\flat\mathrm{Disc}S := \mathrm{Disc}\Gamma\mathrm{Disc}S \to \mathrm{Disc}S$ is also an equivalence, we have

$$\begin{aligned} \flat_{\mathrm{dR}}\mathrm{Disc}S &:= * \prod_{\mathrm{Disc}S} \flat\mathrm{Disc}S \\ &\simeq * \prod_{\mathrm{Disc}S} \mathrm{Disc}S \;\; , \\ &\simeq * \end{aligned}$$

since the pullback of an equivalence is an equivalence. □

Proposition 3.9.18. *For every* X *in a cohesive* ∞*-topos* $\mathbf{H}$*, the object* $\mathbf{\Pi}_{\mathrm{dR}}X$ *is globally connected in that* $\pi_0\mathbf{H}(*, \mathbf{\Pi}_{\mathrm{dR}}X) = *$.

If X *has at least one point* ($\pi_0(\Gamma X) \neq \emptyset$) *and is geometrically connected* ($\pi_0(\Pi X) = *$) *then* $\mathbf{\Pi}_{\mathrm{dR}}(X)$ *is also locally connected:* $\tau_0\mathbf{\Pi}_{\mathrm{dR}} \simeq * \in \mathbf{H}$.

Proof. Since Γ preserves ∞-colimits in a cohesive ∞-topos we have

$$\begin{aligned} \mathbf{H}(*, \mathbf{\Pi}_{\mathrm{dR}}X) &\simeq \Gamma\mathbf{\Pi}_{\mathrm{dR}}X \\ &\simeq * \coprod_{\Gamma X} \mathbf{\Gamma\Pi X} \;\; , \\ &\simeq * \coprod_{\Gamma X} \mathbf{\Pi X} \end{aligned}$$

where in the last step we used that Disc is full and faithful, so that there is an equivalence $\Gamma\mathbf{\Pi}X := \Gamma\mathrm{Disc}\Pi X \simeq \Pi X$.

To analyse this ∞-pushout we present it by a homotopy pushout in $\mathrm{sSet}_{\mathrm{Quillen}}$. Denoting by ΓX and ΠX any represetatives in $\mathrm{sSet}_{\mathrm{Quillen}}$ of the objects of the same name in $\infty\mathrm{Grpd}$, this may be computed by the ordinary pushout of simplicial sets

$$\begin{array}{ccc} \Gamma X & \longrightarrow & (\Gamma X) \times \Delta[1] \coprod_{\Gamma X} * \;\; , \\ \downarrow & & \downarrow \\ \Pi X & \longrightarrow & Q \end{array}$$

where on the right we have inserted the cone on ΓX in order to turn the top morphism into a cofibration. From this ordinary pushout it is clear that the connected components of Q are obtained from those of ΠX by identifying all those in the image of a connected component of ΓX. So if the left morphism is surjective on π_0 then $\pi_0(Q) = *$. This is precisely the condition that *pieces have points* in $\mathbf{H}$.

For the local analysis we consider the same setup objectwise in the injective model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}$. For any $U \in C$ we then have the pushout Q_U in

$$\begin{array}{ccc} X(U) & \longrightarrow & (X(U)) \times \Delta[1] \coprod_{X(U)} * \;\; , \\ \downarrow & & \downarrow \\ \mathrm{sSet}(\Gamma(U), \Pi X) & \longrightarrow & Q_U \end{array}$$

as a model for the value of the simplicial presheaf presenting $\mathbf{\Pi}_{\mathrm{dR}}(X)$. If X is geometrically connected then $\pi_0 \mathrm{sSet}(\Gamma(U), \Pi(X)) = *$ and hence for the left morphism to be surjective on π_0 it suffices that the top left object is not empty. Since the simplicial set $X(U)$ contains at least the vertices $U \to * \to X$ of which there is by assumption at least one, this is the case. □

Remark 3.9.19. In summary we see that in any cohesive ∞-topos the objects $\mathbf{\Pi}_{\mathrm{dR}}(X)$ of def. 3.9.12 have the essential abstract properties of pointed *geometric de Rham homotopy types* ([Toën06], section 3.5.1). In section 4 we will see that, indeed, the intrinsic de Rham cohomology of the cohesive ∞ -topos $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$

$$H_{\mathrm{dR}}(X, A) := \pi_0 \mathbf{H}(\mathbf{\Pi}_{\mathrm{dR}} X, A)$$

reproduces ordinary de Rham cohomology in degree $d > 1$.

In degree 0 the intrinsic de Rham cohomology is necessarily trivial, while in degree 1 we find that it reproduces closed 1-forms, not divided out by exact forms. This difference to ordinary de Rham cohomology in the lowest two degrees may be understood in terms of the obstruction-theoretic meaning of de Rham cohomology by which we essentially characterized it above: we have that the intrinsic $H^n_{\mathrm{dR}}(X, K)$ is the home for the obstructions to flatness of $\mathbf{B}^{n-2}K$-principal ∞-bundles. For $n = 1$ this are groupoid-principal bundles over the *groupoid* with K as its space of objects. But the 1-form curvatures of groupoid bundles are not to be regarded modulo exact forms.

We turn now to identifying certain de Rham cocycles that are adapted to intrinsic manifolds, as discussed in 3.9.2. In general a cocycle $\omega : X \to \flat_{\mathrm{dR}} \mathbf{B} A$ is to be thought of as what traditionally is called a cocycle in de Rham *hypercohomology*. The following definition models the idea of picking in de Rham hypercohomology over a manifold those cocycles that are given by globally defined differential forms.

Fix a line object $\mathbb{A}^1 \in \mathbf{H}$ which *exhibits the cohesion* of $\mathbf{H}$ in the sense of def. 3.9.2.

Definition 3.9.20. For $A \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, a choice of *A-valued differential forms* is a morphism

$$\Omega_{\mathrm{cl}}(-, A) \to \flat_{\mathrm{dR}} \mathbf{B} A$$

in $\mathbf{H}$, which is an *atlas over manifolds* of $\flat_{\mathrm{dR}} \mathbf{B} A$, in that:

1. $\Omega_{\mathrm{cl}}(-, A)$ is 0-truncated;
2. for each intrinsic $\mathbb{A}^1$-manifold Σ, def. 3.9.9, the morphism $[\Sigma, \Omega^n_{\mathrm{cl}}(-, A)] \to [\Sigma, \flat_{\mathrm{dR}} \mathbf{B}^n A]$ is an effective epimorphism, def. 2.3.3.

Remark 3.9.21. We discuss below in 4.4.50 how in the standard model of smooth cohesion this notion reproduces the traditional notion of smooth differential forms.

3.9.4 Exponentiated ∞-Lie algebras

We consider an intrinsic notion of *exponentiated* ∞-Lie algebras in every cohesive ∞-topos. In order to have a general abstract notion of the ∞-Lie algebras themselves we need the further axiomatics of *infinitesimal cohesion*, discussed below in 3.5 and 3.10.9.

Definition 3.9.22. For a connected object $\mathbf{B} \exp(\mathfrak{g})$ in $\mathbf{H}$ that is *geometrically contractible*

$$\Pi(\mathbf{B} \exp(\mathfrak{g})) \simeq *$$

we call its loop space object (see 3.6.8) $\exp(\mathfrak{g}) := \Omega_* \mathbf{B} \exp(\mathfrak{g})$ a *Lie integrated ∞-Lie algebra* in $\mathbf{H}$.

Definition 3.9.23. Set

$$\exp \mathrm{Lie} := \mathbf{\Pi}_{\mathrm{dR}} \circ \flat_{\mathrm{dR}} : */\mathbf{H} \to */\mathbf{H} \, .$$

Observation 3.9.24. If $\mathbf{H}$ is cohesive, then $\exp \mathrm{Lie}$ is a left adjoint.

Proof. By the construction in def. 3.4.3. □

Example 3.9.25. For all $X \in \mathbf{H}$ the object $\mathbf{\Pi}_{\mathrm{dR}}(X)$ is geometrically contractible.

Proof. Since on the locally ∞-connected and ∞-connected $\mathbf{H}$ the functor Π preserves ∞-colimits and the terminal object, we have

$$\begin{aligned} \Pi \mathbf{\Pi}_{\mathrm{dR}} X := & \Pi(*) \coprod_{\Pi X} \Pi \mathbf{\Pi} X \\ \simeq & * \coprod_{\Pi X} \Pi \mathrm{Disc} \Pi X \\ \simeq & * \coprod_{\Pi X} \Pi X \qquad \simeq * \end{aligned},$$

where we used that on the ∞-connected $\mathbf{H}$ the functor Disc is full and faithful. □

Corollary 3.9.26. *We have for every $(* \to A) \in */\mathbf{H}$ that $\exp \mathrm{Lie} A$ is geometrically contractible.*

We shall write $\mathbf{B}\exp(\mathfrak{g})$ for $\exp \mathrm{Lie}\mathbf{B}G$, when the context is clear.

Proposition 3.9.27. *Every de Rham cocycle (3.9.3) $\omega : \mathbf{\Pi}_{\mathrm{dR}} X \to \mathbf{B}G$ factors through the Lie integrated ∞-Lie algebra of G*

$$\begin{array}{ccc} & & \mathbf{B}\exp(\mathfrak{g}) \\ & \nearrow & \downarrow \\ \mathbf{\Pi}_{\mathrm{dR}} X & \xrightarrow{\omega} & \mathbf{B}G \end{array}.$$

Proof. By the universality of the $(\mathbf{\Pi}_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}})$-counit we have that ω factors through the counit $\epsilon : \exp \mathrm{Lie}\mathbf{B}G \to \mathbf{B}G$

$$\begin{array}{ccc} & \mathbf{\Pi}_{\mathrm{dR}} X & \\ {}^{\mathbf{\Pi}_{\mathrm{dR}}\tilde{\omega}}\swarrow & & \searrow^{\omega} \\ \mathbf{\Pi}_{\mathrm{dR}} \flat_{\mathrm{dR}} \mathbf{B}G & \xrightarrow{\epsilon} & \mathbf{B}G \end{array},$$

where $\tilde{\omega} : X \to \flat_{\mathrm{dR}} \mathbf{B}G$ is the adjunct of ω. □

Therefore instead of speaking of a G-valued de Rham cocycle, it is less redundant to speak of an $\exp(\mathfrak{g})$-valued de Rham cocycle. In particular we have the following.

Corollary 3.9.28. *Every morphism $\mathbf{B}\exp(\mathfrak{h}) := \exp \mathrm{Lie}\mathbf{B}H \to \mathbf{B}G$ from a Lie integrated ∞-Lie algebra to an ∞-group factors through the Lie integrated ∞-Lie algebra of that ∞-group*

$$\begin{array}{ccc} \mathbf{B}\exp(\mathfrak{h}) & \longrightarrow & \mathbf{B}\exp(\mathfrak{g}) \\ & \searrow & \downarrow \\ & & \mathbf{B}G \end{array}.$$

3.9.5 Maurer-Cartan forms and curvature characteristic forms

In the intrinsic de Rham cohomology of the cohesive ∞-topos $\mathbf{H}$ there exist canonical cocycles that we may identify with *Maurer-Cartan forms* and with universal *curvature characteristic forms*.

Definition 3.9.29. For $G \in \mathrm{Group}(\mathbf{H})$ an ∞-group in the cohesive ∞-topos $\mathbf{H}$, write

$$\theta : G \to \flat_{\mathrm{dR}}\mathbf{B}G$$

for the G-valued de Rham cocycle on G induced by this pasting of ∞-pullbacks

$$\begin{array}{ccc} G & \longrightarrow & * \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & \flat\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

using prop. 3.9.27.

We call θ the *Maurer-Cartan form* on G.

Remark 3.9.30. For any object X, postcomposition the Maurer-Cartan form sends G-valued functions on X to $\mathfrak{g}$-valued forms on X

$$[\theta_*] : H^0(X,G) \to H^1_{\mathrm{dR}}(X,G)\,.$$

Remark 3.9.31. For $G = \mathbf{B}^n A$ an Eilenberg-MacLane object, we also write

$$\mathrm{curv} : \mathbf{B}^n A \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}A$$

for its intrinsic Maurer-Cartan form and call this the intrinsic *universal curvature characteristic form* on $\mathbf{B}^n A$.

These curvature characteristic forms serve to define differential cohomology in the next section.

3.9.6 Differential cohomology

We discuss an intrinsic realization of *differential cohomology* with coefficients in braided ∞-groups in any cohesive ∞-topos.

We first give a general discussion in 3.9.6.1 and then consider a special class of cases in 3.9.6.2. Finally we discuss issues of constructing differential moduli objects in 3.9.6.4.

3.9.6.1 General

Definition 3.9.32. For $\mathbb{G}$ a braided ∞-group, def. 3.6.119, write

$$\mathrm{curv}_{\mathbb{G}} := \theta_{\mathbf{B}\mathbb{G}} : \mathbf{B}\mathbb{G} \to \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$$

for the Maurer-Cartan form, def. 3.9.29, on its delooping ∞-group $\mathbf{B}\mathbb{G}$. We call this the *universal curvature characteristic* of $\mathbb{G}$.

We say that the cohomology in the slice ∞-topos $\mathbf{H}_{/\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}}$ with coefficients in $\mathrm{curv}_{\mathbb{G}}$ is the *differential cohomology* with coefficients in $\mathbf{B}\mathbb{G}$.

Remark 3.9.33. A domain object $(X,F) \in \mathbf{H}_{/\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}}$ is an object $X \in \mathbf{H}$ equipped with a de Rham cocycle $F : X \to \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$, to be thought of as a prescribed *curvature differential form.*

A differential cocycle $\nabla \in \mathbf{H}_{/\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}}((X,F),\mathrm{curv}_G)$ on such a pair is a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & \mathbf{B}G \\ & {\scriptstyle \swarrow\!\!\swarrow \nabla} & \\ {\scriptstyle F}\searrow & & \swarrow{\scriptstyle \mathrm{curv}_{\mathbb{G}}} \\ & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} & \end{array}$$

in $\mathbf{H}$. This is

1. a cocycle $g : X \to \mathbf{B}\mathbb{G}$ in $\mathbf{H}$ for a $\mathbb{G}$-principal ∞-bundle over X;
2. a choice of equivalence

$$g^* \mathrm{curv}_{\mathbb{G}} \xrightarrow[\simeq]{\nabla} F$$

between the pullback of the universal $\mathbb{G}$-curvature characteristic, def. 3.9.32 and the prescribed curvature differential form.

This choice of equivalence is to be interpreted as a *connection* on the $\mathbb{G}$-principal bundle modulated by g.

Often one is of interested in demanding that the curvature $F : X \to \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$ in the above factors through a prescribed morphism $C \to \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$, notably through an inclusion of differential forms as in def. 3.9.20. This means that one is interested in cocycles as in remark 3.9.33 above which factor as diagrams

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & \mathbf{B}G \\ {}^{F}\downarrow & \swArrow_{\nabla} & \swarrow_{\mathrm{curv}_{\mathbb{G}}} \\ C & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \end{array} .$$

This in turn means equivalently that the cocycle is given by a morphism $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ into the ∞-pullback $\mathbf{B}\mathbb{G}_{\mathrm{conn}} \simeq C \times_{\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}} \mathbf{B}G$. This object we may then regard as a *moduli stack for differential cohomology* with coefficients in A and curvatures in C.

This we now discuss in 3.9.6.2 below.

3.9.6.2 Global curvature forms We consider the subcase of the general notion of differential cohomology as in 3.9.6.1 above, where now the curvatures are required to be globally defined differential forms according to def. 3.9.20. The resulting definition essentially reproduces that of differential cohomology in terms of homotopy pullbacks as discussed in [HoSi05], but is formulated entirely internal to a cohesive ∞-topos. Therefore it refines the construction of [HoSi05] in two ways[10]:

1. The coefficient object may be a cohesive ∞-groupoid, where in [HoSi05] it is just a topological space, hence, as explained below in 4.1, a *discrete* ∞-groupoid.
2. The domain object may also be a cohesive ∞-groupoid, where in [HoSi05] it is restricted to be a manifold. In particular it can be an orbifold, or itself a moduli stack.

We give below an intrinsic characterization of domain objects that are manifolds in the sense of def. 3.9.9. On more general objects our definition subsumes also a notion of *equivariant* differential cohomology.

Definition 3.9.34. For $\mathbb{G}$ a braided ∞-group in $\mathbf{H}$, def. 3.6.119, the *moduli of closed 2-forms* with values in $\mathbb{G}$ is a morphism

$$\Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$$

characterized as follows:

1. $\Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \in \mathbf{H}$ is 0-truncated;
2. for every $\mathbb{A}^1$-manifold $\Sigma \in \mathbf{H}$, def. 3.6.119, we have that

$$[\Sigma, \iota] : \; [\Sigma, \Omega^2_{\mathrm{cl}}(-,\mathbb{G})] \longrightarrow [\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}]$$

is an epimorphism

[10] After we had proposed this refinement, in [Ho11] it says that this is the context to which the article [HoSi05] was intended to be refined.

3. ι is universal with the above two properties.

A morphism $\omega X : \Omega^2_{\mathrm{cl}}(-,\mathbb{G})$ we call a *closed* Lie($\mathbb{G}$)*-valued differential 2-form* on X, or a *presymplectic structure* on X, with values in Lie($\mathbb{G}$).

Definition 3.9.35. For $\mathbb{G}$ a braided ∞-group, we write

$$\mathbf{B}\mathbb{G}_{\mathrm{conn}} := \mathbf{B}\mathbb{G} \underset{\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}}{\times} \Omega^2(-,\mathbb{G})$$

for the ∞-fiber product in

$$\begin{array}{ccc} \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \xrightarrow{F_{(-)}} & \Omega^2(-,\mathbb{G}) \\ \downarrow^{U} & & \downarrow \\ \mathbf{B}\mathbb{G} & \xrightarrow{\mathrm{curv}_{\mathbb{G}}} & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \end{array} .$$

We say that

1. $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ is the *moduli object* for *differential cocycles with coefficients in* $\mathbb{G}$ or equivalently for $\mathbb{G}$*-principal connections*;
2. For $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ we say that
   (a) $F_\nabla : X \to \Omega^2(-,\mathbb{G})$ is the *curvature form* of ∇
   (b) that $U(\nabla) : X \to \mathbf{B}\mathbb{G}$ is (the morphism modulation) the *underlying* $\mathbb{G}$-principal bundle of ∇.

Proposition 3.9.36. *For $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ a braided ∞-group, the loop space object, def. 3.6.114, of $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ is equivalent to the flat coefficient object $\flat G$*

$$\Omega\mathbf{B}\mathbb{G}_{\mathrm{conn}} \simeq \flat\mathbb{G} .$$

Proof. Using that $\Omega_{\mathrm{cl}}(-,\mathbb{G})$ is 0-truncated by definition, using that $\flat$ is right adjoint and hence commutes with ∞-pullbacks and repeatedly using the pasting law, prop. 2.3.2, we find a pasting diagram of ∞-pullbacks of the form

$$\begin{array}{ccccccc} \flat\mathbb{G} & \to & \mathbb{G} & \to & * & & \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \to & \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G} & \to & \flat\mathbf{B}\mathbb{G} & \to & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \to & * & \to & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \to & \Omega_{\mathrm{cl}}(-,\mathbb{G}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & * & \to & \mathbf{B}\mathbb{G} & \to & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \end{array} .$$

□

3.9.6.3 Ordinary differential cohomology

We now spell out the constructions of 3.9.6.2 in more detail for the special case that $\mathbb{G}$ is an Eilenberg-MacLane object, hence for the case there is a 0-truncated abelian group object $A \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H}) \hookrightarrow \mathbf{H}$ and $n \in \mathbb{N}$ such that

$$\mathbf{B}\mathbb{G} \simeq \mathbf{B}^n A\,.$$

This is the case of *ordinary differential cohomology* that refines what the *ordinary cohomology* with coefficients in A, according to remark 3.6.138. The explicit realization of this construction in smooth cohesion is discussed below in 4.4.16.

By the discussion in 3.6.8 we have for all $n \in \mathbf{N}$ the corresponding Eilenberg-MacLane object $\mathbf{B}^n A$. By the discussion in 3.6.10 this classifies $\mathbf{B}^{n-1}A$-principal ∞-bundles in that for any $X \in \mathbf{H}$ we have an equivalence of n-groupoids

$$\mathbf{B}^{n-1}A\mathrm{Bund}(X) \simeq \mathbf{H}(X, \mathbf{B}^n A)$$

whose objects are $\mathbf{B}^{n-1}A$-principal ∞-bundles on X, whose morphisms are gauge transformations between these, and so on. The following definition refines this by equipping these ∞-bundles with the structure of a *connection*.

Let $\mathbb{A}^1 \in \mathbf{H}$ be a line object *exhibiting the cohesion* of $\mathbf{H}$ in the sense of def. 3.9.2. Let then furthermore for each $n \in \mathbb{N}$

$$\Omega^n_{\mathrm{cl}}(-, A) \to \flat_{\mathrm{dR}}\mathbf{B}^n A$$

be a choice of differential form objects, according to def. 3.9.20.

Definition 3.9.37. For $X \in \mathbf{H}$ any object and $n \geq 1$ write

$$\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) := \mathbf{H}(X, \mathbf{B}^n A) \prod_{\mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^n A)} H^{n+1}_{\mathrm{dR}}(X, A)$$

for the cocycle ∞-groupoid of *twisted cohomology*, 3.6.12, of X with coefficients in A relative to the canonical curvature characteristic morphism $\mathrm{curv} : \mathbf{B}^n A \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}A$ (3.9.5). By prop. 3.6.221 this is the ∞-pullback

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) & \xrightarrow{[F]} & H^{n+1}_{\mathrm{dR}}(X, A) \\ \downarrow^{c} & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n A) & \xrightarrow{\mathrm{curv}_*} & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1}A) \end{array},$$

where the right vertical morphism $\pi_0\mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1}A) \to \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1}A)$ is the unique, up to equivalence, effective epimorphism out of a 0-truncated object: a choice of cocycle representative in each cohomology class, equivalently a choice of point in every connected component.

We call

$$H^n_{\mathrm{diff}}(X, A) {:=} \pi_0\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$$

the degree-n *differential cohomology* of X with coefficient in A.

For $\nabla \in \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$ a cocycle, we call

- $[c(\nabla)] \in H^n(X, A)$ the *characteristic class* of the *underlying* $\mathbf{B}^{n-1}A$*-principal* ∞*-bundle*;
- $[F](\nabla) \in H^{n+1}_{\mathrm{dR}}(X, A)$ the *curvature* class of c (this is the *twist*).

We also say that ∇ is an ∞*-connection* on the principal ∞-bundle $\eta(\nabla)$.

Observation 3.9.38. The differential cohomology $H^n_{\mathrm{diff}}(X, A)$ does not depend on the choice of morphism $H^{n+1}_{\mathrm{dR}}(X, A) \to \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1}A)$ (as long as it is an isomorphism on π_0, as required). In fact, for different choices the corresponding cocycle ∞-groupoids $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$ are equivalent.

Proof. The set

$$H^{n+1}_{\mathrm{dR}}(X,A) = \coprod_{H^{n+1}_{\mathrm{dR}}(X,A)} *$$

is, as a 0-truncated ∞-groupoid, an ∞-coproduct of the terminal object ∞Grpd. By universal colimits in this ∞-topos we have that ∞-colimits are preserved by ∞-pullbacks, so that $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$ is the coproduct

$$\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) \simeq \coprod_{H^{n+1}_{\mathrm{dR}}(X,A)} \left(\mathbf{H}(X, \mathbf{B}^n A) \underset{\mathbf{H}_{\mathrm{dR}} \times (X, \mathbf{B}^{n+1} A)}{*} \right)$$

of the homotopy fibers of curv_* over each of the chosen points $* \to \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1} A)$. These homotopy fibers only depend, up to equivalence, on the connected component over which they are taken. □

Proposition 3.9.39. *When restricted to vanishing curvature, differential cohomology coincides with flat differential cohomology, 3.8.5,*

$$H^n_{\mathrm{diff}}(X,A)|_{[F]=0} \simeq H_{\mathrm{flat}}(X, \mathbf{B}^n A)\,.$$

Moreover this is true at the level of cocycle ∞-groupoids

$$\left(\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) \underset{H^{n+1}_{\mathrm{dR}}(X,A)}{\times} \{[F]=0\} \right) \simeq \mathbf{H}_{\mathrm{flat}}(X, \mathbf{B}^n A)\,,$$

hence there is a canonical embedding by a full and faithful morphism

$$\mathbf{H}_{\mathrm{flat}}(X, \mathbf{B}^n A) \hookrightarrow \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$$

Proof. By the pasting law for ∞-pullbacks, prop. 2.3.2, the claim is equivalently that we have a pasting of ∞-pullback diagrams

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{flat}}(X, \mathbf{B}^n A) & \longrightarrow & * \\ \downarrow & & \downarrow^{[F]=0} \\ \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) & \xrightarrow{[F]} & H^{n+1}_{\mathrm{dR}}(X,A) \\ \downarrow^{\eta} & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n A) & \xrightarrow{\mathrm{curv}_*} & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1} A) \end{array}\,.$$

By definition of flat cohomology, def. 3.8.13 and of intrinsic de Rham cohomology, def. 3.9.15, in $\mathbf{H}$, the outer rectangle is

$$\begin{array}{ccc} \mathbf{H}(X, \flat \mathbf{B}^n A) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n A) & \xrightarrow{\mathrm{curv}_*} & \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A) \end{array}\,.$$

Since the hom-functor $\mathbf{H}(X, -)$ preserves ∞-limits this is a pullback if

$$\begin{array}{ccc} \flat \mathbf{B}^n A & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}^n A & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A \end{array}$$

is. Indeed, this is one step in the fiber sequence

$$\cdots \to \flat \mathbf{B}^n A \to \mathbf{B}^n A \overset{\mathrm{curv}}{\to} \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A \to \flat \mathbf{B}^{n+1} A \to \mathbf{B}^{n+1} A$$

that defines curv (using that $\flat$ preserves limits and hence looping and delooping).

Finally, $* \xrightarrow{[F]=0} H_{\mathrm{dR}}^{n-1}(X,A)$ is, trivially, a monomorphism of sets, hence a full and faithfull morphism of ∞-groupoids, and since these are stable under ∞-pullback, it follows that the canonical inclusion of flat ∞-connections into all ∞-connections is also full and faithful. □

The following establishes the characteristic short exact sequences that characterizes intrinsic differential cohomology as an extension of curvature forms by flat ∞-bundles and of bare ∞-bundles by connection forms.

Proposition 3.9.40. *Let* $\mathrm{im}F \subset H_{\mathrm{dR}}^{n+1}(X,A)$ *be the image of the curvatures. Then the differential cohomology group* $H_{\mathrm{diff}}^n(X,A)$ *fits into a short exact sequence*

$$0 \to H_{\mathrm{flat}}^n(X,A) \to H_{\mathrm{diff}}^n(X,A) \to \mathrm{im}F \to 0$$

Proof. Form the long exact sequence in homotopy groups of the fiber sequence

$$\mathbf{H}_{\mathrm{flat}}(X,\mathbf{B}^nA) \to \mathbf{H}_{\mathrm{diff}}(X,\mathbf{B}^nA) \stackrel{[F]}{\to} H_{\mathrm{dR}}^{n+1}(X,A)$$

of prop. 3.9.39 and use that $H_{\mathrm{dR}}^{n+1}(X,A)$ is, as a set – a homotopy 0-type – to get the short exact sequence on the bottom of this diagram

$$\begin{array}{ccccccc} \pi_1(H_{\mathrm{dR}}(X,A)) & \longrightarrow & \pi_0(\mathbf{H}_{\mathrm{flat}}(X,\mathbf{B}^nA)) & \longrightarrow & \pi_0(\mathbf{H}_{\mathrm{diff}}(X,\mathbf{B}^nA)) & \xrightarrow{[F]} & \pi_0(H_{\mathrm{dR}}^{n+1}(X,A)) \\ \| & & \| & & \| & & \downarrow \\ 0 & \longrightarrow & H_{\mathrm{flat}}^n(X,A) & \longrightarrow & H_{\mathrm{diff}}^n(X,A) & \longrightarrow & \mathrm{im}[F] \end{array} .$$

□

Proposition 3.9.41. *The differential cohomology group* $H_{\mathrm{diff}}^n(X,A)$ *fits into a short exact sequence of abelian groups*

$$0 \to H_{\mathrm{dR}}^n(X,A)/H^{n-1}(X,A) \to H_{\mathrm{diff}}^n(X,A) \stackrel{c}{\to} H^n(X,A) \to 0\,.$$

Proof. We claim that for all $n \geq 1$ we have a fiber sequence

$$\mathbf{H}(X,\mathbf{B}^{n-1}A) \to \mathbf{H}_{\mathrm{dR}}(X,\mathbf{B}^nA) \to \mathbf{H}_{\mathrm{diff}}(X,\mathbf{B}^nA) \to \mathbf{H}(X,\mathbf{B}^nA)$$

in ∞Grpd. This implies the short exact sequence using that by construction the last morphism is surjective on connected components (because in the defining ∞-pullback for $\mathbf{H}_{\mathrm{diff}}$ the right vertical morphism is by assumption surjective on connected components).

To see that we do have the fiber sequence as claimed, consider the pasting composite of ∞-pullbacks

$$\begin{array}{ccccc} \mathbf{H}_{\mathrm{dR}}(X,\mathbf{B}^{n-1}A) & \longrightarrow & \mathbf{H}_{\mathrm{diff}}(X,\mathbf{B}^nA) & \longrightarrow & H_{\mathrm{dR}}(X,\mathbf{B}^{n+1}A) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{H}(X,\mathbf{B}^nA) & \xrightarrow{\mathrm{curv}} & \mathbf{H}_{\mathrm{dR}}(X,\mathbf{B}^{n+1}A) \end{array} .$$

The square on the right is a pullback by def. 3.9.37. Since also the square on the left is assumed to be an ∞-pullback it follows by the pasting law for ∞-pullbacks, prop. 2.3.2, that the top left object is the ∞-pullback of the total rectangle diagram. That total diagram is

$$\begin{array}{ccc} \Omega\mathbf{H}(X,\flat_{\mathrm{dR}}\mathbf{B}^{n+1}A) & \longrightarrow & H(X,\flat_{\mathrm{dR}}\mathbf{B}^{n+1}A) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{H}(X,\flat_{\mathrm{dR}}\mathbf{B}^{n+1}A) \end{array} ,$$

because, as before, this ∞-pullback is the coproduct of the homotopy fibers, and they are empty over the connected components not in the image of the bottom morphism and are the loop space object over the single connected component that is in the image.

Finally using that

$$\Omega \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A) \simeq \mathbf{H}(X, \Omega \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A)$$

and

$$\Omega \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A \simeq \flat_{\mathrm{dR}} \Omega \mathbf{B}^{n+1} A$$

since both $\mathbf{H}(X, -)$ as well as $\flat_{\mathrm{dR}}$ preserve ∞-limits and hence formation of loop space objects, the claim follows. □

Often it is desireable to restrict attention to differential cohomology over domains on which the twisting cocycles can be chosen functorially. This we consider now.

Definition 3.9.42. For any $n \in \mathbb{N}$ write $\mathbf{B}^n A_{\mathrm{conn}}$ for the ∞-pullback

$$\begin{array}{ccc} \mathbf{B}^n A_{\mathrm{conn}} & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(-, A) \\ \downarrow & & \downarrow \\ \mathbf{B}^n A & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A \end{array}$$

in $\mathbf{H}$.

For X an A-dR-projective object we write

$$H^n_{\mathrm{conn}}(X, A) := \pi_0 \mathbf{H}(X, \mathbf{B}^n A_{\mathrm{conn}})$$

for the cohomology group on X with coefficients in $\mathbf{B}^n A_{\mathrm{conn}}$.

The objects $\mathbf{B}^n A_{\mathrm{conn}}$ represent differential cohomology in the following sense.

Observation 3.9.43. For every A-dR-projective object X there is a full and faithful morphism

$$\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) \hookrightarrow \mathbf{H}(X, \mathbf{B}^n A_{\mathrm{conn}}),$$

hence in particular an inclusion

$$H^n_{\mathrm{diff}}(X, A) \hookrightarrow H^n_{\mathrm{conn}}(X, A).$$

Proof. Since $\Omega^{n+1}_{\mathrm{cl}}(X, A) \to H^{n+1}_{\mathrm{dR}}(X, A)$ is a surjection by definition, there exists a factorization

$$H^{n+1}_{\mathrm{dR}}(X, A) \hookrightarrow \Omega^{n+1}_{\mathrm{cl}}(X, A) \to \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A)$$

of the canonical effective epimorphism (well defined up to homotopy), where the first morphism is an injection of sets, hence a monomorphism of ∞-groupoids. Since these are stable under ∞-pullback, it follows that also the top left morphism in the pasting diagram of ∞-pullbacks

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A) & \longrightarrow & H^{n+1}_{\mathrm{dR}}(X, A) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n A_{\mathrm{conn}}) & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(X, A) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n A) & \xrightarrow{\mathrm{curv}} & \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A) \end{array}$$

is a monomorphism.

Notice that here the bottom square is indeed an ∞-pullback, by def. 3.9.42 combined with the fact that the hom-functor $\mathbf{H}(X,-) : \mathbf{H} \to \infty\mathrm{Grpd}$ preserves ∞-pullbacks, and that with the top square defined to be an ∞-pullback the total outer rectangle is an ∞-pullback by prop. 2.3.2. This identifies the top left object as $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$ by def. 3.9.37. □

The reason that prop. 3.9.43 gives in inclusion is that $H^n_{\mathrm{conn}}(X, A)$ contains connections for all possible curvature forms, while $H^n_{\mathrm{diff}}(X, A)$ contains only connections for one curvature representative in each de Rham cohomology class. This is made precise by the following refinement of the exact sequences from prop. 3.9.40 and prop. 3.9.41.

Definition 3.9.44. Write

$$\Omega^n_{\mathrm{cl,int}}(-, A) \hookrightarrow \Omega^n_{\mathrm{cl}}(-, A)$$

for the image factorization of the canonical morphism $\mathbf{B}^n A_{\mathrm{conn}} \to \Omega^n_{\mathrm{cl}}(-, A)$ from def. 3.9.42.

Proposition 3.9.45. *For X an A-dR-projective object we have a short exact sequence of groups*

$$H^n_{\mathrm{flat}}(X, A) \longrightarrow H^n_{\mathrm{conn}}(X, A) \xrightarrow{\mathrm{curv}} \Omega^{n+1}_{\mathrm{cl,int}}(X, A) \; .$$

Proof. As in the proof of prop. 3.9.39 we have a pasting diagram of ∞-pullbacks

$$\begin{array}{ccccccc}
* & \longrightarrow & \mathbf{H}(X, \flat \mathbf{B}^n A) & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \downarrow^{0} & & \\
* & \longrightarrow & \mathbf{H}(X, \mathbf{B}^n A_{\mathrm{conn}}) & \longrightarrow & \Omega^{n+1}_{\mathrm{cl,int}}(X, A) & \hookrightarrow & \Omega^{n+1}_{\mathrm{cl}}(X, A) \\
 & & \downarrow & & & & \downarrow \\
 & & \mathbf{H}(X, \mathbf{B}^n A) & & \xrightarrow{\quad\mathrm{curv}\quad} & & \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1} A)
\end{array} .$$

After passing to connected components, this implies the claim. □

Details on how traditional ordinary differential cohomology is recovered by implementing the above in the context of smooth cohesion are discussed in 4.4.16.

3.9.6.4 Differential moduli We discuss issues related to the formulation of *moduli objects* in a cohesive ∞-topos for differential cocycles as discussed above, over a fixed base object.

To motivate this consider the following. Given a coefficient object $\mathbf{B}\mathbb{G}_{\mathrm{conn}} \in \mathbf{H}$ for differential cohomology as discussed above, and given any object $X \in \mathbf{H}$, the mapping space object $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \in \mathbf{H}$ is a kind of moduli object for $\mathbb{G}$-differential cocycles on X, in that its global points are precisely such cocycles. However, for any $U \in \mathbf{H}$ a U-plot of $[X, \mathbf{B}\mathbb{G}_{conn}]$ may be more general than just a cohesively parameterized U-collection of such cocycles on X, because it is actually a differential cocycle on $U \times X$ and hence may contain nontrivial differential/connection data along U, not just along X.

In some applications this behaviour of $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]$ is exactly what is needed. This is notably the case for the construction of extended Chern-Simons action functionals in all codimensions, discussed below in 3.9.11. But in other applications, such as the construction of the extended phase spaces of Chern-Simons functionals, one rather needs to have a object of genuine *differential moduli*, which is such that its U-plots are genuine U-parameterized collections of differential cocycles (and their gauge transformations) just on X. This issue is discussed in more detail with illustrative examples in the model of smooth cohesion below in 4.4.16.3.

Here we discuss how such differential moduli objects are obtained general abstractly in a cohesive ∞-topos from a *degreewise concretification* of the mapping space objects $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]$ in the sense of 3.7.2.

Definition 3.9.46. Let $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ be a braided ∞-group, def. 3.6.119, which is exactly $n-1$-truncated, def. 3.6.22. Then for $k \leq n+1 \in \mathbb{N}$ write $\mathbf{B}\mathbb{G}_{\mathrm{conn}^k}$ for the ∞-pullback in

$$\begin{array}{ccc} \mathbf{B}\mathbb{G}_{\mathrm{conn}^k} & \longrightarrow & \Omega^{n+1\leq \bullet \leq k}(-,\mathbb{G}) \\ \downarrow & & \downarrow \\ \mathbf{B}\mathbb{G} & \xrightarrow{\mathrm{curv}_{\mathbb{G}}} & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \end{array}$$

Remark 3.9.47. For A a 0-truncated abelian group and $\mathbb{G} \simeq \mathbf{B}A$, the objects $\mathbf{B}^2 A_{\mathrm{conn}^1}$ of def. 3.9.46 modulates what in the literature is oftn known as a *bundle gerbe with connective data but without curving.* In this context then the structures modulated by $\mathbf{B}^2_{\mathrm{conn}^2} \simeq \mathbf{B}^2 A_{\mathrm{conn}}$ would be called *bundles gerbes with connective data and with curving.* We discuss this in more detail in 4.4.16 below.

Remark 3.9.48. The objects $\mathbf{B}\mathbb{G}_{\mathrm{conn}^k}$ of def. 3.9.46 play two key roles:

1. They appear as an ingredient in the construction of differential moduli stacks in def. 3.9.50 below. Here their role is mainly a technical one: the object of interest is really $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ and the $\mathbf{B}\mathbb{G}_{\mathrm{conn}^k}$ just serve to refine its structure.
2. They appear as variant differential coefficients in their own right in various contexts, for instance in the context of higher Atiyah Lie algebroids and Courant Lie algebroids in 3.9.13.8 below.

Remark 3.9.49. By the universal property of the ∞-pullback, the canonical tower of morphisms

$$\Omega^{n+1}_{\mathrm{cl}} \longrightarrow \Omega^{n+1\leq\bullet\leq n}_{\mathrm{cl}} \longrightarrow \cdots \longrightarrow \Omega^{n+1\leq\bullet\leq 1}_{\mathrm{cl}} \xrightarrow{\simeq} \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$$

induces a tower of morphisms

$$\mathbf{B}\mathbb{G}_{\mathrm{conn}} \xrightarrow{\simeq} \mathbf{B}\mathbb{G}_{\mathrm{conn}^n} \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}^0} \xrightarrow{\simeq} \mathbf{B}\mathbb{G} \; .$$

Definition 3.9.50. For $X \in \mathbf{H}$ and $n \in \mathbb{N}$, $n \geq 1$, $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ a braided ∞-group which is precisely $(n-1)$-truncated, then the *moduli of $\mathbb{G}$-principal connections* on X is the iterated ∞-fiber product

$$\begin{aligned} &\mathbb{G}\mathbf{Conn}(X) \\ &:= \sharp_1[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^n}] \underset{\sharp_1[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}}]}{\times} \sharp_2[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}}] \underset{\sharp_2[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-2}}]}{\times} \cdots \underset{\sharp_n[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^0}]}{\times} [X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^0}] \, , \end{aligned}$$

of the morphisms

$$\sharp_k[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-k+1}}] \longrightarrow \sharp_k[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-k}}]$$

which are the image under $\sharp_k$, def. 3.7.6, of the image under the internal hom $[X,-]$ of the canonical projections of remark 3.9.49, and of the morphisms

$$\sharp_{k+1}[X,\mathbf{B}^n U(1)_{\mathrm{conn}^{n-k}}] \longrightarrow \sharp_k[X,\mathbf{B}^n U(1)_{\mathrm{conn}^{n-k}}]$$

of def. 3.7.6.

Remark 3.9.51. By the universal property of the ∞-pullback, the commuting naturality diagrams

$$\begin{array}{ccc} \sharp_{k_2}[X,\mathbf{B}G_{\mathrm{conn}^{n_2}}] & \longrightarrow & \sharp_{k_2}[X,\mathbf{B}G_{\mathrm{conn}^{n_1}}] \\ \downarrow & & \downarrow \\ \sharp_{k_1}[X,\mathbf{B}G_{\mathrm{conn}^{n_2}}] & \longrightarrow & \sharp_{k_1}[X,\mathbf{B}G_{\mathrm{conn}^{n_1}}] \end{array}$$

induce a canonical projection

$$\mathrm{conc}: [X, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow G\mathbf{Conn}(X)$$

from the mapping space object into the object of differential moduli. We call this *differential concretification.*

We need the analogous construction also for the $\mathbf{B}\mathbb{G}_{\mathrm{conn}^k}$ regarded as coefficient objects themselves. The following straightforwardly generalizes def. 3.9.50 from $k = n$ to arbitrary $k \leq n$.

Definition 3.9.52. For $X \in \mathbf{H}$ and $n \in \mathbb{N}$, $n \geq 1$, $0 \leq k \leq n$, $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ a braided ∞-group which is precisely $(n-1)$-truncated, then the *moduli of* $\mathbb{G}$*-principal* k*-connections* on X is the iterated ∞-fiber product

$$\begin{aligned}&\mathbb{G}\mathbf{Conn}_k(X)\\ &:= \sharp_{n-k+1}[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}^k}] \underset{\sharp_{n-k+1}[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{k-1}}]}{\times} \sharp_{n-k+2}[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}^{k-1}}] \underset{\sharp_{n-k+2}[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^{k-2}}]}{\times} \cdots \underset{\sharp_n[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^0}]}{\times} [X, \mathbf{B}\mathbb{G}_{\mathrm{conn}^0}] \,.\end{aligned}$$

Remark 3.9.53. The projection maps out of the iterated ∞-pullbacks induce a canonical sequence of projections

$$\mathbb{G}\mathbf{Conn}(X) \simeq \mathbb{G}\mathbf{Conn}_n(X) \longrightarrow \mathbb{G}\mathbf{Conn}_{n-1}(X) \longrightarrow \cdots \longrightarrow \mathbb{G}\mathbf{Conn}_1(X) \longrightarrow \mathbb{G}\mathbf{Conn}_0(X) \simeq \mathbf{B}\mathbb{G} \;.$$

3.9.6.5 Flat Differential moduli

We now turn to defining moduli for *flat* differential cocycles.

Definition 3.9.54. For $\mathbb{G}$ a braided ∞-group which is precisely $(n-1)$-truncated, and for any $X \in \mathbf{H}$, we call the iterated ∞-fiber product

$$\begin{aligned}&\mathbb{G}\mathbf{FlatConn}(X)\\ &:= \sharp[X, \flat\mathbf{B}\mathbb{G}] \underset{\sharp[X,\Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}}])}{\times} \sharp_1[X, \Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}})] \underset{\sharp_1[X,\Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-2}})]}{\times} \cdots \underset{\sharp_n[X,\Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^0})]}{\times} [X, \mathbb{G}]\end{aligned}$$

the *moduli object for flat* $\mathbb{G}$*-connections on* X.

Proposition 3.9.55. *For* $\mathbf{B}\mathbb{G}$ *a truncated braided* ∞*-group we have a natural equivalence*

$$\mathbb{G}\mathbf{FlatConn}\,(X) \simeq \Omega_0\left((\mathbf{B}\mathbb{G})\,\mathbf{Conn}\,(X)\right) \,.$$

Moreover, if $\mathbf{H}$ *has a set of generators being concrete objects (in particular if is has an* ∞*-cohesive site of definition, def. 3.4.8) then for* $\mathbb{G}$ *a 0-truncated* ∞*-group and* X *geometrically connected (meaning that* $\tau_0\Pi(X) \simeq *$*), we have*

$$\mathbb{G} \simeq \Omega_0\left(\mathbb{G}\mathbf{Conn}(X)\right)$$

Proof. Since forming loops is an ∞-pullback operation, it commutes with the iterated ∞-fiber product. Moreover, by prop. 3.6.47 it passes through the $\sharp_k$, while lowering their degree by one. Finally by prop. 3.9.36 we have

$$\Omega\left(\mathbf{B}^2 G_{\mathrm{conn}}\right) \simeq \flat\mathbf{B}\mathbb{G}\,.$$

This gives the first claim. For the second, observe that with the same reasoning we obtain

$$\begin{aligned}\Omega\left(\mathbb{G}\mathbf{Conn}(X)\right) &\simeq \Omega\left(\sharp_1[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \underset{\sharp_1[X,\mathbf{B}\mathbb{G}]}{\times} [X, \mathbf{B}\mathbb{G}]\right) \\ &\simeq \sharp[X, \flat\mathbb{G}] \underset{\sharp[X,\mathbb{G}]}{\times} [X, \mathbb{G}]\end{aligned}\,.$$

Hence for any concrete $U \in \mathbf{H}$ we have

$$\begin{aligned}
\mathbf{H}(U, \Omega(\mathbb{G}\mathbf{Conn}(X))) &\simeq \infty\mathrm{Grpd}(\Gamma(U), \mathbf{H}(X, \flat\mathbb{G})) \underset{\infty\mathrm{Grpd}(\Gamma(U),\mathbf{H}(X,\mathbb{G}))}{\times} \mathbf{H}(U \times X, \mathbb{G}) \\
&\simeq \infty\mathrm{Grpd}(\Gamma(U) \times \Pi(X), \Gamma(\mathbb{G})) \underset{\infty\mathrm{Grpd}(\Gamma(U),\mathbf{H}(X,\mathbb{G}))}{\times} \mathbf{H}(U \times X, \mathbb{G}) \\
&\simeq \mathrm{Set}(\tau_0\Gamma(U), \Gamma(\mathbb{G})) \underset{\mathrm{Set}(\tau_0\Gamma(U),\mathbf{H}(X,\mathbb{G}))}{\times} \mathbf{H}(U \times X, \mathbb{G}) \\
&\simeq \mathbf{H}(U, \mathbb{G})
\end{aligned} .$$

Here we used the defining adjunctions of cohesion and that $\mathbb{G}$ is 0-truncated by assumption, so that $\mathbf{H}(-, \mathbb{G})$ takes values in sets. In the last step we used that U is concrete so that maps out of it are determined by their value on all global points of U. So the second but last row says in words "those maps out of $U \times X$ which for every point of U are independent of X" and the last equivalence identifies that with the maps out of just U. Since these equivalences are all natural in U the claim follows by the assumption that the Us range over a set of generators (hence with the ∞-Yoneda lemma, prop. 2.1.10, if the Us range over the objects of a site of definition). □

3.9.7 Chern-Weil homomorphism

We discuss an intrinsic realization of the Chern-Weil homomomorphism [GHV] in cohesive ∞-toposes.

Definition 3.9.56. For G an ∞-group and

$$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n A$$

a representative of a characteristic class $[\mathbf{c}] \in H^n(\mathbf{B}G, A)$ we say that the composite

$$\mathbf{c}_{\mathrm{dR}} : \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^n A \xrightarrow{\mathrm{curv}} \flat_{\mathrm{dR}}\mathbf{B}^{n+1}A$$

represents the *curvature characteristic class* $[\mathbf{c}_{\mathrm{dR}}] \in H^{n+1}_{\mathrm{dR}}(\mathbf{B}G, A)$. The induced map on cohomology

$$(\mathbf{c}_{\mathrm{dR}})_* : H^1(-, G) \to H^{n+1}_{\mathrm{dR}}(-, A)$$

we call the (unrefined) ∞-*Chern-Weil homomorphism* induced by $\mathbf{c}$.

The following construction universally lifts the ∞-Chern-Weil homomorphism from taking values in the de Rham cohomology to values in the differential cohomology of $\mathbf{H}$.

Definition 3.9.57. For $X \in \mathbf{H}$ any object, define the ∞-groupoid $\mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}G)$ as the ∞-pullback

$$\begin{array}{ccc}
\mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}G) & \xrightarrow{(\hat{\mathbf{c}}_i)_i} & \prod_{[\mathbf{c}_i] \in H^{n_i}(\mathbf{B}G, A); i \geq 1} \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^{n_i}A) \\
\downarrow^{\eta} & & \downarrow \\
\mathbf{H}(X, \mathbf{B}G) & \xrightarrow{(\mathbf{c}_i)_i} & \prod_{[\mathbf{c}_i] \in H^{n_i}(\mathbf{B}G, A); i \geq 1} \mathbf{H}(X, \mathbf{B}^{n_i}A)
\end{array} .$$

We say

- a cocycle in $\nabla \in \mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}G)$ is an ∞-*connection*
- on the principal ∞-bundle $\eta(\nabla)$;

- a morphism in $\mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}G)$ is a *gauge transformation* of connections;
- for each $[\mathbf{c}] \in H^n(\mathbf{B}G, A)$ the morphism

$$[\hat{\mathbf{c}}] : H_{\mathrm{conn}}(X, \mathbf{B}G) \to H^n_{\mathrm{diff}}(X, A)$$

is the (full/refined) ∞-*Chern-Weil homomorphism* induced by the characteristic class $[\mathbf{c}]$.

Observation 3.9.58. Under the curvature projection $[F] : H^n_{\mathrm{diff}}(X, A) \to H^{n+1}_{\mathrm{dR}}(X, A)$ the refined Chern-Weil homomorphism for $\mathbf{c}$ projects to the unrefined Chern-Weil homomorphism.

Proof. This is due to the existence of the pasting composite

$$\begin{array}{ccccc}
\mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}G) & \xrightarrow{(\hat{\mathbf{c}}_i)_i} & \prod\limits_{[\mathbf{c}_i] \in H^{n_i}(\mathbf{B}G, A); i \geq 1} \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^{n_i} A) & \xrightarrow{[F]} & \prod\limits_{[\mathbf{c}_i] \in H^{n_i}(\mathbf{B}G, A); i \geq 1} H^{n_i+1}_{\mathrm{dR}}(X, A) \\
\downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow \\
\mathbf{H}(X, \mathbf{B}G) & \xrightarrow{(\mathbf{c}_i)_i} & \prod_{[\mathbf{c}_i] \in H^{n_i}(\mathbf{B}G, A); i \geq 1} \mathbf{H}(X, \mathbf{B}^{n_i} A) & \xrightarrow{\mathrm{curv}_*} & \prod_{[\mathbf{c}_i] \in H^{n_i}(\mathbf{B}G, A); i \geq 1} \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n_i+1}, A)
\end{array}$$

of the defining ∞-pullback for $\mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}G)$ with the products of the definition ∞-pullbacks for the $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^{n_i} A)$. □

As before for abelian differential cohomology in 3.9.6, nonabelian differential cohomology is in general not representable, but becomes representable on a suitable collection of domains. To reflect this we expand def. 3.9.42 as follows.

Definition 3.9.59. Let $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n A$ be a characteristic map, and let $\mathbf{B}^n A_{\mathrm{conn}}$ be a differential refinement as in def. 3.9.42. Then we write $\mathbf{B}G_{\mathrm{conn}}$ for an object that fits into a factorization

$$\begin{array}{ccc}
\flat \mathbf{B}G & \xrightarrow{\flat \mathbf{c}} & \flat \mathbf{B}^n A \\
\downarrow & & \downarrow \\
\mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^n A_{\mathrm{conn}} \\
\downarrow & & \downarrow \\
\mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}^n A
\end{array}$$

of the naturality diagram of the $(\mathrm{Disc} \dashv \Gamma)$-counit.

Warning 3.9.60. The object $\mathbf{B}G_{\mathrm{conn}}$ here depends, in general, on the choices involved. But for the moment we find it convenient not to indicate this in the notation but have it be implied by the corresponding context.

3.9.8 Twisted differential structures

We discuss the differential refinement of *twisted cohomology*, def. 3.6.12. Following [SSS09c] we speak of *twisted differential* $\mathbf{c}$*-structures*.

Definition 3.9.61. For $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n A$ a characteristic map in a cohesive ∞-topos $\mathbf{H}$, define for any $X \in \mathbf{H}$ the ∞-groupoid $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ to be the ∞-pullback

$$\begin{array}{ccc}
\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & H^n(X, A) \\
\downarrow & & \downarrow \\
\mathbf{H}(X, \mathbf{B}G) & \xrightarrow{\mathbf{c}} & \mathbf{H}(X, \mathbf{B}^n A)
\end{array} \quad ,$$

where the vertical morphism on the right is the essentially unique effective epimorphism that picks on point in every connected component.

Let now $\mathbf{H}$ be a cohesive ∞-topos that canonically contains the circle group $A = U(1)$, such as Smooth∞Grpd and its variants. Then by 4.4.16 the intrinsic differential cohomology with $U(1)$-coefficients reproduces traditional ordinary differential cohomology and by 4.4.17 we have models for the ∞-connection coefficients $\mathbf{B}G_{\mathrm{conn}}$. Using this we consider the differential refinement of def. 3.9.61 as follows.

Definition 3.9.62. For $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ a characteristic map as above, and for $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ a differential refinement, we write $\hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X)$ for the corresponding twisted cohomology, def. 3.6.225,

$$\begin{array}{ccc} \hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & H^n_{\mathrm{diff}}(X, U(1)) \\ \downarrow^{\chi} & & \downarrow \\ \mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \end{array} ,$$

We say $\hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X)$ is the ∞-groupoid of *twisted differential* $\mathbf{c}$*-structures* on X.

3.9.9 Higher holonomy

The notion of ∞-connections in a cohesive ∞-topos induces a notion of *higher holonomoy.*

Definition 3.9.63. We say an object $\Sigma \in \mathbf{H}$ has *cohomological dimension* $\leq n \in \mathbb{N}$ if for all Eilenberg-MacLane objects $\mathbf{B}^{n+1}A$ the corresponding cohomology on Σ is trivial

$$H(\Sigma, \mathbf{B}^{n+1}A) \simeq * \,.$$

Let $\dim(\Sigma)$ be the maximum n for which this is true.

Observation 3.9.64. If Σ has cohomological dimension $\leq n$ then its de Rham cohomology, def. 3.9.15, vanishes in degree $k > n$

$$H^{k>n}_{\mathrm{dR}}(\Sigma, A) \simeq * \,.$$

Proof. Since $\flat$ is a right adjoint it preserves delooping and hence $\flat \mathbf{B}^k A \simeq \mathbf{B}^k \flat A$. It follows that

$$\begin{aligned} H^k_{\mathrm{dR}}(\Sigma, A) &:= \pi_0 \mathbf{H}(\Sigma, \flat_{\mathrm{dR}} \mathbf{B}^k A) \\ &\simeq \pi_0 \mathbf{H}(\Sigma, * \prod_{\mathbf{B}^k A} \mathbf{B}^k \flat A) \\ &\simeq \pi_0 \left(\mathbf{H}(\Sigma, *) \prod_{\mathbf{H}(\Sigma, \mathbf{B}^k A)} \mathbf{H}(\Sigma, \mathbf{B}^k \flat A) \right) \\ &\simeq \pi_0(*) \end{aligned} .$$

□

Let now A be fixed as in 3.9.6.

Definition 3.9.65. Let $\Sigma \in \mathbf{H}$, $n \in \mathbf{N}$ with $\dim\Sigma \leq n$. We say that the composite

$$\int_\Sigma : \mathbf{H}_{\mathrm{flat}}(\Sigma, \mathbf{B}^n A) \xrightarrow{\simeq} \infty\mathrm{Gprd}(\Pi(\Sigma), \Pi(\mathbf{B}^n A)) \xrightarrow{\tau_{\leq n - dim(\Sigma)}} \tau_{n - dim(\Sigma)} \infty\mathrm{Gprd}(\Pi(\Sigma), \Pi(\mathbf{B}^n A))$$

of the adjunction equivalence followed by truncation as indicated is the *flat holonomy* operation on flat ∞-connections.

More generally, let

- $\nabla \in \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n A)$ be a differential coycle on some $X \in \mathbf{H}$

- $\phi : \Sigma \to X$ a morphism.

Write

$$\phi^* : \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^{n+1}A) \to \mathbf{H}_{\mathrm{diff}}(\Sigma, \mathbf{B}^n A) \simeq \mathbf{H}_{\mathrm{flat}}(\Sigma, \mathbf{B}^n A)$$

(using prop. 3.9.39) for the morphism on ∞-pullbacks induced by the morphism of diagrams

$$\begin{array}{ccccc} \mathbf{H}(X, \mathbf{B}^n A) & \longrightarrow & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1}A) & \longleftarrow & H^{n+1}_{\mathrm{dR}}(X, A) \\ \downarrow^{\phi^*} & & \downarrow^{\phi^*} & & \downarrow \\ \mathbf{H}(\Sigma, \mathbf{B}^n A) & \longrightarrow & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1}A) & \longleftarrow & * \end{array}$$

The *holonomomy* of ∇ over σ is the flat holonomy of $\phi^*\nabla$:

$$\int_\phi \nabla := \int_\Sigma \phi^* \nabla \,.$$

This is a special case of the more general notion of transgression, 3.9.10.

3.9.10 Transgression

We discuss an intrinsic notion of *transgression* of differential cocycles to mapping spaces. This generalizes the notion of holonomy from 3.9.9 to the case of higher codimension.

Let $A \in \infty\mathrm{Grp}(\mathbf{H})$ be an abelian group object and $\mathbf{B}^n A_{\mathrm{conn}}$ a differential coefficient object, as in 3.9.6, for $n \in \mathbb{N}$.

Let $\Sigma \in \mathbf{H}$ be of cohomological dimension $k \leq n$, def. 3.9.63.

Definition 3.9.66. For $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n A_{\mathrm{conn}}$ a differentia characteristic map as in def. 3.9.59, we say that the *transgression* of $\hat{\mathbf{c}}$ to $[\Sigma, \mathbf{B}G_{\mathrm{conn}}]$ is the composite

$$\mathrm{tg}_\Sigma \hat{\mathbf{c}} : \; [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma, \hat{\mathbf{c}}]} [\Sigma, \mathbf{B}^n A_{\mathrm{conn}}] \longrightarrow \mathrm{conc}_{n-k} \tau_{n-k} [\Sigma, \mathbf{B}^n A_{\mathrm{conn}}] \;,$$

where $[-,-] : \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ is the cartesian internal hom, where τ_{n-k} is $(n-k)$-truncation, prop. 3.6.25, and where conc_{n-k} is $(n-k)$-concretification from def. 3.7.7.

Remark 3.9.67. In the models we consider we find inclusions

$$\mathbf{B}^{n-k} A_{\mathrm{conn}} \hookrightarrow \mathrm{conc}_{n-k} \tau_{n-k} [\Sigma, \mathbf{B}^n A_{\mathrm{conn}}] \,.$$

In these cases truncation takes A-principal n-connections $\hat{\mathbf{c}}$ on $\mathbf{B}G_{\mathrm{conn}}$ to A-principal $(n-k)$-connections $\mathrm{tg}_\Sigma \hat{\mathbf{c}}$ on $[\Sigma, \mathbf{B}G_{\mathrm{conn}}]$.

In particular for $k = n$ in this case the transgression is of the form

$$\mathrm{tg}_\Sigma \hat{\mathbf{c}} : [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \to A \,.$$

3.9.11 Chern-Simons functionals

Combining the refined ∞-Chern-Weil homomorphism, 3.9.7 with the higher holonomy, 3.9.9, of the resulting ∞-connections produces a notion of higher *Chern-Simons functionals* internal to any cohesive ∞-topos. For a review of standard Chern-Simons functionals see [Fre].

Definition 3.9.68. Let $\Sigma \in \mathbf{H}$ be of cohomological dimension $\dim \Sigma = n \in \mathbb{N}$ and let $\mathbf{c} : X \to \mathbf{B}^n A$ a representative of a characteristic class $[\mathbf{c}] \in H^n(X, A)$ for some object X. We say that the composite

$$\exp(S_{\mathbf{c}}(-)) : \mathbf{H}(\Sigma, X) \stackrel{\hat{\mathbf{c}}}{\to} \mathbf{H}_{\mathrm{diff}}(\Sigma, \mathbf{B}^n A) \stackrel{\simeq}{\to} \mathbf{H}_{flat}(\Sigma, \mathbf{B}^n A) \stackrel{\int_\Sigma}{\to} \tau_{\leq 0} \infty\mathrm{Grpd}(\Pi(\Sigma), \Pi \mathbf{B}^n A)$$

is the ∞-*Chern-Simons functional* induced by $\mathbf{c}$ on Σ.

Here $\hat{\mathbf{c}}$ denotes the refined Chern-Weil homomorphism, 3.9.7, induced by $\mathbf{c}$, and $\int_\Sigma$ is the holonomy over Σ, 3.9.9, of the resulting n-bundle with connection.

Remark 3.9.69. In the language of σ-model quantum field theory the ingredients of this definition have the following interpretation

- Σ is the *worldvolume of a fundamental* $(\dim\Sigma - 1)$*-brane* ;
- X is the *target space*;
- $\hat{\mathbf{c}}$ is the *background gauge field* on X;
- the external hom $\mathbf{H}_{\mathrm{conn}}(\Sigma, X)$ is the *space of worldvolume field configurations* $\phi : \Sigma \to X$ or *trajectories* of the brane in X;
- $\exp(S_{\mathbf{c}}(\phi)) = \int_\Sigma \phi^* \hat{\mathbf{c}}$ is the value of the action functional on the field configuration ϕ.

Traditionally, σ-models have been considered for X an ordinary (Riemannian) manifold, or at most an orbifold, see for instance [DEFJKMMW]. The observation that it makes sense to allow target objects X to be more generally a gerbe, 3.6.15, is explored in [PaSh05] [HeSh10]. Here we see that once we pass to fully general (higher) stacks, then also all (higher) gauge theories are subsumed as σ-models.

For if there is an ∞-group G such that the target space object X is the moduli ∞-stack of G-∞-connections, def. 3.9.59, $X \simeq \mathbf{B}G_{\mathrm{conn}}$, then a "trajectory" $\Sigma \to X \simeq \mathbf{B}G_{\mathrm{conn}}$ is in fact a G-gauge field on Σ. Hence in the context of ∞-stacks, the notions of gauge theories and of σ-models unify.

More in detail, assume that $\mathbf{H}$ has a canonical line object $\mathbb{A}^1$ and a natural numbers object $\mathbb{Z}$. Then the action functional $\exp(iS(-))$ may lift to the internal hom with respect to the canonical cartesian closed monoidal structure on any ∞-topos to a morphism of the form

$$\exp(iS_{\mathbf{c}}(-)) : [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}^{n-\dim\Sigma}\mathbb{A}^1/\mathbb{Z}\,.$$

We call the internal hom $[\Sigma, \mathbf{B}G_{\mathrm{conn}}]$ the *moduli ∞-stack* of field configurations on Σ of the ∞*-Chern-Simons theory* defined by $\mathbf{c}$ and $\exp(iS_{\mathbf{c}}(-))$ the action functional in codimension $(n - \dim\Sigma)$ defined on it.

A list of examples of Chern-Simons action functionals defined on moduli stacks obtained this way is given in 4.4.19.

3.9.12 Wess-Zumino-Witten functionals

We discuss an canonical realization of Wess-Zumino-Witten action functionals and their higher analogs in every cohesive ∞-topos. More precisely, to every ∞-Chern-Simons Lagrangian on $\mathbf{B}G_{\mathrm{conn}}$ as in 3.9.11 above is associated a corresponding ∞*-Wess-Zumino-Witten Lagrangian* on G, given by a differentially refined looping.

For a review of standard WZW functionals see for instance [Ga00].

Before giving the definition of intrinsic WZW functionals, it is useful to restate the the concept of bundle extensions in the following way.

Definition 3.9.70. Let $G \in \infty\mathrm{Grp}(\mathbf{H})$ be an ∞-group and

$$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+1}A$$

a characteristic map classifying a *Chern-Simons* $(\mathbf{B}^n A)$*-bundle* $\mathbf{B}\hat{G} \to \mathbf{B}G$.

We say that its image $\hat{G} \to G$ under forming loop space objects is the corresponding *Wess-Zumino-Witten* $(\mathbf{B}^n A)$-principal bundle.

Remark 3.9.71. By prop. 3.6.142, the WZW ∞-bundle sits in the pasting diagram of ∞-pullbacks

$$\begin{array}{ccccc} \hat{G} & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \\ G & \stackrel{\Omega \mathbf{c}}{\longrightarrow} & \mathbf{B}^n A & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\ & & \downarrow & & \downarrow^{\mathbf{c}} \\ & & * & \longrightarrow & \mathbf{B}^{n+1}A \end{array} .$$

In the following, the WZW action functional arises from a differential refinement of this situation. First consider the following differential refinement of the codomain of $\Omega\mathbf{c}$.

Definition 3.9.72. For $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{n+1}A_{\mathrm{conn}}$ a differential refinement of $\mathbf{c}$, write $\mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)}$ for the homotopy fiber

$$\begin{array}{ccccccc} \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)} & \longrightarrow & & & & & * \\ \downarrow & & & & & & \downarrow \\ \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & \flat\mathbf{B}G & \longrightarrow & \mathbf{B}G_{\mathrm{conn}} & \stackrel{\hat{\mathbf{c}}}{\longrightarrow} & \mathbf{B}^{n+1}A_{\mathrm{conn}} \end{array} ,$$

where the left bottom morphism is the canonical one, and the middle bottom morphism that induced by prop. 3.9.39.

We say that $\mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)}$ is the coefficient object for *WZW A-principal n-bundles with connection*. The notation here is motivated by the discussion to follow.

Definition 3.9.73. Write

$$\eta : \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)} \to \mathbf{B}^n A$$

for the morphism into the ∞-pullback

$$\begin{array}{ccccccc} \mathbf{B}^n A & \longrightarrow & \mathbf{B}\hat{G} & \longrightarrow & & & * \\ \downarrow & & \downarrow & & & & \downarrow \\ * & \longrightarrow & \mathbf{B}G & \stackrel{\mathrm{id}}{\longrightarrow} & \mathbf{B}G & \stackrel{\mathbf{c}}{\longrightarrow} & \mathbf{B}^{n+1}A \end{array}$$

induced by the morphism of pullback diagrams given by

$$\begin{array}{ccccccccc} \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & \flat\mathbf{B}G & \longrightarrow & \mathbf{B}G_{\mathrm{conn}} & \stackrel{\hat{\mathbf{c}}}{\longrightarrow} & \mathbf{B}^{n+1}A_{\mathrm{conn}} & \longleftarrow & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G & \stackrel{\mathrm{id}}{\longrightarrow} & \mathbf{B}G & \stackrel{\mathbf{c}}{\longrightarrow} & \mathbf{B}^{n+1}A & \longleftarrow & * \end{array} .$$

For a given WZW connection $X \to \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)}$, we say that the composite $X \to \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)} \stackrel{\eta}{\to} \mathbf{B}^n A$ is the *underlying WZW n-bundle*.

Definition 3.9.74. For $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{n+1}A_{\mathrm{conn}}$ a differential refinement of $\mathbf{c}$, def. 3.9.59, inducing an ∞-Chern-Simons functional, by 3.9.11, we say that the morphism

$$\mathrm{WZW}_{\hat{\mathbf{c}}} : G \to \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)}$$

in the pasting diagram of ∞-pullbacks

$$
\begin{array}{ccccc}
G & \xrightarrow{\mathrm{WZW}_{\mathbf{c}}} & \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)} & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}G \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \cdot & \longrightarrow & \flat\mathbf{B}G \\
 & & \downarrow & & \downarrow \\
 & & & & \mathbf{B}G_{\mathrm{conn}} \\
 & & \downarrow & & \downarrow{\scriptstyle \hat{\mathbf{c}} =: \mathrm{CS}_{\mathbf{c}}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1}A_{\mathrm{conn}}
\end{array}
$$

is the corresponding *WZW Lagrangian*.

Here the total top rectangle is the ∞-pullback that defines the canonical form $\theta : G \to \flat_{\mathrm{dR}}\mathbf{B}G$, def. 3.9.29, the morphism $\flat\mathbf{B}G \to \mathbf{B}G_{\mathrm{conn}}$ is that induced by prop. 3.9.39, and the object $\mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)}$ is from def. 3.9.72.

Proposition 3.9.75. *The WZW Lagrangian* $\mathrm{WZW}_{\mathbf{c}}$ *from def. 3.9.74 is a differential refinement of the morphism* $\Omega\mathbf{c}$*, from remark 3.9.71, which classifies the WZW* n*-bundle, in that we have a commuting diagram*

$$
\begin{array}{ccc}
G & \xrightarrow{\mathrm{WZW}_{\mathbf{c}}} & \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \eta} \\
G & \xrightarrow{\Omega\mathbf{c}} & \mathbf{B}^n A
\end{array}
$$

in $\mathbf{H}$*, where the right vertical morphism is from def. 3.9.73.*

Proof. Paste the morphism of diagrams that defines η in def. 3.9.73 to the right total rectangle in def. 3.9.74. Pulling the result back one more step to the left, there appears in the top left a diagram of the form as in the claim, whose top and right morphism are as in the claim. It remains to see that the morphism $G \to G$ appearing is the identity. Since the way it it appears under this pullback it is a morphism of pullbacks induced by a morphism of pullback diagrams, there is, up to equivalence, only a unique such morphism which makes all diagrams in sight commute. One sees that the identity morphism has this property, and hence by uniqueness it must be the morphism in question. □

Definition 3.9.76. For Σ of dimension n we say that the composition with the holonomy over Σ, def. 3.9.65,

$$
\exp(S_{\mathrm{WZW}_{\hat{\mathbf{c}}}}) : \mathbf{H}(\Sigma, G) \xrightarrow{\mathrm{WZW}_{\hat{\mathbf{c}}}} \mathbf{H}(\Sigma, \mathbf{B}^n A_{\mathrm{conn}}|_{F=\mathbf{c}_{\mathrm{dR}}(\theta)}) \hookrightarrow \mathbf{H}(\Sigma, \mathbf{B}^{n+1}A_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} A
$$

is the corresponding exponentiated *WZW action functional* induced by $\hat{\mathbf{c}}$.

3.9.13 Prequantum geometry

Traditional *prequantum geometry* (see for instance [EMRV98] for a standard account) is the differential geometry of smooth manifolds which are "twisted" by circle-principal bundles and circle-principal connections – thought of as "prequantum bundles" – or equivalently is the *contact geometry* [Et] of the total spaces of these bundles thought of as *regular contact manifolds* [BW]. Prequantum geometry studies the automorphisms of prequantum bundles covering diffeomorphisms of the base – the *prequantum operators* – and the action of these on the space of sections of the associated line bundle – the *prequantum states*. This is an intermediate step in the genuine *geometric quantization* of the curvature 2-form of these bundles, which is obtained by dividing the above data in half, important for instance in the *orbit method*. But prequantum geometry is of interest already in its own right. For instance the above automorphism group naturally provides the Lie integration of the *Poisson Lie algebra* of the underlying symplectic manifold. Moreover, it is canonically included into the group of bisections of the Lie integration of the Atiyah Lie algebroid of the given circle bundle.

We now formulate *geometric prequantum theory* internally to any cohesive ∞-topos to obtain *higher prequantum geoemtry*.

- 3.9.13.1 – Survey
- 3.9.13.2 – Prequantization;
- 3.9.13.3 – Symplectomorphism group;
- 3.9.13.4 – Contactomorphism group;
- 3.9.13.5 – Quantomorphism group;
- 3.9.13.6 – Heisenberg group;
- 3.9.13.7 – Poisson and Heisenberg Lie algebra;
- 3.9.13.8 – Courant Lie algebroid;
- 3.9.13.9 – Prequantum states;
- 3.9.13.10 – Prequantum operators.

3.9.13.1 Survey *Quantization* is supposed to be a process that reads in an action functional and produces from it, possibly non-uniquely, a quantum field theory. There are two main formalizations of what this means. One is *algebraic (deformation) quantization*, one is *geometric quantization*. The geometric and differential-cohomological nature of the latter makes it well suited for formalizatin in cohesive ∞-toposes. See for instance [EMRV98] for a standard account.

Geometric quantization involves two steps, the first called *geometric prequantization* the second being the genuine geometric quantization. The first step refines de Rham coefficient structures, notably symplectic structures, to differential cohomology and interprets automorphisms of differential cocycles as (pre-)quantum operators.

For discussion of background and classical references see 4.4.20 below, where the general theory is worked out in the context of smooth cohesion. In 4.4.20.1 we make the connection to the traditional theory of symplectic prequantization, in 4.4.20.2 to the younger theory of multisymplectic prequantization, and in 5.6, we discuss further and higher examples.

Here we formulate axiomatically the ingredients of prequantum geometry. The following table gives a quick overview over the key constructions discussed in detail in the following.

∞-geometric quantization	cohesive homotopy type theory	twisted hyper-sheaf cohomology
pre-n-plectic cohesive ∞-groupoid	$\omega : X \to \Omega^2_{\mathrm{cl}}(-,\mathbb{G})$ (e.g. $\mathbb{G} = \mathbf{B}^{n-1}U(1)$ or $= \mathbf{B}^{n-1}\mathbb{C}^\times$)	twisting cocycle in de Rham cohomology
symplectomorphisms	$\mathbf{Aut}_{\mathbf{H}}(\omega) = \left\{ \begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {}_{\omega}\searrow \quad \swarrow_{\omega} & \\ & \Omega^2_{\mathrm{cl}}(-,\mathbb{G}) & \end{array} \right\}$	twist automorphism ∞-group
prequantum bundle	$\begin{array}{ccc} & & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \\ & {}^{\nabla}\nearrow & \downarrow F_{(-)} \\ X & \xrightarrow{\omega} & \Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \end{array}$	twisting cocycle in differential cohomology
Planck's constant $\hbar$	$\frac{1}{\hbar}\nabla : X \to \mathbf{B}^n\mathbb{G}_{\mathrm{conn}}$	divisibility of twist class
quantomorphism group $\supset$ Heisenberg group	$\mathbf{Aut}_{\mathbf{H}}(\nabla) = \left\{ \begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {}_{\nabla}\searrow \; \swarrow\!\!\!\!\!\!\swarrow_{\simeq} \; \swarrow_{\nabla} & \\ & \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} & \end{array} \right\}$	twist automorphism ∞-group
Hamiltonian G-action	$\mu : \mathbf{B}G \to \mathbf{Aut}_{\mathbf{H}}(\nabla)$	G-∞-action on the twisting cocycle
gauge reduction	$\nabla//G \; : \; X//G \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$	G-∞-quotient of the twisting cocycle
Hamiltonian observables with Poisson bracket	$\mathrm{Lie}(\mathbf{Aut}_{\mathbf{H}}(\nabla))$	infinitesimal twist automorphisms
Hamiltonian symplectomorphisms	$\mathrm{image}\Big(\mathbf{Aut}_{\mathbf{H}}(\nabla) \longrightarrow \mathbf{Aut}(X) \Big)$	twists in de Rham cohomology that lift to differential cohomology
$\mathbb{G}$-representation	$\begin{array}{ccc} V & \longrightarrow & V//\mathbb{G} \\ & & \downarrow \rho \\ & & \mathbf{B}\mathbb{G} \end{array}$	local coefficient ∞-bundle
prequantum space of states	$\mathbf{\Gamma}_X(E) = \left\{ \begin{array}{ccc} X & \xrightarrow{\sigma} & V//\mathbb{G} \\ & {}_{\mathbf{c}}\searrow \; \swarrow\!\!\!\!\!\!\swarrow_{\simeq} \; \swarrow_{\rho} & \\ & \mathbf{B}\mathbb{G} & \end{array} \right\}$	cocycles in $[\mathbf{c}]$-twisted cohomology
prequantum operator action	$\widehat{(-)} : \mathbf{\Gamma}_X(E) \times \mathbf{Aut}_{\mathbf{H}} \to \mathbf{\Gamma}_X(E)$	∞-action of twist automorphisms on twisted cocycles
transgression	composition with: $\begin{array}{ccc} [S^1, V//\mathbb{G}_{\mathrm{conn}}] & \xrightarrow{\mathrm{tr}\,\mathrm{hol}_{S^1}} & V//\Omega\mathbb{G}_{\mathrm{conn}} \\ \downarrow \rho^V_{\mathrm{conn}} & & \downarrow \rho^V_{\mathrm{conn}} \\ \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \xrightarrow{\exp(2\pi i \int_{S^1}(-))} & \mathbb{G}_{\mathrm{conn}} \end{array}$	fiber integration in (nonabelian) differential cohomology

3.9.13.2 Prequantization Let $X \in \mathbf{H}$ be a cohesive homotopy type. Let $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ be a braided cohesive group, def. 3.6.119. In the present context we say

Definition 3.9.77. 1. A morphism (def. 3.9.34)

$$\omega : X \longrightarrow \Omega^2_{\mathrm{cl}}(-, \mathbb{G})$$

is a *presymplectic structure* on X.

2. Given a presymplectic structure, a lift ∇ in

$$\begin{array}{ccc} & & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \\ & \overset{\nabla}{\nearrow} \;\swArrow & \downarrow{\scriptstyle F_{(-)}} \\ X & \xrightarrow[\omega]{} & \Omega^2_{\mathrm{cl}}(-, \mathbb{G}) \end{array}$$

is a *prequantization* of (X, ω).

3.9.13.3 Symplectomorphisms Let $X \in \mathbf{H}$ be a cohesive homotopy type. Let $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ be a braided cohesive group, def. 3.6.119. Let

$$\omega : X \longrightarrow \Omega^2_{\mathrm{cl}}(-, \mathbb{G}) \; .$$

be a presymplectic structure, def. 3.9.34.

Definition 3.9.78. The *symplectomorphism group* $\mathbf{Sympl}(\omega)$ of the presymplectic geometry (X, ω) is the $\mathbf{H}$-valued automorphism group, def. 3.6.11, of $\omega \in \mathbf{H}_{/\Omega^2_{\mathrm{cl}}(-,\mathbb{G})}$:

$$\mathbf{Sympl}(\omega) := \mathbf{Aut}_{\mathbf{H}}(\omega) := \prod_{\Omega^2_{\mathrm{cl}}(-,\mathbb{G})} \mathbf{Aut}(\omega) \, .$$

Remark 3.9.79. According to remark 3.6.7 a global element of $\mathbf{Sympl}(\omega)$ corresponds to a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} X & \underset{\simeq}{\overset{\phi}{\longrightarrow}} & X \\ & {\scriptstyle \omega}\searrow \quad \swarrow{\scriptstyle \omega} & \\ & \Omega^2_{\mathrm{cl}}(-, \mathbb{G}) & \end{array} \; .$$

This is a diffeomorphism ϕ of X which preserves the presymplectic structure in that

$$\phi^* \omega = \omega \, .$$

Definition 3.9.80. Write

$$p_{\Omega^2_{\mathrm{cl}}(-,\mathbb{G})} : \mathbf{Sympl}(\omega) \longrightarrow \mathbf{Aut}(X)$$

for the canonical morphism induced by restriction of the morphism of prop. 3.6.9.

Proposition 3.9.81. *The morphism $p_{\Omega^2_{\mathrm{cl}}(-,\mathbb{G})}$ of def. 3.9.80 is a monomorphism*

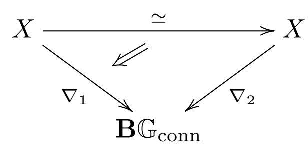
Proof. By direct generalization of the proof of prop. 3.6.16 we find that for each $U \in \mathbf{H}$ the fibers of $p_{\Omega^2_{\mathrm{cl}}(-,\mathbb{G})}$ are path space objects of $[X, \Omega^2_{\mathrm{cl}}(-, \mathbb{G})]$. But since $\Omega^2_{\mathrm{cl}}(-, \mathbb{G})$ is 0-truncated by def. 3.9.34, also $[X, \Omega^2_{\mathrm{cl}}(-, \mathbb{G})]$ is 0-truncated, and so its path spaces are either contractible or empty. □

3.9.13.4 Contactomorphisms

Definition 3.9.82. Given two $\mathbb{G}$-principal connections $\nabla_1 : X_1 \to \mathbf{B}\mathbb{G}_{\text{conn}}$ and $\nabla_2 : X_2 \to \mathbf{B}\mathbb{G}_{\text{conn}}$, a (strict) *contactomorphism between regular contact spaces* from ∇_1 to ∇_2 is a morphism between them in the slice $\mathbf{H}_{/\mathbf{B}\mathbb{G}_{\text{conn}}}$. The *$\infty$-groupoid of contactomorphisms* betwen ∇_1 and ∇_2 is

$$\text{ContactMorph}(\nabla_1, \nabla_2) := \Gamma\left([\nabla_1, \nabla_2]_{\mathbf{H}}\right) := \Gamma \prod_{\mathbf{B}\mathbb{G}_{\text{conn}}} [\nabla_1, \nabla_2] \,,$$

Remark 3.9.83. This means that a single contactomorphism from ∇_1 to ∇_2 is given by a diagram

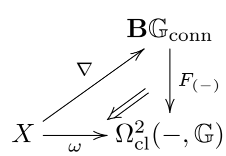


in $\mathbf{H}$. However, in order to obtain the correct cohesive structure on the collection of all contactomorphisms we need to *concretify* the object $[\nabla_1, \nabla_2]_{\mathbf{H}}$, as in the discussion at 3.9.6.4.

3.9.13.5 Quantomorphism group Famously, quantum theory is governed by the appearance of a group of quantum observables/operators called a *Heisenberg group*. But in fact the Heisenberg group is but the subgroup on *linear* translations in phase space of the full group of prequantum operators. In standard textbooks on geometric quantization the latter is called the *quantomorphism group*. While standard in geometric quantization, that term is rather less wide-spread in the physics literature. Many physics textbooks know the quantomorphism group, if at all, just as *the Fréchet-Lie group which integrates the Poisson bracket*.

Here we take the opposite perspective: we give a general abstract formalization of quantomorphism ∞-groups in a cohesive ∞-topos. We work out concrete differential-geometric incarnations of this in the context of smooth cohesion in section 4.4.20 Then, further below in section 3.9.13.7, we *define* the *Poisson ∞-Lie algebra* to be the ∞-Lie algebra of the quantomorphism ∞-group. Our main result below says that this reduces in the appropriate special cases not only to the traditional Poisson bracket Lie algebra in symplectic geometry, but also to the Poisson Lie-n-algebras of n-plectic geometry.

For a given presymplectic structure $\omega : X \to \Omega^2_{\text{cl}}(-, \mathbb{G})$, let then moreover

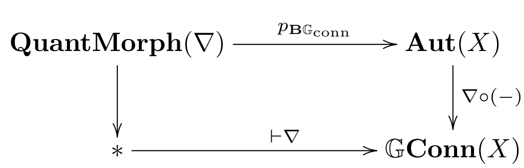


be a *prequantization* of (X, ω), def. 3.9.77.

Definition 3.9.84. A *quantomorphism* of the prequantum geometry (X, ∇) is an invertible *contactomorphism* from ∇ to itself, def. 3.9.82. The *discrete quantomorphism group* of the prequantum geometry (X, ∇) is the ∞-group of automorphisms

$$\text{QuantMorph}(\nabla) := \text{Aut}(\nabla) := \mathbf{H}_{/\mathbf{B}\mathbb{G}_{\text{conn}}}(\nabla, \nabla)_{\simeq}$$

of ∇ in the slice. The *cohesive quantomorphism group* of ∇ is the ∞-fiber is the ∞-pullback square



where the bottom right item is the object of $\mathbb{G}$-differential moduli from def. 3.9.50.

Remark 3.9.85. The ∞-pullback defining $\mathbf{QuantMorph}(\nabla)$ is a direct variant of the ∞-pullback diagram

$$\begin{array}{ccc} \mathbf{Aut}(\nabla)_{\mathbf{H}} & \longrightarrow & \mathbf{Aut}(X) \\ \downarrow & & \downarrow^{\nabla\circ(-)} \\ * & \xrightarrow{\vdash\nabla} & [X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \end{array},$$

which characterizes the ∞-group $\mathbf{Aut}(\nabla)_{\mathbf{H}}$ of def. 3.6.11 according to prop. 3.6.12. The difference between the two versions is that in the latter case the homotopy filling the diagram takes values in $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]$ and hence has, by remark **??**, U-plots of autoequivalences which are *stricter* than it should be for the desired automorphisms of ∇. Definition 3.9.84 precisely fixes this defect by exchanging the naive differential moduli stack by its concretification, according to the discussion in 3.9.6.4.

Proposition 3.9.86. *The discrete quantomorphism ∞-group is the underlying group of global points of the cohesive quantomorphism group, def. 3.9.84:*

$$\mathrm{QuantMorph}(\nabla) \simeq \Gamma\left(\mathbf{QuantMorph}\left(\nabla\right)\right) \simeq \mathbf{H}\left(*, \mathbf{QuantMorph}\left(\nabla\right)\right).$$

Proof. By prop. 3.6.8. ☐

Remark 3.9.87. So a global element of $\mathbf{QuantMorph}(\nabla)$ is a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & \swArrow & \\ {}_{\nabla}\searrow & & \swarrow_{\nabla} \\ & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \end{array}$$

and a U-plot of $\mathbf{QuantMorp}(\nabla)$ is a U-parameterized collection of such diagrams.

Notice that therefore the quantomorphism ∞-group is similar to an ∞-group of *bisections*, as in def. 3.6.95, of the atlas of a groupoid object, only that here the morphism in question is ∇ and not required to be 1-epimorphic as for an atlas, and that here we apply differential concretification to the result.

Definition 3.9.88. The *Hamiltonian symplectomorphism group* $\mathbf{HamSymp}(\nabla)$ of the prequantum geometry (X, ∇) is the 1-image, def. 3.6.37, of the morphism $p_{\mathbf{B}\mathbb{G}_{\mathrm{conn}}}$ of def. 3.9.84, hence the factorization of as a 1-empimorphism followed by a 1-monomorphism

$$p_{\mathbf{B}\mathbb{G}_{\mathrm{conn}}} : \mathbf{QuantMorph}(\nabla) \xrightarrow{p} \mathbf{HamSympl}(\nabla) \hookrightarrow \mathbf{Aut}(X).$$

Proposition 3.9.89. *There is a fiber sequence*

$$(\Omega\mathbb{G})\mathbf{FlatConn}(X) \hookrightarrow \mathbf{QuantMorph}(\nabla) \xrightarrow{p} \mathbf{HamSympl}(\nabla)$$

which exhibits $\mathbf{QuantMorph}(\nabla)$ *as an ∞-group extension of* $\mathbf{HamSympl}(\nabla)$ *by the differential moduli for flat $\Omega\mathbb{G}$-connections, def. 3.9.54*

Proof. By prop. 3.6.44 the homotopy fiber over the identity in $\mathbf{HamSympl}(\nabla)$ is equivalently that over the identity in $\mathbf{Aut}(X)$. By the pasting law, prop. 2.3.2, and using def. 3.9.84, we have a pasting diagram of ∞-pullbacks of the form

$$\begin{array}{ccc} \Omega_{\nabla}\left(\mathbb{G}\mathbf{Conn}\left(X\right)\right) & \longrightarrow & * \\ \downarrow & & \downarrow^{\vdash \mathrm{id}_X} \\ \mathbf{QuantMorph}(\nabla) & \longrightarrow & \mathbf{Aut}(X) \\ \downarrow & & \downarrow^{\nabla\circ(-)} \\ * & \xrightarrow{\vdash\nabla} & \mathbb{G}\mathbf{Conn}(X) \end{array}.$$

Since $\mathbb{G}$ is braided, $\mathbb{G}\mathbf{Conn}(X)$ inherits the structure of an ∞-group. Therefore

$$\Omega_\nabla(\mathbb{G}\mathbf{Conn}(X)) \simeq \Omega_0(\mathbb{G}\mathbf{Conn}(X))\,.$$

From this the statement follows with prop. 3.9.55. □

Proposition 3.9.90. *The ∞-group extension of* $\mathbf{HamSympl}(\nabla)$ *by* $\mathbf{QuantMorph}(\nabla)$ *is* central*, def. 3.6.247: the ∞-group cocycle that classifies it by remark 3.6.246*

$$c:\ \mathbf{BQuantMorph}(\nabla) \longrightarrow \mathbf{B}^2((\Omega\mathbb{G})\mathbf{FlatConn}(X))$$

is the delooping of the ∞-pullback of $\nabla \circ (-)$ along the canonical inclusion $\mathbf{B}\Omega(\mathbb{G}\mathbf{Conn}(X)) \to \mathbb{G}\mathbf{Conn}(X)$.

Proof. Consider the 1-image factorization, def. 3.6.31, of the defining ∞-pullback square in def. 3.9.84:

$$\begin{array}{ccccc}
\mathbf{QuantMorph}(\nabla) & \twoheadrightarrow & \mathbf{HamSympl}(\nabla) & \hookrightarrow & \mathbf{Aut}(X) \\
\downarrow & & \downarrow{\scriptstyle \Omega c} & & \downarrow{\scriptstyle \nabla\circ(-)} \\
* & \twoheadrightarrow & \mathbf{B}\left((\Omega\mathbb{G})\,\mathbf{FlatConn}\,(X)\right) & \hookrightarrow & \mathbb{G}\mathbf{Conn}(X)
\end{array}
\quad .$$

(with the bottom composite $* \to \mathbb{G}\mathbf{Conn}(X)$ labeled $\vdash\nabla$)

Here in the bottom row in the middle we identified the 1-image of the name of ∇ in $\mathbb{G}\mathbf{Conn}(X)$ with

$$\mathbf{B}\left(\Omega_\nabla\left(\mathbb{G}\mathbf{Conn}(X)\right)\right) \simeq \mathbf{B}\left(\Omega_0\left(\mathbb{G}\mathbf{Conn}(X)\right)\right) \simeq \mathbf{B}\left((\Omega\mathbb{G})\,\mathbf{FlatConn}(X)\right)$$

again by the ∞-group structure on $\mathbb{G}\mathbf{Conn}(X)$ and by prop. 3.9.55, respectively.

Observe then that both sub-squares appearing here are again ∞-pullbacks themselves: by the pasting law, prop. 2.3.2, the total ∞-pullback is computed by two consecutive ∞-pullback squares and both 1-monomorphisms as well as 1-epimorphisms are preserved under pullback (the first because the are the right class of an orthogonal factorization system, prop. 3.6.33, the second by prop. 2.3.5), so that the pullback along the bottom 1-image factorization produces the top 1-image factorization (due to essential uniqueness of the factorization).

This means, again with the pasting law, that we have a total pasting diagram of ∞-pullback squares of the form

$$\begin{array}{ccccc}
(\Omega\mathbb{G})\mathbf{FlatConn}(X) & \longrightarrow & * & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{QuantMorph}(\nabla) & \twoheadrightarrow & \mathbf{HamSympl}(\nabla) & \hookrightarrow & \mathbf{Aut}(X) \\
\downarrow & & \downarrow{\scriptstyle \Omega c} & & \downarrow{\scriptstyle \nabla\circ(-)} \\
* & \twoheadrightarrow & \mathbf{B}\left((\Omega\mathbb{G})\,\mathbf{FlatConn}\,(X)\right) & \hookrightarrow & \mathbb{G}\mathbf{Conn}(X)
\end{array}
\quad .$$

(with the bottom composite $* \to \mathbb{G}\mathbf{Conn}(X)$ labeled $\vdash\nabla$)

In the left half this now exhibits the long homotopy fiber sequence of ∞-group homomorphisms

$$(\Omega\mathbb{G})\,\mathbf{FlatConn}\,(X) \longrightarrow \mathbf{QuantMorph}\,(\nabla) \longrightarrow \mathbf{HamSympl}\,(\nabla) \xrightarrow{\Omega c} \mathbf{B}\left((\Omega\mathbb{G})\,\mathbf{FlatConn}\,(X)\right)\ .$$

Finally, this sequence is indeed deloopable, by prop 3.6.47. □

Remark 3.9.91. Under restriction of the quantomorphism ∞-group to the *Heisenberg ∞-group* which we discuss below in 3.9.13.6, the cocycle c above generalizes the traditional *Heisenberg group cocycle* to higher cohesive geometry.

3.9.13.6 Heisenberg group

Definition 3.9.92. For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group and

$$\phi : G \to \mathbf{HamSympl}(\nabla)$$

a group homomorphism, hence an ∞-action of G on X by Hamiltonian symplectomorphisms, consider the ∞-pullback of the quantomorphism ∞-group extension along this map, hence the left vertical fiber sequence in the pasting diagram of ∞-pullbacks

$$\begin{array}{ccc} \mathbb{G} & \longrightarrow & \mathbb{G} \\ \downarrow & & \downarrow \\ \phi^*\mathbf{QuantMorph}(\nabla) & \longrightarrow & \mathbf{QuantMorph}(\nabla) \\ \downarrow & & \downarrow \\ G & \stackrel{\phi}{\longrightarrow} & \mathbf{HamSympl}(\nabla) \end{array} .$$

If ϕ is a 1-monomorphism this is called the*Heisenberg group* induced by ϕ.

Remark 3.9.93. If $\mathbf{c} : \mathbf{BHamSympl} \to \mathbf{B}^2\mathbb{G}$ is the ∞-group cocycle that classifies the quantomorphism group extension, then

$$\phi^*\mathbf{c} : \ \mathbf{B}G \xrightarrow{\mathbf{B}\phi} \mathbf{BHamSympl} \xrightarrow{\mathbf{c}} \mathbf{B}^2\mathbb{G}$$

is the group cocylce that classifies the Heisenberg group extension relative ϕ. This is due to the pasting law for ∞-pullbacks.

$$\begin{array}{ccccc} \mathbf{B}\phi^*\mathbf{QuantMorp} & \longrightarrow & \mathbf{QuantMorph} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}G & \stackrel{\mathbf{B}\phi}{\longrightarrow} & \mathbf{BHamSympl} & \stackrel{\mathbf{c}}{\longrightarrow} & \mathbf{B}^2\mathbb{G} \end{array}$$

3.9.13.7 Poisson and Heisenberg Lie algebra

We consider now the ∞-Lie algebras of these ∞-groups in prequantum geometry.

Definition 3.9.94.

- The ∞-Lie algebra

  $$\mathfrak{poisson}(X, \hat{\omega}) := \mathrm{Lie}(\mathbf{QuantMorph}(\nabla))$$

  of the quantomorphism group we call the *Poisson ∞-Lie algebra* of the prequantum geometry (X, ∇).

- The ∞-Lie algebra of the Hamiltonian symplectomorphisms

  $$\mathcal{X}_{\mathrm{Ham}}(X, \hat{\omega}) := \mathrm{Lie}(\mathbf{HamSympl}(\nabla))$$

  we call the ∞-Lie algebra of *Hamiltonian vector fields* of the prequantum geometry.

Remark 3.9.95. If X has a linear structure (the structure of a vector space) and ω is constant on X, then we can consider the sub ∞-Lie algebra of $\mathfrak{poisson}(X, \hat{\omega})$ on the constant and linear elements. We discuss realizations of this below in 4.4.20.1. This sub ∞-Lie algebra we call the *Heisenberg ∞-Lie algebra*

$$\mathfrak{heis}(\nabla) \hookrightarrow \mathfrak{poisson}(\nabla) .$$

The corresponding sub-∞-group we call the *Heisenberg ∞-group*

$$\mathbf{Heis}(\nabla) \hookrightarrow \mathbf{QuantMorph}(\nabla) .$$

3.9.13.8 Courant groupoids We consider the evident variant of *Atiyah groupoids*, def. 3.6.104, induced by differential cocycles, taking differential concretification into accont. The resulting notion reproduces, under Lie differentiation, the traditional notion of *Courant Lie algebroids*, and so we speak of *Courant groupoids*.

Let $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ be a braided ∞-group, def. 3.6.119, which is precisely $(n-1)$-truncated, def. 3.6.22. Write

$$\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}} \in \mathbf{H}$$

for the differential coefficient object of def. 3.9.46 and write

$$\mathbf{B}\mathbb{G}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}}$$

for the canonical projection morphism of remark 3.9.49.

Definition 3.9.96. For $n \in \mathbb{N}$, $\mathbb{G}$ a braided ∞-group which is precisely $(n-1)$-truncated, given a map

$$\nabla_{n-1} : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}}$$

in $\mathbf{H}$, its *cohesive Atiyah-Courant bisection group* $\mathbf{AtiyahCourant}(\nabla_{n-1}) \in \mathrm{Grp}(\mathbf{H})$ is the ∞-pullback in

$$\begin{array}{ccc} \mathbf{AtiyahCourant}(\nabla_{n-1}) & \longrightarrow & \mathbf{Aut}(X) \\ \downarrow & & \downarrow^{\nabla\circ(-)} \\ * & \xrightarrow{\vdash \nabla_{n-1}} & \mathbb{G}\mathbf{Conn}_{n-1}(X) \end{array} ,$$

where the bottom right object is the differential moduli stack of $(n-1)$-$\mathbb{G}$-principal connections of def. 3.9.52.

Remark 3.9.97. Given a $\mathbb{G}$-princpal ∞-connection $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ write ∇_{n-1} specifically for its truncation, hence for the composite

$$\nabla_{n-1} : \; X \xrightarrow{\nabla} \mathbf{B}\mathbb{G}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}} \;\; ,$$

where the second map is the canonical projection. By the universal property/functoriality of ∞-pullbacks, the canonical projection

$$\mathbb{G}\mathbf{Conn}(X) \to \mathbb{G}\mathbf{Conn}_{n-1}(X)$$

of remark 3.9.53 induces a canonical ∞-group homomorphism

$$\mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{AtiyaCourant}(\nabla_{n-1})$$

from the cohesive quantomorphism ∞-group, def. 3.9.84, to the cohesive Atiyah-Courant ∞-group, def. 3.9.96.

3.9.13.9 Prequantum states Given a prequantum geometry

$$X \xrightarrow{\nabla} \mathbf{B}\mathbb{G}_{\mathrm{conn}} \xrightarrow{F_{(-)}} \Omega^2_{\mathrm{cl}}(-, \mathbb{G})$$

as above, choose now finally a representation, def. 3.6.152, of $\mathbb{G}$, hence a fiber sequence in $\mathbf{H}$ of the form

$$V \longrightarrow V/\!/\mathbb{G} \xrightarrow{\rho} \mathbf{B}\mathbb{G} \; .$$

For $U_{\mathbf{B}\mathbb{G}} : \; \mathbf{B}\mathbb{G}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathbb{G}$ the forgetful morphism, we obtain from the prequantum connection $\nabla \in \mathbf{H}_{/\mathbf{B}\mathbb{G}_{\mathrm{conn}}}$ the underlying modulus

$$\sum_{U_{\mathbf{B}\mathbb{G}}} \nabla \in \mathbf{H}_{/\mathbf{B}\mathbb{G}}$$

of the prequantum bundle proper.

Definition 3.9.98. The ρ-associated V-fiber bundle

$$E := \left(\sum_{U_{\mathbf{B}\mathbb{G}}} \nabla\right)^* \rho \in \mathbf{H}_{/X}$$

to $\sum\limits_{U_{\mathbf{B}\mathbb{G}}} \nabla$, def. 3.6.209, we call the *prequantum V-bundle* (or just *prequantum line bundle* if V is equipped compatibly with a ring structure).

Remark 3.9.99. If we write $P \to X$ for the total space projection of the prequantum bundle, sitting in the ∞-pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sum_{U_{\mathbf{B}\mathbb{G}}} \nabla} & \mathbf{B}\mathbb{G} \end{array} ,$$

then by prop. 3.6.209 the total space projection of the prequantum line bundle is the left morphism in the ∞-pullback diagram

$$\begin{array}{ccc} P \times_{\mathbb{G}} V & \longrightarrow & V//\mathbb{G} \\ \downarrow & & \downarrow^{\rho} \\ X & \xrightarrow{\sum_{U_{\mathbf{B}\mathbb{G}}} \nabla} & \mathbf{B}\mathbb{G} \end{array} .$$

Definition 3.9.100. The space of sections, def. 3.6.220, of the prequantum line bundle

$$\mathbf{\Gamma}_X(E) \in \mathbf{H}$$

we call the *prequantum space of states.*

Remark 3.9.101. By prop. 3.6.230 the prequantum space of states is equivalently expressed as

$$\mathbf{\Gamma}_X(E) \simeq \prod_{\mathbf{B}\mathbb{G}} \left[\sum_U \nabla, \rho\right] .$$

3.9.13.10 Prequantum operators

Definition 3.9.102. The *prequantum operator action* of the quantomorphism group $\mathbf{QuantMorph}(\nabla)$, def. 3.9.84, on the space of prequantum states $\mathbf{\Gamma}_X(E)$, def. 3.9.100, is the action, def. 3.6.152,

$$\begin{array}{ccc} \mathbf{\Gamma}_X(E) & \longrightarrow & \mathbf{\Gamma}_X(E)//\mathbf{QuantMorph}(\nabla) \\ & & \downarrow^{\rho_{\mathrm{prequant}}} \\ & & \mathbf{BQuantMorph}(\nabla) \end{array}$$

given by the canonical precomposition action, example 3.6.241, of $\mathbf{Aut}_{\mathbf{H}}(\sum\limits_U \nabla)$ on $\mathbf{\Gamma}_X(E) \simeq \prod\limits_{\mathbf{B}\mathbb{G}} \left[\sum\limits_U \nabla, \rho\right]_{\mathbf{H}}$ (remark 3.9.101) restricted to a $\mathbf{QuantMorph}(\nabla) := \mathbf{Aut}_{\mathbf{H}}(\nabla)$-action, def. 3.6.238, along the canonical morphism $p_U : \mathbf{Aut}_{\mathbf{H}}(\nabla) \to \mathbf{Aut}_{\mathbf{H}}(\sum\limits_U \nabla)$.

Remark 3.9.103. The prequantum operator action of def. 3.9.102 is exhibited by the following pasting diagram of ∞-pullback squares.

$$
\begin{array}{ccccc}
\mathbf{\Gamma}_X(E) \simeq \prod\limits_{\mathbf{B}\mathbb{G}}[\sum\limits_U \nabla, \rho] & \longrightarrow & \prod\limits_{\mathbf{B}\mathbb{G}}\left(\left[\sum\limits_U\nabla,\rho\right]/\!/\prod\limits_U \mathbf{Aut}(\nabla)\right) & \longrightarrow & \prod\limits_{\mathbf{B}\mathbb{G}}\left(\left[\sum\limits_U\nabla,\rho\right]/\!/\mathbf{Aut}(\sum\limits_U\nabla)\right) \; .\\
\downarrow & & \downarrow & & \downarrow \\
 & & \mathbf{B}\prod\limits_{\mathbf{B}\mathbb{G}}\left(\prod\limits_U \mathbf{Aut}\,(\nabla)\right) & \xrightarrow{\mathbf{B}\prod_{\mathbf{B}\mathbb{G}}(p_U)} & \mathbf{B}\prod\limits_{\mathbf{B}\mathbb{G}}\left(\mathbf{Aut}\left(\sum\limits_U\nabla\right)\right)\\
\downarrow & & \| & & \| \\
* & \longrightarrow & \mathbf{B}\,(\mathbf{Aut}_{\mathbf{H}}(\nabla)) & \longrightarrow & \mathbf{B}\left(\mathbf{Aut}_{\mathbf{H}}(\sum\limits_U\nabla)\right)\\
\| & & \| & & \\
* & \longrightarrow & \mathbf{B}\,(\mathbf{QuantMorph}(\nabla)) & &
\end{array}
$$

This uses that the dependent product is right adjoint and hence preserves ∞-pullbacks (as well as group structure).

Remark 3.9.104. A prequantum state is given by a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & V/\!/\mathbb{G} \\
 & \swarrow\!\!\!\swarrow & \\
{\scriptstyle \sum_U \nabla} \searrow & & \swarrow {\scriptstyle \rho} \\
 & \mathbf{B}\mathbb{G} &
\end{array}
$$

and a prequantum operator by a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & X \; . \\
 & \swarrow\!\!\!\swarrow_{O} & \\
{\scriptstyle \nabla} \searrow & & \swarrow {\scriptstyle \rho} \\
 & \mathbf{B}\mathbb{G}_{\mathrm{conn}} &
\end{array}
$$

Then the result of the action is the new prequantum state $O(\psi)$ given by the pasting diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\phi} & X & \xrightarrow{\psi} & V/\!/\mathbb{G} \\
 & {\scriptstyle \nabla}\searrow & \downarrow{\scriptstyle \nabla} & & \\
{\scriptstyle \sum_U \nabla} & & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & & \swarrow{\scriptstyle \rho} \\
 & \searrow & \downarrow{\scriptstyle U} & \swarrow & \\
 & & \mathbf{B}\mathbb{G} & &
\end{array}
$$

(where all the 2-cells are notationally suppressed, for readability).

3.10 Structures in a differential ∞-topos

We discuss a list of differential geometric notions that can be formulated in the presence of the axioms for infinitesimal cohesion, 3.9. These structures parallel the structures in a general cohesive ∞-topos, 3.9.

- 3.10.1 – Infinitesimal path ∞-groupoid and de Rham spaces;
- 3.10.2 – Crystalline cohomology, flat infinitesimal ∞-connections and local systems;
- 3.10.3 – Jet ∞-bundles;
- 3.10.4 – Infinitesimal Galois theory / Formally étale morphisms;
- 3.10.5 – Formally étale groupoids;
- 3.10.6 – Manifolds (separated);
- 3.10.8 – Critical loci, variational calculus and BV-BRST complexes;
- 3.10.9 – Formal cohesive ∞-groupoids.

3.10.1 Infinitesimal path ∞-groupoid and de Rham spaces

We discuss the infinitesimal analog of the *path ∞-groupoid*, 3.8.3, which exists in a context of infinitesimal cohesion, def. 3.5.1.

Let $(i_! \dashv i^* \dashv i_* \dashv i^1) : \mathbf{H} \to \mathbf{H}_{\mathrm{th}}$ be an infinitesimal neighbourhood of a cohesive ∞-topos.

Definition 3.10.1. Write

$$(\mathbf{Red} \dashv \mathbf{\Pi}_{\mathrm{inf}} \dashv \flat_{\mathrm{inf}}) : (i_! i^* \dashv i_* i^* \dashv i_* i^!) : \mathbf{H}_{\mathrm{th}} \to \mathbf{H}_{\mathrm{th}}$$

for the adjoint triple induced by the adjoint quadruple that defines the differential cohesion. For $X \in \mathbf{H}_{\mathrm{th}}$ we say that

- $\mathbf{\Pi}_{\mathrm{inf}}(X)$ is the *infinitesimal path ∞-groupoid* of X;

  The $(i^* \dashv i_*)$-unit

  $$X \to \mathbf{\Pi}_{\mathrm{inf}}(X)$$

  we call the *constant infinitesimal path inclusion*.

- $\mathbf{Red}(X)$ is the *reduced cohesive ∞-groupoid* underlying X.

  The $(i_* \dashv i^*)$-counit

  $$\mathbf{Red} X \to X$$

  we call the *inclusion of the reduced part* of X.

Remark 3.10.2. This is an abstraction of the setup considered in [SiTe]. In traditional contexts as considered there, the object $\mathbf{\Pi}_{\mathrm{inf}}(X)$ is called the *de Rham space* of X or the *de Rham stack* of X. Here we may tend to avoid this terminology, since by 3.9.3 we have a good notion of intrinsic de Rham cohomology in every cohesive ∞-topos already without equipping it with infinitesimal cohesion, which, over some $X \in \mathbf{H}$ is co-represented by the object $\mathbf{\Pi}_{\mathrm{dR}}(X)$, the cohesive de Rham homotopy type of remark 3.9.19. On the other hand, $\mathbf{\Pi}_{\mathrm{inf}}$ co-represents instead what is called *crystalline cohomology*, 3.10.2 below.

Proposition 3.10.3. *In the notation of def. 3.5.4, there is a canonical natural transformation*

$$\mathbf{\Pi}_{\mathrm{inf}}(X) \to \mathbf{\Pi}(X)$$

that factors the finite path inclusion through the infinitesimal path inclusion

$$\begin{array}{ccc} & & \mathbf{\Pi}_{\mathrm{inf}}(X) \\ & \nearrow & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}(X) \end{array} .$$

Dually there is a canonical natural transformation

$$\flat A \to \flat A$$

that factors the $\flat$*-counits*

$$\begin{array}{ccc} \flat A & & \\ \downarrow & \searrow & \\ \flat_{\mathrm{inf}} A & \longrightarrow & A \end{array} .$$

Proof. By def. 3.5.4 this is just the formula for the unit of the composite adjunction

$$(\mathbf{\Pi}_{\mathbf{H}_{\mathrm{th}}} \dashv \flat_{\mathbf{H}_{\mathrm{th}}}) : \ \mathbf{H}_{\mathrm{th}} \underset{\mathrm{Disc}_{\mathrm{inf}}}{\overset{\Pi_{\mathrm{inf}}}{\rightleftarrows}} \mathbf{H} \underset{\mathrm{Disc}}{\overset{\Pi}{\rightleftarrows}} \infty\mathrm{Grpd} \ ,$$

more explicitly given by

$$\begin{array}{ccc} & & \mathrm{Disc}_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}(X) \\ & \nearrow & \downarrow \\ X & \longrightarrow & \mathrm{Disc}_{\mathrm{inf}} \circ \mathrm{Disc}_{\mathrm{H}} \circ \Pi_{\mathbf{H}} \circ \Pi_{\mathrm{inf}}(X) \end{array} .$$

The case for $\flat$ is formally dual. □

3.10.2 Crystalline cohomology, flat infinitesimal connections and local systems

Definition 3.10.4. For $X \in \mathbf{H}_{\mathrm{th}}$ an object, we call the cohomology, def. 3.6.137 of $\mathbf{\Pi}_{\mathrm{inf}}(X)$ the *crystalline cohomology* of X.

We discuss now the infinitesimal analog of intrinsic flat cohomology, 3.8.5.

Definition 3.10.5. For $X \in \mathbf{H}_{\mathrm{th}}$ an object, we call the cohomology, def. 3.6.137 of $\mathbf{\Pi}_{\mathrm{inf}}(X)$ the *crystalline cohomology* of X. More specificaly, for $A \in \mathbf{H}_{\mathrm{th}}$ we say that

$$H_{\mathrm{infflat}}(X, A) := \pi_0 \mathbf{H}(\mathbf{\Pi}_{\mathrm{inf}}(X), A) \simeq \pi_0 \mathbf{H}(X, \flat_{\mathrm{inf}} A)$$

is the *infinitesimal flat cohomology* of X with coefficient in A.

Remark 3.10.6. That traditional crystalline cohomology is the cohomology of the "de Rham stack", see remark 3.10.2 above with coefficients in a suitable stack is discussed in [Lurie09c], above theorem 0.4. The relation to de Rham cohomology in traditional contexts is discussed for instance in [SiTe].

Remark 3.10.7. By observation 3.10.3 we have canonical natural morphisms

$$\mathbf{H}_{\mathrm{flat}}(X, A) \to \mathbf{H}_{\mathrm{infflat}}(X, A) \to \mathbf{H}(X, A)$$

The objects on the left are principal ∞-bundles equipped with flat ∞-connection. The first morphism forgets their higher parallel transport along finite volumes and just remembers the parallel transport along infinitesimal volumes. The last morphism finally forgets also this connection information.

Definition 3.10.8. For $A \in \mathbf{H}_{\mathrm{th}}$ a 0-truncated abelian ∞-group object we say that the *de Rham theorem* for A-coefficients holds in $\mathbf{H}_{\mathrm{th}}$ if for all $X \in \mathbf{H}_{\mathrm{th}}$ the infinitesimal path inclusion of observation 3.10.3

$$\mathbf{\Pi}_{\mathrm{inf}}(X) \to \mathbf{\Pi}(X)$$

is an equivalence in A-cohomology, hence if for all $n \in \mathbb{N}$ we have that

$$\pi_0 \mathbf{H}_{\mathrm{th}}(\mathbf{\Pi}(X), \mathbf{B}^n A) \to \pi_0 \mathbf{H}_{\mathrm{th}}(\mathbf{\Pi}_{\mathrm{inf}}(X), \mathbf{B}^n A)$$

is an isomorphism.

If we follow the notation of remark 3.10.6 and moreover write $|X| = |\Pi X|$ for the intrinsic geometric realization, def. 3.8.2, then this becomes

$$H^\bullet_{\mathrm{dR,th}}(X, A) \simeq H^\bullet(|X|, A_{\mathrm{disc}}) \,,$$

where on the right we have ordinary cohomology in Top (for instance realized as singular cohomology) with coefficients in the discrete group $A_{\mathrm{disc}} := \Gamma A$ underlying the cohesive group A.

In certain contexts of infinitesimal neighbourhoods of cohesive ∞-toposes the de Rham theorem in this form has been considered in [SiTe]. We discuss a realization below in 4.5.3.

3.10.3 Jet bundles

In the presence of infinitesimal cohesion there is a canonical higher analog notion of *jet bundles*: the generalization of tangent bundles to higher order infinitesimals (higher order tangents).

Definition 3.10.9. For any object $X \in \mathbf{H}$ write

$$\mathrm{Jet} : \mathbf{H}_{/X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathbf{H}_{/\mathbf{\Pi}_{\mathrm{inf}}(X)}$$

for the base change geometric morphism, prop. 3.6.13, induced by the constant infinitesimal path inclusion $i : X \to \mathbf{\Pi}_{\mathrm{inf}}(X)$, def. 3.10.1.

For $(E \to X) \in \mathbf{H}_{/X}$ we call $\mathrm{Jet}(E) \to \mathbf{\Pi}_{\mathrm{inf}}(X)$ as well as its pullback $i^*\mathrm{Jet}(E) \to X$ (if the context is clear) the *jet ∞-bundle* of $E \to X$.

Remark 3.10.10. In the context over an algebraic site the construction of def. 3.10.9 reduces to the construction in section 2.3.2 of [BeDr04], see [Paug11] for a review.

3.10.4 Infinitesimal Galois theory / Formally étale morphisms

In every context of infinitesimal cohesion, there are canonical induced notions of morphisms being *formally étale*, meaning that at least on infinitesimal neighbourhoods of every point they behave like the analog of what in topology is a *local homeomorphism/étale map*. Close cousins of this are the notions of *formally smooth* and of *formally unramified* morphisms.

We first discuss formal étaleness in $\mathbf{H}$. Below in def. 3.10.19 we discuss the notion more generally in $\mathbf{H}_{\mathrm{th}}$.

Definition 3.10.11. We say an object $X \in \mathbf{H}_{\mathrm{th}}$ is *formally smooth* if the constant infinitesimal path inclusion, def. 3.10.1,

$$X \to \mathbf{\Pi}_{\mathrm{inf}}(X)$$

is an effective epimorphism, def. 2.3.3.

Remark 3.10.12. In this form this is the direct ∞-categorical analog of the characterization of formal smoothness in [SiTe]. The following equivalent reformulation corresponds in turn to the discussion in section 4.1 of [RoKo04].

Definition 3.10.13. Write

$$\phi : i_! \to i_*$$

for the canonical natural transformation given as the composite

$$i_! \xrightarrow{\eta i_!} \mathbf{\Pi}_{\mathrm{inf}} i_! \xrightarrow{:=} i_* i^* i_! \xrightarrow{\simeq} i_* \ .$$

Since the last composite on the right here is an equivalence due to $i_!$ being fully faithful we have:

Proposition 3.10.14. *An object $X \in \mathbf{H} \stackrel{i_!}{\hookrightarrow} \in \mathbf{H}_{\mathrm{th}}$ is formally smooth according to def. 3.10.11 precisely if the canonical morphism*

$$\phi : i_! X \to i_* X$$

is an effective epimorphism.

Remark 3.10.15. In this form this characterization of formal smoothness is the evident generalization of the condition given in section 4.1 of [RoKo04]. (Notice that the notation there is related to the one used here by $u^* = i_!$, $u_* = i^*$ and $u^! = i_*$.)

Therefore with [RoKo04] we have the following more general definitions.

Definition 3.10.16. For $f : X \to Y$ a morphism in $\mathbf{H}$, we say that

1. f is a *formally smooth morphism* if the canonical morphism

   $$i_! X \to i_! Y \prod_{i_* Y} i_* Y$$

   is an effective epimorphism;

2. f is a *formally étale morphism* if this morphism is an equivalence, equivalently if the naturality square

   $$\begin{array}{ccc} i_! X & \xrightarrow{i_! f} & i_! Y \\ \downarrow^{\phi_X} & & \downarrow^{\phi_Y} \\ i_* X & \xrightarrow{i_* f} & i_* Y \end{array}$$

   is an ∞-pullback square.

3. f is a *formally unramified morphism* if this is a (-1)-truncated morphism. More generally, f is an *order-k formally unramified morphisms* for $(-2) \leq k \leq \infty$ if this is a k-truncated morphism ([LuHTT], 5.5.6).

Remark 3.10.17. An order-(-2) formally unramified morphism is equivalently a formally étale morphism. Only for 0-truncated X does formal smoothness together with formal unramifiedness imply formal étaleness.

Remark 3.10.18. The idea of characterizing étale morphisms with respect to a notion of *infinitesimal extension* as those making certain naturality squares into pullback squares goes back to lectures by André Joyal in the 1970s, as is recalled in the introduction of [Dub00]. Notice that in sections 3 and 4 there the analog of our functor $i_!$ is assumed to be the inverse image of a geometric morphism, whereas here we only require it to be a left adjoint and to preserve finite products, as opposed to all finite limits. Indeed, it will fail to preserve general pullbacks in most models for infinitesimal cohesion of interest, such as the one discussed below in 4.5. In [**?**] a different kind of axiomatization, by way of closure properties. This we discuss further below, see remark 3.10.30.

The characterization of formal étaleness by cartesian naturality squares induced specifically by adjoint triples of functors, as in our def. 3.10.11, appears around prop. 5.3.1.1 of [RoKo04].

But in view of prop. 3.10.11, which applies to objects in $\mathbf{H}_{\mathrm{th}}$ not necessarily in the image of the inclusion $i_!$, and in view of def. 3.10.13 it is natural to generalize further:

Definition 3.10.19. A morphism $f : X \to Y$ in $\mathbf{H}_{\mathrm{th}}$ is a *formally étale morphism* if the naturality diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(X) \\ \downarrow^{f} & & \downarrow^{\mathbf{\Pi}_{\mathrm{inf}}(f)} \\ Y & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(Y) \end{array}$$

of the infinitesimal path inclusion, def. 3.10.1, is an ∞-pullback.

Remark 3.10.20. Def. 3.10.19 is compatible with def. 3.10.16 in that a morphism $f \in \mathbf{H}$ is formally étale in the sense of the former precisely if $i_! f \in \mathbf{H}_{\mathrm{th}}$ is formally étale in the sense of the latter.

Remark 3.10.21. This condition is the immediate infinitesimal analog of the notion of $\mathbf{\Pi}$-*closure* in def. 3.8.21: we may say equivalently that a morphism $f \in \mathbf{H}_{\mathrm{th}}$ is formally étale precisely if it is $\mathbf{\Pi}_{\mathrm{inf}}$-*closed*. Moreover, by the discussion in 3.8.6 the $\mathbf{\Pi}$-closed morphisms into some X are interpreted as the total space projections of *locally constant* ∞*-stacks* over X by general abstract Galois theory. Accordingly here we may think of $\mathbf{\Pi}_{\mathrm{inf}}$-closed morphisms into X as total space projections of more general ∞-stacks over X by what we may call general abstract *infinitesimal Galois theory*. This perspective we develop below in 3.10.7.

In particular, we have the following immediate infinitesimal analogs of properties of $\mathbf{\Pi}$-closure.

Definition 3.10.22. Call a morphism $f : X \to Y$ in $\mathbf{H}_{\mathrm{th}}$ a $\mathbf{\Pi}_{\mathrm{inf}}$*-equivalence* if $\mathbf{\Pi}_{\mathrm{inf}}(f)$ is an equivalence.

Proposition 3.10.23. *For $i : \mathbf{H} \to \mathbf{H}_{\mathrm{th}}$ a differentially cohesive ∞-topos, the pair of classes of morphisms*

$$(\mathbf{\Pi}_{\mathrm{inf}}\text{-}\textit{equivalences},\ \textit{formally étale morphisms}) \subset \mathrm{Mor}(\mathbf{H}_{\mathrm{th}}) \times \mathrm{Mor}(\mathbf{H}_{\mathrm{th}})$$

constitutes an orthogonal factorization system.

Proof. Since $\mathbf{\Pi}_{\mathrm{inf}}$ has the left adjoint **Red** it preserves all ∞-pullbacks and hence in particular those over objects of the form $\mathbf{\Pi}_{\mathrm{inf}}(X)$. Therefore factorization follows as in the proof of prop. 3.8.25. Accordingly, orthogonality follows as in the proof of prop. 3.8.26. □

This and the fact that $\mathbf{\Pi}_{\mathrm{inf}}$ preserves ∞-limits implies a wealth of stability properties of formally étale maps.

Corollary 3.10.24. *Formally étale morphisms in $\mathbf{H}_{\mathrm{th}}$, def. 3.10.19, satisfy the following stability properties*

1. *Every equivalence is formally étale.*
2. *The composite of two formally étale morphisms is itself formally étale.*

3. *If*

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & & \searrow^{g} \\ X & \xrightarrow{h} & Z \end{array}$$

is a diagram such that g and h are formally étale, then also f is formally étale.

4. *Any retract of a formally étale morphisms is itself formally étale.*

5. *The ∞-pullback of a formally étale morphisms is formally étale.*

But since the embedding functor $i_!$ does not preserve ∞-limits in general, closure under pullback in $\mathbf{H}$ requires a condition on the codomain:

Proposition 3.10.25. *The collection of formally étale morphisms in $\mathbf{H}$, def. 3.10.16, is closed under the following operations.*

1. *Every equivalence is formally étale.*

2. *The composite of two formally étale morphisms is itself formally étale.*

3. *If*

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & & \searrow^{g} \\ X & \xrightarrow{h} & Z \end{array}$$

is a diagram such that g and h are formally étale, then also f is formally étale.

4. *Any retract of a formally étale morphisms is itself formally étale.*

5. *The ∞-pullback of a formally étale morphisms is formally étale if the pullback is preserved by $i_!$.*

Remark 3.10.26. The statements about closure under composition and pullback appears as prop. 5.4, prop. 5.6 in [RoKo04]. The extra assumption that $i_!$ preserves the pullback is implicit in their setup.

Proof. The first statement follows trivially because ∞-pullbacks are well defined up to equivalence. The second two statements follow by the pasting law for ∞-pullbacks, prop. 2.3.2: let $f : X \to Y$ and $g : Y \to Z$ be two morphisms and consider the pasting diagram

$$\begin{array}{ccccc} i_! X & \xrightarrow{i_! f} & i_! Y & \xrightarrow{i_! g} & Z \\ \downarrow & & \downarrow & & \downarrow \\ i_* X & \xrightarrow{i_* f} & i_* Y & \xrightarrow{i_* g} & i_* Z \end{array} .$$

If f and g are formally étale then both small squares are pullback squares. Then the pasting law says that so is the outer rectangle and hence $g \circ f$ is formally étale. Similarly, if g and $g \circ f$ are formally étale then the right square and the total reactangle are pullbacks, so the pasting law says that also the left square is a pullback and so also f is formally étale.

For the fourth claim, let $\mathrm{Id} \simeq (g \to f \to g)$ be a retract in he arrow ∞-category $\mathbf{H}^I$. By applying the natural transformation $\phi : i_! \to i_*$ this becomes a retract

$$\mathrm{Id} \simeq ((i_! g \to i_* g) \to (i_! f \to i_* f) \to (i_! g \to i_* g))$$

in the category of squares $\mathbf{H}^{\square}$. By assumption the middle square is an ∞-pullback square and we need to show that the also the outer square is. This follows generally: a retract of an ∞-limiting cone is itself ∞-limiting. To see this, we invoke the presentation of ∞-limits by *derivators* (thanks to Mike Shulman for this argument): we have

1. ∞-limits in $\mathbf{H}$ are computed by homotopy limits in an presentation by a model category $K := [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{loc}}$ [LuHTT];

2. for $j : J \to J^{\triangleleft}$ the inclusion of a diagram into its cone (the join with an initial element), the homotopy limit over C is given by forming the right Kan extension $j_* : \mathrm{Ho}(K^J(W^J)^{-1}) \to \mathrm{Ho}(K^{J^{\triangleleft}}(W^{J^{\triangleleft}})^{-1})$,

3. a $J^{\triangleleft}$-diagram F is a homotopy limiting cone precisely if the unit

$$F \to j_* j^* F$$

us an isomorphism.

Therefore we have a retract in $[\Delta[1], [\Box, K$

$$\begin{array}{ccccc} (i_! g \to i_! g) & \longrightarrow & (i_! f \to i_! f) & \longrightarrow & (i_! g \to i_! g) \\ \downarrow & & \downarrow & & \downarrow \\ j^* j_* (i_! g \to i_! g) & \longrightarrow & j^* j_* (i_! f \to i_! f) & \longrightarrow & j^* j_* (i_! g \to i_! g) \end{array} ,$$

where the middle morphism is an isomorphism. Hence so is the outer morphism and therefore also g is formally étale.

For the last claim, consider an ∞-pullback diagram

$$\begin{array}{ccc} A \times_Y X & \longrightarrow & X \\ \downarrow^{p} & & \downarrow^{f} \\ A & \longrightarrow & Y \end{array}$$

where f is formally étale. Applying the natural transformation $\phi : i_! \to i_*$ to this yields a square of squares. Two sides of this are the pasting composite

$$\begin{array}{ccccc} i_! A \times_Y X & \longrightarrow & i_! X & \xrightarrow{\phi_X} & i_* X \\ \downarrow^{i_! p} & & \downarrow^{i_! f} & & \downarrow^{i_* f} \\ i_! A & \longrightarrow & i_! Y & \xrightarrow{\phi_Y} & i_* Y \end{array}$$

and the other two sides are the pasting composite

$$\begin{array}{ccccc} i_! A \times_Y X & \xrightarrow{\phi_{A \times_Y X}} & i_* A \times_Y A & \longrightarrow & i_* X \\ \downarrow^{i_! p} & & \downarrow^{i_* p} & & \downarrow^{i_* f} \\ i_! A & \xrightarrow{\phi_A} & i_* A & \longrightarrow & i_* Y \end{array} .$$

Counting left to right and top to bottom, we have that

- the first square is a pullback by assumption that $i_!$ preserves the given pullback;
- the second square is a pullback, since f is formally étale.
- the total top rectangle is therefore a pullback, by the pasting law;
- the fourth square is a pullback since i_* is right adjoint and so also preserves pullbacks;
- also the total bottom rectangle is a pullback, since it is equal to the top total rectangle;

- therefore finally the third square is a pullback, by the other clause of the pasting law. Hence p is formally étale.

$\square$

We consider now types of ∞-pullbacks that are preserved by $i_!$.

Proposition 3.10.27. *If $U \longrightarrow\!\!\!\!\!\rightarrow X$ is an effective epimorphism in $\mathbf{H}$ that it is addition formally étale, def. 3.10.16, then also its image $i_! U \to i_! X$ in $\mathbf{H}_{\mathrm{th}}$ is an effective epimorphism.*

Proof. Because i_* is left and right adjoint it preserves all small ∞-limits and ∞-colimits and therefore preserves effective epimorphisms. Since these are stable under ∞-pullback, it follows by definition of formal étaleness that with $i_* U \to i_* X$ also $i_! U \to i_! X$ is an effective epimorphism. $\square$

Proposition 3.10.28. *If in a differentially cohesive ∞-topos $i : \mathbf{H} \hookrightarrow \mathbf{H}_{\mathrm{th}}$ both $\mathbf{H}$ as well as $\mathbf{H}_{\mathrm{th}}$ have an ∞-cohesive site of definition, then the functor $i_!$ preserves pullbacks over discrete objects.*

Proof. Since it preserves finite products by assumption, the claim follows as in the proof of theorem 3.8.19. $\square$

Proposition 3.10.29. *If in an infinitesimal cohesive neighbourhood $i : \mathbf{H} \hookrightarrow \mathbf{H}_{\mathrm{th}}$ both $\mathbf{H}$ as well as $\mathbf{H}_{\mathrm{th}}$ have an ∞-cohesive site of definition, then the morphism $E \to X$ in $\mathbf{H}$ out of the total space of a locally constant ∞-stack over X, 3.8.6, is formally étale.*

Proof. First observe that every discrete morphism $\mathrm{Disc}(A \stackrel{f}{\to} B)$ is formally étale: since every discrete ∞-groupoid is an ∞-colimit over the ∞-functor constant on the point, $\phi_* : i_! * \to i_* *$ is an equivalence, and $i_! \to i_*$ preserves ∞-colimits, so we have that $\phi_{\mathrm{Dic}(A)}$ and $\phi_{\mathrm{Disc}(B)}$ are equivalences. Therefore the relevant diagram is an ∞-pullback.

Next, by definition, $E \to X$ is a pullback of a discrete morphism. By prop. 3.10.28 this pullback is preserved by $i_!$ and so by prop. 3.10.25 also $E \to X$ is locally étale. $\square$

Remark 3.10.30. The properties listed in prop. 3.10.24 imply in particular that étale morphisms in $\mathbf{H}_{\mathrm{th}}$ are "admissible maps" modelling a notion of *local homeomorphism* in a *geometry for structured ∞-toposes* according to def. 1.2.1 of [Lurie09a]. In the terminology used there this means that $\mathbf{H}_{\mathrm{th}}$ equipped with its canonical topology and with this notion of admissible maps is a *geometry*, see remark 3.10.43 below.

Another proposal for an axiomatization of *open maps* and étale maps has been proposed in [**?**], and the above list of properties covers most, but not necessarily all of these axioms.

In order to interpret the notion of formal smoothness, we close by further discussion of infinitesimal reduction.

Observation 3.10.31. The operation $\mathbf{Red}$ is an idempotent projection of $\mathbf{H}_{\mathrm{th}}$ onto the image of $\mathbf{H}$ under $i_!$:

$$\mathbf{Red}\,\mathbf{Red} \simeq \mathbf{Red}\,.$$

Accordingly also

$$\mathbf{\Pi}_{\mathrm{inf}} \mathbf{\Pi}_{\mathrm{inf}} \simeq \mathbf{\Pi}_{\mathrm{inf}}$$

and

$$\flat_{\mathrm{inf}} \flat_{\mathrm{inf}} \simeq \flat_{\mathrm{inf}}\,.$$

Proof. By definition of infinitesimal neighbourhood we have that $i_!$ is a full and faithful ∞-functor. It follows that $i^* i_! \simeq \mathrm{id}$ and hence

$$\begin{aligned} \mathbf{Red}\mathbf{Red} &\simeq i_! i^* i_! i^* \\ &\simeq i_! i^* \\ &\simeq \mathbf{Red} \end{aligned} .$$

□

Observation 3.10.32. For every $X \in \mathbf{H}_{\mathrm{th}}$, we have that $\mathbf{\Pi}_{\mathrm{inf}}(X)$ is formally smooth according to def. 3.10.11.

Proof. By prop. 3.10.31 we have that

$$\mathbf{\Pi}_{\mathrm{inf}}(X) \to \mathbf{\Pi}_{\mathrm{inf}}\mathbf{\Pi}_{\mathrm{inf}}(X)$$

is an equivalence. As such it is in particular an effective epimorphism. □

3.10.5 Formally étale groupoids

We discuss an intrinsic realization of the notion of *formally étale groupoids* internal to a differential ∞-topos. In typical models, for instance that discussed below in 4.5, formal étaleness automatically implies global étaleness, and so the following formulation captures the notion of *étale groupoid* objects in a differential ∞-topos. For a classical texts on étale 1-groupoids see [MoMr03].

Recall from 3.6.7 that groupoid objects $\mathcal{G}$ in an ∞-topos $\mathbf{H}$ are equivalent to effective epimorphisms $U \xrightarrow{p} X$ in $\mathbf{H}$, which we think of as being an *atlas* for $X \in \mathbf{H}$.

Definition 3.10.33. For $\mathbf{H} \overset{i}{\hookrightarrow} \mathbf{H}_{\mathrm{th}}$ a differential ∞-topos, def. 3.5.1, we say that a groupoid object is *formally étale* if the corresponding atlas $U \xrightarrow{p} X$ is a formally étale morphism, def. 3.10.16.

Remark 3.10.34. When $\mathbf{H}$ is presented by a category of simplicial (pre)sheaves, 2.2.3, then for any simplicial presheaf X there is, by remark 2.3.29, a canonical atlas, given by the inclusion $\mathrm{const} X_0 \to X$. If the presentation of X and the induced canonical atlas is understood explicitly, we often speak just of X itself being a formally étale groupoid or a *formally étale ∞-stack*.

Observation 3.10.35. If $U \xrightarrow{p} X$ is a formally étale groupoid, then both $i_* U \xrightarrow{i_* p} i_* X$ and $i_! U \xrightarrow{i_! p} i_! X$ are effective epimorphisms in $\mathbf{H}_{\mathrm{th}}$.

Proof. Since i_* is both left and right ∞-adjoint, it preserves all the ∞-limits and ∞-colimits that define effective epimorphisms. Then since these are stable under ∞-pullback, and since $p : U \to X$ being formally étale by definition means that $i_! p$ is an ∞-pullback of i_*, it follows that also $i_! p$ is an effective epimorphism. □

3.10.6 Manifolds (separated)

We discuss a formalization of the notion of *separated manifold* (Hausdorff manifold) in a context of differential cohesion.

Let $\mathbb{A}^1 \in \mathbf{H}$ be a line object exhibiting the cohesion of $\mathbf{H}$ according to def. 3.9.2.

Definition 3.10.36. An (unseparated) manifold $X \in \mathbf{H}_{\mathrm{th}}$, def. 3.9.9, is *separated* if it admits a defining cover $\phi : \coprod_j \mathbb{A}^n \to X$ such that the induced Čech nerve is a formally étale grouoid over $\coprod_j \mathbb{A}^n$, def. 3.10.33.

Remark 3.10.37. In the standard synthetic differential model for differential cohesion, $\mathbb{A}^n \simeq \mathbb{R}^n$ is the standard Cartesian space (by prop. 4.3.33) and formal étaleness makes the components of the face maps be local diffeomorphisms (prop. 4.5.53 below). These are in particular open maps, which ensures that the corresponding space X is a smooth Hausdorff manifold in the traditional sense. This is prop. 4.5.38 below.

3.10.7 Structure sheaves

For $X \in \mathbf{H}_{\mathrm{th}}$ an object in a differential cohesive ∞-topos, we formulate

- the ∞-topos $\mathrm{Sh}_{\mathbf{H}}(\mathcal{X})$ of *∞-sheaves over X*, or rather of *formally étale maps into X*;
- the *structure sheaf $\mathcal{O}_X$* of X.

The resulting pair $(\mathrm{Sh}_{\mathbf{H}}, \mathcal{O}_X)$ is essentially a $\mathbf{H}_{\mathrm{th}}$-*structured ∞-topos* in the sense of [Lurie09a].

One way to motivate the following construction, is to notice that for $G \in Grp(\mathbf{H}_{\mathrm{th}})$ a differential cohesive ∞-group with de Rham coefficient object $\flat_{dR}\mathbf{B}G$ and for $X \in \mathbf{H}_{\mathrm{th}}$, def. 3.9.12 any differential homotopy type, the product projection

$$X \times \flat_{\mathrm{dR}}\mathbf{B}G \to X$$

regarded as an object of the slice ∞-topos $(\mathbf{H}_{\mathrm{th}})_{/X}$ *almost* qualifies as a "bundle of flat $\mathfrak{g}$-valued differential forms" over X: for $U \to X$ a cover (a 1-epimorphism) regarded in $(\mathbf{H}_{\mathrm{th}})_{/X}$, a U-plot of this product projection is a U-plot of X together with a flat $\mathfrak{g}$-valued de Rham cocycle on X.

This is indeed what the sections of a corresponding bundle of differential forms over X are supposed to look like – but only *if* $U \to X$ is sufficiently "spread out" over X, hence sufficiently étale. Because, on the extreme, if X is the point (the terminal object), then there should be no non-trivial section of differential forms relative to U over X, but the above product projection instead reproduces all the sections of $\flat_{\mathrm{dR}}\mathbf{B}G$.

In order to obtain the correct cotangent-like bundle from the product with the de Rham coefficient object, it needs to be *restricted* to plots out of sufficiently étale maps into X. In order to correctly test differential form data, "suitable" here should be "formally", namely infinitesimally. Hence the restriction should be along the full inclusion

$$(\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X} \hookrightarrow (\mathbf{H}_{\mathrm{th}})_{/X}$$

of the formally étale maps into X. Since on formally étale covers the sections should be those given by $\flat_{\mathrm{dR}}\mathbf{B}G$, one finds that the corresponding *sheaf of flat forms $\mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}G)$* must be the *coreflection* of the given projection along this map.

Definition 3.10.38. For $X \in \mathbf{H}_{\mathrm{th}}$ an object, write

$$(\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X} \hookrightarrow (\mathbf{H}_{\mathrm{th}})_{/X}$$

for the full sub-∞-category of the slice over X, def. 3.6.1, on the formally étale morphisms into X, def. 3.10.19.

Proposition 3.10.39. *The inclusion of def. 3.10.38 is both reflective as well as coreflective: we have a left and a right adjoint*

$$(\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X} \underset{\mathrm{Et}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\hookrightarrow}}} (\mathbf{H}_{\mathrm{th}})_{/X} \ .$$

Proof. The reflection is given by the factorization of prop. 3.10.23. This exhibits $(\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X}$ as a presentable ∞-category and hence, by the adjoint ∞-functor theorem, the coreflection exists precisely if the inclusion preserves all small ∞-colomits. Since the inclusion is full, for this it is sufficient to show that an ∞-colimit in $(\mathbf{H}_{\mathrm{th}})_{/X}$ of a diagram A that factors through the inclusion,

$$A : I \to (\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X} \hookrightarrow (\mathbf{H}_{\mathrm{th}})_{/X} \ ,$$

is again in the inclusion. Since moreover ∞-colimits in a slice are preserved and detected by the dependent sum, prop. 3.6.2, we are, by def. 3.10.19, reduced to showing that for the above diagram the square

$$\begin{array}{ccc} \varinjlim_{i \in I} A_i & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}} \varinjlim_{i \in I} A_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(X) \end{array}$$

is an ∞-pullback square in $\mathbf{H}_{\mathrm{th}}$. Since $\mathbf{\Pi}_{\mathrm{inf}}$ is a left adjoint by def. 3.10.1, this square is equivalent to

$$\begin{array}{ccc} \varinjlim_{i \in I} A_i & \longrightarrow & \varinjlim_{i \in I} \mathbf{\Pi}_{\mathrm{inf}} A_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(X) \end{array} .$$

Now that this square is an ∞-pullback follows since ∞-colimits are preserved by ∞-pullback in the ∞-topos $\mathbf{H}_{\mathrm{th}}$, def. 2.2.2, and the fact that every component square

$$\begin{array}{ccc} A_i & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}} A_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{\Pi}_{\mathrm{inf}}(X) \end{array}$$

is an ∞-pullback by the assumption that the diagram factored through the inclusion of the étale morphisms into the slice. □

Proposition 3.10.40. *For $X \in \mathbf{H}_{\mathrm{th}}$, the ∞-category $(\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X}$ of def. 3.10.38 is an ∞-topos, and the defining inclusion into the slice $(\mathbf{H}_{\mathrm{th}})_{/X}$ is a geometric embedding.*

Proof. By prop. 3.10.39 the ∞-category $\mathrm{Sh}_{\mathbf{H}}(X)$ is the sub-slice induced by a reflective factorization system. This is a stable factorization system (in that the left class of $\mathbf{\Pi}_{\mathrm{inf}}$-equivalences is stable under ∞-pullback) and reflective factorization systems are stable precisely if the corresponding reflector preserves finite ∞-limits. Hence the embedding is a geometric embedding of a sub-∞-topos. □

Definition 3.10.41. For $\mathbf{H}_{\mathrm{th}}$ a differential cohesive ∞-topos and $X \in \mathbf{H}_{\mathrm{th}}$, we call the ∞-topos

$$\mathrm{Sh}_{\mathbf{H}}(X) := (\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X}$$

the *petit ∞-topos* of $X \in \mathbf{H}_{\mathrm{th}}$. An object of $\mathrm{Sh}_{\mathbf{H}}(X)$ we also call an *∞-sheaf over X*. The composite functor

$$\mathcal{O}_X : \mathbf{H}_{\mathrm{th}} \xrightarrow{(-)\times X} (\mathbf{H}_{\mathrm{th}})_{/X} \xrightarrow{\mathrm{Et}} (\mathbf{H}_{\mathrm{th}})_{/X} =: \mathrm{Sh}_{\mathbf{H}}(X) ,$$

with Et the right adjoint of prop. 3.10.39, we call the *structure ∞-sheaf* of X. For $A \in \mathbf{H}_{\mathrm{th}}$ we say that

$$\mathcal{O}_X(A) \in \mathrm{Sh}_{\mathbf{H}}(X)$$

is the *∞-sheaf of A-valued functions on X*.

Proposition 3.10.42. *The functor $\mathcal{O}_X$ is right adjoint to the forgetful functor*

$$\mathrm{Sh}_{\mathbf{H}}(X) := (\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X} \hookrightarrow (\mathbf{H}_{\mathrm{th}})_{/X} \xrightarrow{\sum_X} \mathbf{H}_{\mathrm{th}} .$$

In particular it preserves all small ∞-limits.

Proof. By essential uniqueness of ∞-adjoints, it is sufficient to observe that the the component maps are pairwise adjoint. For the first this is prop. 3.6.2, for the second it is prop. 3.10.39. □

Remark 3.10.43. The triple $(\mathbf{H}_{\mathrm{th}}, \mathrm{can}, \mathrm{fet})$ of the differential cohesive ∞-topos equipped with

1. its *canonical topology* (a collection $\{U_i \to X\}_i$ of morphisms in $\mathbf{H}_{\mathrm{th}}$ is covering precisely if $\coprod_i U_i \to X$ is a 1-epimorphism, def. 2.3.3);
2. its class of formally étale morphisms, def. 3.10.19.

is a (large) *geometry* in the sense of [Lurie09a]. For $X \in \mathbf{H}_{\mathrm{th}}$, the pair $(\mathrm{Sh}_{\mathbf{H}}(X), \mathcal{O}_X)$ of def. 3.10.41 is a *structured ∞-topos* with respect to this geometry in the sense of [Lurie09a]. In fact, it is essentially the structured ∞-topos associated to X in the geometry $\mathbf{H}_{\mathrm{th}}$ by def. 2.2.9 there.

We close this section by making explicit the special case of ∞-sheaves of *flat de Rham coefficients* over X.

Definition 3.10.44. For $G \in \mathrm{Grp}(\mathbf{H}_{\mathrm{th}})$ a differential cohesive ∞-group and for $X \in \mathbf{H}_{\mathrm{th}}$ any object, we say that the *∞-sheaf of flat* $\exp(\mathfrak{g})$*-valued differential forms* over X is

$$\mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}G) \in (\mathbf{H}_{\mathrm{th}})^{\mathrm{fet}}_{/X} \hookrightarrow (\mathbf{H})_{\mathrm{th}})_{/X} \,,$$

where $\mathcal{O}_X$ is given by def. 3.10.41 and where $\flat_{\mathrm{dR}}\mathbf{B}G$ is given by def. 3.9.12.

Definition 3.10.45. The canonical point $0 : * \to \flat_{\mathrm{dR}}\mathbf{B}G$ induces a section

$$(\mathrm{id}_X, 0) : X \to X \times \flat_{\mathrm{dR}}\mathbf{B}G$$

of the projection map. The image of this section under the coreflection of prop. 3.10.39

$$\begin{array}{ccc} & & \mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}G) \\ & {}^{0:=\mathrm{Et}(\mathrm{id},0)}\nearrow & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

we call the *0-section* of the ∞-sheaf of flat differential forms.

3.10.8 Critical loci, variational calculus and BV-BRST complexes

We give a general abstract formulation of the notion of *critical locus* of a function, the local of its domain where its first derivative vanishes. Applied to functions that are regarded as *action functionals* and with a constraint that the differential is trivial on certain boundaries, this critical locus is known as the space of solutions of the *Euler-Lagrange equations* of the action functional. If the ambient cohesive ∞-topos is ∞-localic, then this critical locus is what is called a *derived critical locus*, whose complex of functions is known as the *BV-BRST complex* of the action functional

Let $G \in \mathrm{Grp}(\mathbf{H}_{\mathrm{th}})$ be a differential cohesive ∞-group. Write

$$\theta_G : G \to \flat_{\mathrm{dR}}\mathbf{B}G$$

for its canonical differential form, def. 3.9.29.

Definition 3.10.46. For $X \in \mathbf{H}_{\mathrm{th}}$ any differential cohesive homotopy type and for

$$S : \; X \longrightarrow G$$

any morphism, write

$$\mathbf{d}S := S^*\theta : \; X \xrightarrow{S} G \xrightarrow{\theta_G} \flat_{\mathrm{dR}}\mathbf{B}G$$

for its composite with the canonical differential form on G, def. 3.9.29. We call this the *de Rham derivative* of S.

By def. 3.10.44 this corresponds to a section

$$\begin{array}{ccc} & & \mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}G) \\ & {}^{\mathbf{d}S}\nearrow & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

of the ∞-sheaf of flat G-valued forms over X, which we denote by the same symbols.

Definition 3.10.47. The *critical locus* of $S : X \to G$ is the object

$$\sum_{x:X}(\mathbf{d}S(x) \simeq 0) \in \mathbf{H}_{\mathrm{th}}$$

in the ∞-pullback

$$\begin{array}{ccc} \sum_{x:X}(\mathbf{d}S(x) \simeq 0) & \longrightarrow & X \\ \downarrow & & \downarrow^{0} \\ X & \xrightarrow{\mathbf{d}S} & \mathcal{O}_X(\flat_{\mathrm{dR}}\mathbf{B}G) \end{array}$$

in $\mathrm{Sh}_{\mathbf{H}_{\mathrm{th}}}(X)$, where the horizontal section is the de Rham differential of S from def. 3.10.46, and where the right vertical morphism is the 0-section of def. 3.10.45.

If X here is itself a space of functions, then for *variational calculus* one wants to constrain the differential of S to vary the data in X only away from the boundary. This is what the following construction achieves.

Definition 3.10.48. Let $\Sigma \in \mathbf{H}_{\mathrm{th}}$ be a manifold, def. 3.10.36, with boundary $\partial\Sigma \hookrightarrow \Sigma$. Let $A \in \mathbf{H}_{\mathrm{th}}$ be any object. Then the *variational domain*

$$[\Sigma, A]_{\partial\Sigma} \in \mathbf{H}_{\mathrm{th}}$$

is the ∞-pullback in

$$\begin{array}{ccc} [\Sigma, A]_{\partial\Sigma} & \longrightarrow & \flat[\partial\Sigma, A] \\ \downarrow & & \downarrow \\ [\Sigma, A] & \longrightarrow & \flat[\Sigma, A] \end{array} .$$

For

$$S : [\Sigma, A]_{\partial\Sigma} \to G$$

a map, we say that its critical locus, def. 3.10.47

$$\sum_{\phi:\Sigma \to A}(\mathbf{d}S(\phi) \simeq 0)$$

is the space of solutions to the *Euler-Lagrange equations* of S.

3.10.9 Formal groupoids

The infinitesimal analog of an exponentiated ∞-Lie algebra, 3.9.4, is a formal cohesive ∞-group.

Definition 3.10.49. An object $X \in \mathbf{H}_{\mathrm{th}}$ is a *formal cohesive ∞-groupoid* if $\mathbf{\Pi}_{\mathrm{inf}} X \simeq *$.

An ∞-group object $\mathfrak{g} \in \mathbf{H}_{\mathrm{th}}$ that is infinitesimal we call a *formal ∞-group.*

For $X \in \mathbf{H}$ any object, we say $\mathfrak{a} \in \mathbf{H}_{\mathrm{th}}$ is an *formal cohesive ∞-groupoid over X* if $\mathbf{\Pi}_{\mathrm{inf}}(\mathfrak{a}) \simeq \mathbf{\Pi}_{\mathrm{inf}}(X)$; equivalently: if there is a morphism

$$\mathfrak{a} \to \mathbf{\Pi}_{\mathrm{inf}}(X)$$

equivalent to the infinitesimal path inclusion, def. 3.10.1, for $\mathfrak{a}$.

Proposition 3.10.50. *An infinitesimal cohesive ∞-groupoid, def. 3.10.49 – $X \in \mathbf{H}_{\mathrm{th}}$ with $\mathbf{\Pi}_{\mathrm{inf}}(X) \simeq *$ – is both geometrically contractible and has as underlying discrete ∞-groupoid the point:*

- $\Pi X \simeq *$
- $\Gamma X \simeq *$.

Proof. The first statement is implied by the fact both $i_!$ as well as i_* are full and faithful, by definition of infinitesimal neighbourhood. This means that if $\mathbf{\Pi}_{\mathrm{inf}}(X) \simeq *$ then already $i^* X = \Pi_{\mathrm{inf}}(X) \simeq *$. Since $\Pi_{\mathbf{H}_{\mathrm{th}}} \simeq \Pi_{\mathbf{H}} \Pi_{\mathrm{inf}}$ and $\Pi_{\mathbf{H}}$ preserves the terminal object by cohesiveness, this implies the first claim.

The second statement follows by

$$\begin{aligned} \Gamma X &\simeq \mathbf{H}_{\mathrm{th}}(*, X) \\ &\simeq \mathbf{H}_{\mathrm{th}}(\mathbf{Red}*, X) \\ &\simeq \mathbf{H}_{\mathrm{th}}(*, \mathbf{\Pi}_{\mathrm{inf}}(X))\,. \\ &\simeq \mathbf{H}_{\mathrm{th}}(*, *) \\ &\simeq * \end{aligned}$$

□

Observation 3.10.51. For all $X \in \mathbf{H}$, we have that X and $\mathbf{\Pi}_{\mathrm{inf}}(X)$ are formal cohesive ∞-groupoids over X, X by the constant infinitesimal path inclusion and $\mathbf{\Pi}_{\mathrm{inf}}(X)$ by the identity.

Proof. For X this is tautological, for $\mathbf{\Pi}(X)$ it follows from prop. 3.10.31 and the $(i^* \dashv i_*)$-zig-zag-identity. □

Proposition 3.10.52. *The delooping $\mathbf{B}\mathfrak{g}$ of a formal ∞-group $\mathfrak{g}$, def. 3.10.49, is a formal ∞-groupoid over the point.*

Proof. Since both i^* and i_* are right adjoint, $\mathbf{\Pi}_{\mathrm{inf}}$ commutes with delooping. Therefore

$$\begin{aligned} \mathbf{\Pi}_{\mathrm{inf}} \mathbf{B}\mathfrak{g} &\simeq \mathbf{B} \mathbf{\Pi}_{\mathrm{inf}} \mathfrak{g} \\ &\simeq \mathbf{B}* \\ &\simeq * \\ &\simeq \mathbf{\Pi}_{\mathrm{inf}} * \end{aligned} \, .$$

□

4 Models

In this section we construct specific cohesive ∞-toposes, 3.4, and differential cohesive ∞-toposes, 3.5, and discuss the realization the general abstract structures of 3.9 in these models.

- 4.1 – discrete cohesion;
- 4.2 – diagrams;
- 4.3 – Euclidean-topological cohesion;
- 4.4 – smooth cohesion;
- 4.5 – synthetic differential cohesion;
- 4.6 – supergeometric cohesion.

Six of the cohesive ∞-toposes that we discuss fit into a diagram of geometric morphisms of the following form:

$$
\begin{array}{cccccc}
 & \textbf{cohesion} & & \textbf{differential cohesion} & & \begin{array}{c}\textbf{base}\\ \infty\textbf{-topos}\end{array} \\
\textbf{supergeometry} & \mathrm{SmoothSuper}\infty\mathrm{Grpd} & \hookrightarrow & \mathrm{SynthDiffSuper}\infty\mathrm{Grpd} & \longrightarrow & \infty\mathrm{SuperGrpd} \\
 & \downarrow & & \downarrow & & \downarrow \\
\textbf{differential geometry} & \mathrm{Smooth}\infty\mathrm{Grpd} & \hookrightarrow & \mathrm{SynthDiff}\infty\mathrm{Grpd} & \longrightarrow & \infty\mathrm{Grpd}
\end{array}
.
$$

In the bottom right we have plain ∞-groupoids, modelling *discrete* cohesion, 4.1. The bottom left is the cohesive ∞-topos of *smooth ∞-groupoids*, 4.4 and the middle entry on the bottom is the cohesive ∞-topos *synthetic differential cohesion*, 4.5. The total bottom row exhibits the latter as a model for *differential cohesion* in the sense of 3.5. This we regard as the standard model for *higher differential geometry*. The bottom row shows the supergeometric refinement of this situation. See below in 4.6 for more discussion of the top row of this diagram.

4.1 Discrete ∞-groupoids

For completeness, and because it serves to put some concepts into a useful perspective, we record aspects of the case of *discrete* cohesion.

Observation 4.1.1. The terminal ∞-sheaf ∞-topos ∞Grpd is trivially a cohesive ∞-topos, where each of the defining four ∞-functors $(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) : \infty Grpd \to \infty Grpd$ is an equivalence of ∞-categories.

Definition 4.1.2. In the context of cohesive ∞-toposes we say that ∞Grpd defines *discrete cohesion* and refer to its objects as *discrete ∞-groupoids*.

More generally, given any other cohesive ∞-topos

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) : \mathbf{H} \to \infty\mathrm{Grpd}$$

the inverse image Disc of the global section functor is a full and faithful ∞-functor and hence embeds ∞Grpd as a full sub-∞-category of $\mathbf{H}$. We say $X \in \mathbf{H}$ is a *discrete ∞-groupoid* if it is in the image of Disc.

This generalizes the traditional use of the terms *discrete space* and *discrete group*:

- a *discrete space* is equivalently a 0-truncated discrete ∞-groupoid;
- a *discrete group* is equivalently a 0-truncated group object in discrete ∞-groupoids.

We now discuss some of the general abstract structures in cohesive ∞-toposes, 3.9, in the context of discrete cohesion.

- 4.1.1 – Geometric homotopy
- 4.1.2 – Groups
- 4.1.3 – Cohomology
- 4.1.4 – Principal bundles
- 4.1.5 – Twisted cohomology
- 4.1.6 – Representations and associated bundles

4.1.1 Geometric homotopy

We discuss geometric homotopy and path ∞-groupoids, 3.8.1, in the context of discrete cohesion, 4.1. Using $\mathrm{sSet}_{\mathrm{Quillen}}$ as a presentation for ∞Grpd this is entirely trivial, but for the equivalent presentation by $\mathrm{Top}_{\mathrm{Quillen}}$ it becomes effectively a discussion of the classical Quillen equivalence $\mathrm{Top}_{\mathrm{Quillen}} \simeq \mathrm{sSet}_{\mathrm{Quillen}}$ from the point of view of cohesive ∞-toposes. It may be useful to make this explicit.

By the homotopy hypothesis-theorem the ∞-toposes Top and ∞Grpd are equivalent, hence indistinguishable by general abstract constructions in ∞-topos theory. However, in practice it can be useful to distinguish them as two different presentations for an equivalence class of ∞-toposes. For that purposes consider the following

Definition 4.1.3. Define the quasi-categories

$$\mathrm{Top} := N(\mathrm{Top}_{\mathrm{Quillen}})^\circ$$

and

$$\infty\mathrm{Grpd} := N(\mathrm{sSet}_{\mathrm{Quillen}})^\circ\,.$$

Here on the right we have the standard model structure on topological spaces, $\mathrm{Top}_{\mathrm{Quillen}}$, and the standard model structure on simplicial sets, $\mathrm{sSet}_{\mathrm{Quillen}}$, and $N((-)^\circ)$ denotes the homotopy coherent nerve of the simplicial category given by the full sSet-subcategory of these simplicial model categories on fibrant-cofibrant objects.

For

$$(|-| \dashv \mathrm{Sing}) : \mathrm{Top}_{\mathrm{Quillen}} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Quillen}}$$

the standard Quillen equivalence given by the singular simplicial complex-functor and geometric realization, write

$$(\mathbb{L}|-| \dashv \mathbb{R}\mathrm{Sing}) : \mathrm{Top} \underset{\mathbb{R}\mathrm{Sing}}{\overset{\mathbb{L}|-|}{\rightleftarrows}} \infty\mathrm{Grpd}$$

for the corresponding derived ∞-functors (the image under the homotopy coherent nerve of the restriction of $|-|$ and Sing to fibrant-cofibrant objects followed by functorial fibrant-cofibrant replacement) that constitute a pair of adjoint ∞-functors modeled as morphisms of quasi-categories.

Since this is an equivalence of ∞-categories either functor serves as the left adjoint and right ∞-adjoint and so we have

Observation 4.1.4. Top is exhibited as a cohesive ∞-topos by

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) : \mathrm{Top} \begin{array}{c} \xrightarrow{\mathbb{L}\mathrm{Sing}} \\ \xleftarrow{\mathbb{R}|-|} \\ \xrightarrow{\mathbb{L}\mathrm{Sing}} \\ \xleftarrow{\mathbb{R}|-|} \end{array} \infty\mathrm{Grpd}$$

In particular a presentation of the intrinsic fundamental ∞-groupoid is given by the familiar singular simplicial complex construction

$$\Pi(X) \simeq \mathbb{R}\mathrm{Sing}X\,.$$

Notice that the topology that enters the explicit construction of the objects in Top here does *not* show up as cohesive structure. A topological space here is a model for a *discrete* ∞-groupoid, the topology only serves to allow the construction of $\mathrm{Sing}X$. For discussion of ∞-groupoids equipped with genuine *topological cohesion* see 4.3.

4.1.2 Groups

Discrete ∞-groups may be presented by simplicial groups. See 3.6.8.2.

(...)

4.1.3 Cohomology

We discuss the general notion of cohomology in cohesive ∞-toposes, 3.6.9, in the context of discrete cohesion.

Cohomology in Top is the ordinary notion of (nonabelian) cohomology. The equivalence to ∞Grpd makes manifest in which way this is equivalently the *cohomology of groups* for connected, homotopy 1-types, the *cohomology of groupoids* for general 1-types and generally, of course, the cohomology of ∞-groups.

4.1.3.1 Group cohomology

Proposition 4.1.5. *For G a (discrete) group, A a (discrete) abelian group, the group cohomology of G with coefficients in the trivial G-module A is*

$$H^n_{\mathrm{grp}}(G,A) \simeq \pi_0 \mathrm{Disc}\infty\mathrm{Grpd}(\mathbf{B}G, \mathbf{B}^n A)\,.$$

The case of group cohomology with coefficients in a non-trivial module is a special case of *twisted cohomology* in Disc∞Grpd. This is discussed below in 4.1.5.1.

4.1.4 Principal bundles

We discuss the general notion of principal ∞-bundles in cohesive ∞-toposes, 3.6.10, in the context of discrete cohesion.

There is a traditional theory of *strictly* principal Kan simplicial bundles, i.e. simplicial bundles with G action for which the shear map is an *isomorphism* instead of more generally a weak equivalence. A classical reference for this is [May67]. A standard modern reference is section V of [GoJa99]. We now compare this classical theory of strictly principal simplicial bundles to the theory of weakly principal simplicial bundles from 3.6.10.4.

Definition 4.1.6. Let G be a simplicial group and X a Kan simplicial set. A *strictly G-principal bundle* over X is a morphism of simplicial sets $P \to X$ equipped with a G-action on P over X such that

1. the G action is degreewise free;
2. the canonical morphism $P/G \to X$ out of the ordinary (1-categorical) quotient is an isomorphism of simplicial sets.

A morphism of stricly G-principal bundles over X is a map $P \to P'$ respecting both the G-action as well as the projection to X.

Write $\mathrm{s}G\mathrm{Bund}(X)$ for the category of strictly G-principal bundles.

In [GoJa99] this is definition V3.1, V3.2.

Lemma 4.1.7. *Every morphism in* $\mathrm{s}G\mathrm{Bund}(X)$ *is an isomorphism.*

In [GoJa99] this is remark V3.3.

Observation 4.1.8. Every strictly G-principal bundle is evidently also a weakly G-principal bundle, def. 3.6.185. In fact the strictly principal G-bundles are precisely those weakly G-principal bundles for which the shear map is an isomorphism. This identification induces a full inclusion of categories

$$\mathrm{s}G\mathrm{Bund}(X) \hookrightarrow \mathrm{w}G\mathrm{Bund}(X)\,.$$

Lemma 4.1.9. *Every morphism of weakly principal Kan simplicial bundles is a weak equivalence on the underlying Kan complexes.*

Proposition 4.1.10. *For G a simplicial group, the category* sSet_G *of G-actions on simplicial sets and G-equivariant morphisms carries the structure of a simplicial model category where the fibrations and weak equivalences are those of the underlying simplicial sets.*

This is theorem V2.3 in [GoJa99].

Corollary 4.1.11. *For G a simplicial group and X a Kan complex, the slice category* sSet_G/X *carries a simplicial model structure where the fibrations and weak equivalences are those of the underlying simplicial sets after forgetting the map to X.*

Lemma 4.1.12. *Let G be a simplicial group and $P \to X$ a weakly G-principal simplicial bundle. Then the loop space* $\Omega_{(P\to X)}\mathrm{Ex}^\infty N(\mathrm{w}G\mathrm{Bund}(X))$ *has the same homotopy type as the derived hom space* $\mathbb{R}\mathrm{Hom}_{\mathrm{sSet}_G/X}(P,P)$.

Proof. By theorem V2.3 of [GoJa99] and lemma 4.1.9 the free resolution P^f of P from corollary 3.6.203 is a cofibrant-fibrant resolution of P in the slice model structure of corollary 4.1.11. Therefore the derived hom space is presented by the simplicial set of morphisms $\mathrm{Hom}_{\mathrm{sSet}_G/X}(P^f \cdot \Delta^\bullet, P^f)$ and all these morphisms are equivalences. Therefore by prop. 2.3 in [DwKa84a] this simplicial set is equivalent to the loop space of the nerve of the subcategory of sSet_G/X on the weak equivalences connected to P^f. By lemma 4.1.9 this subcategory is equivalent (isomorphic even) to the connected component of $\mathrm{w}G\mathrm{Bund}(X)$ on P. □

Proposition 4.1.13. *Under the simplicial nerve, the inclusion of observation 4.1.8 yields a morphism*

$$N\mathrm{s}G\mathrm{Bund}(X) \to N\mathrm{w}G\mathrm{Bund}(X) \quad \in \mathrm{sSet}_{\mathrm{Quillen}}$$

which is

- *for all G and X an isomorphism on connected components;*
- *not in general a weak equivalence.*

Proof. Let $P \to X$ be a weakly G-principal bundle. To see that it is connected in $\mathrm{w}G\mathrm{Bund}(X)$ to some strictly G-principal bundle, first observe that by corollary 3.6.203 it is connected via a morphism $P^f \to P$ to the bundle

$$P^f := \mathrm{Rec}(X \leftarrow P/_hG \stackrel{f}{\to} \overline{W}G)\,,$$

which has free G-action, but does not necessarily satisfy strict principality. Since, by theorem 3.6.197, the morphism $P/_hG \to X$ is an acyclic fibration of simplicial sets it has a section $\sigma : X \to P/_hG$ (every simplicial set is cofibrant in $\mathrm{sSet}_{\mathrm{Quillen}}$). The bundle

$$P^s := \mathrm{Rec}(X \stackrel{\mathrm{id}}{\leftarrow} X \stackrel{f\circ\sigma}{\to} \overline{W}G)$$

is strictly G-principal, and with the morphism

$$(P^s \to P^f) := \mathrm{Rec}\left(\begin{array}{ccccc} & & P/_hG & & \\ & {}^{\sim}\swarrow & \uparrow & \searrow^{f} & \\ X & & \sigma & & \overline{W}G \\ & {}_{\mathrm{id}}\nwarrow & & \nearrow_{f\circ\sigma} & \\ & & X & & \end{array} \right)$$

we obtain (non-naturally, due to the choice of section) in total a morphism $P^s \to P^f \to P$ of weakly G-principal bundles from a strictly G-principal replacement P^s to P.

To see that the full embedding of strictly G-principal bundles is also injective on connected components, notice that by lemma 4.1.12 if a weakly G-principal bundle P with degreewise free G-action is connected by a zig-zag of morphisms to some other weakly G-principal bundle P, then there is already a direct morphism $P \to P'$. Since all strictly G-principal bundles have free action by definition, this shows that two of them that are connected in $\mathrm{w}G\mathrm{Bund}(X)$ are already connected in $\mathrm{s}G\mathrm{Bund}(X)$.

To see that in general $N\mathrm{s}G\mathrm{Bund}(X)$ nevertheless does not have the correct homotopy type, it is sufficient to notice that the category $\mathrm{s}G\mathrm{Bund}(X)$ is always a groupoid, by lemma 4.1.7. Therefore $N\mathrm{s}G\mathrm{Bund}(X)$ it is always a homotopy 1-type. But by theorem 3.6.201 the object $N\mathrm{w}G\mathrm{Bund}(X)$ is not an n-type if G is not an $(n-1)$-type. □

Corollary 4.1.14. *For all Kan complexes X and simplicial groups G there is an isomorphism*

$$\pi_0 N\mathrm{s}G\mathrm{Bund} \simeq H^1(X,G) := \pi_0 \infty\mathrm{Grpd}(X, \mathbf{B}G)$$

between the isomorphism classes of strictly G-principal bundles over X and the first nonabelian cohomology of X with coefficients in G.

But this isomorphism on cohomology does not in general lift to an equivalence on cocycle spaces.

Proof. By prop. 4.1.13 and remark 3.6.202. □

Remark 4.1.15. The first statement of corollary 4.1.14 is the classical classification result for strictly principal simplicial bundles, for instance theorem V3.9 in [GoJa99].

4.1.5 Twisted cohomology

We discuss the notion of twisted cohomology, 3.6.12, in the context of discrete cohesion.

4.1.5.1 Group cohomology with coefficients in nontrivial modules

We discuss ∞-group cohomology for discrete ∞-groups with coefficients in a module according to 3.6.13.

For G a (discrete) group and A a (discrete) group equipped with a G-action, write $\mathbf{B}^n A//G$ for the n-groupoid which is given by the crossed complex, def. 1.2.60 of groups

$$\mathbf{B}^n A//G := [A \to 1 \to \cdots \to 1 \to G]$$

coming from the given G-action on A. There is a canonical morphism

$$\mathbf{B}^n A//G \to \mathbf{B}G\,.$$

Proposition 4.1.16. *We have a fiber sequence*

$$\mathbf{B}^n A \to \mathbf{B}^n A//G \to \mathbf{B}G$$

in $\mathrm{Disc}\infty\mathrm{Grpd}$*.*

In view of remark 3.6.209 this fiber sequence exhibits a $\mathbf{B}^n A$-fiber bundle which is associated to the universal G-principal ∞-bundle, 4.1.4.

In generalization of prop. 4.1.5 we have

Proposition 4.1.17. *The group cohomology of G with coefficients in the module A is naturally identified with the* id*-twisted cohomology of $\mathbf{B}G$, relative to $\mathbf{B}^n A//G$,*

$$H^n_{\mathrm{grp}}(G, A) \simeq \pi_0 \mathrm{Disc}\infty\mathrm{Grpd}_{[\mathrm{id}]}(\mathbf{B}G, \mathbf{B}^n A//G)\,.$$

Remark 4.1.18. Equivalently this says that group cohomology with coefficients in nontrivial modules A describes the sections of the bundle $\mathbf{B}^n A//G$.

4.1.6 Representations and associated bundles

We discuss canonical representations of automorphism ∞-groups in Disc∞Grpd, following 3.6.13.

For all of the following, fix a regular uncountable cardinal κ.

Definition 4.1.19. Write $\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa$ for the core (the maximal ∞-groupoid inside) the full sub-∞-category of ∞Grpd on the κ-small ∞-groupoids, [LuHTT] def. 5.4.1.3. We regard this canonically as an object

$$\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa \in \infty\mathrm{Grpd}\,.$$

Remark 4.1.20. We have

$$\mathrm{Core}\,\infty\mathrm{Grpd}_\kappa \simeq \coprod_i \mathbf{B}\mathrm{Aut}(F_i)\,,$$

where the coproduct ranges over all κ-small homotopy types $[F_i]$ and where $\mathrm{Aut}(F_i)$ is the automorphism ∞-group of any representative F_i of $[F_i]$.

Lemma 4.1.21. *For X a κ-small ∞-groupoid, and $f : Y \to X$ a morphism in $\infty\mathrm{Grpd}$, the following are equivalent*

1. *for all objects $x \in X$ the homotopy fiber $Y_x := Y \times_X \{x\}$ of f is κ-small;*
2. *Y is κ-small.*

Proof. The implication 1. $\Rightarrow$ 2. is stated for ∞-categories, and assuming that f is presented by a Cartesian fibration of simplicial sets, as prop. 5.4.1.4 in [LuHTT]. But by prop. 2.4.2.4 there, every Cartesian fibration between Kan complexes is a right fibration; and by prop. 2.1.3.3 there over a Kan complex every right fibration is a Kan fibration. Finally, by the Quillen model structure every morphism of ∞-groupoids is presented by a Kan fibration. Therefore the condition that f be presented by a Cartesian morphism is automatic in our case.

For the converse, assume that all homotopy fibers are κ-small. We may write X as the ∞-colimit of the functor constant on the point, over itself ([LuHTT], corollary 4.4.4.9)

$$X \simeq \varinjlim_{x\in X} \{x\}\,.$$

Since ∞Grpd is an ∞-topos, its ∞-colimits are preserved by ∞-pullback. Therefore we have an ∞-pullback diagram

$$\begin{array}{ccc} \varinjlim_{x\in X} Y_x & \xrightarrow{\simeq} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \varinjlim_{x\in X} \{x\} & \xrightarrow{\simeq} & X \end{array} .$$

that exhibits Y as the ∞-colimit over X of the homotopy fibers of f. By corollary 5.4.1.5 in [LuHTT], the κ-small ∞-groupoids are precisely the κ-compact objects of ∞Grpd. By corollary 5.3.4.15 there, κ-compact objects are closed under κ-small ∞-colimits. Therefore the above ∞-colimit exhibits Y as a κ-small ∞-groupoid. $\square$

Definition 4.1.22. Write $\widehat{\mathrm{Core}\infty\mathrm{Grpd}}_\kappa \to \mathrm{Core}\,\infty\mathrm{Grpd}_\kappa$ for the ∞-pullback

$$\begin{array}{ccc} \widehat{\mathrm{Core}\infty\mathrm{Grpd}}_\kappa & \longrightarrow & Z|_{\infty\mathrm{Grpd}} \\ \downarrow & & \downarrow \\ \mathrm{Core}\,\infty\mathrm{Grpd} & \longrightarrow & \infty\mathrm{Grpd} \end{array}$$

of the universal right fibration $Z|_{\infty\mathrm{Grpd}} \to \infty\mathrm{Grpd}$, as in [LuHTT] above prop. 3.3.2.5., along the canonical map that embeds κ-small ∞-groupoids into all ∞-groupoids.

Proposition 4.1.23. *The morphism* $\widehat{\mathrm{Core}\infty\mathrm{Grpd}}_\kappa \to \mathrm{Core}\,\infty\mathrm{Grpd}_\kappa$ *is the κ-compact object-classifier, section 6.1.6 of [LuHTT], in* $\infty\mathrm{Grpd}$.

Proof. By prop. 3.3.2.5 in [LuHTT] the universal right fibration classifies right fibrations; and for $[X] : * \to \infty\mathrm{Grpd}$ the name of an ∞-groupoid X, the homotopy fiber

$$Z \times_{\infty\mathrm{Grpd}} \{[X]\} \simeq X$$

is equivalent to X. As in the proof of lemma 4.1.21, every morphism between ∞-groupoids is represented by a Cartesian fibration. Since moreover every morphism out of an ∞-groupoid into $\infty\mathrm{Grpd}$ factors essentially unqiquely through $\mathrm{Core}\,\infty\mathrm{Grpd}$ it follows that $\widehat{\mathrm{Core}\infty\mathrm{Grpd}}_\kappa \to \mathrm{Core}\,\infty\mathrm{Grpd}_\kappa$ classifies morphisms of ∞-groupoids with κ-small homotopy fibers. By lemma 4.1.21 and using again that κ-compact objects in $\infty\mathrm{Grpd}$ are κ-small ∞-groupoids, these are precisely the relatively κ-compact morphisms from def. 6.1.6.4 of [LuHTT]. □

Remark 4.1.24. By remark 4.1.20 we have that $\widehat{\mathrm{Core}\infty\mathrm{Grpd}}_\kappa \to \mathrm{Core}\,\infty\mathrm{Grpd}_\kappa$ decomposes as a coproduct of morphisms $\coprod_{[F_i]} \rho_i$ indexed by the κ-small homotopy types. According to prop. 4.1.23 the (essentially unique) homotopy fiber of ρ_i is equivalent to the κ-small ∞-groupoid F_i itself. Therefore by def. 3.6.152 we may write

$$\rho_i : F_i // \mathrm{Aut}(F_i) \to \mathbf{B}\mathrm{Aut}(F_i)$$

and identify this with the canonical representation of $\mathrm{Aut}(F_i)$ on F_i, exhibited, by example 3.6.209, as the universal F_i-fiber bundle which is ρ_i-associated to the universal $\mathrm{Aut}(F_i)$-principal bundle.

In terms of this perspective we have the following classical result.

Corollary 4.1.25. *For X a connected ∞-groupoid, every morphism $P \to X$ in* $\infty\mathrm{Grpd}$ *with κ-small small homotopy fibers F (over one and hence, up to equivalence, over each object $x \in X$) arises as the F-fiber bundle ρ-associated to an* $\mathrm{Aut}(F)$*-principal ∞-bundle, 3.6.10, given by an ∞-pullback of the form*

$$\begin{array}{ccc} P & \longrightarrow & F//\mathrm{Aut}(F) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{B}\mathrm{Aut}(F) \end{array} .$$

More discussion of discrete principal and discrete associated ∞-bundles is in 3.8.6 and 4.1.4.

4.2 Diagrams of cohesive ∞-groupoids

We discuss here models of cohesion given by ∞-*presheaf* ∞-toposes, hence ∞-toposes "of diagrams" in a given ambient cohesive ∞-topos.

- 4.2.1 – Bundles of geometrically contractible ∞-groupoids
- 4.2.2 – Simplicial objects in an ∞-topos

The first of these mainly serves an illustrative purpose. It is a simple but non-trivial model of cohesion that illuminates the central notions, such as cohesive homotopy types, by elementary combinatorial reasoning. The second is also simple as far as cohesion goes, but is central relevance for further constructions

4.2.1 Bundles of geometrically contractible ∞-groupoids

We discuss a class of examples of cohesive ∞-toposes that are obtained from a given cohesive ∞-topos $\mathbf{H}$ by passing to the ∞-topos $\mathbf{H}^D$ of interval-shaped diagrams in it. The cohesive interpretation of an object in $\mathbf{H}^D$ is as a bundle of $\mathbf{H}$-cohesive ∞-groupoids all whose fibers are regarded as being geometrically contractible.

Proposition 4.2.1. *Let $\mathbf{H}$ be a cohesive ∞-topos. Let D be a small category with initial object $\bot$ and terminal object $\top$.*

There is an adjoint triple of ∞-functors

$$D \underoverset{\top}{\bot}{\begin{smallmatrix}\longrightarrow \\ \xleftarrow{p} \\ \longrightarrow\end{smallmatrix}} *$$

obtained from the inclusion of the terminal and the initial object.

The ∞-functor ∞-category $\mathbf{H}^D$ (D-shaped diagrams in $\mathbf{H}$) is a cohesive ∞-topos, exhibited by the composite adjoint quadruple

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) \ : \ \mathbf{H}^D \underoverset{\bot_*}{\top^*}{\begin{smallmatrix}\longrightarrow \\ \xleftarrow{p^*} \\ \xrightarrow{\bot^*} \\ \hookleftarrow\end{smallmatrix}} \mathbf{H} \underoverset{\mathrm{coDisc}_{\mathbf{H}}}{\Pi_{\mathbf{H}}}{\begin{smallmatrix}\longrightarrow \\ \xleftarrow{\mathrm{Disc}_{\mathbf{H}}} \\ \xrightarrow{\Gamma_{\mathbf{H}}} \\ \hookleftarrow\end{smallmatrix}} \infty\mathrm{Grpd} \ .$$

Proof. Each of the first three functors induces an adjoint triple $(p_! \dashv p^* \dashv p_*)$, etc., where p^* is given by precomposition, $p_!$ by left ∞-Kan extension and p_* by right ∞-Kan extension (use for instance [LuHTT], A.2.8). In particular therefore $\top^*$ preserves finite products (together with all small ∞-limits). The adjointness $(\bot \dashv p \dashv \top)$ implies that $p_! \simeq \top^*$ and $\bot_! \simeq p^*$. This yields the adjoint quadruple as indicated. Finally it is clear that $\top^* p^* \simeq \mathrm{id}$, which means that p^* is full and faithful, and by adjointness so is $\bot_*$. □

The following simple example not only illustrates the above proposition, but also serves as a useful toy example for the notion of cohesion itself.

Example 4.2.2. For $\mathbf{H}$ any cohesive ∞-topos, also its arrow category $\mathbf{H}^{\Delta[1]}$ is cohesive.

In particular, for $\mathbf{H} = \infty\mathrm{Grpd}$ (see 4.1 below for a discussion of $\infty\mathrm{Grpd}$ as a cohesive ∞-topos), the arrow ∞-category $\infty\mathrm{Grpd}^{\Delta[1]}$ is cohesive. This is equivalently the ∞-category of ∞-presheaves on the interval $\Delta[1]$, which in turn is equivalent to the ∞-category of ∞-sheaves on the topological spaces called the *Sierpinski space*

$$\mathrm{Sierp} = (\{0,1\}, \mathrm{Opens} = (\emptyset \hookrightarrow \{1\} \hookrightarrow \{0,1\}))$$

(see for instance [John03], B.3.2.11):

$$\infty\mathrm{Grpd}^{\Delta[1]} \simeq \mathrm{PSh}_\infty(\Delta[1]) \simeq \mathrm{Sh}_\infty(\mathrm{Sierp}) \,.$$

We call this the *Sierpinski ∞-topos*.

Notice that the Sierpinski space, as a topological space,

1. is contractible;
2. is locally contractible;
3. has a focal point (a point whose only open neighbourhood is the entire space).

The Sierpinski ∞-topos is 0-localic, being the image of the Sierpinski space under the embedding of topological spaces into ∞-toposes. Accordingly the cohesion of $\mathrm{Sh}_\infty(\mathrm{Sierp})$ may be traced back to these three properties, which imply, in this order, that $\mathrm{Sh}_\infty(\mathrm{Sierp})$ is, as an ∞-topos,

1. ∞-connected;

2. locally ∞-connected;

3. local.

So the Sierpinski space is the "abstract cohesive blob" on which the cohesion of $\mathrm{Sh}_\infty(\mathrm{Sierp})$ is modeled: it is the abstract "point with an open neighbourhood".

While the cohesion encoded by the Sierpinski ∞-topos is very simple, it may be instructive to make the geometric interpretation fully explicit (the reader may want to compare the following with the more detailed discussions of the meaning of the functor Π on a cohesive ∞-topos below in 3.8.1):

an object of $\mathrm{Sh}_\infty(\mathrm{Sierp})$ is a morphism $[P \to X]$ in $\infty\mathrm{Grpd}$. The functor Π sends this to its domain

$$\Pi([P \to X]) \simeq X\,.$$

In particular

$$\Pi([P \to *]) \simeq *\,.$$

Therefore Π sees $[P \to *]$ as being cohesively/geometrically contractible and sees a bundle $[P \to X]$ as having cohesively/geometrically contractible fibers. At the same time, for $X \in \infty\mathrm{Grpd}$, we have

$$\mathrm{Disc}(X) \simeq [X \stackrel{id}{\to} X]\,,$$

which says that the base of such a bundle is regarded by the cohesion of the Sierpinski ∞-topos as being discrete. Accordingly, we may interpret $[P \to X]$ as describing a discrete ∞-groupoid X to which are attached cohesively contractible blobs, being the fibers of the morphism $P \to X$.

Even though they are geometrically contractible, these fibers have inner structure: this is seen by Γ, which takes the underlying ∞-groupoid to be the total space of the bundle

$$\Gamma([P \to X]) \simeq P\,.$$

Finally a codiscrete object is one of the form

$$\mathrm{coDisc}(Q) \simeq [Q \to *]\,,$$

which is entirely cohesively contractible, for any inner structure.

Observation 4.2.3. Let $\mathbf{H}$ be a cohesive ∞-topos and regard the Sierpinski ∞-topos $\mathbf{H}^I$, def. 4.2.2, as a cohesive ∞-topos over $\mathbf{H}$. Then

1. the full sub-∞-category of $\mathbf{H}^I$ on those objects for which *pieces have points*, def. 3.4.5, is canonically identified with the ∞-category of effective epimorphisms in $\mathbf{H}$, hence with the ∞-category of groupoid objects in $\mathbf{H}$, def. 3.6.88;

2. the full sub-∞-category of $\mathbf{H}^I$ on those objects which have *one point per piece*, def. 3.4.5, is canonically identified with $\mathbf{H}$ itself.

4.2.2 Simplicial cohesive ∞-groupoids

For $\mathbf{H}$ an ∞-topos, the ∞-topos $\mathbf{H}^{\Delta^{op}}$ of simplicial objects in $\mathbf{H}$ is cohesive over $\mathbf{H}$.

(...)

4.3 Euclidean-topological ∞-groupoids

We discuss *Euclidean-topological cohesion*, modeled on Euclidean topological spaces and continuous maps between them. This subsumes the homotopy theory of simplicial topological spaces.

Definition 4.3.1. Let $\mathrm{CartSp}_{\mathrm{top}}$ be the site whose underlying category has as objects the Cartesian spaces $\mathbb{R}^n$, $n \in \mathbb{N}$ equipped with the standard Euclidean topology and as morphisms the continuous maps between them; and whose coverage is given by good open covers.

Proposition 4.3.2. *The site* $\mathrm{CartSp}_{\mathrm{top}}$ *is an* ∞*-cohesive site (def 3.4.8).*

Proof. Clearly $\mathrm{CartSp}_{\mathrm{loc}}$ has finite products, given by $\mathbb{R}^k \times \mathbb{R}^l \simeq \mathbb{R}^{k+l}$, and clearly every object has a point $* = \mathbb{R}^0 \to \mathbb{R}^n$. In fact $\mathrm{CartSp}_{\mathrm{top}}(*, \mathbb{R}^n)$ is the underlying set of the Cartesian space $\mathbb{R}^n$.

Let $\{U_i \to U\}$ be a good open covering family in $\mathrm{CartSp}_{\mathrm{top}}$. By the very definition of *good cover* it follows that the Čech nerve $C(\coprod_i U_i \to U) \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ is degreewise a coproduct of representables.

The condition $\varinjlim C(\coprod_i U_i) \stackrel{\simeq}{\to} \varinjlim U = *$ follows from the nerve theorem [Bors48], which asserts that $\varinjlim C(\coprod_i U_i \to U) \simeq \mathrm{Sing} U$, and using that, as a topological space, every Cartesian space is contractible.

The condition $\varprojlim C(\coprod_i U_i) \stackrel{\simeq}{\to} \varprojlim U = \mathrm{CartSp}_{\mathrm{loc}}(*, U)$ is immediate. Explicitly, for $(x_{i_0} \in U_{i_0}, \cdots, x_{i_n} \in U_{i_n})$ a sequence of points in the covering patches of U such that any two consecutive ones agree in U, then they all agree in U. So the morphism of simplicial sets in question has the right lifting property against all boundary inclusions $\partial\Delta[n] \to \Delta[n]$ and is therefore is a weak equivalence. □

Definition 4.3.3. Define

$$\mathrm{ETop}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp}_{\mathrm{top}})$$

to be the ∞-category of ∞-sheaves on $\mathrm{CartSp}_{\mathrm{top}}$.

Proposition 4.3.4. *The* ∞*-category* $\mathrm{ETop}\infty\mathrm{Grpd}$ *is a cohesive* ∞*-topos.*

Proof. This follows with prop. 4.3.2 by prop. 3.4.9. □

Definition 4.3.5. We say that $\mathrm{ETop}\infty\mathrm{Grpd}$ defines *Euclidean-topological cohesion*. An object in $\mathrm{ETop}\infty\mathrm{Grpd}$ we call a *Euclidean-topological* ∞*-groupoid*.

Definition 4.3.6. Write TopMfd for the category whose objects are topological manifolds that are

- finite-dimensional;
- paracompact;
- with an arbitrary set of connected components (hence not assumed to be second-countable);

and whose morphisms are continuous functions between these. Regard this as a (large) site with the standard open-cover coverage.

Proposition 4.3.7. *The* ∞*-topos* $\mathrm{ETop}\infty$*Grpd is equivalently that of hypercomplete* ∞*-sheaves ([LuHTT], section 6.5) on* TopMfd

$$\mathrm{ETop}\infty\mathrm{Grpd} \simeq \hat{\mathrm{Sh}}_\infty(\mathrm{TopMfd})\,.$$

Proof. Since every topological manifold admits an cover by open balls homeomorphic to a Cartesian space, we have that $\mathrm{CartSp}_{\mathrm{top}}$ is a dense sub-site of TopMfd. By theorem C.2.2.3 in [John03] it follows that the sheaf toposes agree

$$\mathrm{Sh}(\mathrm{CartSp}_{\mathrm{top}}) \simeq \mathrm{Sh}(\mathrm{TopMfd})\,.$$

From this it follows directly that the Joyal model structures on simplicial sheaves over both sites (see [Jard87]) are Quillen equivalent. By [LuHTT], prop 6.5.2.14, these present the hypercompletions

$$\hat{\mathrm{Sh}}_\infty(\mathrm{CartSp}_{\mathrm{top}}) \simeq \hat{\mathrm{Sh}}_\infty(\mathrm{TopMfd})\,.$$

of the corresponding ∞-sheaf ∞-toposes. But by corollary 3.4.4 we have that ∞-sheaves on $\mathrm{CartSp}_{\mathrm{top}}$ are already hypercomplete, so that

$$\mathrm{Sh}_\infty(\mathrm{CartSp}_{\mathrm{top}}) \simeq \hat{\mathrm{Sh}}_\infty(\mathrm{TopMfd})\,.$$

□

Definition 4.3.8. Let $\mathrm{Top}_{\mathrm{cgH}}$ be the 1-category of compactly generated and Hausdorff topological spaces and continuous functions between them.

Proposition 4.3.9. *The category* $\mathrm{Top}_{\mathrm{cgH}}$ *is cartesian closed.*

See [Stee67]. We write $[-,-] : \mathrm{Top}_{\mathrm{cgH}}^{\mathrm{op}} \times \mathrm{Top}_{\mathrm{cgH}} \to \mathrm{Top}_{\mathrm{cgH}}$ for the corresponding internal hom-functor.

Definition 4.3.10. There is an evident functor

$$j : \mathrm{Top}_{\mathrm{cgH}} \to \mathrm{ETop}\infty\mathrm{Grpd}$$

that sends each topological space X to the 0-truncated ∞-sheaf (ordinary sheaf) represented by it

$$j(X) : (U \in \mathrm{CartSp}_{\mathrm{top}}) \mapsto \mathrm{Hom}_{\mathrm{Top}\,\mathrm{cgH}}(U, X) \in \mathrm{Set} \hookrightarrow \infty\mathrm{Grpd}\,.$$

Corollary 4.3.11. *The functor j exhibits* TopMfd *as a full sub-∞-category of* ETop∞Grpd

$$j : \mathrm{TopMfd} \hookrightarrow \mathrm{ETop}\infty\mathrm{Grpd}$$

Proof. By prop. 4.3.7 this is a special case of the ∞-Yoneda lemma. □

Remark 4.3.12. While, according to prop. 4.3.7, the model categories $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ and $[\mathrm{TopMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ are both presentations of ETop∞Grpd, they lend themselves to different computations: in the former there are more fibrant objects, fewer cofibrant objects than in the latter, and vice versa.

In 3.4.2.2 we gave a general discussion concerning this point, here we amplify specific detail for the present case.

Proposition 4.3.13. *Let $X \in [\mathrm{TopMfd}^{\mathrm{op}}, \mathrm{sSet}]$ be an object that is* globally fibrant*,* separated *and* locally trivial*, meaning that*

1. *$X(U)$ is a non-empty Kan complex for all $U \in \mathrm{TopMfd}$;*
2. *for every covering $\{U_i \to U\}$ in* TopMfd *the descent morphism $X(U) \to [\mathrm{TopMfd}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), X)$ is a full and faithful ∞-functor;*
3. *for contractible U we have $\pi_0[\mathrm{TopMfd}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), X) \simeq *$.*

Then the restriction of X along $\mathrm{CartSp}_{\mathrm{top}} \hookrightarrow \mathrm{TopMfd}$ is a fibrant object in the local model structure $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Proof. The fibrant objects in the local model structure are precisely those that are Kan complexes over every object and for which the descent morphism is an equivalence for all covers. The first condition is given by the first assumption. The second and third assumptions imply the second condition over contractible manifolds, such as the Cartesian spaces. □

Example 4.3.14. Let G be a topological group, regarded as the presheaf over TopMfd that it represents. Write $\bar{W}G$ for the simplicial presheaf on TopMfd given by the nerve of the topological groupoid $(G \overset{\to}{\to} *)$. (We discuss this in more detail in 4.3.2 below.)

The fibrant resolution of $\bar{W}G$ in $[\mathrm{TopMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ is (the rectification of) its stackification: the stack $G\mathrm{Bund}$ of topological G-principal bundles. But the canonical morphism

$$\bar{W}G \to G\mathrm{Bund}$$

is a full and faithful functor (over each object $U \in \mathrm{TopMfd}$): it includes the single object of $\bar{W}G$ as the trivial G-principal bundle. The automorphisms of the single object in $\bar{W}G$ over U are G-valued continuous functions on U, which are precisely the automorphisms of the trivial G-bundle. Therefore this inclusion is full and faithful, the presheaf $\bar{W}G$ is a separated prestack.

Moreover, it is locally trivial: every Čech cocycle for a G-bundle over a Cartesian space is equivalent to the trivial one. Equivalently, also $\pi_0 G\mathrm{Bund}(\mathbb{R}^n) \simeq *$. Therefore $\bar{W}G$, when restricted $\mathrm{CartSp}_{\mathrm{top}}$, does become a fibrant object in $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

On the other hand, let $X \in \mathrm{TopMfd}$ be any non-contractible manifold. Since in the projective model structure on simplicial presheaves every representable is cofibrant, this is a cofibrant object in $[\mathrm{Mfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.However, it fails to be cofibrant in $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Instead, there a cofibrant replacement is given by the Čech nerve $C(\{U_i\})$ of any good open cover $\{U_i \to X\}$.

This yields two different ways for computing the first nonabelian cohomology

$$H^1_{\mathrm{ETop}}(X, G) := \pi_0 \mathrm{ETop}\infty\mathrm{Grpd}(X, \mathbf{B}G)$$

in ETop∞Grpd on X with coefficients in G:

1. $\cdots \simeq \pi_0[\mathrm{Mfd}^{\mathrm{op}}, \mathrm{sSet}](\mathrm{X}, G\mathrm{Bund}) \simeq \pi_0 G\mathrm{Bund}(X)$;
2. $\cdots \simeq \pi_0[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \bar{W}G) \simeq H^1(X, G)$.

In the first case we need to construct the fibrant replacement GBund. This amounts to constructing G-principal bundles over *all* paracompact manifolds and then evaluate on the given one, X, by the 2-Yoneda lemma. In the second case however we cofibrantly replace X by a good open cover, and then find the Čech cocycles with coefficients in G on that.

For ordinary G-bundles the difference between the two computations may be irrelevant in practice, because ordinary G-principal bundles are very well understood. However, for more general coefficient objects, for instance general topological simplicial groups G, the first approach requires to find the full ∞-sheafification to the ∞-sheaf of all principal ∞-bundles, while the second approach requires only to compute specific coycles over one specific base object. In practice the latter is often all that one needs.

We discuss a few standard techniques for constructing *cofibrant* resolutions in $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Proposition 4.3.15. *Let*

$$X \in \mathrm{TopMfd} \hookrightarrow [\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

be a topological manifold and let $\{U_i \to X\}$ be a good open cover. Then the Čech nerve

$$C(\{U_i\}) := \int^{[n]\in\Delta} \Delta[n] \cdot \coprod_{i_0, \cdots, i_n} j(U_{i_0}) \cap \cdots \cap j(U_{i_n})$$

(where $j : \mathrm{TopMfd} \hookrightarrow [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ is the Yoneda embedding) equipped with the canonical projection $C(\{U_i\}) \to X$ is a cofibrant resolution of X.

Proof. The morphism is clearly a stalkwise weak equivalence. Therefore it is a weak equivalence in the local model structure by theore, 2.2.12.

Moreover, by the very definition of *good* open cover the non-empty finite intersections of the U_i are themselves represented by objects in $\mathrm{CartSp}^{\mathrm{op}}$. Therefore the Čech nerve is degreewise a coproduct of representables. Also, its degeneracies split off as a direct summand in each degree. By [Dugg01] this means that it is cofibrant in the global projective model structure. But the cofibrations do not change under left Bousfield localization to the local model structure, therefore it is cofibrant also there. □

Proposition 4.3.16.

$$X_\bullet \in \mathrm{TopMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

be a simplicial manifold, such that there is a choice $\mathcal{U}$ *of good open covers* $\{U_{n,i} \to X_n\}_i$ *in each degree which are simplicially compatible in that they arrange into a morphism of bisimplicial presheaves*

$$C(\mathcal{U})_{\bullet,\bullet} \to X_\bullet \,.$$

Then

$$\int^{[n]\in\Delta} \mathbf{\Delta}[n] \cdot C(\mathcal{U})_{n,\bullet} \;\to\; X_\bullet \,,$$

where $\mathbf{\Delta} : \Delta^{\mathrm{op}} \to \mathrm{sSet}$ *is given by* $\mathbf{\Delta}[n] := N(\Delta/[n])$*, is a cofibrant resolution in* $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}]_{\mathrm{proj,loc}}$.

Proof. First consider

$$\int^{[n]\in\Delta} \Delta[n] \cdot C(\mathcal{U})_{n,\bullet} \;\to\; X_\bullet$$

with the ordinary simplex in the integrand. Over ach object $U \in \mathrm{CartSp}_{\mathrm{top}}$ the coend appearing here is isomorphic to the diagonal of the given bisimplicial set. Since the diagonal sends degreewise weak equivalences to weak equivalences, prop. 4.3.15 implies that this is a weak equivalence in the local model structure.

Let $\mathbf{\Delta} \to \Delta$ be the canonical projection. We claim that the induced morphism

$$\int^{[n]\in\Delta} \mathbf{\Delta}[n] \cdot C(\mathcal{U})_{n,\bullet} \to \int^{[n]\in\Delta} \Delta[n] \cdot C(\mathcal{U})_{n,\bullet}$$

is a global projective weak equivalence, and hence in particular also a local projective weak equivalence. This follows from the fact that

$$\int^{\Delta} (-)\cdot(-) : [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}} \times [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}_{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{Reedy}} \to [\mathrm{CartSp}^{\mathrm{op}_{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{Reedy}}$$

is a left Quillen bifunctor prop. 2.3.17. Since every object in $[\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}_{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{Reedy}}$ is cofibrant, and since $\mathbf{\Delta} \to \Delta$ is a Reedy equivalence between Reedy cofibrant objects, the coend over the tensoring preserves this weak equivalence and produces a global injective weak equivalence which is also a global projective weak equivalence.

This shows that the morphism is question is a weak equivalence. To see that it is a cofibrant resolution use that $\mathbf{\Delta}$ is also cofibrant in $[\Delta, \mathrm{sSet}]_{\mathrm{proj}}$ and that also

$$\int^{\Delta} (-)\cdot(-) : [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}} \times [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}_{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{inj}} \to [\mathrm{CartSp}^{\mathrm{op}_{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{proj}}]$$

is a left Quillen bifunctor, prop. 2.3.17. By prop. 4.3.15 we have a cofibration $\emptyset \hookrightarrow C(\mathcal{U})_{\bullet,\bullet}$ in $[\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}_{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{inj}}$, which is therefore preserved by $\int^{\Delta} \mathbf{\Delta} \cdot (-)$. Again using that global projective cofibrations are also local projective cofibrations, the claim follows. □

We now discuss some of the general abstract structures in any cohesive ∞-topos, 3.9, realized in ETop∞Grpd.

- 4.3.1 – Stalks
- 4.3.2 – Groups
- 4.3.4 – Geometric homotopy
- 4.3.5 – $\mathbb{R}^1$-homotopy / The standard continuum
- 4.3.6 – Manifolds
- 4.3.7 – Paths and geometric Postnikov towers
- 4.3.8 – Cohomology
- 4.3.9 – Principal ∞-bundles
- 4.3.11 – Universal coverings and geometric Whitehead towers

4.3.1 Stalks

We discuss the points of ETop∞Grpd.

Proposition 4.3.17. *For every $n \in \mathbb{N}$ there is a topos point*

$$p(n) : \ \mathrm{Set} \underset{p(n)_*}{\overset{p(n)^*}{\rightleftarrows}} \mathrm{Sh}(\mathrm{Mfd})$$

as well as a corresponding ∞-topos point

$$p(n) : \ \infty\mathrm{Grpd} \underset{p(n)_*}{\overset{p(n)^*}{\rightleftarrows}} \mathrm{ETop}\infty\mathrm{Grpd} \ ,$$

where the inverse image $p(n)^*$ forms the stalk at the origin of $\mathbb{R}^n$:

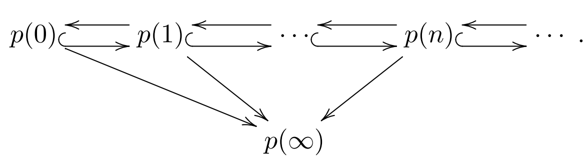
$$p(n)^* : X \mapsto \varinjlim_{k \in \mathbb{N}} X(D^n(1/k)) \,.$$

Here for $r \in \mathbb{R}_{\geq 0}$ we denote by $D^n(r) \hookrightarrow \mathbb{R}^n$ the inclusion of the standard open n-disk of radius r. In particular

$$p(0) \simeq (\Gamma \dashv \mathrm{coDisc}) \,.$$

The collection of topos points $\{p(n)\}_{n \in \mathbb{N}}$ exhibits the topos Sh(Mfd) *and the ∞-topos* ETop∞Grpd *(hence the sites* CartSp *and* Mfd*) as having* enough points*, def. 2.2.9.*

These points form a tower of retractions



The inductive limit $p(\infty) := \varinjlim_n p(n)$ over the tower of inclusions is the topos point whose inverse image is given by

$$p(\infty)^* X = \varinjlim_n \varinjlim_k X(D^n(1/k)) \,.$$

This point alone forms a set of enough points: a morphism $f : X \to Y$ is an equivalence precisely if $p(\infty)^* f$ is.

Proof. For convenience, we discuss this in terms of the 1-topos. The discussion for the ∞-topos is verbatim the same.

First it is clear that for all $n \in \mathbb{N}$ the functor $p(n)^*$ is indeed the inverse image of a geometric morphism: being given by a filtered colimit, it commutes with all colimits and with finite limits.

To see that these points are enough to detect isomorphisms of sheaves, notice the following construction. For $A \in \mathrm{Sh}(\mathrm{Mfd})$ and $X \in \mathrm{Mfd}$, we obtain a sheaf $\tilde{A} \in \mathrm{Sh}(\mathrm{Mfd}/_{\mathrm{op}}X)$ on the slice site of open embeddings into X by restriction of A. The topos $\mathrm{Sh}(\mathrm{Mfd}/_{\mathrm{op}}X)$ clearly has enough points, given by the ordinary stalks at the ordinary points $x \in X$, formed as

$$p_x(n)^*\tilde{A} = \varinjlim_k \tilde{A}(D^n_x(1/k)),$$

where $D^n_x(r) \hookrightarrow \mathbb{R}^n \overset{\phi}{\hookrightarrow} X$ is a disk of radius r around x in any coordinate patch ϕ containing X. (Because if a morphism of sheaves on $\mathrm{Mfd}/_{\mathrm{op}}X$ is an isomorphism on an open disk around every point of X, then it is an isomorphism on the covering given by the union of all these disks, hence is an isomorphism of sheaves). Notice that by defintion of $\tilde{A}$ the above stalk is in fact independent of the point x and coincides with $p(n)^*$ applied to the original A:

$$\cdots \simeq \varinjlim_k A(D^n(1/k)) =: p(n)^*A.$$

So if for a morphism $f : A \to B$ in $\mathrm{Sh}(\mathrm{Mfd})$ all the $p(n)^*f$ are isomorphisms, then for every $X \in \mathrm{Mfd}$ the induced morphism $\tilde{f} : \tilde{A} \to \tilde{B}$ is an isomorphism, hence is an isomorphism $\tilde{f}(X) = f(X)$ on global sections. Since this is true for all X, it follows that f is already an isomorphism. This shows that $\{p(n)\}_{n\in\mathbb{N}}$ is a set of enough points of $\mathrm{Sh}(\mathrm{Mfd})$.

To see that these points sit in a sequence of retractions as stated, choose a tower of inclusions

$$\mathbb{R}^0 \hookrightarrow \mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \cdots \quad \in \mathrm{Mfd},$$

where each morphism is isomorphic to $\mathbb{R}^n \times \mathbb{R}^0 \overset{(\mathrm{id},0)}{\to} \mathbb{R}^n \times \mathbb{R}^1$.

This induces for each $n \in \mathbb{N}$ and $r \in \mathbb{R}$ an inclusion of disks $D^n(r) \to D^{n+1}(r)$, which regards $D^n(r)$ as an equatorial plane of $D^{n+1}(r)$, and it induces a projection $D^{n+1}(r)$, which together exhibit a retraction

$$D^n \longrightarrow D^{n+1} \longrightarrow D^n, \quad \text{composite} = \mathrm{id}.$$

All this is natural with respect to the inclusions $D^n(\frac{1}{k+1}) \to D^n(\frac{1}{k})$. Therefore we have induced morphisms

$$\varinjlim_k X(D^n(1/k)) \longrightarrow \varinjlim_k X(D^{n+1}(1/k)) \longrightarrow \varinjlim_k X(D^n(1/k)), \quad \text{composite} = \mathrm{id}.$$

Since these are natural in X, they consistute natural transformations

$$p(n)^* \longrightarrow p(n+1)^* \longrightarrow p(n)^*, \quad \text{composite} = \mathrm{id}$$

of inverse images, hence morphisms

$$p(n) \longrightarrow p(n+1) \longrightarrow p(n), \quad \text{composite} = \mathrm{id}$$

of geometric morphisms.

Finally, since equivalences are stable under retract, it follows that $p(n)^* f$ is an equivalence if $p(n+1)^*$ is. Similarly, for every $n \in \mathbb{N}$ we have a retract

$$p(n) \longrightarrow p(\infty) \longrightarrow p(n), \qquad \text{composite} = \mathrm{id}$$

seen by noticing that each $p(n)$ naturally forms a co-cone under the above tower of inclusions. So an isomorphism under $p(\infty)^*$ implies one under all the $p(n)$. □

4.3.2 Groups

We discuss cohesive ∞-group objects, def 3.6.8, realized in ETop∞Grpd: *Euclidean-topological ∞-groups.*

Recall that by prop. 3.6.134 every ∞-group object in ETop∞Grpd has a presentation by a presheaf of simplicial groups. Among the presentations for concrete ∞-groups in ETop∞Grpd are therefore *simplicial topological groups.*

Write $\mathrm{sTop}_{\mathrm{cgH}}$ for the category of simplicial objects in $\mathrm{Top}_{\mathrm{cgH}}$, def. 4.3.8. For $X, Y \in \mathrm{sTop}_{\mathrm{cgH}}$, write

$$\mathrm{sTop}_{\mathrm{cgH}}(X,Y) := \int_{[k]\in\Delta} [X_k, Y_k] \quad \in \mathrm{Top}_{\mathrm{cgH}}$$

for the hom-object, where in the integrand of the end $[-,-]$ is the internal hom of $\mathrm{Top}_{\mathrm{cgH}}$.

Definition 4.3.18. We say a morphism $f : X \to Y$ of simplicial topological spaces is a *global Kan fibration* if for all $n \in \mathbb{N}$ and $0 \leq k \leq n$ the canonical morphism

$$X_n \to Y_n \ \times_{\mathrm{sTop}_{\mathrm{cgH}}(\Lambda[n]_i, Y)} \ \mathrm{sTop}_{\mathrm{cgH}}(\Lambda[n]_i, X)$$

in $\mathrm{Top}_{\mathrm{cgH}}$ has a section, where $\Lambda[n]_i \in \mathrm{sSet} \hookrightarrow \mathrm{sTop}_{\mathrm{cgH}}$ is the ith n-horn regarded as a discrete simplicial topological space.

We say a simplicial topological space $X_\bullet$ is a *(global) Kan simplicial space* if the unique morphism $X_\bullet \to *$ is a global Kan fibration, hence if for all $n \in \mathbb{N}$ and all $0 \leq i \leq n$ the canonical continuous function

$$X_n \to \mathrm{sTop}_{\mathrm{cgH}}(\Lambda[n]_i, X)$$

into the topological space of ith n-horns admits a section.

This global notion of topological Kan fibration is considered for instance in [BrSz89], def. 2.1, def. 6.1. In fact there a stronger condition is imposed: a Kan complex in Set automatically has the lifting property not only against all full horn inclusions but also against sub-horns; and in [BrSz89] all these fillers are required to be given by global sections. This ensures that with X globally Kan also the internal hom $[Y, X] \in \mathrm{sTop}_{\mathrm{cgH}}$ is globally Kan, for any simplicial topological space Y. This is more than we need and want to impose here. For our purposes it is sufficient to observe that if f is globally Kan in the sense of [BrSz89], def. 6.1, then it is so also in the above sense.

For G a simplicial group, there is a standard presentation of its universal simplicial bundle by a morphism of Kan complexes traditionally denoted $WG \to \bar{W}G$. This construction has an immediate analog for simplicial topological groups. A review is in [RoSt12].

Proposition 4.3.19. *Let G be a simplicial topological group. Then*

1. *G is a globally Kan simplicial topological space;*

2. *$\bar{W}G$ is a globally Kan simplicial topological space;*

3. *$WG \to \bar{W}G$ is a global Kan fibration.*

Proof. The first and last statement appears as [BrSz89], theorem 3.8 and lemma 6.7, respectively, the second is noted in [RoSt12]. ∎

Let for the following $\mathrm{Top}_s \subset \mathrm{Top}_{\mathrm{cgH}}$ be any small full subcategory. Under the degreewise Yoneda embedding $\mathrm{sTop}_s \hookrightarrow [\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]$ simplicial topological spaces embed into the category of simplicial presheaves on Top_s. We equip this with the projective model structure on simplicial presheaves $[\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proposition 4.3.20. *Under this embedding a global Kan fibration, def. 4.3.18, $f : X \to Y$ in sTop_s maps to a fibration in $[\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.*

Proof. By definition, a morphism $f : X \to Y$ in $[\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is a fibration if for all $U \in \mathrm{Top}_s$ and all $n \in \mathbb{N}$ and $0 \leq i \leq n$ diagrams of the form

$$\begin{array}{ccc} \Lambda[n]_i \cdot U & \longrightarrow & X \\ \downarrow & & \downarrow^{f} \\ \Delta[n] \cdot U & \longrightarrow & Y \end{array}$$

have a lift. This is equivalent to saying that the function

$$\mathrm{Hom}(\Delta[n] \cdot U, X) \to \mathrm{Hom}(\Delta[n] \cdot U, Y) \times_{\mathrm{Hom}(\Lambda[n]_i \cdot U, Y)} \mathrm{Hom}(\Lambda[n]_i \cdot U, X)$$

is surjective. Notice that we have

$$\begin{aligned} \mathrm{Hom}_{[\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]}(\Delta[n] \cdot U, X) &= \mathrm{Hom}_{\mathrm{sTop}_s}(\Delta[n] \cdot U, X) \\ &= \int_{[k] \in \Delta} \mathrm{Hom}_{\mathrm{Top}_s}(\Delta[n]_k \times U, X_k) \\ &= \int_{[k] \in \Delta} \mathrm{Hom}_{\mathrm{Top}_s}(U, [\Delta[n]_k, X_k]) \\ &= \mathrm{Hom}_{\mathrm{Top}}(U, \int_{[k] \in \Delta} [\Delta[n]_k, X_k]) \\ &= \mathrm{Hom}_{\mathrm{Top}_s}(U, \mathrm{sTop}(\Delta[n], X)) \\ &= \mathrm{Hom}_{\mathrm{Top}_s}(U, X_n) \end{aligned}$$

and analogously for the other factors in the above morphism. Therefore the lifting problem equivalently says that the function

$$\mathrm{Hom}_{\mathrm{Top}}(U,\ X_n \to Y_n \times_{\mathrm{sTop}_s(\Lambda[n]_i, Y)} \mathrm{sTop}_s(\Lambda[n]_i, X)\ )$$

is surjective. But by the assumption that $f : X \to Y$ is a global Kan fibration of simplicial topological spaces, def. 4.3.18, we have a section $\sigma : Y_n \times_{\mathrm{sTop}_s(\Lambda[n]_i), Y} \mathrm{sTop}_s(\Lambda[n]_i, X) \to X_n$. Therefore $\mathrm{Hom}_{\mathrm{Top}_s}(U, \sigma)$ is a section of our function. ∎

In section 4.3.4 we use this in the discussion of geometric realization of simplicial topological groups.

In summary, we find that $WG \to \bar{W}G$ is a presentation of the universal G-principal ∞-bundle, 1.2.5.4.).

Proposition 4.3.21. *Let $G \in$ ETop∞Grpd be a group object presented in $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ by a simplicial topological group (to be denoted by the same symbol) which is degreewise a topological manifold. Then its delooping $\mathbf{B}G$, def. 3.6.116, is presented by $\bar{W}G$.*

Proof. By prop. 4.3.19 and prop. 4.3.20 the morphism $WG \to \bar{W}G$ is a fibration presentation of $* \to \mathbf{B}G$ in $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Since $\bar{W}G$ is evidently connected, and since we have an ordinary pullback diagram

$$\begin{array}{ccc} G & \longrightarrow & WG \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bar{W}G \end{array} ,$$

it follows with the discussion in 2.3.2.1 that this presents in ETop∞Grpd the ∞-pullback

$$\begin{array}{ccc} G & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

that defines the delooping $\mathbf{B}G$. □

4.3.3 Representations

We discuss the intrinsic notion of ∞-group representations, 3.6.13, realized in the context ETop∞Grpd.

We make precise the role of *topological action groupoids*, introduced informally in 1.2.5.1.

Proposition 4.3.22. *Let X be a toplogical manifold, and G a topological group. Then the category of continuous G-actions on X in the traditional sense is equivalent to the category of G-actions on X in the cohesive ∞-topos* ETop∞Grpd*, according to def. 3.6.152.*

Proof. For $\rho : X \times G \to X$ a given G-action, define the *action groupoid*

$$X//G := (\ X \times G \underset{p_1}{\overset{\rho}{\rightrightarrows}} X\ )$$

with the evident composition operation. This comes with the evident morphism of topological groupoids

$$X//G \to *//G \simeq \mathbf{B}G\,,$$

with $\mathbf{B}G$ as in prop. 4.4.19. It is immediate that regarding this as a morphism in $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ in the canonical way, this is a fibration. Therefore, by 2.3.13, the homotopy fiber of this morphism in Smooth∞Grpds is given by the ordinary fiber of this morphism in simplicial presheaves. This is manifestly X.

Accordingly this construction constitutes an embedding of the traditional G actions on X into the category $\mathrm{Rep}_G(X)$ from def. 3.6.152. By turning this argument around, one finds that this embedding is essentially surjective. □

Remark 4.3.23. Let $X, \in\in$ TopMfd, G a topological group, and let $\rho : X \times G \to X$ be a continuous action. Write $X//G \in$ ETop∞Grpd for the corresponding action groupoid. As a simplicial topological space the action groupoid is

$$X//G = \left(\cdots\cdots\ X \times G \times G \begin{array}{c} \xrightarrow{(\rho,\mathrm{id})} \\ \xrightarrow{(\mathrm{id},\cdot)} \\ \xrightarrow[(p_1,p_2)]{} \end{array} X \times G \underset{p_1}{\overset{\rho}{\rightrightarrows}} X \right)$$

4.3.4 Geometric homotopy

We discuss the intrinsic geometric homotopy, 3.8.1, in ETop∞Grpd.

4.3.4.1 Geometric realization of topological ∞-groupoids

We start by recalling some facts about geometric realization of simplicial topological spaces.

Definition 4.3.24. For $X_\bullet \in \mathrm{sTop}_{\mathrm{cgH}}$ a simplicial topological space, write

- $|X_\bullet| := \int^{[k]\in\Delta} \Delta^k_{\mathrm{Top}} \times X_k$ for its *geometric realization*;

- $\|X_\bullet\| := \int^{[k]\in\Delta_+} \Delta^k_{\mathrm{Top}} \times X_k$ for its *fat geometric realization*,

where in the second case the coend is over the subcategory $\Delta_+ \hookrightarrow \Delta$ spanned by the face maps.

See [RoSt12] for a review.

Proposition 4.3.25. *Ordinary geometric realization* $|-| : \mathrm{sTop}_{\mathrm{cgH}} \to \mathrm{Top}_{\mathrm{cgH}}$ *preserves pullbacks. Fat geometric realization preserves pullbacks when regarded as a functor* $\|-\| : \mathrm{sTop}_{\mathrm{cgH}} \to \mathrm{Top}_{\mathrm{cgH}}/\|*\|$.

Definition 4.3.26. We say

- a simplicial topological space $X \in \mathrm{sTop}_{\mathrm{cgH}}$, def. 4.3.8, is *good* if all degeneracy maps $s_i : X_n \to X_{n+1}$ are closed Hurewicz cofibrations;
- a simplicial topological group G is *well pointed* if all units $i_n : * \to G_n$ are closed Hurewicz cofibrations.

The notion of good simplicial topological spaces goes back to [Sega73]. For a review see [RoSt12].

Proposition 4.3.27. *For* $X \in \mathrm{sTop}_s$ *a good simplicial topological space, its ordinary geometric realization is equivalent to its homotopy colimit, when regarded as a simplicial diagram:*

$$\mathrm{sTop}_s \hookrightarrow [\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \xrightarrow{\mathrm{hocolim}} \mathrm{Top}_{\mathrm{Quillen}}\,.$$

Proof. Write $\|-\|$ for the fat geometric realization. By standard facts about geometric realization of simplicial topological spaces [Sega70] we have the following zig-zag of weak homotopy equivalences

$$\begin{array}{ccccc} \|X_\bullet\| & \xleftarrow{\simeq} & \|\,|\mathrm{Sing}(X_\bullet)|\,\| & & \\ \downarrow^{\simeq} & & \downarrow^{\simeq} & & \\ |X_\bullet| & & |\,|\mathrm{Sing}(X_\bullet)|\,| \underset{\mathrm{iso}}{=\!=} |\mathrm{diagSing}(X_\bullet)_\bullet| & \xrightarrow{\simeq} & |\mathrm{hocolim}_n \mathrm{Sing} X_n| \end{array}.$$

By the Bousfield-Kan map, the object on the far right is manifestly a model for the homotopy colimit $\mathrm{hocolim}_n X_n$. □

Proposition 4.3.28. *For* $X \in \mathrm{TopMfd}$ *and* $\{U_i \to X\}$ *a good open cover, the Čech nerve* $C(\{U_i\}) := \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_n} U_{i_0} \times_X \cdots \times U_{i_n}$ *is cofibrant in* $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *and the canonical projection* $C(\{U_i\}) \to X$ *is a weak equivalence.*

Proof. Since the open cover is good, the Čech nerve is degreewise a coproduct of representables, hence is a *split hypercover* in the sense of [DuHoIs04], def. 4.13. Moreover $\coprod_i U_i \to X$ is directly seen to be a *generalized cover* in the sense used there (below prop. 3.3) By corollary A.3 there, $C(\{U_i\}) \to X$ is a weak equivalence. □

Proposition 4.3.29. *Let* X *be a paracompact topological space that admits a good open cover by open balls (for instance a topological manifold). Write* $i(X) \in \mathrm{ETop}\infty\mathrm{Grpd}$ *for its incarnation as a 0-truncatd Euclidean-topological* ∞*-groupoid. Then* $\Pi(X) := \Pi(i(X)) \in \infty\mathrm{Grpd}$ *is equivalent to the standard fundamental* ∞*-groupoid of* X*, presented by the singular simplicial complex* $\mathrm{Sing}X : [k] \mapsto \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cgH}}}(\Delta^k, X)$

$$\Pi(X) \simeq \mathrm{Sing}X\,.$$

Equivalently, under geometric realization $\mathbb{L}|-| : \infty\mathrm{Grpd} \to \mathrm{Top}$ *we have that there is a weak homotopy equivalence*

$$X \simeq |\Pi(X)|\,.$$

Proof. By the proof of prop. 3.4.9 we have an equivalence $\Pi(-) \simeq \mathbb{L}\varinjlim$ to the derived functor of the sSet-colimit functor $\varinjlim : [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}} \to \mathrm{sSet}_{\mathrm{Quillen}}$.

To compute this derived functor, let $\{U_i \to X\}$ be a good open cover by open balls, hence homeomorphically by Cartesian spaces. By goodness of the cover the Čech nerve $C(\coprod_i U_i \to X) \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ is degreewise a coproduct of representables, hence a split hypercover. By [DuHoIs04] we have that in this case the canonical morphism

$$C(\coprod_i U_i \to X) \to X$$

is a cofibrant resolution of X in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Accordingly we have

$$\Pi(X) \simeq (\mathbb{L}\varinjlim)(X) \simeq \varinjlim C(\coprod_i U_i \to X)\,.$$

Using the equivalence of categories $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}] \simeq [\Delta^{op}, [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Set}]$ and that colimits in presheaf categories are computed objectwise, and finally using that the colimit of a representable functor is the point (an incarnation of the Yoneda lemma) we have that $\Pi(X)$ is presented by the Kan complex that is obtained by contracting in the Čech nerve $C(\coprod_i U_i)$ each open subset to a point.

The classical nerve theorem [Bors48] asserts that this implies the claim. □

Regarding Top itself as a cohesive ∞-topos by 4.1.1, the above proposition may be stated as saying that for X a paracompact topological space with a good covering, we have

$$\Pi_{\mathrm{ETop}\infty\mathrm{Grpd}}(X) \simeq \Pi_{\mathrm{Top}}(X)\,.$$

Proposition 4.3.30. *Let $X_\bullet$ be a good simplicial topological space that is degreewise paracompact and degreewise admits a good open cover, regarded naturally as an object $X_\bullet \in \mathrm{sTop}_{\mathrm{cgH}} \to \mathrm{ETop}\infty\mathrm{Grpd}$.*

We have that the intrinsic $\Pi(X_\bullet) \in \infty\mathrm{Grpd}$ coincides under geometric realization $\mathbb{L}|-| : \infty\mathrm{Grpd} \stackrel{\simeq}{\to} \mathrm{Top}$ with the ordinary geometric realization of simplicial topological spaces $|X_\bullet|_{\mathrm{Top}^{\Delta^{\mathrm{op}}}}$ from def. 4.3.25:

$$|\Pi(X_\bullet)| \simeq |X_\bullet|\,.$$

Proof. Write Q for Dugger's cofibrant replacement functor, prop. 2.2.18, on $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. On a simplicially constant simplicial presheaf X it is given by

$$QX := \int^{[n]\in\Delta} \Delta[n] \cdot \left(\coprod_{U_0 \to \cdots \to U_n \to X} U_0 \right)\,,$$

where the coproduct in the integrand of the coend is over all sequences of morphisms from representables U_i to X as indicated. On a general simplicial presheaf $X_\bullet$ it is given by

$$QX_\bullet := \int^{[k]\in\Delta} \Delta[k] \cdot QX_k\,,$$

which is the simplicial presheaf that over any $\mathbb{R}^n \in \mathrm{CartSp}$ takes as value the diagonal of the bisimplicial set whose (n,r)-entry is $\coprod_{U_0 \to \cdots \to U_n \to X_k} \mathrm{CartSp}_{\mathrm{top}}(\mathbb{R}^n, U_0)$. Since coends are special colimits, the colimit functor itself commutes with them and we find

$$\begin{aligned} \Pi(X_\bullet) &\simeq (\mathbb{L}\varinjlim) X_\bullet \\ &\simeq \varinjlim QX_\bullet \\ &\simeq \int^{[n]\in\Delta} \Delta[k] \cdot \varinjlim(QX_k)\,. \end{aligned}$$

By general facts about the Reedy model structure on bisimplicial sets, this coend is a homotopy colimit over the simplicial diagram $\varinjlim QX_\bullet : \Delta \to \mathrm{sSet}_{\mathrm{Quillen}}$

$$\cdots \simeq \mathrm{hocolim}_\Delta \varinjlim QX_\bullet \,.$$

By prop. 4.3.29 we have for each $k \in \mathbb{N}$ weak equivalences $\varinjlim QX_k \simeq (\mathbb{L}\varinjlim)X_k \simeq \mathrm{Sing}X_k$, so that

$$\begin{aligned} \cdots &\simeq \mathrm{hocolim}_\Delta \mathrm{Sing}X_\bullet \\ &\simeq \int^{[k]\in\Delta} \Delta[k]\cdot \mathrm{Sing}X_k \,\cdot \\ &\simeq \mathrm{diag}\,\mathrm{Sing}(X_\bullet)_\bullet \end{aligned}$$

By prop. 4.3.27 this is the homotopy colimit of the simplicial topological space $X_\bullet$, given by its geometric realization if $X_\bullet$ is proper. □

4.3.4.2 Examples

We discuss some examples related to the geometric realization of topological ∞-groupoids.

Proposition 4.3.31. *Let K and G be topological groups whose underlying topological space is a manifold. Consider a morphism of topological groups $f : K \to G$ that is a homotopy equivalence of the underlying topological manifolds. Then*

$$\Pi\mathbf{B}f :\ \Pi(\mathbf{B}K) \longrightarrow \Pi(\mathbf{B}G)$$

is a weak equivalence.

Proof. By prop. 4.3.21 the delooping $\mathbf{B}G$ is presented in $[\mathrm{CartSp}_{\mathrm{top}^{\mathrm{op}}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ by $(\mathbf{B}G_{\mathrm{ch}}) : n \mapsto G^{\times n}$. Therefore $\Pi(K^{\times n}) \to \Pi(G^{\times n})$ is an equivalence in ∞Grpd. By the discussion in 3.6.8 we have that the delooping $\mathbf{B}K$ is the ∞-colimit

$$\mathbf{B}K \simeq \lim_{\to\, n} K^{\times n}$$

and similarly for $\mathbf{B}G$. The morphism of moduli stacks is the ∞-colimit of the component inclusions

$$\mathbf{c} \simeq \lim_{\to\, n} (K^{\times n} \to G^{\times n})\,.$$

Since Π is left adjoint, it commutes with these colimits, so that $\Pi(\mathbf{c})$ is exhibited as an ∞-colimit over equivalences, hence as an equivalence. □

Proposition 4.3.32. *Let X be a topological manifold, equipped with a continuous action $\rho : X \times G \to X$ of a group in* TopMfd. *Then the geometric realization of the corresponding action groupoid, def. 4.3.22, is the Borel space*

$$\Pi(X//G) \simeq |X//G| = X \times_G EG\,.$$

Proof. By remark 4.3.23 the action groupoid as an object in $\mathrm{TopMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{Top}}, \mathrm{sSet}]$ is

$$X//G = \left(\cdots\cdots\; X \times G \times G \underset{(p_1,p_2)}{\overset{(\rho,\mathrm{id})}{\underset{(\mathrm{id},\cdot)}{\rightrightarrows\!\!\to}}} X \times G \underset{p_1}{\overset{\rho}{\rightrightarrows}} X \right).$$

Accordingly

$$\mathbf{E}G := G//G = \left(\cdots\cdots\; G \times G \times G \underset{(p_1,p_2)}{\overset{(\cdot,\mathrm{id})}{\underset{(\mathrm{id},\cdot)}{\rightrightarrows\!\!\to}}} \qquad G \times G \underset{p_1}{\overset{\cdot}{\rightrightarrows}} X \right).$$

Therefore we have an isomorphism

$$X//G = X \times_G \mathbf{E}G\,.$$

By prop. 4.3.25 geometric realization preserves the product involved here, and, being given by a coend, it preserves the quotient involved, so that we have isomorphisms

$$|X//G| = |X \times_G \mathbf{E}G| = X \times_G EG\,.$$

□

Below in 4.3.8.3 we discuss how the cohomology of the Borel space is related to the equivariant cohomology of X.

4.3.5 $\mathbb{R}^1$-homotopy / The standard continuum

We discuss that the standard continuum real line $\mathbb{R} \in \mathrm{SmthMfd} \hookrightarrow \mathrm{ETop}\infty\mathrm{Grpd}$ regarded in Euclidean-topological cohesion is indeed a contiuum $\mathbb{A}^1$-line object in the general abstract sense of 3.8.1.

Proposition 4.3.33. *The real line* $\mathbb{R}^1 \in \mathrm{TopMfd} \hookrightarrow \mathrm{ETop}\infty\mathrm{Grpd}$ *is a geometric interval, def. 3.9.2, exhibiting the cohesion of* $\mathrm{ETop}\infty\mathrm{Grpd}$*.*

Proof. Since $\mathrm{CartSp}_{\mathrm{top}}$ is a site of definition for $\mathrm{ETop}\infty\mathrm{Grpd}$ and is both ∞-cohesive (prop. 4.3.2) and the syntactic category of a Lawvere algebraic theory, with

$$\mathbb{A}^1 = \mathbb{R}^1\,,$$

the claim follows with prop. 3.9.4. □

Remark 4.3.34. The statement of prop. 4.3.33 is the central claim of the notes [Dugg99], where it essentially appears stated as theorem 3.4.3.

4.3.6 Manifolds

We discuss the realization of the general abstract notion of manifolds in a cohesive ∞-topos in 3.9.2 realized in Euclidean-topological cohesion.

With $\mathbb{A} := \mathbb{R} \in \mathrm{TopMfd} \hookrightarrow \mathrm{ETop}\infty\mathrm{Grpd}$ the standard line object exhibiting the cohesion of $\mathrm{ETop}\infty\mathrm{Grpd}$ according to prop. 4.3.33, def. 3.9.9 is equivalent to the traditional definition of topological manifolds.

4.3.7 Paths and geometric Postnikov towers

We discuss the general abstract notion of path ∞-groupoid, 3.8.3, realized in $\mathrm{ETop}\infty\mathrm{Grpd}$.

Proposition 4.3.35. *Let X be a paracompact topological space, canonically regarded as an object of* $\mathrm{ETop}\infty\mathrm{Grpd}$*, then the path ∞-groupoid* $\mathbf{\Pi}(X)$ *is presented by the simplicial presheaf* $\mathrm{Disc}\,\mathrm{Sing}X \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ *which is constant on the singular simplicial complex of X:*

$$\mathrm{Disc}\,\mathrm{Sing}X : (U,[k]) \mapsto \mathrm{Sing}X\,.$$

Proof. By definition we have $\mathbf{\Pi}(X) = \mathrm{Disc}\,\Pi(X)$. By prop. 4.3.29 $\Pi(X) \in \infty\mathrm{Grpd}$ is presented by $\mathrm{Sing}X$. By prop. 3.4.9 the ∞-functor Disc is presented by the left derived functor of the constant presheaf functor. Since every object in $\mathrm{sSet}_{\mathrm{Quillen}}$ is cofibrant this is just the plain constant presheaf functor. □
A more natural presentation of the idea of a topological path ∞-groupoid may be one that remembers the topology on the space of k-dimensional paths:

Definition 4.3.36. For X a paracompact topological space, write $\mathbf{Sing}X \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ for the simplicial presheaf given by

$$\mathbf{Sing}X : (U, [k]) \mapsto \mathrm{Hom}_{\mathrm{Top}}(U \times \Delta^k, X) \,.$$

Proposition 4.3.37. *Also* $\mathrm{Sing}X$ *is a presentation of* $\mathbf{\Pi}X$*.*

Proof. For each $U \in \mathrm{CartSp}$ the canonical inclusion of simplicial sets

$$\mathrm{Sing}X \to \mathbf{Sing}(X)(U)$$

is a weak homotopy equivalence, because U is continuously contractible. Therefore the canonical inclusion of simplicial presheaves

$$\mathrm{Disc}\,\mathrm{Sing}X \to \mathbf{Sing}X$$

is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. □

Remark 4.3.38. Typically one is interested in mapping out of $\mathbf{\Pi}(X)$. While $\mathrm{Disc}\,\mathrm{Sing}X$ is always cofibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, the relevant resolutions of $\mathbf{Sing}(X)$ may be harder to determine.

4.3.8 Cohomology

We dicuss aspects of the intrinsic cohomology (3.6.9) in ETop∞Grpd.

4.3.8.1 Čech cohomology We expand on the way that the intrinsic cohomology in ETop∞Grpd is expressed in terms of traditional Čech cohomology over manifolds, further specializing the general discussion of 2.2.5.

Proposition 4.3.39. *For* $X \in \mathrm{TopMfd}$ *and* $A \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *a fibrant representative of an object in* ETop∞Grpd*, the intrinsic cocycle* ∞*-groupoid* ETop∞Grpd *is given by the Čech cohomology cocycles on* X *with coefficients in* A*.*

Proof. Let $\{U_i \to X\}$ be a good open cover. By prop. 4.3.28 its Čech nerve $C(\{U_i\}) \stackrel{\simeq}{\to} X$ is a cofibrant replacement for X (it is a split hypercover [Dugg01] and hence cofibrant because the cover is good, and it is a weak equivalence because it is a *generalized cover* in the sense of [DuHoIs04]). Since $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ is a simplicial model category, it follows that the cocycle ∞-groupoid in question is given by the Kan complex $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), A)$. One checks that its vertices are Čech cocycles as claimed, its edges are Čech homotopies, and so on. □

4.3.8.2 Nonabelian cohomology with constant coefficients

Definition 4.3.40. Let $A \in \infty\mathrm{Grpd}$ be any discrete ∞-groupoid. Write $|A| \in \mathrm{Top}_{\mathrm{cgH}}$ for its geometric realization. For X any topological space, the nonabelian cohomology of X with coefficients in A is the set of homotopy classes of maps $X \to |A|$

$$H_{\mathrm{Top}}(X, A) := \pi_0 \mathrm{Top}(X, |A|) \,.$$

We say $\mathrm{Top}(X, |A|)$ itself is the cocycle ∞-groupoid for A-valued nonabelian cohomology on X.

Similarly, for $X, \mathbf{A} \in$ ETop∞Grpd two Euclidean-topological ∞-groupoids, write

$$H_{\mathrm{ETop}}(X, \mathbf{A}) := \pi_0 \mathrm{ETop}\infty\mathrm{Grpd}(X, \mathbf{A})$$

for the intrinsic cohomology of ETop∞Grpd on X with coefficients in $\mathbf{A}$.

Proposition 4.3.41. *Let $A \in \infty\mathrm{Grpd}$, write $\mathrm{Disc}A \in \mathrm{ETop}\infty\mathrm{Grpd}$ for the corresponding discrete topological ∞-groupoid. Let X be a paracompact topological space admitting a good open cover, regarded as 0-truncated Euclidean-topological ∞-groupoid.*

We have an isomorphism of cohomology sets

$$H_{\mathrm{Top}}(X, A) \simeq H_{\mathrm{ETop}}(X, \mathrm{Disc}A)$$

and in fact an equivalence of cocycle ∞-groupoids

$$\mathrm{Top}(X, |A|) \simeq \mathrm{ETop}\infty\mathrm{Grpd}(X, \mathrm{Disc}A) .$$

Proof. By the $(\Pi \dashv \mathrm{Disc})$-adjunction of the locally ∞-connected ∞-topos $\mathrm{ETop}\infty\mathrm{Grpd}$ we have

$$\mathrm{ETop}\infty\mathrm{Grpd}(X, \mathrm{Disc}A) \simeq \infty\mathrm{Grpd}(\Pi(X), A) \xrightarrow[\simeq]{|-|} \mathrm{Top}(|\Pi X|, |A|) .$$

From this the claim follows by prop. 4.3.29. □

4.3.8.3 Equivariant cohomology

Proposition 4.3.42. *Given an action $\rho : X \times G \to X$ of a topological group G on a topological manifold X, as in prop. 4.3.32, $n \in \mathbb{N}$ and K a discrte group, abelian if $n \geq 2$, then the G-equivariant cohomology, def. 3.6.139, of X with coefficients in K is the cohomology of the Borel space, prop. 4.3.32, with values in K*

$$H^n_G(X, K) \simeq H^n(X \times_G EG, K) .$$

Proof. The equivariant cohomology is the cohomology of the action groupoid

$$H^n_G(X, K) \simeq \pi_0 \mathrm{ETop}\infty\mathrm{Grpd}(X//G, \mathbf{B}^n K) .$$

Since K is assumed discrete, this is equivalently, as in prop. 4.3.41,

$$\cdots \simeq \pi_0 \infty\mathrm{Grpd}(\Pi(X//G), \mathbf{B}^n K)$$

By prop. 4.3.32 this is

$$\cdots \simeq \pi_0 \mathrm{Top}(X \times_G EG, B^n K) \simeq H^n(X \times_G EG, K) .$$

□

4.3.9 Principal bundles

We discuss principal ∞-bundles, 3.6.10, with topological structure and presented by topological simplicial principal bundles.

Proposition 4.3.43. *If G is a well-pointed simplicial topological group, def. 4.3.26, then both WG and $\bar{W}G$ are good simplicial topological spaces.*

Proof. For $\bar{W}G$ this is [RoSt12] prop. 19. For WG this follows with their lemma 10, lemma 11, which says that $WG = \mathrm{Dec}_0 \bar{W}G$ and the observations in the proof of prop. 16 that $\mathrm{Dec}_0 X$ is good if X is. □

Proposition 4.3.44. *For G a well-pointed simplicial topological group, the geometric realization of the universal simplicial principal bundle $WG \to \bar{W}G$*

$$|WG| \to |\bar{W}G|$$

is a fibration resolution in $\mathrm{Top}_{\mathrm{Quillen}}$ of the point inclusion $* \to B|G|$ into the classifying space of the geometric realization of G.

This is [RoSt12], prop. 14.

Proposition 4.3.45. *Let $X_\bullet$ be a good simplicial topological space and G a well-pointed simplicial topological group. Then for every morphism*

$$\tau : X \to \bar{W}G$$

the corresponding topological simplicial principal bundle P over X is itself a good simplicial topological space.

Proof. The bundle is the pullback $P = X \times_{\bar{W}G} WG$ in $\mathrm{sTop}_{\mathrm{cgH}}$

$$\begin{array}{ccc} P & \longrightarrow & \bar{W}G \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & \bar{W}G \end{array} .$$

By assumption on X and G and using prop. 4.3.43 we have that X, $\bar{W}G$ and WG are all good simplicial spaces. This means that the degeneracy maps of $P_\bullet$ are induced degreewise by morphisms between pullbacks in $\mathrm{Top}_{\mathrm{cgH}}$ that are degreewise closed cofibrations, where one of the morphisms in each pullback is a fibration. This implies that also these degeneracy maps of $P_\bullet$ are closed cofibrations. □

Proposition 4.3.46. *The homotopy colimit operation*

$$\mathrm{sTop}_s \hookrightarrow [\mathrm{Top}_s^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \stackrel{\mathrm{hocolim}}{\to} \mathrm{Top}_{\mathrm{Quillen}}$$

preserves homotopy fibers of morphisms $\tau \colon X \to \bar{W}G$ with X good and G well-pointed (def. 4.3.26) and globally Kan (def. 4.3.18).

Proof. By prop. 4.3.19 and prop. 4.3.20 we have that $WG \to \bar{W}G$ is a fibration resolution of the point inclusion $* \to \bar{W}G$ in $[\mathrm{Top}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. By general properties of homotopy limits this means that the homotopy fiber of a morphism $\tau \colon X \to \bar{W}G$ is computed as the ordinary pullback P in

$$\begin{array}{ccc} P & \longrightarrow & WG \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & \bar{W}G \end{array}$$

(since all objects X, $\bar{W}G$ and WG are fibrant and at least one of the two morphisms in the pullback diagram is a fibration) and hence

$$\mathrm{hofib}(\tau) \simeq P .$$

By prop. 4.3.19 and prop. 4.3.45 it follows that all objects here are good simplicial topological spaces. Therefore by prop. 4.3.27 we have

$$\mathrm{hocolim} P_\bullet \simeq |P_\bullet|$$

in $\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}})$. By prop. 4.3.25 we have that

$$\cdots = |X_\bullet| \times_{|\bar{W}G|} |WG| .$$

But prop. 4.3.44 says that this is again the presentation of a homotopy pullback/homotopy fiber by an ordinary pullback

$$\begin{array}{ccc} |P| & \longrightarrow & |WG| \\ \downarrow & & \downarrow \\ |X| & \xrightarrow{\tau} & |\bar{W}G| \end{array} ,$$

because $|WG| \to |\bar{W}G|$ is again a fibration resolution of the point inclusion. Therefore

$$\mathrm{hocolim} P_\bullet \simeq \mathrm{hofib}(|\tau|)\,.$$

Finally by prop. 4.3.27 and using the assumption that X and $\bar{W}G$ are both good, this is

$$\cdots \simeq \mathrm{hofib}(\mathrm{hocolim}\tau)\,.$$

In total we have shown

$$\mathrm{hocolim}(\mathrm{hofib}(\tau)) \simeq \mathrm{hofib}(\mathrm{hocolim}(\tau))\,.$$

□

We now generalize the model of *discrete* principal ∞-bundles by simplicial principal bundles over simplicial groups, from 4.1.3, to Euclidean-topological cohesion.

Recall from theorem 3.8.19 that over any ∞-cohesive site Π preserves homotopy pullbacks over discrete objects. The following proposition says that on ETop∞Grpd it preserves also a large class of ∞-pullbacks over non-discrete objects.

Theorem 4.3.47. *Let G be a well-pointed simplicial group object in* TopMfd*. Then the ∞-functor Π :* ETop∞Grpd $\to$ ∞Grpd *preserves homotopy fibers of all morphisms of the form $X \to \mathbf{B}G$ that are presented in* $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ *by morphism of the form $X \to \bar{W}G$ with X fibrant*

$$\Pi(\mathrm{hofib}(X \to \bar{W}G)) \simeq \mathrm{hofib}(\Pi(X \to \bar{W}G))\,.$$

Proof. By prop. 2.3.13 we may discuss the homotopy fiber in the global model structure on simplicial presheaves. Write $QX \overset{\simeq}{\to} X$ for the global cofibrant resolution given by $QX : [n] \mapsto \coprod_{\{U_{i_0} \to \cdots \to U_{i_n} \to X_n\}} U_{i_0}$, where the U_{i_k} range over $\mathrm{CartSp}_{\mathrm{top}}$ [Dugg01]. This has degeneracies splitting off as direct summands, and hence is a good simplicial topological space that is degreewise in TopMfd. Consider then the pasting of two pullback diagrams of simplicial presheaves

$$\begin{array}{ccccc} P' & \xrightarrow{\simeq} & P & \longrightarrow & WG \\ \downarrow && \downarrow && \downarrow \\ QX & \xrightarrow{\simeq} & X & \longrightarrow & \bar{W}G \end{array}\,.$$

Here the top left morphism is a global weak equivalence because $[\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is right proper. Since the square on the right is a pullback of fibrant objects with one morphism being a fibration, P is a presentation of the homotopy fiber of $X \to \bar{W}G$. Hence so is P', which is moreover the pullback of a diagram of good simplicial spaces. By prop. 4.3.30 we have that on the outer diagram Π is presented by geometric realization of simplicial topological spaces $|-|$. By prop. 4.3.44 we have a pullback in $\mathrm{Top}_{\mathrm{Quillen}}$

$$\begin{array}{ccc} |P| & \longrightarrow & |WG| \\ \downarrow && \downarrow \\ |QX| & \longrightarrow & |\bar{W}G| \end{array}$$

which exhibits $|P|$ as the homotopy fiber of $|QX| \to |\bar{W}G|$. But this is a model for $|\Pi(X \to \bar{W}G)|$. □

4.3.10 Gerbes

We discuss ∞-gerbes, 3.6.15, in the context of Euclidean-topological cohesion, with respect to the cohesive ∞-topos $\mathbf{H} := \mathrm{ETop}\infty\mathrm{Grpd}$ from def. 4.3.3.

For $X \in \mathrm{TopMfd}$ write

$$\mathcal{X} := \mathbf{H}/X$$

for the slice of $\mathbf{H}$ over X, as in remark 3.6.140. This is equivalently the ∞-category of ∞-sheaves on X itself

$$\mathcal{X} \simeq \mathrm{Sh}_\infty(X)\,.$$

By remark 3.6.140 this comes with the canonical étale essential geometric morphism

$$(X_! \dashv X^* \dashv X_*) : \ \mathbf{H}/X \underset{X_*}{\overset{\overset{X_!}{\longrightarrow}}{\underset{\longrightarrow}{\xleftarrow{X^*}}}} \mathbf{H}\ .$$

Any topological group G is naturally an object $G \in \mathrm{Grp}(\mathbf{H}) \subset \infty\mathrm{Grp}(\mathbf{H})$ and hence as an object

$$X^*G \in \mathrm{Grp}(\mathcal{X})\,.$$

Under the identification $\mathcal{X} \simeq \mathrm{Sh}_\infty(X)$ this is the sheaf of grpups which assigns sets of continuous functions from open subsets of X to G:

$$X^*G : (U \subset X) \mapsto C(U, G)\,.$$

Since the inverse image X^* commutes with looping and delooping, we have

$$X^*\mathbf{B}G \simeq \mathbf{B}X^*G\,.$$

On the left $\mathbf{B}G$ is the abstract stack of topological G-principal bundles, regarded over X, on the right is the stack over X of X^*G-torsors.

More generally, an arbitrary group object $G \in \mathrm{Grp}(\mathcal{X})$ is (up to equivalence) any sheaf of groups on X, and $\mathbf{B}G \in \mathcal{X}$ is the corresponding stack of G-torsors over X. (A detailed discussion of these is for instance in [Br06].)

Definition 4.3.48. Let $G = U(1) := \mathbb{R}/\mathbb{Z}$ and $n \in \mathbb{N}$, $n \geq 1$. Write $\mathbf{B}^{n-1}U(1) \in \infty\mathrm{Grp}(\mathbf{H})$ for the topological *circle n-group.*

A $\mathbf{B}^{n-1}U(1)$-n-gerbe we call a *circle n-gerbe.*

Proposition 4.3.49. *The automorphism ∞-groups, def. 3.6.212, of the circle n-groups, def. 4.3.48, are given by the following crossed complexes (def. 1.2.61)*

$$\mathrm{AUT}(U(1)) \simeq [U(1) \xrightarrow{0} \mathbb{Z}_2]\,,$$

$$\mathrm{AUT}(\mathbf{B}U(1)) \simeq [U(1) \xrightarrow{0} U(1) \xrightarrow{0} \mathbb{Z}_2]\,.$$

Here $\mathbb{Z}_2$ acts on the $U(1)$ by the canonical action via $\mathbb{Z}_2 \simeq \mathrm{Aut}_{\mathrm{Grp}}(U(1))$.

The outer automorphism ∞-groups, def. 3.6.270 are

$$\mathrm{Out}(U(1)) \simeq \mathbb{Z}_2\,;$$

$$\mathrm{Out}(\mathbf{B}U(1)) \simeq [U(1) \xrightarrow{0} \mathbb{Z}_2]\,.$$

Hence both ∞-groups are, of course, their own center.

With prop. 3.6.267 it follows that

$$\pi_0 U(1)\mathrm{Gerbe}(X) \simeq H^1(X, [U(1) \stackrel{0}{\to} \mathbb{Z}_2)$$

$$\pi_0 \mathbf{B}U(1)\mathrm{Gerbe}(X) \simeq H^1(X, [U(1) \stackrel{0}{\to} U(1) \stackrel{0}{\to} \mathbb{Z}_2)\,.$$

Notice that this classification is different (is richer) than that of $U(1)$ *bundle gerbes* and $U(1)$ *bundle 2-gerbes*. These are really models for $\mathbf{B}U(1)$-principal 2-bundles and $\mathbf{B}^2U(1)$-principal 3-bundles on X, and hence instead have the classification of prop. 3.6.170:

$$\pi_0 \mathbf{B}U(1)\mathrm{Bund}(X) \simeq H^1(X, [U(1) \to 1]) \simeq H^2(X, U(1))\,,$$

$$\pi_0 \mathbf{B}^2 U(1)\mathrm{Bund}(X) \simeq H^1(X, [U(1) \to 1 \to 1]) \simeq H^3(X, U(1))\,.$$

Alternatively, this is the classification of the $U(1)$-1-gerbes and $\mathbf{B}U(1)$-2-gerbes with trivial *band*, def. 3.6.274, in $H^1(X, \mathrm{Out}(U(1)))$ and $H^1(X, \mathrm{Out}(\mathbf{B}U(1)))$.

$$\pi_0 U(1)\mathrm{Gerbe}_{*\in H^1(X,\mathrm{Out}(U(1)))}(X) \simeq H^2(X, U(1))\,,$$

$$\pi_0 \mathbf{B}U(1)\mathrm{Gerbe}_{*\in H^1(X,\mathrm{Out}(U(1)))}(X) \simeq H^3(X, U(1))\,.$$

4.3.11 Universal coverings and geometric Whitehead towers

We discuss geometric Whitehead towers (3.8.4) in ETop∞Grpd.

Proposition 4.3.50. *Let X be a pointed paracompact topological space that admits a good open cover. Then its ordinary Whitehead tower $X^{(\infty)} \to \cdots X^{(2)} \to X^{(1)} \to X^{(0)} = X$ in* Top *coincides with the image under the intrinsic fundamental ∞-groupoid functor $|\Pi(-)|$ of its geometric Whitehead tower $* \to \cdots X^{(\mathbf{2})} \to X^{(\mathbf{1})} \to X^{(\mathbf{0})} = X$ in* ETop∞Grpd*:*

$$\begin{aligned} |\Pi(-)| : (X^{(\infty)} \to \cdots X^{(\mathbf{2})} \to X^{(\mathbf{1})} \to X^{(\mathbf{0})} = X) &\in \mathrm{ETop}\infty\mathrm{Grpd} \\ \mapsto (* \to \cdots X^{(2)} \to X^{(1)} \to X^{(0)} = X) &\in \mathrm{Top} \end{aligned}\,.$$

Proof. The geometric Whitehead tower is characterized for each n by the fiber sequence

$$X^{(\mathbf{n})} \to X^{(\mathbf{n-1})} \to \mathbf{B}^n \pi_n(X) \to \mathbf{\Pi}_n(X) \to \mathbf{\Pi}_{(n-1)}(X)\,.$$

By the above prop. 4.3.29 we have that $\mathbf{\Pi}_n(X) \simeq \mathrm{Disc}(\mathrm{Sing}X)$. Since Disc is right adjoint and hence preserves homotopy fibers this implies that $\mathbf{B}\pi_n(X) \simeq \mathbf{B}^n \mathrm{Disc}\pi_n(X)$, where $\pi_n(X)$ is the ordinary nth homotopy group of the pointed topological space X.

Then by prop. 4.3.47 we have that under $|\Pi(-)|$ the space $X^{(\mathbf{n})}$ maps to the homotopy fiber of $|\Pi(X^{(\mathbf{n-1})})| \to B^n|\mathrm{Disc}\pi_n(X)| = B^n\pi_n(X)$.

By induction over n this implies the claim. □

4.4 Smooth ∞-groupoids

We discuss *smooth* cohesion.

Definition 4.4.1. Write SmoothMfd for the category whose objects are smooth manifolds that are

- finite-dimensional;
- paracompact;
- with arbitrary set of connected components;

and whose morphisms are smooth functions between these.

Notice the evident forgetful functor

$$i : \mathrm{SmoothMfd} \to \mathrm{TopMfd}$$

to the category of topological manifolds, from def. 4.3.6.

Definition 4.4.2. For $X \in \mathrm{SmoothMfd}$, say an open cover $\{U_i \to X\}$ is a *differentiably good open cover* if each non-empty finite intersection of the U_i is *diffeomorphic* to a Cartesian space $\mathbb{R}^n$.

Proposition 4.4.3. *Every paracompact smooth manifold admits a differentiably good open cover.*

Proof. This is a folk theorem. A detailed proof is in the appendix of [FSS10]. □

Notice that the statement here is a bit stronger than the familiar statement about topologically good open covers, where the intersections are only required to be homeomorphic to a ball.

Definition 4.4.4. Regard SmoothMfd as a large site equipped with the coverage of differentiably good open covers. Write $\mathrm{CartSp}_{\mathrm{smooth}} \hookrightarrow \mathrm{SmoothMfd}$ for the full sub-site on Cartesian spaces.

Observation 4.4.5. Differentiably good open covers do indeed define a coverage and the Grothendieck topology generated from it is the standard open cover topology.

Proof. For X a paracompact smooth manifold, $\{U_i \to X\}$ an open cover and $f : Y \to X$ any smooth function from a paracompact manifold Y, the inverse images $\{f^{-1}(U_i) \to Y\}$ form an open cover of Y. Since $\coprod_i f^{-1}(U_1)$ is itself a paracompact smooth manifold, there is a differentiably good open cover $\{K_j \to \coprod_i U_i\}$, hence a differentiably good open cover $\{K_j \to Y\}$ such that for all j there is an $i(j)$ such that we have a commuting square

$$\begin{array}{ccc} K_j & \longrightarrow & U_{i(j)} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} .$$

□

Proposition 4.4.6. $\mathrm{CartSp}_{\mathrm{smooth}}$ *is an ∞-cohesive site.*

Proof. By the same kind of argument as in prop. 4.3.2. □

Definition 4.4.7. The ∞-topos of *smooth ∞-groupoids* is the ∞-sheaf ∞-topos on $\mathrm{CartSp}_{\mathrm{smooth}}$:

$$\mathrm{Smooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp}_{\mathrm{smooth}}) .$$

Since $\mathrm{CartSp}_{\mathrm{smooth}}$ is similar to the site $\mathrm{CartSp}_{\mathrm{top}}$ from def. 4.3.1, various properties of $\mathrm{Smooth}\infty\mathrm{Grpd}$ are immediate analogs of the corresponding properties of $\mathrm{ETop}\infty\mathrm{Grpd}$ from def. 4.3.3.

Proposition 4.4.8. Smooth∞Grpd *is a cohesive ∞-topos.*

Proof. With prop. 4.4.6 this follows by prop. 3.4.9. □

Proposition 4.4.9. Smooth∞Grpd *is equivalent to the hypercompletion of the ∞-sheaf ∞-topos over* SmoothMfd*:*

$$\mathrm{Smooth}\infty\mathrm{Grpd} \simeq \hat{\mathrm{Sh}}_\infty(\mathrm{SmoothMfd})\,.$$

Proof. Observe that $\mathrm{CartSp}_{\mathrm{smooth}}$ is a small dense sub-site of SmoothMfd. With this the claim follows as in prop. 4.3.7. □

Corollary 4.4.10. *The canonical embedding of smooth manifolds as 0-truncated objects of* Smooth∞Grpd *extends to a full and faithful ∞-functor*

$$\mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}.$$

Proof. With prop. 4.4.9 this follows from the ∞-Yoneda lemma. □

Remark 4.4.11. By example 2.2.21 there is an equivalence of ∞-categories

$$\mathrm{Smooth}\infty\mathrm{Grpd} \simeq L_W \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}}\,,$$

where on the right we have the simplicial localization of the category of simplicial smooth manifolds (with arbitrary set of connected components) at the stalkwise weak equivalences.

This says that every smooth ∞-groupoid has a presentation by a simplicial smooth manifold (not in general a locally Kan simplicial manifold, though) and that this identification is even homotopy-full and faithful.

Consider the canonical forgetful functor

$$i : \mathrm{CartSp}_{\mathrm{smooth}} \to \mathrm{CartSp}_{\mathrm{top}}$$

to the site of definition for the cohesive ∞-topos ETop∞Grpd of Euclidean-topological ∞-groupoids, def. 4.3.3.

Proposition 4.4.12. *The functor i extends to an essential geometric morphism*

$$(i_! \dashv i^* \dashv i_*) : \mathrm{Smooth}\infty\mathrm{Grpd} \underset{i_*}{\overset{i_!}{\underset{\longleftarrow i^* \longrightarrow}{\longrightarrow}}} \mathrm{ETop}\infty\mathrm{Grpd}$$

such that the ∞-Yoneda embedding is factored through the induced inclusion $\mathrm{SmoothMfd} \overset{i}{\hookrightarrow} \mathrm{Mfd}$ *as*

$$\begin{array}{ccc} \mathrm{SmoothMfd} & \hookrightarrow & \mathrm{Smooth}\infty\mathrm{Grpd} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_!} \\ \mathrm{Mfd} & \hookrightarrow & \mathrm{ETop}\infty\mathrm{Grpd} \end{array}$$

Proof. Using the observation that i preserves coverings and pullbacks along morphism in covering families, the proof follows the steps of the proof of prop. 3.5.3. □

Corollary 4.4.13. *The essential global section* ∞*-geometric morphism of* Smooth∞Grpd *factors through that of* ETop∞Grpd

$$(\Pi_{\mathrm{Smooth}} \dashv \mathrm{Disc}_{\mathrm{Smooth}} \dashv \Gamma_{\mathrm{Smooth}}) : \mathrm{Smooth}\infty\mathrm{Grpd} \underset{i_*}{\overset{i_!}{\underset{\leftarrow i^* -}{\longrightarrow}}} \mathrm{ETop}\infty\mathrm{Grpd} \underset{\Gamma_{\mathrm{ETop}}}{\overset{\Pi_{\mathrm{ETop}}}{\underset{\leftarrow \mathrm{Disc}_{\mathrm{ETop}} -}{\longrightarrow}}} \infty\mathrm{Grpd}$$

Proof. This follows from the essential uniqueness of the global section ∞-geometric morphism, prop 2.2.4, and of adjoint ∞-functors. □

The functor $i_!$ here is the forgetful functor that *forgets smooth structure* and only *remembers Euclidean topology-structure.*

We now discuss the various general abstract structures in a cohesive ∞-topos, 3.9, realized in Smooth∞Grpd.

- 4.4.1 – Concrete objects
- 4.4.2 – Groups
- 4.4.3 – Groupoids
- 4.4.4 – Geometric homotopy
- 4.4.5 – Paths and geometric Postnikov towers
- 4.4.6 – Cohomology
- 4.4.7 – Principal ∞-bundles
- 4.4.8 – Twisted cohomology
- 4.4.9 – ∞-Group representations
- 4.4.10 – Associated bundles
- 4.4.11 – Manifolds
- 4.4.12 – Flat ∞-connections and local systems
- 4.4.13 – de Rham cohomology
- 4.4.14 – Exponentiated ∞-Lie algebras
- 4.4.15 – Maurer-Cartan forms and curvature characteristic forms
- 4.4.16 – Differential cohomology
- 4.4.17 – ∞-Chern-Weil homomorphism
- 4.4.18 – Higher holonomy
- 4.4.19 – ∞-Chern-Simons functionals
- 4.4.20 – Geometric prequantization

4.4.1 Concrete objects

We discuss the general notion of *concrete objects* in a cohesive ∞-topos, 3.7.2, realized in Smooth∞Grpd.

The following definition generalizes the notion of smooth manifold and has been used as a convenient context for differential geometry. It goes back to [Sour79] and, in a slight variant, to [Chen77]. The formulation of differential geometry in this context is carefully exposed in [Igle]. The sheaf-theoretic formulation of the definition that we state is amplified in [BaHo09].

Definition 4.4.14. A sheaf X on $\mathrm{CartSp}_{\mathrm{smooth}}$ is a *diffeological space* if it is a *concrete sheaf* in the sense of [Dub79]: if for every $U \in \mathrm{CartSp}_{\mathrm{smooth}}$ the canonical function

$$X(U) \simeq \mathrm{Sh}(U, X) \xrightarrow{\Gamma} \mathrm{Set}(\Gamma(U), \Gamma(X))$$

is an injection.

The following observations are due to [CarSch].

Proposition 4.4.15. *Write* $\mathrm{Conc}(\mathrm{Smooth}\infty\mathrm{Grpd})_{\leq 0}$ *for the full subcategory on the 0-truncated concrete objects, according to def. 3.7.7. This is equivalent to the the full subcategory of* $\mathrm{Sh}(\mathrm{CartSp}_{\mathrm{smooth}})$ *on the diffeological spaces:*

$$\mathrm{DiffeolSpace} \simeq \mathrm{Conc}(\mathrm{Smooth}\infty\mathrm{Grpd})_{\leq 0}\,.$$

Proof. Let $X \in \mathrm{Sh}(\mathrm{CartSp}_{\mathrm{smooth}}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ be a sheaf. The condition for it to be a concrete object according to def. 3.7.7 is that the $(\Gamma \dashv \mathrm{coDisc})$-unit

$$X \to \mathrm{coDisc}\Gamma X$$

is a monomorphism. Since monomorphisms of sheaves are detected objectwise this is equivalent to the statement that for all $U \in \mathrm{CartSp}_{\mathrm{smooth}}$ the morphism

$$X(U) \simeq \mathrm{Smooth}\infty\mathrm{Grpd}(U, X) \to \mathrm{Smooth}\infty\mathrm{Grpd}(U, \mathrm{coDisc}\Gamma X) \simeq \infty\mathrm{Grpd}(\Gamma U, \Gamma X)$$

is a monomorphism of sets, where in the first step we used the ∞-Yoneda lemma and in the last one the $(\Gamma \dashv \mathrm{coDisc})$-adjunction. This is manifestly the defining condition for concrete sheaves that define diffeological spaces. □

Corollary 4.4.16. *The canonical embedding* $\mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ *from prop. 4.4.10 factors through diffeological spaces: we have a sequence of full and faithful* ∞*-functors*

$$\mathrm{SmoothMfd} \hookrightarrow \mathrm{DiffeolSpace} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}\,.$$

Definition 4.4.17. Write $\mathrm{DiffeolGrpd} \hookrightarrow \mathrm{SmoothGrpd}$ for the full sub-∞-category on those smooth ∞-groupoids that are represented by a groupoid object internal to diffeological spaces.

Proposition 4.4.18. *There is a canonical equivalence*

$$\mathrm{DiffeolGrpd} \simeq \mathrm{Conc}(\mathrm{Smooth}\infty\mathrm{Grpd})_{\leq 1}$$

identifying diffeological groupoids with the concrete 1-truncated smooth ∞*-groupoids.*

Proof. By definition, an object $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ is concrete precisely if there exists a 0-concrete object U, and an effective epimorphism $U \to X$ such that $U \times_X U$ is itself 0-concrete. By prop. 4.4.15 both U and $U \times_X U$ are equivalent to diffeological spaces. Therefore the groupoid object $(U \times_X U \rightrightarrows U)$ internal to $\mathrm{Smooth}\infty\mathrm{Grpd}$ comes from a groupoid object internal to diffeological spaces. By Giraud's axioms for ∞-toposes, X is equivalent to (the ∞-colimit over) this groupoid object:

$$X \simeq \varinjlim(U \times_X U \rightrightarrows U)\,.$$

□

4.4.2 Groups

We discuss some cohesive ∞-group objects, according to 3.6.8, in $\mathrm{Smooth}\infty\mathrm{Grpd}$.

Let $G \in \mathrm{SmoothMfd}$ be a Lie group. Under the embedding $\mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ this is canonically identifed as a 0-truncated ∞-group object in $\mathrm{Smooth}\infty\mathrm{Grpd}$. Write $\mathbf{B}G \in \mathrm{Smooth}\infty\mathrm{Grpd}$ for the corresponding delooping object.

Proposition 4.4.19. *A fibrant presentation of the delooping object* $\mathbf{B}G$ *in the projective local model structure on simplicial presheaves* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *is given by the simplicial presheaf that is the nerve of the one-object Lie groupoid*

$$\mathbf{B}G_{\mathrm{ch}} := (G \rightrightarrows *)$$

regarded as a simplicial manifold and canonically embedded into simplicial presheaves:

$$\mathbf{B}G_{\mathrm{ch}} : U \mapsto N(C^\infty(U, G) \rightrightarrows *)\,.$$

Proof. This is essentially a special case of prop. 4.3.13. The presheaf is clearly objectwise a Kan complex, being objectwise the nerve of a groupoid. It satisfies descent along good open covers $\{U_i \to \mathbb{R}^n\}$ of Cartesian spaces, because the descent ∞-groupoid $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}G)$ is $\cdots \simeq G\mathrm{Bund}(\mathbb{R}^n) \simeq G\mathrm{TrivBund}(\mathbb{R}^n)$: an object is a Čech 1-cocycle with coefficients in G, a morphism a Čech coboundary. This yields the groupoid of G-principal bundles over U, which for the Cartesian space U is however equivalent to the groupoid of trivial G-bundles over U.

To show that $\mathbf{B}G$ is indeed the delooping object of G it is sufficient by prop. 2.3.13 to compute the ∞-pullback $G \simeq * \times_{\mathbf{B}G} * \in \mathrm{Smooth}\infty\mathrm{Grpd}$ in the global model structure $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. This is accomplished by the ordinary pullback of the fibrant replacement diagram

$$
\begin{array}{ccc}
G & \longrightarrow & N(G \times G \underset{p_1}{\overset{p_1 \cdot p_2}{\rightrightarrows}} G) \\
\downarrow & & \downarrow{\scriptstyle p_2} \\
* & \longrightarrow & N(G \rightrightarrows *)
\end{array} .
$$

□

Proposition 4.4.20. *For G a Lie group, $\mathbf{B}G$ is a 1-concrete object in $\mathbf{H}$.*

Proof. Since $\mathbf{B}G_{\mathrm{ch}}$ is fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ and since G presents a concrete sheaf, this follows with prop. 3.7.8. □

Definition 4.4.21. Write equivalently

$$U(1) = S^1 = \mathbb{R}/\mathbb{Z}$$

for the *circle Lie group*, regarded as a 0-truncated ∞-group object in $\mathrm{Smooth}\infty\mathrm{Grpd}$ under the embedding prop. 4.4.10.

For $n \in \mathbb{N}$ the n-fold delooping $\mathbf{B}^n U(1) \in \mathrm{Smooth}\infty\mathrm{Grpd}$ we call the circle *Lie $(n+1)$-group*.

Write

$$U(1)[n] := [\cdots \to 0 \to C^\infty(-, U(1)) \to 0 \to \cdots \to 0] \in [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{Ch}_{\bullet \geq 0}]$$

for the chain complex of sheaves concentrated in degree n on $U(1)$. Recall the right Quillen functor $\Xi : [\mathrm{CartSp}^{op}_{\mathrm{smooth}}, \mathrm{Ch}^+]_{\mathrm{proj}} \to [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]_{\mathrm{proj}}$ from prop. 2.2.31.

Proposition 4.4.22. *The simplicial presheaf $\Xi(U(1)[n])$ is a fibrant representative in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ of the circle Lie $(n+1)$-group $\mathbf{B}^n U(1)$.*

Proof. First notice that since $U(1)[n]$ is fibrant in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{Ch}_\bullet]_{\mathrm{proj}}$ we have that $\Xi U(1)[n]$ is fibrant in the global model structure $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. By prop. 2.3.13 we may compute the ∞-pullback that defines the loop space object in $\mathrm{Smooth}\infty\mathrm{Grpd}$ in terms of a homotopy pullback in this global model structure.

To that end, consider the global fibration resolution of the point inclusion $* \to \Xi(U(1)[n])$ given under Ξ by the morphism of chain complexes

$$
\begin{array}{ccccccccc}
[C^\infty(-, U(1)) & \xrightarrow{\mathrm{Id}} & C^\infty(-, U(1)) & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0] \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & & & \downarrow \\
[C^\infty(-, U(1)) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0]
\end{array} .
$$

The underlying morphism of chain complexes is clearly degreewise surjective, hence a projective fibration, hence its image under Ξ is a projective fibration. Therefore the homotopy pullback in question is given by the ordinary pullback

$$\begin{array}{ccc} \Xi[0 \to C^\infty(-,U(1)) \to 0 \to \cdots \to 0] & \longrightarrow & \Xi[C^\infty(-,U(1)) \stackrel{\mathrm{Id}}{\to} C^\infty(-,U(1)) \to 0 \to \cdots \to 0] \\ \downarrow & & \downarrow \\ \Xi[0 \to 0 \to 0 \to \cdots \to 0] & \longrightarrow & \Xi[C^\infty(-,U(1)) \to 0 \to 0 \to \cdots \to 0] \end{array} ,$$

computed in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Ch}^+]$ and then using that Ξ is the right adjoint and hence preserves pullbacks. This shows that the loop object $\Omega\Xi(U(1)[n])$ is indeed presented by $\Xi(U(1)[n-1])$.

Now we discuss the fibrancy of $U(1)[n]$ in the local model structure. We need to check that for all differentiably good open covers $\{U_i \to U\}$ of a Cartesian space U we have that the mophism

$$C^\infty(U, U(1))[n] \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \Xi(U(1)[n]))$$

is an equivalence of Kan complexes, where $C(\{U_i\})$ is the Čech nerve of the cover. Observe that the Kan complex on the right is that whose vertices are cocycles in degree-n Čech cohomology (see [FSS10] for more on this) with coefficients in $U(1)$ and whose morphisms are coboundaries between these.

We proceed by induction on n. For $n = 0$ the condition is just that $C^\infty(-, U(1))$ is a sheaf, which clearly it is. For general n we use that since $C(\{U_i\})$ is cofibrant, the above is the derived hom-space functor which commutes with homotopy pullbacks and hence with forming loop space objects, so that

$$\pi_1[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}](C(\{U_i\}), \Xi(U(1)[n])) \simeq \pi_0[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}](C(\{U_i\}), \Xi(U(1)[n-1]))$$

by the above result on delooping. So we find that for all $0 \leq k \leq n$ that $\pi_k[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \Xi(U(1)[n]))$ is the Čech cohomology of U with coefficients in $U(1)$ in degree $n - k$. By standard facts about Čech cohomology (using the short exact sequence of abelian groups $\mathbb{Z} \to U(1) \to \mathbb{R}$ and the fact that the cohomology with coefficients in $\mathbb{R}$ vanishes in positive degree, for instance by a partition of unity argument) we have that this is given by the integral cohomology groups

$$\pi_0[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \Xi(U(1)[n])) \simeq H^{n+1}(U, \mathbb{Z})$$

for $n \geq 1$. For the contractible Cartesian space all these cohomology groups vanish.

So we find that $\Xi(U(1)[n])(U)$ and $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}](C(\{U_i\}), \Xi U(1)[n])$ both have homotopy groups concentrated in degree n on $U(1)$. The above looping argument together with the fact that $U(1)$ is a sheaf also shows that the morphism in question is an isomorphism on this degree-n homotopy group, hence is indeed a weak homotopy equivalence. □

Notice that in the equivalent presentation of Smooth∞Grpd by simplicial presheaves on the large site SmoothMfd the objects $\Xi(U(1)[n])$ are far from being locally fibrant. Instead, their locally fibrant replacements are given by the n-stacks of circle n-bundles.

4.4.3 Groupoids

We discuss aspects of the general abstract theory of *groupoid objects*, 3.6.7, realized in the context of smooth cohesion.

4.4.3.1 Group of bisections We discuss the general notion of groups of bisections of 3.6.7.1.2, realized in smooth cohesion.

Let

$$X = \; X_1 \rightrightarrows X_0 \; \in \mathrm{Grpd}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$$

be a Lie groupoid, regarded canonically as smooth ∞-groupoid and equipped with the atlas given by the canonical inclusion

$$i_X : X_0 \twoheadrightarrow X$$

of the manifold of objcts.

Proposition 4.4.23. *The group of bisections* $\mathbf{BiSect}_X(X_0) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ *of this groupoid object, according to def, 3.6.95, is equivalent to the traditional diffeological group of bisections of Lie groupoid theory and the canonical morphism of def. 3.6.97.*

Proof. First observe that the hom-groupoid $\mathbf{Smooth}\infty\mathrm{Grpd}_X(X_0, X_0)$ is equivalently given by that of $\mathrm{Grpd}(\mathrm{SmoothMfd})_{/X}(X_0, X_0)$. This follows for instance from prop. 3.6.5, according to which we have a homotopy pullback diagram

$$\begin{array}{ccc} \mathbf{H}_{/X}(U \times X_0, X_0) & \longrightarrow & \mathbf{H}(U \times X_0, X_0) \\ \downarrow & & \downarrow^{\mathbf{H}(U \times X_0, i_X)} \\ * & \xrightarrow{\vdash i_X} & \mathbf{H}(U \times X_0, X) \end{array}$$

for each $U \in \mathrm{CartSp} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$. Here the top right morphism set is equivalent to $\mathrm{SmoothMfd}(U \times X_0, X_0)$. The bottom right morphism set is a priori given by morphisms out of the Cech nerve of a good open over of $U \times X_0$. But since the right and bottom morphism both hit elements in there which come from direct maps out of $U \times X_0$, also the gauge transformations between them are given by globally defined smooth functions $U \times X_0 \to X_1$.

With this now it remains to observe that a diagram

$$\begin{array}{ccc} U \times X_0 & \xrightarrow{\phi} & X_0 \\ & {}_{i_X}\searrow \;\swArrow\; \swarrow_{i_X} & \\ & X_0 & \end{array}$$

of smooth groupoids is equivalently

1. a smoothly U-paramaterized collection of smooth function $\phi_u : X_0 \to X_0$;
2. for each such a smooth choice of morphisms $x \to \phi(x)$ in $X_1$for all $x \in x_0$.

This is precisely the traditional description of the group of bisections of X. □

4.4.3.2 Atiyah groupoids

We discuss the general notion of Atiyah groupoids, 3.6.7.1.4, realized in smooth cohesion.

Let $G \in \mathrm{Grp}(\mathrm{Top}) \hookrightarrow \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ be a Lie group, and write $\mathbf{B}G \in \mathrm{ETop}\infty\mathrm{Grpd}$ for its internal delooping, as in 4.4.2 above. Let $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ be a smooth manifold. Let $P \to X$ be any G-principal bundle over X and write $g : X \to \mathbf{B}G$ for the, essentially unique, morphism that modulates it (discussed in more detail in 4.4.7 below).

The following definition is traditional

Definition 4.4.24. The *Atiyah Lie groupoid* of the G-principal bundle $P \to X$ is the Lie groupoid

$$\mathrm{At}(P) := \left(P \times_G P \rightrightarrows X \right) ,$$

with composition defined by the evident composition of pairs of representatives. $[s_2, s_3] \circ [s_1, s_2] := [s_1, s_3]$.

Remark 4.4.25. Here $P \times_{U(1)} P = (P \times P)/U(1)$ is the quotient of the cartesian product of the total space of the bundle with itself by the diagonal action of G on both factors. So if $(x_1, x_2) \in X \times X$ is fixed then the morphisms in $\mathrm{At}(P)_{x_1,x_2}$ with this source and target form the space $(P_{x_1} \times P_{x_2})/G$. But this is canonically isomorphic to the space of G-torsor homomorphisms (over the point) $P_{x_1} \to P_{x_2}$:

$$\mathrm{At}(P)_{x_1,x_2} = G\mathrm{Tor}(P_{x_1}, P_{x_2})\,.$$

We now discuss that this traditional construction is indeed a special case of the general discussion in 3.6.7.1.4.

Proposition 4.4.26. *For $P \to X$ a smooth G-principal bundle with modulating map $g : X \to \mathbf{B}G$ as above, its Atiyah groupoid in* Smooth∞Grpd *in the sense of def. 3.6.104 is canonically represented by the traditional Atiyah groupoid construction of def. 4.4.24, under the canonical embedding* LieGrpd → Smooth∞Grpd.

Proof. By prop. 3.6.40 we have that $\mathrm{im}_1(g)$ is given by the ∞-colimit over its Čech nerve. Since $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ is 0-truncated and $\mathbf{B}G \in \mathrm{Smooth}\infty\mathrm{Grpd}$ is 1-truncated, this Čech nerve is given by a 2-coskeletal simplicial smooth manifold:

$$\mathrm{im}_1(g) \simeq \varinjlim \left(\cdots \rightrightarrows X \underset{\mathbf{B}G}{\times} X \rightrightarrows X \right).$$

Therefore by prop. 2.3.21 this simplicial diagram, regarded under the embedding $\mathrm{SmthMfd}^{\Delta^{\mathrm{op}}} \to \mathrm{Smooth}\infty\mathrm{Grpd}$, is equivalently the 1-image of g. It is then sufficient to observe that

$$X \underset{\mathbf{B}G}{\times} X \simeq P \times_G P\,.$$

To see this, observe that (since the ∞-hom functor $\mathbf{H}(U, -)$ preserves homotopy limits) for every $U \in \mathrm{CartSp}$ the U-plots of the object on the left are equivalently pairs of smooth functions $r, l : U \to X$ equipped with a morphism of G-principal bundles $l^*P \to r^*P$. By remark 4.4.25 this are equivalently the U-plots of $P \times_G P$. □

4.4.4 Geometric homotopy

We discuss the intrinsic fundamental ∞-groupoid construction, 3.8.1, and the induced notion of geometric realization, realized in Smooth∞Grpd.

Proposition 4.4.27. *If $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ is presented by $X_\bullet \in \mathrm{SmoothMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$, then its image $i_!(X) \in \mathrm{ETop}\infty\mathrm{Grpd}$ under the relative topological cohesion morphism, prop. 4.4.12, is presented by the underlying simplicial topological space $X_\bullet \in \mathrm{TopMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{top}}^{\mathrm{op}}, \mathrm{sSet}]$.*

Proof. Let first $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{SmoothMfd}^{\Delta^{\mathrm{op}}}$ be simplicially constant. Then there is a differentiably good open cover, 4.4.3, $\{U_i \to X\}$ such that the Čech nerve projection

$$\left(\int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\cdots,i_k} U_{i_0} \times_X \cdots \times_X U_{i_k} \right) \xrightarrow{\simeq} X$$

is a cofibrant resolution in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ which is degreewise a coproduct of representables. That means that the left derived functor $\mathbb{L}\mathrm{Lan}_i$ on X is computed by the application of Lan_i on this coend, which by the fact that this is defined to be the left Kan extension along i is given degreewise by i, and since i preserves pullbacks along covers, this is

$$\begin{aligned}(\mathbb{L}\mathrm{Lan}_i)X &\simeq \mathrm{Lan}_i\left(\int^{[k]\in\Delta}\Delta[k]\cdot\coprod_{i_0,\cdots,i_k}U_{i_0}\times_X\cdots\times_X U_{i_k}\right)\\ &= \int^{[k]\in\Delta}\Delta[k]\cdot\coprod_{i_0,\cdots,i_k}\mathrm{Lan}_i(U_{i_0}\times_X\cdots\times_X U_{i_k})\\ &\simeq \int^{[k]\in\Delta}\Delta[k]\cdot\coprod_{i_0,\cdots,i_k}i(U_{i_0}\times_X\cdots\times_X U_{i_k})\\ &\simeq \int^{[k]\in\Delta}\Delta[k]\cdot\coprod_{i_0,\cdots,i_k}(i(U_{i_0})\times_{i(X)}\cdots\times_{i(X)} i(U_{i_k}))\\ &\simeq i(X)\end{aligned},$$

The last step follows from observing that we have manifestly the Čech nerve as before, but now of the underlying topological spaces of the $\{U_i\}$ and of X.

The claim then follows for general simplicial spaces by observing that $X_\bullet = \int^{[k]\in\Delta}\Delta[k]\cdot X_k \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ presents the ∞-colimit over $X_\bullet : \Delta^{\mathrm{op}} \to \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ and the left adjoint ∞-functor $i_!$ preserves these. □

Corollary 4.4.28. *If* $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *is presented by* $X_\bullet \in \mathrm{SmoothMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$, *then the image of* X *under the fundamental* ∞*-groupoid functor, 3.8.1,*

$$\mathrm{Smooth}\infty\mathrm{Grpd} \xrightarrow{\Pi} \infty\mathrm{Grpd} \xrightarrow[\simeq]{|-|} \mathrm{Top}$$

is weakly homotopy equivalent to the geometric realization of (a Reedy cofibrant replacement of) the underlying simplicial topological space

$$|\Pi(X)| \simeq |QX_\bullet|\,.$$

In particular if X *is an ordinary smooth manifold then*

$$\Pi(X) \simeq \mathrm{Sing}X$$

is equivalent to the standard fundamental ∞*-groupoid of* X*.*

Proof. By prop. 4.4.13 the functor Π factors as $\Pi X \simeq \Pi_{\mathrm{ETop}} i_! X$. By prop. 4.4.27 this is Π_{Etop} applied to the underlying simplicial topological space. The claim then follows with prop. 4.3.30. □

Corollary 4.4.29. *The* ∞*-functor* $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}$ *preserves homotopy fibers of morphisms that are presented in* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ *by morphisms of the form* $X \to \bar W G$ *with* X *fibrant and* G *a simplicial group in* SmoothMfd.

Proof. By prop. 4.4.13 the functor factors as $\Pi_{\mathrm{Smooth}} \simeq \Pi_{\mathrm{ETop}} \circ i_!$. By prop. 4.4.27 $i_!$ assigns the underlying topological spaces. If we can show that this preserves the homotopy fibers in question, then the claim follows with prop. 4.3.47. We find this as in the proof of the latter proposition, by considering the pasting diagram of pullbacks of simplicial presheaves

$$\begin{array}{ccccc} P' & \xrightarrow{\simeq} & P & \longrightarrow & WG \\ \downarrow && \downarrow && \downarrow \\ QX & \xrightarrow{\simeq} & X & \longrightarrow & \bar W G \end{array}.$$

Since the component maps of the right vertical morphisms are surjective, the degreewise pullbacks in SmoothMfd that define P' are all along transversal maps, and thus the underlying objects in TopMfd are the pullbacks of the underlying topological manifolds. Therefore the degreewise forgetful functor SmoothMfd $\to$ TopMfd presents $i_!$ on the outer diagram and sends this homotopy pullback to a homotopy pullback. $\square$

4.4.5 Paths and geometric Postnikov towers

We discuss the general abstract notion of path ∞-groupoid, 3.8.3, realized in *Smooth*∞Grpd.

The presentation of $\mathbf{\Pi}(X)$ in ETop∞Grpd, 4.3.7 has a direct refinement to smooth cohesion:

Definition 4.4.30. For $X \in \mathrm{SmthMfd}$ write $\mathbf{Sing}X \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ for the simplicial presheaf given by

$$\mathbf{Sing}X : (U,[k]) \mapsto \mathrm{Hom}_{\mathrm{SmthMfd}}(U \times \Delta^k, X)\,.$$

Proposition 4.4.31. *The simplicial presheaf* $\mathbf{Sing}X$ *is a presentation of* $\mathbf{\Pi}(X) \in \mathrm{Smooth}\infty\mathrm{Grpd}$.

Proof. This reduces to the argument of prop. 4.3.37 after using the Steenrod approximation theorem [Wock09] to refine continuous paths to smooth paths $\square$

4.4.6 Cohomology

We discuss the intrinsic cohomology, 3.6.9, in Smooth∞Grpd.

- 4.4.6.1 – Cohomology with constant coefficients;
- 4.4.6.2 – Refined Lie group cohomology.

4.4.6.1 Cohomology with constant coefficients

Proposition 4.4.32. *Let* $A \in \infty\mathrm{Grpd}$*, write* $\mathrm{Disc}A \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *for the corresponding discrete smooth* ∞*-groupoid. Let* $X \in \mathrm{SmoothMfd} \overset{i}{\hookrightarrow} \mathrm{Smooth}\infty\mathrm{Grpd}$ *be a paracompact topological space regarded as a 0-truncated Euclidean-topological* ∞*-groupoid.*

We have an isomorphism of cohomology sets

$$H_{\mathrm{Top}}(X, A) \simeq H_{\mathrm{Smooth}}(X, \mathrm{Disc}A)$$

and in fact an equivalence of cocycle ∞*-groupoids*

$$\mathrm{Top}(X, |A|) \simeq \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathrm{Disc}A)\,.$$

More generally, for $X_\bullet \in \mathrm{SmoothMfd}^{\Delta^{op}}$ *presenting an object* $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *we have*

$$H_{\mathrm{Smooth}}(X_\bullet, \mathrm{Disc}A) \simeq H_{\mathrm{Top}}(|X|, |A|)\,.$$

Proof. This follows from the ($\Pi \dashv$ Disc)-adjunction and prop. 4.4.28. $\square$

4.4.6.2 Refined Lie group cohomology The cohomology of a Lie group G with coefficients in a Lie group A was historically originally defined in terms of cocycles given by smooth functions $G^{\times n} \to A$, by naive analogy with the situation discussed in 4.1.3.1. In the language of simplicial presheaves on CartSp these are morphisms of simplicial presheaves of the form $\mathbf{B}G_{\mathrm{ch}} \to \mathbf{B}^n A$, with the notation as in 4.4.2. This is clearly not a good definition, in general, since while $\mathbf{B}^n A$ will be fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, the object $\mathbf{B}G_{\mathrm{ch}}$ in general fails to be cofibrant, hence the above naive definition in general misses cocycles.

A refined definition of Lie group cohomology was proposed in [Sega70] and later independently in [Bry00]. The following theorem asserts that the definitions given there do coincide with the intrinsic cohomology of the stack $\mathbf{B}G$ in the cohesive ∞-topos Smooth∞Grpd.

Theorem 4.4.33. *For $G \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ a Lie group and A either*

1. *a discrete abelian group*
2. *the additive Lie group of real numbers $\mathbb{R}$*

the intrinsic cohomology of G in Smooth∞Grpd *coincides with the refined Lie group cohomology of Segal [Sega70][Bry00]*

$$H^n_{\mathrm{Smooth}\infty\mathrm{Grpd}}(\mathbf{B}G, A) \simeq H^n_{\mathrm{Segal}}(G, A)\,.$$

In particular we have in general

$$H^n_{\mathrm{Smooth}\infty\mathrm{Grpd}}(\mathbf{B}G, \mathbb{Z}) \simeq H^n_{\mathrm{Top}}(BG, \mathbb{Z})$$

and for G compact and $n \geq 1$ also

$$H^n_{\mathrm{Smooth}\infty\mathrm{Grpd}}(\mathbf{B}G, U(1)) \simeq H^{n+1}_{\mathrm{Top}}(BG, \mathbb{Z})\,.$$

Proof. The statement about constant coefficients is a special case of prop. 4.4.32. The statement about real coefficients is a special case of a more general statement in the context of synthetic differential ∞-groupoids that will be proven as prop. 4.5.43. The last statement finally follows from this using that $H^n_{\mathrm{Segal}}(G, \mathbb{R}) \simeq 0$ for positive n and G compact and using the fiber sequence, def. 3.6.141, induced by the short sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \simeq U(1)$. □

4.4.7 Principal bundles

We discuss principal ∞-bundles, 3.6.10, realized in smooth ∞-groupoids.

The following proposition asserts that the notion of smooth principal ∞-bundle reproduces traditional notions of smooth bundles and smooth higher bundles.

Proposition 4.4.34. *For G a Lie group and $X \in \mathrm{SmoothMfd}$, we have that*

$$\mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}G) \simeq G\mathrm{Bund}(X)$$

is equivalent to the groupoid of smooth principal G-bundles and smooth morphisms between these, as traditionally defined, where the equivalence is established by sending a morphism $g : X \to \mathbf{B}G$ in Smooth∞Grpd *to the corresponding principal ∞-bundle $P \to X$ according to prop. 3.6.159.*

For $n \in \mathbb{N}$ and $G = \mathbf{B}^{n-1}U(1)$ the circle Lie n-group, def. 4.4.21, and $X \in \mathrm{SmoothMfd}$, we have that

$$\mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^n U(1)) \simeq U(1)(n-1)\mathrm{BundGerb}(X)$$

is equivalent to the n-groupoid of smooth $U(1)$-bundle $(n-1)$gerbes.

Proof. Presenting Smooth∞Grpd by the local projective model structure $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ on simplicial presheaves over the site of Cartesian spaces, we have that $\mathbf{B}G$ is fibrant, by prop. 4.4.19, and that a cofibrant replacement for X is given by the Čech nerve $C(\{U_i\})$ of any differentiably good open cover $\{U_i \to X\}$. The cocycle ∞-groupoid in question is then presented by the simplicial set $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}G)$ and this is readily seen to be the groupoid of Čech cocycles with coefficients in $\mathbf{B}G$ relative to the chosen cover.

This establishes that the two groupoids are equivalent. That the equivalence is indeed established by forming homotopy fibers of morphisms has been discussed in 1.2.5 (observing that by the discussion in 1.2.5.4 the ordinary pullback of the morphism $\mathbf{E}G \to \mathbf{B}G$ serves as a presentation for the homotopy pullback of $* \to \mathbf{B}G$). □

This establishes the situation for smooth nonabelian cohomology in degree 1 and smooth abelian cohomology in arbitrary degree. We turn now to a discussion of smooth nonabelian cohomology "in degree 2", the case where G is a *Lie 2-group*: *G-principal 2-bundles*.

When $G = \mathrm{AUT}(H)$ the *automorphism 2-group* of a Lie group H (see below) these structures have the same classification as smooth H-1-gerbes, def. 3.6.264. To start with, note the general abstract notion of smooth 2-groups:

Definition 4.4.35. A *smooth 2-group* is a 1-truncated group object in $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{CartSp})$. These are equivalently given by their (canonically pointed) delooping 2-groupoids $\mathbf{B}G \in \mathbf{H}$, which are precisely, up to equivalence, the connected 2-truncated objects of $\mathbf{H}$.

For $X \in \mathbf{H}$ any object, $G2\mathrm{Bund}_{\mathrm{smooth}}(X) := \mathbf{H}(X, \mathbf{B}G)$ is the 2-groupoid of smooth G-principal 2-bundles on G.

We consider the presentation of smooth 2-groups by Lie crossed modules, def. 1.2.45, according to prop. 3.6.136. Write $[G_1 \stackrel{\delta}{\to} G_0]$ for the *2-group* which is the groupoid

$$G_0 \times G_1 \underset{p_1}{\overset{p_1(-)\cdot\delta(p_2(-))}{\rightrightarrows}} G_0$$

equipped with a strict group structure given by the semidirect product group structure on $G_0 \times G_1$ that is induced from the action ρ. The commutativity of the above two diagrams is precisely the condition for this to be consistent. Recall the examples of crossed modules, starting with example 1.2.50.

We discuss sufficient conditions for the delooping of a crossed module of presheaves to be fibrant in the projective model structure. Recall also the conditions from prop. 3.4.23.

Proposition 4.4.36. *Suppose that the smooth crossed module $(G_1 \to G_0)$ is such that the quotient $\pi_0 G = G_0/G_1$ is a smooth manifold and the projection $G_0 \to G_0/G_1$ is a submersion.*

Then $\mathbf{B}(G_1 \to G_0)$ is fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Proof. We need to show that for $\{U_i \to \mathbb{R}^n\}$ a good open cover, the canonical descent morphism

$$B(C^\infty(\mathbb{R}^n, G_1) \to C^\infty(\mathbb{R}^n, G_0)) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}(G_1 \to G_0))$$

is a weak homotopy equivalence. The main point to show is that, since the Kan complex on the left is connected by construction, also the Kan complx on the right is.

To that end, notice that the category CartSp equipped with the open cover topology is a *Verdier site* in the sense of section 8 of [DuHoIs04]. By the discussion there it follows that every hypercover over $\mathbb{R}^n$ can be refined by a split hypercover, and these are cofibrant resolutions of $\mathbb{R}^n$ in both the global and the local model structure $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Since also $C(\{U_i\}) \to \mathbb{R}^n$ is a cofibrant resolution and since $\mathbf{B}G$ is clearly fibrant in the *global* structure, it follows from the existence of the global model structure that morphisms out of $C(\{U_i\})$ into $\mathbf{B}(G_1 \to G_0)$ capture all cocycles over any hypercover over $\mathbb{R}^n$, hence that

$$\pi_0[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}(G_1 \to G_0)) \simeq H^1_{\mathrm{smooth}}(\mathbb{R}^n, (G_1 \to G_0))$$

is the standard Čech cohomology of $\mathbb{R}^n$, defined as a colimit over refinements of covers of equivalence classes of Čech cocycles.

Now by prop. 4.1 of [NW11a] (which is the smooth refinement of the statement of [BSt] in the continuous context) we have that under our assumptions on $(G_1 \to G_0)$ there is a topological classifying space for this smooth Čech cohomology set. Since $\mathbb{R}^n$ is topologically contractible, it follows that this is the singleton set and hence the above descent morphism is indeed an isomorphism on π_0.

Next we can argue that it is also an isomorphism on π_1, by reducing to the analogous local trivialization statement for ordinary principal bundles: a loop in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}(G_1 \to G_0))$ on the trivial cocycle is readily seen to be a $G_0//(G_0 \ltimes G_1)$-principal groupoid bundle, over the action groupoid as indicated. The underlying $G_0 \ltimes G_1$-principal bundle has a trivialization on the contractible $\mathbb{R}^n$ (by classical results or, in fact, as a special case of the previous argument), and so equivalence classes of such loops are given gy G_0-valued smooth functions on $\mathbb{R}^n$. The descent morphism exhibits an isomorphism on these classes.

Finally the equivalence classes of spheres on both sides are directly seen to be smooth $\ker(G_1 \to G_0)$-valued functions on both sides, identified by the descent morphism. □

Corollary 4.4.37. *For $X \in \mathrm{SmoothMfd} \subset \mathbf{H}$ a paracompact smooth manifold, and $(G_1 \to G_0)$ as above, we have for any good open cover $\{U_i \to X\}$ that the 2-groupoid of smooth $(G_1 \to G_0)$-principal 2-bundles is*

$$(G_1 \to G_0)\mathrm{Bund}(X) := \mathbf{H}(X, \mathbf{B}(G_1)) \simeq [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}(G_1 \to G_0))$$

and its set of connected components is naturally isomorphic to the nonabelian Čech cohomology

$$\pi_0 \mathbf{H}(X, \mathbf{B}(G_1 \to G_0)) \simeq H^1_{\mathrm{smooth}}(X, (G_1 \to G_0)) \,.$$

In particular, for $G = \mathrm{AUT}(H)$, $\mathbf{B}G \in \mathbf{H}$ is the moduli 2-stack for smooth H-gerbes, def. 3.6.257.

Proposition 4.4.38. *For $A \to \hat{G} \to G$ a central extension of Lie groups such that $\hat{G} \to G$ is a locally trivial A-bundle, we have a long fiber sequence in* Smooth∞Grpd *of the form*

$$A \to \hat{G} \to G \to \mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G \stackrel{\mathbf{c}}{\to} \mathbf{B}^2 A \,,$$

where the morphism $\mathbf{c}$ is presented by the span of simplicial presheaves

$$\begin{array}{ccccc} \mathbf{B}(A \to \hat{G})_c & \longrightarrow & \mathbf{B}(A \to 1)_c & = & \mathbf{B}^2 A_c \\ \downarrow{\scriptstyle \simeq} & & & & \\ \mathbf{B}G_{\mathrm{ch}} & & & & \end{array}$$

coming from crossed complexes, def. 1.2.60, as indicated.

Proof. We need to show that

$$\begin{array}{ccc} \mathbf{B}\hat{G}_{\mathrm{ch}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}G_{\mathrm{ch}} & \stackrel{\mathbf{c}}{\longrightarrow} & \mathbf{B}^2 A \end{array}$$

is an ∞-pullback. To that end, we notice that we have an equivalence

$$\mathbf{B}(A \to \hat{G})_c \stackrel{\simeq}{\to} \mathbf{B}G_{\mathrm{ch}}$$

and that the morphism of simplicial presheaves $\mathbf{B}(A \stackrel{\mathrm{id}}{\to} A)_c \to \mathbf{B}^2 A_c$ is a fibration replacement of $* \to \mathbf{B}^2 A_c$, both in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

By prop. 2.3.13 it is therefore sufficient to observe the ordinary pullback diagram

$$\begin{array}{ccc} \mathbf{B}(1 \to A)_c & \longrightarrow & \mathbf{B}(A \stackrel{\mathrm{id}}{\to} A)_c \\ \downarrow & & \downarrow \\ \mathbf{B}(A \to \hat{G}) & \longrightarrow & \mathbf{B}(A \to 1)_c \end{array} .$$

□

4.4.8 Twisted cohomology and twisted bundles

We give an extensive discussion of twisted cohomology, 3.6.12, and the corresponding twisted principal ∞-bundles, realized in Smooth∞Grpd, below in 5.4. Most of the discussion there which does not involve differential refinement also goes through verbatim in ETop∞Grpd, 4.3.

Notably in 5.4.2 we discuss as a simple consistency check that the general theory of twisted ∞-bundles as sections of associated ∞-bundles reproduces the ordinary notion of smooth sections of a vector bundle. Then in 5.4.3 we discuss that twisted vector bundles and hence twisted K-cocycles do arise as 2-sections of certain canonically associated 2-bundles to circle 2-bundles. This serves to show how the case of twisted cohomology that traditionally is at the focus the attention is reproduced. After that we discuss in 5.4 a wealth of further examples.

4.4.9 ∞-Group representations

We discuss the intrinsic notion of ∞-group representations, 3.6.13, realized in the context Smooth∞Grpd.

We make precise the role of *action Lie groupoids*, introduced informally in 1.2.5.1.

Proposition 4.4.39. *Let X be a smooth manifold, and G a Lie group. Then the category of smooth G-actions on X in the traditional sense is equivalent to the category of G-actions on X in the cohesive ∞-topos* Smooth∞Grpd*, according to def. 3.6.152.*

Proof. For $\rho : X \times G \to X$ a given G-action, define the *action Lie groupoid*

$$X//G := (\; X \times G \underset{p_1}{\overset{\rho}{\rightrightarrows}} X \;)$$

with the evident composition operation. This comes with the evident morphism of Lie groupoids

$$X//G \to *//G \simeq \mathbf{B}G \,,$$

with $\mathbf{B}G$ as in prop. 4.4.19. It is immediate that regarding this as a morphism in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ in the canonical way, this is a fibration. Therefore, by 2.3.13, the homotopy fiber of this morphism in Smooth∞Grpds is given by the ordinary fiber of this morphism in simplicial presheaves. This is manifestly X.

Accordingly this construction constitutes an embedding of the traditional G actions on X into the category $\mathrm{Rep}_G(X)$ from def. 3.6.152. By turning this argument around, one finds that this embedding is essentially surjective. □

4.4.10 Associated bundles

We discuss aspects of the general notion of *associated* ∞*-bundles*, 3.6.11, realized in the context of smooth cohesion.

We have been discussing the n-stacks $\mathbf{B}^n U(1)$ of *circle* n*-bundles* in 4.4.16, but without any substantial change in the theory we could also use the n-stacks $\mathbf{B}^n \mathbb{C}^\times$ which are the n-fold delooping in $\mathbf{H}$ of the cohesive mutliplicative group of non-zero complex numbers. Under geometric realization $|-| : \mathbf{H} \longrightarrow \infty\mathrm{Grpd}$ the canonical map $\mathbf{B}^n U(1) \to \mathbf{B}^n \mathbb{C}^\times$ becomes an equivalence. Nevertheless, some constructions are more naturally expressed in terms of $U(1)$-principal n-bundles, while other are more naturally expressed in terms of $\mathbb{C}^\times$-principal n-bundles (bundle $(n-1)$-gerbes). Notably the latter is naturally identified with the 2-stack $2\mathbf{Line}_{\mathbb{C}}$ of *complex line 2-bundles*.

To interpret this, we say that for R a ring (or more generally an E_∞-ring), a *2-vector space* over R is, if it admits a 2*2-basis*, a category $A\mathrm{Mod}$ of modules over an R-algebra A (the algebra A is the given 2-basis), and that a *2-linear map* between 2-vector space is a functor $A\mathrm{Mod} \to B\mathrm{Mod}$ which is induced by tensoring with a B-A-bimodule. This identifies a 2-category $2\mathrm{Vect}_R$ of algebras, bimodules and bimodule homomorphisms which we call the 2-category of 2-vector spaces over R (appendix A of [Sc08], section 4.4. of [ScWaIII], section 7 of [1]). This 2-category is naturally braided monoidal. Write then

$$2\mathrm{Line}_R \hookrightarrow 2\mathrm{Vect}_R$$

for the full sub-2-category on those objects which are invertible under this tensor product: the *2-lines* over R. This is necessarily a 2-groupoid, the *Picard 2-groupoid* over R, and with the inherited monoidal structure it is a 3-group, the *Picard 3-group* of R. Its homotopy groups have a familiar algebraic interpretation:

- $\pi_0(2\mathrm{Line}_R)$ is the *Brauer group* of R;
- $\pi_1(2\mathrm{Line}_R)$ is the ordinary *Picard group* of R (of ordinary R-lines);
- $\pi_2(2\mathrm{Line}_R) \simeq R^\times$ is the *group of units*.

If we take the base ring R to be the ring of suitable k-valued functions on some space X, then $2\mathrm{Vect}_R$ is the 2-category of k-2-vector spaces over that vary over X, hence of complex *2-vector bundles*. This construction is natural in R, hence in X, and it restricts to 2-lines and hence to *2-line bundles* over k. Hence there is a 2-stack $2\mathbf{Line}_k \in \mathbf{H}$ of 2-line bundles over k. If k here is algebraically closed, such as $k = \mathbb{C}$, then there is, up to equivalence, only a single 2-line, and only a single invertible bimodule, and hence we find that $2\mathbf{Line}_k \simeq \mathbf{B}^2 k^\times$ In particular we have an equivalence

$$2\mathbf{Line}_{\mathbb{C}} \simeq \mathbf{B}^2 \mathbb{C}^\times .$$

Therefore the 2-stack $2\mathbf{Line}_{\mathbb{C}}$ is of interest in particular in situations where this equivalence no longer holds. This is notably so in the context of supergeometric cohesion; this is discussed below in 4.6.1.

4.4.11 Manifolds

We discuss the realization of the general abstract notion of manifolds in a cohesive ∞-topos in 3.9.2 realized in smooth cohesion.

With $\mathbb{A} := \mathbb{R} \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ the standard line object exhibiting the cohesion of $\mathrm{Smooth}\infty\mathrm{Grpd}$ according to prop. 4.3.33, def. 3.9.9 is equivalent to the traditional definition of smooth manifolds.

4.4.12 Flat connections and local systems

We discuss the intrinsic notion of flat ∞-connections, 3.8.5, realized in Smooth∞Grpd.

Proposition 4.4.40. *Let $X, A \in$ Smooth∞Grpd be any two objects and write $|X| \in$ Top for the intrinsic geometric realization, def. 3.8.2. We have that the flat cohomolog in Smooth∞Grpd of X with coefficients in A is equivalent to the ordinary cohomology in Top of $|X|$ with coefficients in underlying discrete object of A:*

$$H_{\mathrm{Smooth,flat}}(X, A) \simeq H(|X|, |\Gamma A|)\,.$$

Proof. By definition we have

$$H_{\mathrm{flat}}(X, A) \simeq H(\mathbf{\Pi} X, A) \simeq H(\mathrm{Disc}\Pi X, A)\,.$$

Using the (Disc) $\dashv \Gamma$-adjunction this is

$$\cdots \pi_0 \infty\mathrm{Grpd}(\Pi X, \Gamma A)\,.$$

Finally applying the equivalence $|\cdot| : \infty\mathrm{Grpd} \to \mathrm{Top}$ this is

$$\cdots \simeq H(|\Pi X|, |\Gamma A|)\,.$$

The claim hence follows as in prop. 4.4.32. □

Let G be a Lie group regarded as a 0-truncated ∞-group in Smooth∞Grpd. Write $\mathfrak{g}$ for its Lie algebra. Write $\mathbf{B}G \in$ Smooth∞Grpd for its delooping. Recall the fibrant presentation $\mathbf{B}G_{\mathrm{ch}} \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ from prop. 4.4.19.

Proposition 4.4.41. *The object $\flat\mathbf{B}G \in$ Smooth∞Grpd has a fibrant presentation $\flat\mathbf{B}G_{\mathrm{ch}} \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ given by the groupoid of Lie-algebra valued forms*

$$\flat\mathbf{B}G_{\mathrm{ch}} = N\left(C^\infty(-, G) \times \Omega^1_{\mathrm{flat}}(-, \mathfrak{g}) \underset{p2}{\overset{\mathrm{Ad}_{p_1}(p_2) + p_1^{-1} dp_1}{\rightrightarrows}} \Omega^1_{\mathrm{flat}}(-, \mathfrak{g}) \right)$$

and this is such that the canonical morphism $\flat\mathbf{B}G \to \mathbf{B}G$ is presented by the canonical morphism of simplicial presheaves $\flat\mathbf{B}G_{\mathrm{ch}} \to \mathbf{B}G_{\mathrm{ch}}$ which is a fibration in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Remark 4.4.42. This means that a U-parameterized family of objects of $\flat\mathbf{B}G_{\mathrm{ch}}$ is given by a Lie-algebra valued 1-form $A \in \Omega^1(U) \otimes \mathfrak{g}$ whose curvature 2-form $F_A = d_{\mathrm{dR}} A + [A, \wedge A] = 0$ vanishes, and a U-parameterized family of morphisms $g : A \to A'$ is given by a smooth function $g \in C^\infty(U, G)$ such that $A' = \mathrm{Ad}_g A + g^{-1} dg$, where $\mathrm{Ad}_g A = g^{-1} A g$ is the adjoint action of G on its Lie algebra, and where $g^{-1} dg := g^* \theta$ is the pullback of the Maurer-Cartan form on G along g.

Proof. By the proof of prop. 3.4.9 we have that $\flat\mathbf{B}G$ is presented by the simplicial presheaf that is constant on the nerve of the one-object groupoid

$$G_{\mathrm{disc}} \rightrightarrows * \ ,$$

for the discrete group underlying the Lie group G. The canonical morphism of that into $\mathbf{B}G_{\mathrm{ch}}$ is however not a fibration. We claim that the canonical inclusion $N(\, G_{\mathrm{disc}} \rightrightarrows * \,) \to \flat\mathbf{B}G_c$ factors the inclusion into $\mathbf{B}G_{\mathrm{ch}}$ by a weak equivalence followed by a global fibration.

To see the weak equivalence, notice that it is objectwise an equivalence of groupoids: it is essentially surjective since every flat $\mathfrak{g}$-valued 1-form on the contractible $\mathbb{R}^n$ is of the form $g dg^{-1}$ for some function $g : \mathbb{R}^n \to G$ (let $g(x) = P\exp(\int_0^x)A$ be the parallel transport of A along any path from the origin to x). Since the gauge transformation automorphism of the trivial $\mathfrak{g}$-valued 1-form are precisely given by the constant G-valued functions, this is also objectwise a full and faithful functor. Similarly one sees that the map $\flat \mathbf{B}G_{\mathrm{ch}} \to \mathbf{B}G$ is a fibration.

Finally we need to show that $\flat \mathbf{B}G_{\mathrm{ch}}$ is fibrant in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. This is implied by theorem 3.4.16. More explicitly, this can be seen by observing that this sheaf is the coefficient object that in Čech cohomology computes G-principal bundles with flat connection and then reasoning as above: every G-principal bundle with flat connection on a Cartesian space is equivalent to a trivial G-principal bundle whose connection is given by a globally defined $\mathfrak{g}$-valued 1-form. Morphisms between these are precisely G-valued functions that act on the 1-forms by gauge transformations as in the groupoid of Lie-algebra valued forms. □

Let now $\mathbf{B}^n U(1)$ be the circle $(n+1)$-Lie group, def. 4.4.21. Recall the notation and model category presentations as discussed there.

Proposition 4.4.43. *For $n \geq 1$ a fibration presentation in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ of the canonical morphism $\flat \mathbf{B}^n U(1) \to \mathbf{B}^n U(1)$ in* Smooth∞Grpd *is given by the image under $\Xi : [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Ch}^+] \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ of the morphism of chain complexes*

$$\begin{array}{ccccccc} C^\infty(-,U(1)) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1(-) & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow & & & & \downarrow \\ C^\infty(-,U(1)) & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \end{array} ,$$

where at the top we have the flat *Deligne complex.*

Proof. It is clear that the morphism of chain complexes is an objectwise surjection and hence maps to a projective fibration under Ξ. It remains to observe that the flat Deligne complex is a presentation of $\flat \mathbf{B}^n U(1)$:

By the proof of prop. 3.4.9 we have that $\flat = \mathrm{Disc} \circ \Gamma$ is presented in the model category on fibrant objects by first evaluating on the point and then extending back to a constant simplicial presheaf. Since $\Xi U(1)[n]$ is indeed globally fibrant, a fibrant presentation of $\flat \mathbf{B}^n U(1)$ is given by the *constant* presheaf $U(1)_{\mathrm{const}}[n] : U \mapsto \Xi(U(1)[n])$.

The inclusion $U(1)_{\mathrm{const}}[n] \to U(1)[n]$ is not yet a fibration. But by a basic fact of abelian sheaf cohomology – using the Poincaré lemma – we have a global weak equivalence $U(1)_{\mathrm{const}}[n] \stackrel{\simeq}{\to} [C^\infty(-,U(1)) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \stackrel{d_{\mathrm{dR}}}{\to} \Omega^n_{\mathrm{cl}}(-)]$ that factors this inclusion by the above fibration. This completes the proof.

For emphasis, we repeat this argument in more detail. The factorization of $U(1)_{\mathrm{const}}[n] \to U(1)[n]$ into a weak equivalence followed by a fibration that we are looking at is over each object $\mathbb{R}^q \in \mathrm{CartSp}$ in the site given by the morphisms of chain complexes whose components are show on the following diagram.

$$\begin{array}{ccccccccc} U(1) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ C^\infty(\mathbb{R}^q, U(1)) & \xrightarrow{d_{\mathrm{dR}} \log} & \Omega^1(\mathbb{R}^q) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^2(\mathbb{R}^q) & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n_{\mathrm{cl}}(\mathbb{R}^q) \\ \downarrow^{\mathrm{id}} & & \downarrow & & \downarrow & & & & \downarrow \\ C^\infty(\mathbb{R}^q, U(1)) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \end{array} .$$

It is clear that this commutes. It is also clear that the lower vertical morphisms are all surjections, so the lower row exhibits a fibration of chain complexes. In order for the top row to exhibit a weak equivalence of chain complexes – a quasi-isomorphism – we need it to induce an isomorphism on all chain homology groups.

The chain homology of the top complex is evidently concentrated in degree n, where it is $U(1)$, as a discrete group.

The chain homology of the middle complex in degree n is the kernel of the differential $d_{\mathrm{dR}}\log : C^\infty(\mathbb{R}^q, U(1)) \to \Omega^1(\mathbb{R}^q)$. This kernel manifestly consists of the constant $U(1)$-valued functions. Since $\mathbb{R}^q$ is connected, these are naturally identified with the group $U(1)$ itself. This identification is indeed what the top left vertical morphism exhibits.

The chain homology of the middle complex in degree $0 \leq k < n$ is the de Rham cohomology $H^{n-k}_{\mathrm{dR}}(\mathbb{R}^q)$. But this vanishes, since $\mathbb{R}^q$ is smoothly contractible (the Poincaré lemma).

Therefore the homology groups of the top and of the middle chain complex coincide. And by this discussion, the top vertical morphisms induce isomorphisms on these homology groups. □

We discuss presentations of $\flat \mathbf{B}G$ for G more generally the Lie integration of an L_∞-algebra $\mathfrak{g}$ further below in 4.4.14.2.

4.4.13 de Rham cohomology

We discuss intrinsic notion of de Rham cohomology in a cohesive ∞-topos, 3.9.3, realized in the context Smooth∞Grpd. Here it reproduces the traditional notion of de Rham cohomology with abelian and nonabelian group coefficients, as well as its equivariant and simplicial refinements.

Let G be a Lie group. Write $\mathfrak{g}$ for its Lie algebra.

Proposition 4.4.44. *The object* $\flat_{\mathrm{dR}}\mathbf{B}G \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *has a fibrant presentation in* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *by the sheaf* $\flat \mathbf{B}G_{\mathrm{ch}} := \Omega^1_{\mathrm{flat}}(-,\mathfrak{g})$ *of flat Lie algebra-valued forms*

$$\flat \mathbf{B}G_{\mathrm{ch}} : U \mapsto \Omega^1_{\mathrm{flat}}(U, \mathfrak{g})\,.$$

Proof. By prop. 4.4.41 we have a fibration $\flat \mathbf{B}G_{\mathrm{ch}} \to \mathbf{B}G_{\mathrm{ch}}$ in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ given by the morphism of sheaves of groupoids

$$\begin{array}{ccc} C^\infty(-,G) & \xrightarrow{(-)^*\theta} & \Omega^1_{\mathrm{flat}}(-,\mathfrak{g}) \\ \downarrow^{\mathrm{id}} & & \downarrow \\ C^\infty(-,G) & \longrightarrow & 0 \end{array} ,$$

which models the canonical inclusion $\flat \mathbf{B}G \to \mathbf{B}G$. Therefore by prop. 2.3.8 we obtain a presentation for the defining ∞-pullback

$$\flat_{\mathrm{dR}}\mathbf{B}G := * \times_{\mathbf{B}G} \flat \mathbf{B}G$$

in Smooth∞Grpd by the ordinary pullback

$$\flat_{\mathrm{dR}}\mathbf{B}G_{\mathrm{ch}} \simeq * \times_{\mathbf{B}G_{\mathrm{ch}}} \flat \mathbf{B}G_{\mathrm{ch}}$$

in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. This is manifestly equal to $\Omega^1_{\mathrm{flat}}(-,\mathfrak{g})$. This is fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ because it is a sheaf. □

Remark 4.4.45. Another equivalent way to compute the homotopy fiber in prop. 4.4.44 is to produce the fibration resolution specifically by the factorization lemma, prop. 2.3.9. This yields for the de Rham coefficients of the Lie group G the presentation

$$\flat_{\mathrm{dR}}\mathbf{B}G \simeq G/(G_{\mathrm{disc}})\,,$$

where on the right we have the quotient (of sheaves, hence in Smooth∞Grpd) of the Lie group G (the sheaf $C^\infty(-,G)$) by the underlying *geometrically discrete* group (the sheaf constant on the underyling set of G). In other words, over a $U \in \mathrm{CartSp}$ the value of $G/(G_{\mathrm{disc}})$ is the set of equivalence classes of smooth functions $g : U \to G$, where two are regarded as equivalent if they differ by multiplication with a *constant* such function.

By the general theory this sheaf must be equivalent, hence isomorphic, to the one of prop. 4.4.44. Indeed, G_{disc} is the kernel of the map $(-)^*\theta : C^\infty(-,G) \longrightarrow \Omega^1_{\mathrm{flat}}(-,\mathfrak{g})$ which sends $g : U \to G$ to the pullback of the Maurer-Cartan form along g, often writtenm $g^{-1}d_{\mathrm{dR}}g$. Moreover this map is surjective, since for $A \in \Omega^1_{\mathrm{flat}}(U,\mathfrak{g})$ any flat $\mathfrak{g}$-valued form the function $P\exp(\int_{x_0}^{(-)} A) : U \to G$ that sends a point $x \in U$ to the parallel transport of A along any path from any fixed basepoint $x_0 \in U$ is a preimage. Hence we have the image factorization

$$(-)^*\theta : G \twoheadrightarrow G/(G_{\mathrm{disc}}) \xrightarrow{\simeq} \Omega^1_{\mathrm{flat}}(-,\mathfrak{g}) \; .$$

In words this says that a flat differential Lie-algebra valued form on a Cartesian space $\mathbb{R}^k$ is equivalently a smooth function from that space to G "without remembering the origin of this function". What is noteworthy about this is that this second, equivalent, description, no longer refers to *differentials*.

Indeed, this second description of the de Rham coefficient object of a group object is valid for any site, in particular for instance for the Euclidean-topological cohesion of 4.3.

For $n \in \mathbb{N}$, let now $\mathbf{B}^n U(1)$ be the circle Lie $(n+1)$-group of def. 4.4.21. Recall the notation and model category presentations from the discussion there.

Proposition 4.4.46. *A fibrant representative in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *of the de Rham coefficent object* $\flat_{\mathrm{dR}}\mathbf{B}^n U(1)$ *from def. 3.9.12 is given by the truncated ordinary de Rham complex of smooth differential forms*

$$\flat_{\mathrm{dR}}\mathbf{B}^n U(1)_{\mathrm{chn}} := \Xi[\Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^2(-) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \to \Omega^{n-1}(-) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^n_{\mathrm{cl}}(-)] \, .$$

Proof. By definition and using prop. 2.3.13 the object $\flat_{\mathrm{dR}}\mathbf{B}^n U(1)$ is given by the homotopy pullback in $[\mathrm{CartSp}^{\mathrm{op}}, Ch_{\bullet \geq 0}]_{\mathrm{proj}}$ of the inclusion $U(1)_{\mathrm{const}}[n] \to U(1)[n]$ along the point inclusion $* \to U(1)[n]$. We may compute this as the ordinary pullback after passing to a resolution of this inclusion by a fibration. By prop. 4.4.43 such a fibration replacement is given by the map from the flat Deligne complex. Using this we find the ordinary pullback diagram

$$\begin{array}{ccc} \Xi[0 \to \Omega^1(-) \to \cdots \to \Omega^n_{\mathrm{cl}}(-)] & \longrightarrow & \Xi[C^\infty(-,U(1)) \to \Omega^1(-) \to \cdots \to \Omega^n_{\mathrm{cl}}(-)] \\ \downarrow & & \downarrow \\ \Xi[0 \to 0 \to \cdots \to 0] & \longrightarrow & \Xi[C^\infty(-,U(1)) \to 0 \to \cdots \to 0] \end{array} \; .$$

□

Proposition 4.4.47. *Let* X *be a smooth manifold regarded under the embedding* $\mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$*. Write* $H^n_{\mathrm{dR}}(X)$ *for the ordinary de Rham cohomology of* X*.*

For $n \in \mathbb{N}$ *we have isomorphisms*

$$\pi_0 \mathrm{Smooth}\infty\mathrm{Grpd}(X, \flat_{\mathrm{dR}}\mathbf{B}^n U(1)) \simeq \left\{ \begin{array}{ll} H^n_{\mathrm{dR}}(X) & | n \geq 2 \\ \Omega^1_{\mathrm{cl}}(X) & | n = 1 \\ 0 & | n = 0 \end{array} \right.$$

Proof. Let $\{U_i \to X\}$ be a differentiably good open cover. The Čech nerve $C(\{U_i\}) \to X$ is a cofibrant resolution of X in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Therefore we have for all $n \in \mathbb{N}$

$$\mathrm{Smooth}\infty\mathrm{Grpd}(X, \flat_{\mathrm{dR}}\mathbf{B}^n U(1)) \simeq [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \Xi[\Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \to \Omega^n_{\mathrm{cl}}(-)]) \, .$$

The right hand is the ∞-groupoid of cocylces in the Čech hypercohomology of the truncated complex of sheaves of differential forms. A cocycle is given by a collection

$$(C_i, B_{ij}, A_{ijk}, \cdots, Z_{i_1,\cdots,i_n})$$

of differential forms, with $C_i \in \Omega^n_{\mathrm{cl}}(U_i)$, $B_{ij} \in \Omega^{n-1}(U_i \cap U_j)$, etc. , such that this collection is annihilated by the total differential $D = d_{\mathrm{dR}} \pm \delta$, where d_{dR} is the de Rham differential and δ the alternating sum of the pullbacks along the face maps of the Čech nerve.

It is a standard result of abelian sheaf cohomology that such cocycles represent classes in de Rham cohomology of $n \geq 2$. For $n = 1$ and $n = 0$ our truncated de Rham complex degenerates to $\flat_{\mathrm{dR}} \mathbf{B} U(1)_{\mathrm{chn}} = \Xi[\Omega^1_{\mathrm{cl}}(-)]$ and $\flat_{\mathrm{dR}} U(1)_{\mathrm{chn}} = \Xi[0]$, respectively, which obviously has the cohomology as claimed above. □

Remark 4.4.48. Recall from the discussion in 3.9.3 that the failure of the intrinsic de Rham cohomology of Smooth∞ to coincide with traditional de Rham cohomology in degree 0 and 1 is due to the fact that the intrinsic de Rham cohomology in degree n is the home for curvature classes of circle $(n-1)$-bundles. For $n = 1$ these curvatures are not to be taken modulo exact forms. And for $n = 0$ they vanish.

Definition 4.4.49. For $n \in \mathbb{N}$, write $\Omega^n_{\mathrm{cl}} \in \mathrm{Sh}(\mathrm{CartSp}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ for the ordinary sheaf of smooth closed differential n-forms. By prop. 4.4.46 this has a canonical morphism

$$\Omega^n_{\mathrm{cl}} \to \flat_{\mathrm{dR}} \mathbf{B}^n U(1)$$

into the de Rham coefficient object for $\mathbf{B}^{n-1} U(1)$, given in the presentation of the latter as a simplicial presheaf according to prop. 4.4.46 by the inclusion of the simplicial presheaf that is simplicially constant on the degree-0 component.

Proposition 4.4.50. *The morphisms of def. 4.4.49 are differential form objects in the sense of def. 3.9.20 with respect to the standard line object* $\mathbb{R}$*.*

Proof. By the discussion in 4.4.11 the $\mathbb{R}^1$-manifolds are precisely the objects in the inclusion $\mathrm{SmthMfd} \hookrightarrow \mathrm{Sh}_\infty(\mathrm{SmthMfd}) \simeq \mathrm{Smooth}\infty\mathrm{Grpd}$. This means by def. 3.9.20 that we need to check that for each smooth manifold Σ the morphism

$$[\Sigma, \Omega^n_{\mathrm{cl}}] \to [\Sigma, \flat_{\mathrm{dR}} \mathbf{B}^n U(1)]$$

is an effective epimorphism. By prop. 2.3.6 this is equivalent to the 0-truncation of the moprhism being an epimorphism in the sheaf topos Sh(CartSp). By the characterization of internal homs in turn, for this it is sufficient that for each $U \in \mathrm{CartSp}$ the function $\Omega^n_{\mathrm{cl}}(\Sigma \times U) \to \pi_0 \mathbf{H}(\Sigma \times U, \flat_{\mathrm{dR}} \mathbf{B}^n U(1))$ is a surjection. This is the case by prop. 4.4.47. □

We discuss the equivariant version of smooth de Rham cohomology.

Proposition 4.4.51. *Let X be a smooth manifold equipped with a smooth action by a Lie group G. Write $X//G$ for the corresponding action Lie groupoid, prop. 5.4.1. Then for $n \geq 2$ we have an isomorphism*

$$\pi_0 \mathrm{Smooth}\infty\mathrm{Grpd}(X//G, \flat_{\mathrm{dR}} \mathbf{B}^n \mathbb{R}) \simeq H^n_{\mathrm{dR},G}(X)\,,$$

where on the right we have ordinary G-equivariant de Rham cohomology of X.

4.4.14 Exponentiated ∞-Lie algebras

We discuss the intrinsic notion of exponentiated ∞-Lie algebras, 3.9.4, realized in Smooth∞Grpd.

Recall the characterization of L_∞-algebras, def. 1.2.114, by dual dg-algebras, prop. 1.2.116 – their *Chevalley-Eilenberg algebras*–, and the characterization of the category $L_\infty\mathrm{Alg}$ as the full subcategory

$$L_\infty \overset{\mathrm{CE}}{\hookrightarrow} \mathrm{dgAlg}^{\mathrm{op}}\,.$$

We describe now a presentation of the exponentiation of an L_∞ algebra to a smooth ∞-group. The following somewhat technical definition serves to control the smooth structure on these exponentiated objects.

Definition 4.4.52. For $k \in \mathbb{N}$ regard the k-simplex Δ^k as a smooth manifold with corners in the standard way. We think of this embedded into the Cartesian space $\mathbb{R}^k$ in the standard way with maximal rotation symmetry about the center of the simplex, and equip Δ^k with the metric space structure induced this way.

A smooth differential form ω on Δ^k we say has *sitting instants* along the boundary if, for every $(r < k)$-face F of Δ^k there is an open neighbourhood U_F of F in Δ^k such that ω restricted to U is constant in the directions perpendicular to the r-face on its value restricted to that face.

More generally, for any $U \in \mathrm{CartSp}$ a smooth differential form ω on $U \times \Delta^k$ is said to have sitting instants if there is $0 < \epsilon \in \mathbb{R}$ such that for all points $u : * \to U$ the pullback along $(u, \mathrm{Id}) : \Delta^k \to U \times \Delta^k$ is a form with sitting instants on ϵ-neighbourhoods of faces.

Smooth forms with sitting instants form a sub-dg-algebra of all smooth forms. We write $\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)$ for this sub-dg-algebra.

We write $\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^k)$ for the further sub-dg-algebra of vertical differential forms with respect to the projection $p : U \times \Delta^k \to U$, hence the coequalizer

$$\Omega^{\bullet \geq 1}(U) \underset{0}{\overset{p^*}{\rightrightarrows}} \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \longrightarrow \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^k) \,.$$

Definition 4.4.53. For $\mathfrak{g} \in L_\infty$ write $\exp(\mathfrak{g}) \in [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]$ for the simplicial presheaf defined over $U \in \mathrm{CartSp}$ and $n \in \mathbb{N}$ by

$$\exp(\mathfrak{g}) : (U, [n]) \mapsto Hom_{\mathrm{dgAlg}}(\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n), \mathrm{CE}(\mathfrak{g}))$$

with the evident structure maps given by pullback of differential forms.

This definition of the ∞-groupoid associated to an L_∞-algebra realized in the smooth context appears in [FSS10] and in similar form in [Royt10] as the evident generalization of the definition in Banach spaces in [Henr08] and for discrete ∞-groupoids in [Getz09], which in turn goes back to [Hini97].

Proposition 4.4.54. *The objects* $\exp(\mathfrak{g}) \in [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]$ *are*

1. *connected;*
2. *Kan complexes over each* $U \in \mathrm{CartSp}$.

Proof. That $\exp(\mathfrak{g})_0 = *$ follows from degree-counting: $\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^0) = C^\infty(U)$ is entirely in degree 0 and $\mathrm{CE}(\mathfrak{g})$ is in degree 0 the ground field $\mathbb{R}$.

To see that $\exp(\mathfrak{g})$ has all horn-fillers over each $U \in \mathrm{CartSp}$ observe that the standard continuous horn retracts $f : \Delta^k \to \Lambda^k_i$ are smooth away from the preimages of the $(r < k)$-faces of $\Lambda[k]^i$.

For $\omega \in \Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda[k]^i)$ a differential form with sitting instants on ϵ-neighbourhoods, let therefore $K \subset \partial \Delta^k$ be the set of points of distance $\leq \epsilon$ from any subface. Then we have a smooth function

$$f : \Delta^k \setminus K \to \Lambda^k_i \setminus K \,.$$

The pullback $f^* \omega \in \Omega^\bullet(\Delta^k \setminus K)$ may be extended constantly back to a form with sitting instants on all of Δ^k. The resulting assignment

$$(\mathrm{CE}(\mathfrak{g}) \xrightarrow{A} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda^k_i)) \mapsto (\mathrm{CE}(\mathfrak{g}) \xrightarrow{A} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda^k_i) \xrightarrow{f^*} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n))$$

provides fillers for all horns over all $U \in \mathrm{CartSp}$. □

Definition 4.4.55. We say that the loop space object $\Omega \exp(\mathfrak{g})$ is the *smooth ∞-group* exponentiating $\mathfrak{g}$.

Proposition 4.4.56. *The objects* $\exp(\mathfrak{g}) \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *are geometrically contractible:*

$$\Pi \exp(\mathfrak{g}) \simeq * \,.$$

Proof. Observe that every simplicial presheaf X is the homotopy colimit over its component presheaves $X_n \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{Set}] \hookrightarrow [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$

$$X \simeq \mathbb{L}\lim_{\to n} X_n \,.$$

(Use for instance the injective model structure for which $X_\bullet$ is cofibrant in the Reedy model structure $[\Delta^{op}, [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}]_{\mathrm{Reedy}}$). Therefore it is sufficient to show that in each degree n the 0-truncated object $\exp(\mathfrak{g})_n$ is geometrically contractible.

To exhibit a geometric contraction, def. 3.8.4, choose for each $n \in \mathbb{N}$, a smooth retraction

$$\eta_n : \Delta^n \times [0,1] \to \Delta^n$$

of the n-simplex: a smooth map such that $\eta_n(-,1) = \mathrm{Id}$ and $\eta_n(-,0)$ factors through the point. We claim that this induces a diagram of presheaves

$$\begin{array}{ccc} \exp(\mathfrak{g})_n & & \\ \downarrow^{(\mathrm{id},1)} & \searrow^{\mathrm{id}} & \\ \exp(\mathfrak{g})_n \times [0,1] & \xrightarrow{\eta_n^*} & \exp(\mathfrak{g})_n \\ \uparrow^{(\mathrm{id},0)} & & \uparrow \\ \exp(\mathfrak{g})_n & \longrightarrow & * \end{array} ,$$

where over $U \in \mathrm{CartSp}$ the middle morphism is given by

$$\eta_n^* : (\alpha, f) \mapsto (f, \eta_n)^* \alpha \,,$$

where

- $\alpha : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)$ is an element of the set $\exp(\mathfrak{g})_n(U)$,
- f is an element of $[0,1](U)$;
- (f, η_n) is the composite morphism

$$U \times \Delta^n \stackrel{(\mathrm{id},f)\times \mathrm{id}}{\to} U \times [0,1] \times \Delta^n \stackrel{(\mathrm{id},\eta_n)}{\to} U \times \Delta^n$$

- $(f,\eta)^*\alpha$ is the postcomposition of α with the image of (f, η_n) under $\Omega^\bullet_{\mathrm{vert}}(-)$.

Here the last item is well defined given the coequalizer definition of $\Omega^\bullet_{\mathrm{vert}}$ because (f, η_n) is a morphism of bundles over U

$$\begin{array}{ccccc} U \times \Delta^n & \xrightarrow{(\mathrm{id},f)\times\mathrm{id}} & U \times [0,1] \times \Delta^n & \xrightarrow{\mathrm{id}\times\eta_n} & U \times \Delta^n \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\mathrm{id}} & U & \xrightarrow{\mathrm{id}} & U \end{array} .$$

Similarly, for $h : K \to U$ any morphism in $\mathrm{CartSp}_{\mathrm{smooth}}$ the naturality condition for a morphism of presheaves follows from the fact that the composites of bundle morphisms

$$\begin{array}{ccccccc} K \times \Delta^n & \xrightarrow{h\times\mathrm{id}} & U \times \Delta^n & \xrightarrow{(\mathrm{id},f)\times\mathrm{id}} & U \times [0,1] \times \Delta^n & \xrightarrow{(\mathrm{id},\eta_n)} & U \times \Delta^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \xrightarrow{h} & U & \xrightarrow{\mathrm{id}} & U & \xrightarrow{\mathrm{id}} & U \end{array}$$

and

$$\begin{array}{ccccccc} K \times \Delta^n & \xrightarrow{((\mathrm{id}, f\circ h)\times \mathrm{id}} & K \times [0,1] \times \Delta^n & \xrightarrow{\mathrm{id}\times \eta_n} & K \times \Delta^n & \xrightarrow{h \times \mathrm{id}} & U \times \Delta^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \xrightarrow{\mathrm{id}} & K & \xrightarrow{\mathrm{id}} & K & \xrightarrow{h} & U \end{array}$$

coincide.

Moreover, notice that the lower morphism in our diagram of presheaves indeed factors through the point as indicated, because for an L_∞-algebra $\mathfrak{g}$ we have that the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ is in degree 0 the ground field algebra algebra $\mathbb{R}$, so that there is a unique morphism $\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^0) \simeq C^\infty(U)$ in dgAlg.

Finally, since $[0,1]$ is a contractible paracompact manifold, we have that $\Pi([0,1]) \simeq *$ by prop. 4.3.29. Therefore the above diagram of presheaves presents a geometric homotopy in Smooth∞Grpd from the identity map to a map that factors through the point. It follows by prop 3.8.5 that $\Pi(\exp(\mathfrak{g})_n) \simeq *$ for all $n \in \mathbb{N}$. And since Π preserves the homotopy colimit $\exp(\mathfrak{g}) \simeq \mathbb{L}\varinjlim_n \exp(\mathfrak{g})_n$ we have that $\Pi(\exp(\mathfrak{g})) \simeq *$, too. □

We may think of $\exp(\mathfrak{g})$ as the smooth geometrically *∞-simply connected Lie integration* of $\mathfrak{g}$. Notice however that $\exp(\mathfrak{g}) \in$ Smooth∞Grpd in general has nontrivial and interesting homotopy sheaves. The above statement says that its *geometric homotopy groups* vanish .

4.4.14.1 Examples

Let $\mathfrak{g} \in L_\infty$ be an ordinary (finite dimensional) Lie algebra. Standard Lie theory provides a simply connected Lie group G integrating $\mathfrak{g}$. Write $\mathbf{B}G \in$ Smooth∞Grpd for its delooping. According to prop. 4.4.19 this is presented by the simplicial presheaf $\mathbf{B}G_{\mathrm{ch}} \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$.

Proposition 4.4.57. *The operation of parallel transport $P\exp(\int -) : \Omega^1([0,1], \mathfrak{g}) \to G$ yields a weak equivalence (in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$)*

$$P\exp(\int -) : \mathbf{cosk}_3 \exp(\mathfrak{g}) \simeq \mathbf{cosk}_2 \exp(\mathfrak{g}) \simeq \mathbf{B}G_{\mathrm{ch}} \,.$$

Proof. Notice that a flat smooth $\mathfrak{g}$-valued 1-form on a contractible space X is after a choice of basepoint canonically identified with a smooth function $X \to G$. The claim then follows from the observation that by the fact that G is simply connected any two paths with coinciding endpoints have a continuous homotopy between them, and that for smooth paths this may be chose to be smooth, by the Steenrod approximation theorem [Wock09]. □

Let now $n \in \mathbb{N}$, $n \geq 1$.

Definition 4.4.58. Write

$$b^{n-1}\mathbb{R} \in L_\infty$$

for the L_∞-algebra whose Chevalley-Eilenberg algebra is given by a single generator in degree n and vanishing differential. We call this the *line Lie n-algebra*.

Observation 4.4.59. The discrete ∞-groupoid underlying $\exp(b^{n-1}\mathbb{R})$ is given by the Kan complex that in degree k has the set of closed differential n-forms with sitting instants on the k-simplex

$$\Gamma(\exp(b^{n-1}\mathbb{R})) : [k] \mapsto \Omega^n_{\mathrm{si,cl}}(\Delta^k)$$

Definition 4.4.60. We write equivalently

$$\mathbf{B}^n\mathbb{R}_{\mathrm{smp}} := \exp(b^{n-1}\mathbb{R}) \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}] \,.$$

Proposition 4.4.61. *We have that $\mathbf{B}^n\mathbb{R}_{\mathrm{smp}}$ is indeed a presentation of the smooth line n-group $\mathbf{B}^n\mathbb{R}$, from 4.4.21.*

Concretely, with $\mathbf{B}^n\mathbb{R}_{\mathrm{chn}} \in [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]$ the standard presentation given under the Dold-Kan correspondence by the chain complex of sheaves concentrated in degree n on $C^\infty(-,\mathbb{R})$ the equivalence is induced by the fiber integration of differential n-forms over the n-simplex:

$$\int_{\Delta^\bullet} : \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} \stackrel{\simeq}{\to} \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} .$$

Proof. First we observe that the map

$$\int_{\Delta^\bullet} : (\omega \in \Omega^n_{\mathrm{si,vert,cl}}(U \times \Delta^k)) \mapsto \int_{\Delta^k} \omega \in C^\infty(U,\mathbb{R})$$

is indeed a morphism of simplicial presheaves $\exp(b^{n-1}\mathbb{R}) \to \mathbf{B}^n\mathbb{R}_{\mathrm{chn}}$ on. Since it goes between presheaves of abelian simplicial groups, by the Dold-Kan correspondence it is sufficient to check that we have a morphism of chain complexes of presheaves on the corresponding normalized chain complexes.

The only nontrivial degree to check is degree n. Let $\lambda \in \Omega^n_{\mathrm{si,vert,cl}}(\Delta^{n+1})$. The differential of the normalized chains complex sends this to the signed sum of its restrictions to the n-faces of the $(n+1)$-simplex. Followed by the integral over Δ^n this is the piecewise integral of λ over the boundary of the n-simplex. Since λ has sitting instants, there is $0 < \epsilon \in \mathbb{R}$ such that there are no contributions to this integral in an ϵ-neighbourhood of the $(n-1)$-faces. Accordingly the integral is equivalently that over the smooth surface inscribed into the $(n+1)$-simplex. Since λ is a closed form on the n-simplex, this surface integral vanishes, by the Stokes theorem. Hence $\int_{\Delta^\bullet}$ is indeed a chain map.

It remains to show that $\int_{\Delta^\bullet} : \mathbf{cosk}_{n+1}\exp(b^{n-1}\mathbb{R}) \to \mathbf{B}^n\mathbb{R}_{\mathrm{chn}}$ is an isomorphism on simplicial homotopy groups over each $U \in \mathrm{CartSp}$. This amounts to the statement that

- a smooth family of closed $n < k$-forms with sitting instants on the boundary of Δ^{k+1} may be extended to a smooth family of closed forms with sitting instants on Δ^{k+1}
- a smooth family of closed n-forms with sitting instants on the boundary of Δ^{n+1} may be extended to a smooth family of closed forms with sitting instants on Δ^{n+1} precisely if their smooth family of integrals over $\partial Delta^{n+1}$ vanishes.

To demonstrate this, we want to work with forms on the $(k+1)$-ball instead of the $(k+1)$-simplex. To achieve this, choose again $0 < \epsilon \in \mathbb{R}$ and construct the diffeomorphic image of $S^k \times [1-\epsilon, 1]$ inside the $(k+1)$-simplex as indicated by the above construction: outside an ϵ-neighbourhood of the corners the image is a rectangular ϵ-thickening of the faces of the simplex. Inside the ϵ-neighbourhoods of the corners it bends smoothly. By the Steenrod-approximation theorem [Wock09] the diffeomorphism from this ϵ-thickening of the smoothed boundary of the simplex to $S^k \times [0,1]$ extends to a smooth function from the $(k+1)$-simplex to the $(k+1)$-ball. By choosing ϵ smaller than each of the sitting instants of the given n-form on $\partial\Delta^k$, we have that this n-form vanishes on the ϵ-neighbourhoods of the corners and is hence entirely determined by its restriction to the smoothed simplex, identified with the $(k+1)$-ball.

It is now sufficient to show: a smooth family of smooth n-forms $\omega \in \Omega^n_{\mathrm{vert,cl}}(U \times S^k)$ extends to a smooth family of closed n-forms $\hat{\omega} \in \Omega^n_{\mathrm{vert,cl}}(U \times B^{n+1})$ that is radially constant in a neighbourhood of the boundary for all $n < k$ and for $n = k$ precisely if its smooth family of integrals $\int_{S^n} \omega = 0 \in C^\infty(U,\mathbb{R})$ vanishes.

Notice that over the point this is a direct consequence of the de Rham theorem: all $k < n$ forms are exact on S^k and n-forms are exact precisely if their integral vanishes. In that case there is an $(n-1)$-form A with $\omega = dA$. Choosing any smoothing function $f : [0,1] \to [0,1]$ (smooth, surjective non,decreasing and constant in a neighbourhood of the boundary) we obtain a n-form $f \wedge A$ on $(0,1] \times S^n$, vertically constant in a neighbourhood of the ends of the interval, equal to A at the top and vanishing at the bottom. Pushed forward along the canonical $(0,1] \times S^n \to D^{n+1}$ this defines a form on the $(n+1)$-ball, that we denote by the same symbol $f \wedge A$. Then the form $\hat{\omega} := d(f \wedge A)$ solves the problem.

To complete the proof we have to show that this argument does extend to smooth families of forms in that we can find suitable smooth families of the form A in the above discussion. This may be accomplished for instance by invoking Hodge theory: If we equip S^k with a Riemannian metric then the refined form of the Hodge theorem says that we have an equality

$$\mathrm{id} - \pi_{\mathcal{H}} = [d, d^* G]\,,$$

of operators on differential forms, where $\pi_{\mathcal{H}}$ is the orthogonal projection on harmonic forms and G is the Green operator of the Hodge-Laplace operator. For ω an exact form its harmonic projection vanishes so that this gives a homotopy

$$\omega = d(d^* G \omega)\,.$$

This operation $\omega \mapsto d^* G\omega$ depends smoothly on ω. □

4.4.14.2 Flat coefficient objects for exponentiated $L\infty$-algebras. We consider now the flat coefficient object, 3.8.5, $\flat \exp(\mathfrak{g})$ of exponentiated L_∞ algebras $\exp(\mathfrak{g})$, 4.4.14.

Definition 4.4.62. Write $\flat \exp(\mathfrak{g})_{\mathrm{smp}}$ or equivalentl $\exp(\mathfrak{g})_{\mathrm{flat}}$ for the simplicial presheaf given by

$$\flat \exp(\mathfrak{g})_{\mathrm{smp}} : (U, [n]) \mapsto \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^n))\,.$$

Proposition 4.4.63. *The canonical morphism* $\flat \mathbf{B}^n\mathbb{R} \to \mathbf{B}^n\mathbb{R}$ *in* Smooth∞Grpd *is presented in* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$ *by the composite*

$$\mathrm{const}\,\Gamma\ \exp(b^{n-1}\mathbb{R}) \xrightarrow{\simeq} \flat \exp(b^{n-1}\mathbb{R})_{\mathrm{smp}} \twoheadrightarrow \exp(b^{n-1}\mathbb{R})\ ,$$

where the first morphism is a weak equivalence and the second a fibration in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

We discuss the two morphisms in the composite separately in two lemmas.

Lemma 4.4.64. *The canonical inclusion*

$$\mathrm{const}\Gamma(\exp(\mathfrak{g})) \to \flat \exp(\mathfrak{g})_{\mathrm{smp}}$$

is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. The morphism in question is on each object $U \in \mathrm{CartSp}$ the morphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(\Delta^k)) \to \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k))\,,$$

which is given by pullback of differential forms along the projection $U \times \Delta^k \to \Delta^k$.

To show that for fixed U this is a weak equivalence in the standard model structure on simplicial sets we produce objectwise a left inverse

$$F_U : \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^\bullet)) \to \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(\Delta^\bullet))$$

and show that this is an acyclic fibration of simplicial sets. The statement then follows by the 2-out-of-3-property of weak equivalences.

We take F_U to be given by evaluation at $0 : * \to U$, i.e. by postcomposition with the morphisms

$$\Omega^\bullet(U \times \Delta^k) \stackrel{Id \times 0^*}{\to} \Omega^\bullet(* \times \Delta^k) = \Omega^\bullet(\Delta^k)\,.$$

(This is, of course, not natural in U and hence does not extend to a morphism of simplicial presheaves. But for our argument here it need not.) The morphism F_U is an acyclic Kan fibration precisely if all diagrams of the form

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^\bullet)) \\ \downarrow & & \downarrow^{F_U} \\ \Delta[n] & \longrightarrow & \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(\Delta^\bullet)) \end{array}$$

have a lift. Using the Yoneda lemma over the simplex category and since the differential forms on the simplices have sitting instants, we may, as above, equivalently reformulate this in terms of spheres as follows: for every morphism $\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si}}(D^n)$ and morphism $\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si}}(U \times S^{n-1})$ such that the diagram

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}) & \longrightarrow & \Omega^\bullet(U \times S^{n-1}) \\ \downarrow & & \downarrow \\ \Omega^\bullet_{\mathrm{si}}(D^n) & \longrightarrow & \Omega^\bullet(S^{n-1}) \end{array}$$

commutes, this may be factored as

$$\begin{array}{ccccc} \mathrm{CE}(\mathfrak{g}) & & & & \\ & \searrow & & & \\ & & \Omega^\bullet_{\mathrm{si}}(U \times D^n) & \longrightarrow & \Omega^\bullet(U \times S^{n-1}) \\ & & \downarrow & & \downarrow \\ & & \Omega^\bullet(D^n) & \longrightarrow & \Omega^\bullet(S^{n-1}) \end{array} .$$

(Here the subscript "${}_{\mathrm{si}}$" denotes differential forms on the disk that are radially constant in a neighbourhood of the boundary.)

This factorization we now construct. Let first $f : [0,1] \to [0,1]$ be any smoothing function, i.e. a smooth function which is surjective, non-decreasing, and constant in a neighbourhood of the boundary. Define a smooth map $U \times [0,1] \to U$ by $(u, \sigma) \mapsto u \cdot f(1-\sigma)$, where we use the multiplicative structure on the Cartesian space U. This function is the identity at $\sigma = 0$ and is the constant map to the origin at $\sigma = 1$. It exhibits a smooth contraction of U.

Pullback of differential forms along this map produces a morphism

$$\Omega^\bullet(U \times S^{n-1}) \to \Omega^\bullet(U \times S^{n-1} \times [0,1])$$

which is such that a form ω is sent to a form which in a neighbourhood $(1-\epsilon, 1]$ of $1 \in [0,1]$ is constant along $(1-\epsilon, 1] \times U$ on the value $(0, Id_{S^{n-1}})^*\omega$.

Let now $0 < \epsilon \in \mathbb{R}$ some value such that the given forms $CE(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si}}(D^k)$ are constant a distance $d \leq \epsilon$ from the boundary of the disk. Let $q : [0, \epsilon/2] \to [0,1]$ be given by multiplication by $1/(\epsilon/2)$ and $h : D^k_{1-\epsilon/2} \to D^n_1$ the injection of the n-disk of radius $1-\epsilon/2$ into the unit n-disk.

We can then glue to the morphism

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(U \times S^{n-1}) \to \Omega^\bullet(U \times [0,1] \times S^{n-1}) \stackrel{id \times q^* \times id}{\to_\simeq} \Omega^\bullet(U \times [0, \epsilon/2] \times S^{n-1})$$

to the morphism

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(D^n) \to \Omega^\bullet(U \times \{1\} \times D^n) \stackrel{h^*}{\to_\simeq} \Omega^\bullet(U \times \{1\} \times D^n_{1-\epsilon/2})$$

by smoothly identifying the union $[0, \epsilon/2] \times S^{n-1} \coprod_{S^{n-1}} D^n_{1-\epsilon/2}$ with D^n (we glue a disk into an annulus to obtain a new disk) to obtain in total a morphism

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(U \times D^n)$$

with the desired properties: at $u = 0$ the homotopy that we constructed is constant and the above construction hence restricts the forms to radius $\leq 1 - \epsilon/2$ and then extends back to radius ≤ 1 by the constant value that they had before. Away from 0 the homotopy in the rmaining $\epsilon/2$ bit smoothly interpolates to the boundary value. $\square$

Lemma 4.4.65. *The canonical morphism*

$$\flat \exp(\mathfrak{g})_{\mathrm{smp}} \to \exp(\mathfrak{g})$$

is a fibration in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. Over each $U \in \mathrm{CartSp}$ the morphism is induced from the morphism of dg-algebras

$$\Omega^\bullet(U) \to C^\infty(U)$$

that discards all differential forms of non-vanishing degree.

It is sufficient to show that for

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si,vert}}(U \times (D^n \times [0,1]))$$

a morphism and

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si}}(U \times D^n)$$

a lift of its restriction to $\sigma = 0 \in [0,1]$ we have an extension to a lift

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si,vert}}(U \times (D^n \times [0,1]))\,.$$

From these lifts all the required lifts are obtained by precomposition with some evident smooth retractions.

The lifts in question are obtained from solving differential equations with boundary conditions, and exist due to the existence of solutions of first order systems of partial differential equations and the identity $d_{\mathrm{dR}}^2 = 0$. □

We have discussed now two different presentations for the flat coefficient object $\flat \mathbf{B}^n \mathbb{R}$:

1. $\flat \mathbf{B}^n \mathbb{R}_{\mathrm{chn}}$ – prop. 4.4.43;
2. $\flat \mathbf{B}^n \mathbb{R}_{\mathrm{smp}}$ – prop. 4.4.63;

There is an evident degreewise map

$$(-1)^{\bullet+1} \int_{\Delta^\bullet} : \flat \mathbf{B}^n \mathbb{R}_{\mathrm{simp}} \to \flat \mathbf{B}^n \mathbb{R}_{\mathrm{chn}}$$

that sends a closed n-form $\omega \in \Omega^n_{\mathrm{cl}}(U \times \Delta^k)$ to $(-1)^{k+1}$ times its fiber integration $\int_{\Delta^k} \omega$.

Proposition 4.4.66. *This map yields a morphism of simplicial presheaves*

$$\int : \flat \mathbf{B}^n \mathbb{R}_{\mathrm{smp}} \to \flat \mathbf{B}^n \mathbb{R}_{\mathrm{chn}}$$

which is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. First we check that we have a morphism of simplicial sets over each $U \in \mathrm{CartSp}$. Since both objects are abelian simplicial groups we may, by the Dold-Kan correspondence, check the statement for sheaves of normalized chain complexes.

Notice that the chain complex differential on the forms $\omega \in \Omega^n_{\mathrm{cl}}(U \times \Delta^k)$ on simplices sends a form to the alternating sum of its restriction to the faces of the simplex. Postcomposed with the integration map this is the operation $\omega \mapsto \int_{\partial \Delta^k} \omega$ of integration over the boundary.

Conversely, first integrating over the simplex and then applying the de Rham differential on U yields

$$\begin{aligned} \omega \mapsto (-1)^{k+1} d_U \int_{\Delta^k} \omega &= -\int_{\Delta^k} d_U \omega \\ &= \int_{\Delta^k} d_{\Delta^k} \omega \quad , \\ &= \int_{\partial \Delta^k} \omega \end{aligned}$$

where we first used that ω is closed, so that $d_{\mathrm{dR}}\omega = (d_U + d_{\Delta^k})\omega = 0$, and then used Stokes' theorem. Therefore we have indeed objectwise a chain map.

By the discussion of the two objects we already know that both present the homotopy type of $\flat \mathbf{B}^n\mathbb{R}$. Therefore it suffices to show that the integration map is over each $U \in \mathrm{CartSp}$ an isomorphism on the simplicial homotopy group in degree n.

Clearly the morphism

$$\int_{\Delta^n} : \Omega^\bullet_{\mathrm{si,cl}}(U \times \Delta^n) \to C^\infty(U, \mathbb{R})$$

is surjective on degree n homotopy groups: for $f : U \to * \to \mathbb{R}$ constant, a preimage is $f \cdot \mathrm{vol}_{\Delta^n}$, the normalized volume form of the n-simplex times f. Moreover, these preimages clearly span the whole homotopy group $\pi_n(\flat \mathbf{B}^n\mathbb{R}) \simeq \mathbb{R}_{\mathrm{disc}}$ (they are in fact the images of the weak equivalence $\mathrm{const}\Gamma \exp(b^{n-1}\mathbb{R}) \to \flat \mathbf{B}^n\mathbb{R}_{\mathrm{smp}}$) and the integration map is injective on them. Therefore it is an isomorphism on the homotopy groups in degree n. □

4.4.14.3 de Rham coefficients

We now consider the de Rham coefficient object $\flat_{\mathrm{dR}} \exp(\mathfrak{g})$, 3.9.3, of exponentiated L_∞ algebras $\exp(\mathfrak{g})$, def 4.4.53.

Proposition 4.4.67. *For $\mathfrak{g} \in L_\infty$ a representive in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ of the de Rham coefficient object $\flat_{\mathrm{dR}} \exp(\mathfrak{g})$ is given by the presheaf*

$$\flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{smp}} : (U, [n]) \mapsto \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^{\bullet \geq 1, \bullet}_{\mathrm{si}}(U \times \Delta^n)) \,,$$

where the notation on the right denotes the dg-algebra of differential forms on $U \times \Delta^n$ that (apart from having sitting instants on the faces of Δ^n) are along U of non-vanishing degree.

Proof. By the prop. 4.4.63 we may present the defining ∞-pullback $\flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R} := * \times_{\mathbf{B}^n\mathbb{R}} \flat \mathbf{B}^n\mathbb{R}$ in Smooth∞Grpd by the ordinary pullback

$$\begin{array}{ccc} \flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{smp}} & \longrightarrow & \flat \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}^n\mathbb{R} \end{array}$$

in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]$. □

We have discussed now two different presentations for the de Rham coefficient object $\flat \mathbf{B}^n\mathbb{R}$:

1. $\flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{chn}}$ – prop. 4.4.46;
2. $\flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{smp}}$ – prop 4.4.67;

There is an evident degreewise map

$$(-1)^{\bullet+1} \int_{\Delta^\bullet} : \flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{smp}} \to \flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{chn}}$$

that sends a closed n-form $\omega \in \Omega^n_{\mathrm{cl}}(U \times \Delta^k)$ to $(-1)^{k+1}$ times its fiber integration $\int_{\Delta^k} \omega$.

Proposition 4.4.68. *This map yields a morphism of simplicial presheaves*

$$\int : \flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{smp}} \to \flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{chn}}$$

which is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. This morphism is the morphism on pullbacks induced from the weak equivalence of diagrams

$$\begin{array}{ccccc} * & \longrightarrow & \exp(b^{n-1}\mathbb{R}) & \longleftarrow & \flat \mathbf{B}^n \mathbb{R}_{\mathrm{smp}} \\ \downarrow^{=} & & \simeq \downarrow^{\int} & & \simeq \downarrow^{\int} \\ * & \longrightarrow & \mathbf{B}^n \mathbb{R}_{\mathrm{chn}} & \longleftarrow & \flat \mathbf{B}^n \mathbb{R}_{\mathrm{chn}} \end{array} .$$

Since both of these pullbacks are homotopy pullbacks by the above discussion, the induced morphism between the pullbacks is also a weak equivalence. □

4.4.15 Maurer-Cartan forms and curvature characteristic forms

We discuss the universal curvature forms, 3.9.5, in Smooth∞Grpd.

Specifically, we discuss the canonical Maurer-Cartan form on the following special cases of (presentations of) smooth ∞-groups.

- 4.4.15.1 – ordinary Lie groups:
- 4.4.15.2 – circle n-groups $\mathbf{B}^{n-1}U(1)$;
- 4.4.15.3 – simplicial Lie groups.

Notice that, by the discussion in 2.2.6, the case of simplicial Lie groups also subsumes the case of crossed modules of Lie groups, def. 1.2.45, and generally of crossed complexes of Lie groups, def. 1.2.60.

4.4.15.1 Canonical form on an ordinary Lie group

Proposition 4.4.69. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

Under the identification

$$\mathrm{Smooth}\infty\mathrm{Grpd}(X, \flat_{\mathrm{dR}}\mathbf{B}G) \simeq \Omega^1_{\mathrm{flat}}(X, \mathfrak{g})$$

from prop. 4.4.44, for $X \in \mathrm{SmoothMfd}$, we have that the canonical morphism

$$\theta : G \to \flat_{\mathrm{dR}}\mathbf{B}G$$

in Smooth∞Grpd *corresponds to the ordinary Maurer-Cartan form on G.*

Proof. We compute the defining double ∞-pullback

$$\begin{array}{ccc} G & \longrightarrow & * \\ \downarrow^{\theta} & & \downarrow \\ \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & \flat\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

in Smooth∞Grpd as a homotopy pullback in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]_{\mathrm{proj}}$. In prop. 4.4.44 we already modeled the lower ∞-pullback square by the ordinary pullback

$$\begin{array}{ccc} \flat_{\mathrm{dR}}\mathbf{B}G_{\mathrm{ch}} & \longrightarrow & \flat\mathbf{B}G_{\mathrm{ch}} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G_{\mathrm{ch}} \end{array} .$$

A standard fibration replacement of the point inclusion $* \to \flat \mathbf{B}G$ is given by replacing the point by the presheaf that assigns groupoids of the form

$$Q : U \mapsto \left\{ \begin{array}{ccc} & A_0 = 0 & \\ {}^{g_1}\swarrow & & \searrow^{g_2} \\ A_1 & \xrightarrow{h} & A_2 \end{array} \right\},$$

where on the right the commuting triangle is in $(\flat_{\mathrm{dR}}\mathbf{B}G_{\mathrm{ch}})(U)$ and here regarded as a morphism from (g_1, A_1) to (g_2, A_2). And the fibration $Q \to \flat \mathbf{B}G_{\mathrm{ch}}$ is given by projecting out the base of these triangles.

The pullback of this along $\flat_{\mathrm{dR}}\mathbf{B}G_{\mathrm{ch}} \to \flat \mathbf{B}G_{\mathrm{ch}}$ is over each U the restriction of the groupoid $Q(U)$ to its set of objects, hence is the sheaf

$$U \mapsto \left\{ \begin{array}{c} A_0 = 0 \\ \downarrow^{g} \\ g^*\theta \end{array} \right\} \simeq C^\infty(U, G) = G(U)\,,$$

equipped with the projection

$$t_U : G \to \flat_{\mathrm{dR}}\mathbf{B}G_{\mathrm{ch}}$$

given by

$$t_U : (g : U \to G) \mapsto g^*\theta\,.$$

Under the Yoneda lemma (over SmoothMfd) this identifies the morphism t with the Maurer-Cartan form $\theta \in \Omega^1_{\mathrm{flat}}(G, \mathfrak{g})$. □

4.4.15.2 Canonical form on the circle n-group We consider now the canonical differential form on the circle Lie $(n+1)$-group, def. 4.4.21. Below in 4.4.16 this serves as the *universal curvature class* on the universal circle n-bundle.

Definition 4.4.70. For $n \in \mathbb{N}$, write

$$\mathbf{B}^n U(1)_{\mathrm{diff,chn}} := \mathrm{DK}\left(\begin{array}{ccccccccc} & & U(1) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1 & \longrightarrow \cdots \longrightarrow & \Omega^{n-1} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n \\ & \nearrow & \oplus & \nearrow^{-\mathrm{id}} & \oplus & & \oplus & \nearrow^{(-1)^n \mathrm{id}} & \\ 0 & \longrightarrow & \Omega^1 & \xrightarrow[d_{\mathrm{dR}}]{} & \Omega^2 & \longrightarrow \cdots \xrightarrow[d_{\mathrm{dR}}]{} & \Omega^n & & \end{array} \right) \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$$

for the simplicial presheaf which is the image under the Dold-Kan map, prop. 2.2.31, of the chain complex on the right as induicated. (Here we display morphisms between direct sums of presheaves of chain complexes by their matrix components, as usual). Write moreover

$$\mathrm{curv}_{\mathrm{chn}} : \mathbf{B}^n U(1)_{\mathrm{diff,chn}} \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)_{\mathrm{chn}}$$

for the morphism of simplicial presheaves which is the image under the Dold-Kan map, prop. 2.2.31 of the morphism of sheaves of chain complexes which in components is given by

$$
\begin{array}{c}
\mathbf{B}^n U(1)_{\mathrm{diff,chn}} \\
\downarrow^{\mathrm{curv}_{\mathrm{chn}}} \\
\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)_{\mathrm{chn}}
\end{array}
\;:=\; \mathrm{DK}\left(
\begin{array}{ccccccccc}
 & & U(1) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1 & \to \cdots \to & \Omega^{n-1} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n \\
 & \nearrow & \oplus & \nearrow^{-\mathrm{id}} & \oplus & & \oplus & \nearrow^{(-1)^n \mathrm{id}} & \\
0 & \to & \Omega^1 & \xrightarrow{d_{\mathrm{dR}}} & \Omega^2 & \to \cdots \xrightarrow{d_{\mathrm{dR}}} & \Omega^n & & \downarrow^{d_{\mathrm{dR}}} \\
\downarrow & & \downarrow^{(-1)^n\mathrm{id}} & & \downarrow^{(-1)^n\mathrm{id}} & & \downarrow^{(-1)^n\mathrm{id}} & & \\
0 & \to & \Omega^1 & \xrightarrow{d_{\mathrm{dR}}} & \Omega^2 & \to \cdots \to & \Omega^n & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{n+1}_{\mathrm{cl}}
\end{array}
\right)
$$

Proposition 4.4.71. *The evident projection morphism*

$$\mathbf{B}^n U(1)_{\mathrm{diff,chn}} \xrightarrow{\simeq} \mathbf{B}^n U(1)_{\mathrm{chn}}$$

is a weak equivalence in $[\mathrm{CartSp}, \mathrm{sSet}]_{\mathrm{proj}}$. *Moreover, the span*

$$
\begin{array}{ccc}
\mathbf{B}^n U(1)_{\mathrm{diff,chn}} & \xrightarrow{\mathrm{curv}_{\mathrm{chn}}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)_{\mathrm{chn}} \\
\downarrow^{\simeq} & & \\
\mathbf{B}^n U(1)_{\mathrm{chn}} & &
\end{array}
$$

is a presentation in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *of the universal curvature characteristic, def. 3.9.32,* $\mathrm{curv} : \mathbf{B}^n U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ *in* Smooth∞Grpd*.*

Proof. By prop. 2.3.13 we may present the defining ∞-pullback

$$
\begin{array}{ccc}
\mathbf{B}^n U(1) & \longrightarrow & * \\
\downarrow^{\mathrm{curv}} & & \downarrow \\
\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) & \longrightarrow & \flat\mathbf{B}^{n+1}U(1) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{B}^{n+1}U(1)
\end{array}
$$

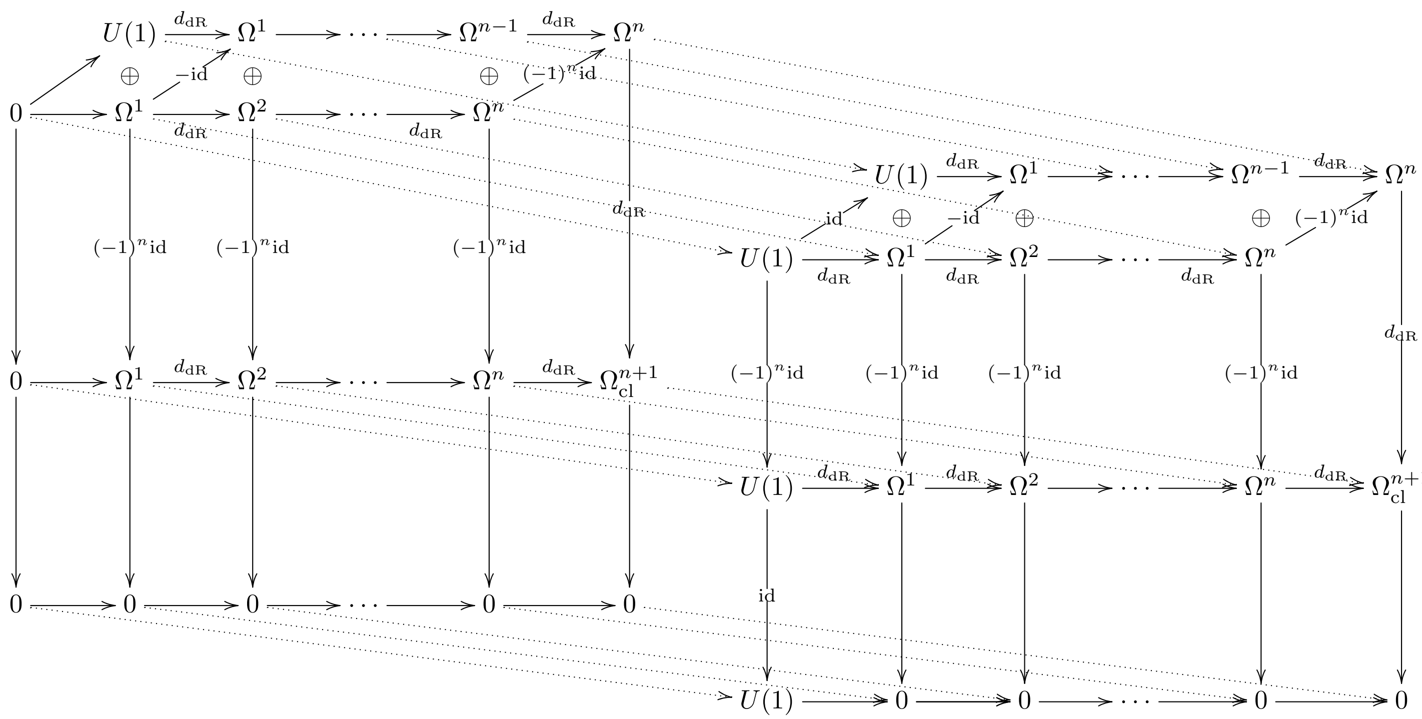
in Smooth∞Grpd by a homotopy pullback in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. We claim that there is a commuting diagram

$$
\begin{array}{ccc}
[0 \to {\scriptstyle C^\infty(-,U(1)) \atop \oplus\Omega^1(-)} \overset{d_{\mathrm{dR}}-\mathrm{Id}}{\to} {\scriptstyle \Omega^1(-) \atop \oplus\Omega^2(-)} \overset{d_{\mathrm{dR}}+\mathrm{Id}}{\to} \cdots \overset{d_{\mathrm{dR}}+\mathrm{Id}}{\to} \Omega^n(-)] & \longrightarrow & [C^\infty(-,U(1)) \overset{d_{\mathrm{dR}}+\mathrm{Id}}{\to} {\scriptstyle C^\infty(-,U(1)) \atop \oplus\Omega^1(-)} \overset{d_{\mathrm{dR}}-\mathrm{Id}}{\to} \cdots {\scriptstyle \Omega^{n-1}(-) \atop \oplus\Omega^n(-)} \overset{d_{\mathrm{dR}}+\mathrm{Id}}{\to} \Omega^n(-)] \\
\downarrow^{(p_2, p_2, \cdots, d_{\mathrm{dR}})} & & \downarrow^{(\mathrm{Id}, p_2, p_2, \cdots, p_2, d_{\mathrm{dR}})} \\
[0 \to \Omega^1(-) \overset{d_{\mathrm{dR}}}{\to} \Omega^2(-) \overset{d_{\mathrm{dR}}}{\to} \cdots \overset{d_{\mathrm{dR}}}{\to} \Omega^{n+1}_{\mathrm{cl}}(-)] & \longrightarrow & [C^\infty(-,U(1)) \overset{d_{\mathrm{dR}}}{\to} \Omega^1(-) \overset{d_{\mathrm{dR}}}{\to} \Omega^2(-) \overset{d_{\mathrm{dR}}}{\to} \cdots \overset{d_{\mathrm{dR}}}{\to} \Omega^{n+1}_{\mathrm{cl}}(-)] \\
\downarrow & & \downarrow \\
[0 \to 0 \to 0 \to \cdots \to 0] & \longrightarrow & [C^\infty(-,U(1)) \to 0 \to 0 \to \cdots \to 0]
\end{array}
$$

in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Ch}^+]_{\mathrm{proj}}$, where

- the objects are fibrant models for the corresponding objects in the above ∞-pullback diagram;
- the two right vertical morphisms are fibrations;
- the two squares are pullback squares.

This implies that under the right adjoint Ξ we have a homotopy pullback as claimed. In full detail, the diagram of morphisms of sheaves that exhibits this diagram of morphisms of complexes of sheaves is



That the lower square here is a pullback is prop. 4.4.46. For the upper square the same type of reasoning applies. The main point is to find the chain complex in the top right such that it is a resolution of the point and maps by a fibration onto our model for $\flat \mathbf{B}^n U(1)$. This is the mapping cone of the identity on the Deligne complex, as indicated. The vertical morphism out of it is manifestly surjective (by the Poincaré lemma applied to each object $U \in \mathrm{CartSp}$) hence this is a fibration. □

In prop. 4.4.67 we had discussed an alternative equivalent presentation of de Rham coefficient objects above. We now formulate the curvature characteristic in this alternative form.

Observation 4.4.72. We may write the simplicial presheaf $\flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}}$ from prop.4.4.67 equivalently as follows

$$\flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}} : (U,[k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^k) & \longleftarrow & 0 \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \longleftarrow & \mathrm{CE}(b^n \mathbb{R}) \end{array} \right\},$$

where on the right we have the set of commuting diagrams in dgAlg of the given form, with the vertical morphisms being the canonical projections.

Definition 4.4.73. Write $\mathrm{W}(b^{n-1}\mathbb{R}) \in \mathrm{dgAlg}$ for the Weil algebra of the line Lie n-algebra, defined to be free commutative dg-algebra on a single generator in degree n, hence the graded commutative algebra on a generator in degree n and a generator in degree $(n+1)$ equipped with the differential that takes the former to the latter.

We write also $\mathrm{inn}(b^{n-1})$ for the L_∞-algebra corresponding to the Weil algebra

$$\mathrm{CE}(\mathrm{inn}(b^{n-1})) := \mathrm{W}(b^{n-1}\mathbb{R})$$

Observation 4.4.74. We have the following properties of $\mathrm{W}(b^{n-1}\mathbb{R})$

1. There is a canonical natural isomorphism
$$\mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{W}(b^{n-1}\mathbb{R}), \Omega^\bullet(U)) \simeq \Omega^n(U)$$
between dg-algebra homomorphisms $A : \mathrm{W}(b^{n-1}\mathbb{R}) \to \Omega^\bullet(X)$ from the Weil algebra of $b^{n-1}\mathbb{R}$ to the de Rham complex and degree-n differential forms, not necessarily closed.

2. There is a canonical dg-algebra homomorphism $\mathrm{W}(b^{n-1}\mathbb{R}) \to \mathrm{CE}(b^{n-1}\mathbb{R})$ and the differential n-form corresponding to A factors through this morphism precisely if the curvature $d_{\mathrm{dR}}A$ of A vanishes.

3. The image under $\exp(-)$
$$\exp(\mathrm{inn}(b^{n-1})\mathbb{R}) \to \exp(b^n\mathbb{R})$$
of the canonical morphism $\mathrm{W}(b^{n-1}\mathbb{R}) \leftarrow \mathrm{CE}(b^n\mathbb{R})$ is a fibration in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ that presents the point inclusion $* \to \mathbf{B}^{n+1}\mathbb{R}$ in Smooth∞Grpd.

Definition 4.4.75. Let $\mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$ be the simplicial presheaf defined by

$$\mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} : (U,[k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(b^{n-1}\mathbb{R}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & \mathrm{W}(b^{n-1}\mathbb{R}) \end{array} \right\},$$

where on the right we have the set of commuting diagrams in dgAlg as indicated.

This means that an element of $\mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}}(U)[k]$ is a smooth n-form A (with sitting instants) on $U \times \Delta^k$ such that its curvature $(n+1)$-form dA vanishes when restricted in all arguments to vector fields tangent to Δ^k. We may write this condition as $d_{\mathrm{dR}}A \in \Omega^{\bullet \geq 1, \bullet}_{\mathrm{si}}(U \times \Delta^k)$.

Observation 4.4.76. There are canonical morphisms

$$\begin{array}{ccc} \mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} & \xrightarrow{\mathrm{curv}_{\mathrm{smp}}} & \flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{R}_{\mathrm{smp}} \\ \downarrow \simeq & & \\ \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} & & \end{array}$$

in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$, where the vertical map is given by remembering only the top horizontal morphism in the above square diagram, and the horizontal morphism is given by forming the pasting composite

$$\mathrm{curv}_{\mathrm{smp}} : \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(b^{n-1}\mathbb{R}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & \mathrm{W}(b^{n-1}\mathbb{R}) \end{array} \right\}$$

$$\mapsto \left\{ \begin{array}{ccccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(b^{n-1}\mathbb{R}) & \longleftarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & \mathrm{W}(b^{n-1}\mathbb{R}) & \longleftarrow & \mathrm{CE}(b^n\mathbb{R}) \end{array} \right\}.$$

Proposition 4.4.77. *This span is a presentation in* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$ *of the universal curvature characteristics* $\mathrm{curv} : \mathbf{B}^n\mathbb{R} \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}$*, def. 3.9.32, in* $\mathrm{Smooth}\infty\mathrm{Grpd}$.

Proof. We need to produce a fibration resolution of the point inclusion $* \to \flat\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}}$ in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and then show that the above is the ordinary pullback of this along $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}} \to \flat\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}}$.

We claim that this is achieved by the morphism

$$(U,[k]) : \{\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \leftarrow \mathrm{W}(b^{n-1}\mathbb{R})\} \mapsto \{\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \leftarrow \mathrm{W}(b^{n-1}\mathbb{R}) \leftarrow \mathrm{CE}(b^n\mathbb{R})\}\,.$$

Here the simplicial presheaf on the left is that which assigns the set of arbitrary n-forms (with sitting instants but not necessarily closed) on $U \times \Delta^k$ and the map is simply given by sending such an n-form A to the $(n+1)$-form $d_{\mathrm{dR}}A$.

It is evident that the simplicial presheaf on the left resolves the point: since there is no condition on the forms every form on $U \times \Delta^k$ is in the image of the map of the normalized chain complex of a form on $U \times \Delta^{k+1}$: such is given by any form that is, up to a sign, equal to the given form on one n-face and 0 on all the other faces. Clearly such forms exist.

Moreover, this morphism is a fibration in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, for instanxce because its image under the normalized chains complex functor is a degreewise surjection, by the Poincaré lemma.

Now we observe that we have over each $(U,[k])$ a double pullback diagram in Set

$$\begin{array}{ccc}
\left\{\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(b^{n-1}\mathbb{R}) \\ \uparrow && \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & W(b^{n-1}\mathbb{R}) \end{array}\right\} & \to & \left\{\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \longleftarrow & \mathrm{W}(b^{n-1}\mathbb{R}) \\ \uparrow && \uparrow^{\mathrm{id}} \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \longleftarrow & \mathrm{W}(b^{n-1}\mathbb{R}) \end{array}\right\} \\
\downarrow && \downarrow \\
\left\{\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \longleftarrow & 0 \\ \uparrow && \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \longleftarrow & \mathrm{CE}(b^n\mathbb{R}) \end{array}\right\} & \to & \left\{\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \longleftarrow & \mathrm{CE}(b^n\mathbb{R}) \\ \uparrow && \uparrow^{\mathrm{id}} \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \longleftarrow & \mathrm{CE}(b^n\mathbb{R}) \end{array}\right\} \\
\downarrow && \downarrow \\
\left\{\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \longleftarrow & 0 \\ \uparrow && \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \longleftarrow & 0 \end{array}\right\} & \to & \left\{\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^k) & \longleftarrow & \mathrm{CE}(b^n\mathbb{R}) \\ \uparrow && \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \longleftarrow & 0 \end{array}\right\}
\end{array}\,,$$

hence a corresponding pullback diagram of simplicial presheaves, that we claim is a presentation for the defining double ∞-pullback for curv.

The bottom square is the one we already discussed for the de Rham coefficients. Since the top right vertical morphism is a fibration, also the top square is a homotopy pullback and hence exhibits the defining ∞-pullback for curv. □

Corollary 4.4.78. *The degreewise map*

$$(-1)^{\bullet+1}\int_{\Delta^\bullet} : \mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} \to \mathbf{B}^n\mathbb{R}_{\mathrm{diff,chn}}$$

that sends an n-form $A \in \Omega^n(U \times \Delta^k)$ and its curvature dA to $(-1)^{k+1}$ times its fiber integration $(\int_{\Delta^k} A, \int_{\Delta^k} dA)$ is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. Since under homotopy pullbacks a weak equivalence of diagrams is sent to a weak equivalence. See the analagous argument in the proof of prop. 4.4.68. □

4.4.15.3 Canonical form on a simplicial Lie group

Above we discussed the canonical differential form on smooth ∞-groups G for the special cases where G is a Lie group and where G is a circle Lie n-group. These are both in turn special cases of the situation where G is a *simplicial Lie group*. This we discuss now.

Proposition 4.4.79. *For G a simplicial Lie group the flat de Rham coefficient object $\flat_{\mathrm{dR}}\mathbf{B}G$ is presented by the simplicial presheaf which in degree k is given by $\Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_k)$, where $\mathfrak{g}_k = \mathrm{Lie}(G_k)$ is the Lie algebra of G_k.*

Proof. Let

$$\Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_\bullet)//G_\bullet = \left(\Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_\bullet) \times C^\infty(-, G_\bullet) \rightrightarrows \Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_\bullet)\right)$$

be the presheaf of simplicial groupoids which in degree k is the groupoid of Lie-algebra valued forms with values in G_k from theorem. 1.2.78. As in the proof of prop. 4.4.44 we have that under the degreewise nerve this is a degreewise fibrant resolution of presheaves of bisimplicial sets

$$N\left(\Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_\bullet)//G_\bullet\right) \to N * //G_\bullet = NB(G_{\mathrm{disc}})_\bullet$$

of the standard presentation of the delooping of the discrete group underlying G. By basic properties of bisimplicial sets [GoJa99] we know that under taking the diagonal

$$\mathrm{diag} : \mathrm{sSet}^\Delta \to \mathrm{sSet}$$

the object on the right is a presentation for $\flat_{\mathrm{dR}}\mathbf{B}G$, because (see the discussion of simplicial groups around prop. 3.6.134)

$$\mathrm{diag} NB(G_{\mathrm{disc}})_\bullet \overset{\simeq}{\rightrightarrows} \bar{W}(G_{\mathrm{disc}}) \simeq \flat \mathbf{B}G\,.$$

Now observe that the morphism

$$\mathrm{diag}(N\Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_\bullet)//G_\bullet) \to \mathrm{diag} N * //G_{\mathrm{disc}}$$

is a fibration in the global model structure. This is in fact true for every morphism of the form

$$\mathrm{diag} N(S_\bullet//G_\bullet) \to \mathrm{diag} * //G_\bullet$$

for $S_\bullet//G_\bullet \to *//G_\bullet$ a simlicial action groupoid projection with G a simplicial group acting on a Kan complex S: we have that

$$(\mathrm{diag} N(S//G))_k = S_k \times (G_k)^{\times_k}\,.$$

On the second factor the horn filling condition is simply that of the identity map $\mathrm{diag} NBG \to \mathrm{diag} NBG$ which is evidently solvable, whereas on the first factor it amounts to $S \to *$ being a Kan fibration, hence to S being Kan fibrant.

But the simplicial presheaf $\Omega^1_{\mathrm{flat}}(-, \mathfrak{g}_\bullet)$ is indeed Kan fibrant: for a given $U \in \mathrm{CartSp}$ we may use parallel transport to (non-canonically) identify

$$\Omega^1_{\mathrm{flat}}(U, \mathfrak{g}_k) \simeq \mathrm{SmoothMfd}_*(U, G_k)\,,$$

where on the right we have smooth functions that send the origin of U to the neutral element. But since $G_\bullet$ is Kan fibrant and has smooth global fillers also $\mathrm{SmoothMfd}_*(U, G_\bullet)$ is Kan fibrant.

In summary this means that the defining homotopy pullback

$$\flat_{\mathrm{dR}}\mathbf{B}G := \flat\mathbf{B}G \times_{\mathbf{B}G} *$$

is presented by the ordinary pullback of simplicial presheaves

$$\mathrm{diag}N\Omega^1_{flat}(-,\mathfrak{g}_\bullet) \times \mathrm{diag}NBG_\bullet * = \Omega^1(-,\mathfrak{g}_\bullet)\,.$$

□

Proposition 4.4.80. *For G a simplicial Lie group the canonical differential form, def. 3.9.29,*

$$\theta : G \to \flat_{\mathrm{dR}}\mathbf{B}G$$

is presented in terms of the above presentation for $\flat_{\mathrm{dR}}\mathbf{B}G$ by the morphism of simplicial presheaves

$$\theta_\bullet : G_\bullet \to \Omega^1_{\mathrm{flat}}(-,\mathfrak{g}_\bullet)$$

which is in degree k the presheaf-incarnation of the Maurer-Cartan form of the ordinary Lie group G_k as in prop. 4.4.69.

Proof. Continuing with the strategy of the previous proof we find a fibration resolution of the point inclusion $* \to \flat\mathbf{B}G$ by applying the construction of the proof of prop. 4.4.69 degreewise and then applying $\mathrm{diag} \circ N$.

The defining homotopy pullback

$$\begin{array}{ccc} G & \longrightarrow & * \\ \downarrow & & \downarrow \\ \flat_{\mathrm{dR}} & \longrightarrow & \flat\mathbf{B}G \end{array}$$

for θ is this way presented by the ordinary pullback

$$\begin{array}{ccc} G_\bullet & \longrightarrow & \mathrm{diag}N(\Omega^1_{\mathrm{flat}}(-,\mathfrak{g}_\bullet))_{\mathrm{triv}}//G_\bullet) \\ \downarrow & & \downarrow \\ \Omega^1_{\mathrm{flat}}(-,\mathfrak{g}_\bullet) & \longrightarrow & \mathrm{diag}N(\Omega^1_{\mathrm{flat}}(-,\mathfrak{g}_\bullet)//G_\bullet) \end{array}$$

of simplicial presheaves, where $\Omega^1_{\mathrm{flat}}(-,\mathfrak{g}_k)$ is the set of flat $\mathfrak{g}$-valued forms A equipped with a gauge transformation $0 \stackrel{g}{\to} A$. As in the above proof one finds that the right vertical morphism is a fibration, hence indeed a resolution of the point inclusion. The pullback is degreewise that from the case of ordinary Lie groups and thus the result follows. □

We can now give a simplicial description of the canonical curvature form $\theta : \mathbf{B}^n U(1) \to \flat_{dR}\mathbf{B}^{n+1}U(1)$ that above in prop. 4.4.71 we obtained by a chain complex model:

Example 4.4.81. The canonical form on the circle Lie n-group

$$\theta : \mathbf{B}^{n-1}U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^n U(1)$$

is presented by the simplicial map

$$\Xi(U(1)[n-1]) \to \Xi(\Omega^1_{cl}(-)[n-1])$$

which is simply the Maurer-Cartan form on $U(1)$ in degree n.

The equivalence to the model we obtained before is given by noticing the equivalence in hypercohomology of chain complexes of abelian sheaves

$$\Omega^1_{\mathrm{cl}}(-)[n] \simeq (\Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \stackrel{d_{\mathrm{dR}}}{\to} \Omega^n_{\mathrm{cl}}(-))$$

on CartSp.

4.4.16 Differential cohomology

We discuss the intrinsic differential cohomology, defined in 3.9.6 for any cohesive ∞-topos, realized in the context Smooth∞Grpd, with coefficients in the circle Lie $(n+1)$-group $\mathbf{B}^n U(1)$, def. 4.4.21.

We show that here the general concept reproduces the Deligne-Beilinson complex, 1.2.102, and generalizes it to a complex for equivariant differential cohomology for ordinary and twisted notions of equivariance.

- 4.4.16.1 – The n-groupoid of circle-principal n-connections;
- 4.4.16.2 – The universal moduli n-stack of circle-principal n-connections;
- 4.4.16.4 – Equivariant circle n-bundles with connection;

4.4.16.1 The smooth n-groupoid of circle-principal n-connections Here we discuss some basic facts about differential cohomology with coefficients in the circle n-group, def. 4.3.48, that are independent of a notion of manifolds and global differential form objects as in 3.9.6.2. Further below in 4.4.16.2 we do consider these structures and show that $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is presented by the Deligne complex.

Here we discuss first that intrinsic differential cohomology in Smooth∞Grpd has the abstract properties of traditional ordinary differential cohomology, [HoSi05], then we establish that both notions indeed coincide in cohomology. The intrinsic definition refines this ordinary differential cohomology to moduli ∞-stacks.

By def. 3.9.37 we are to consider the ∞-pullback

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n U(1)) & \longrightarrow & H_{\mathrm{dR}}(X, \mathbf{B}^{n+1} U(1)) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n U(1)) & \xrightarrow{\mathrm{curv}} & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1} U(1)) \end{array} ,$$

where the right vertical morphism picks one point in each connected component. Moreover, using prop. 4.4.46 in def. 3.9.42, we are entitled to the following bigger object.

Definition 4.4.82. For $n \in \mathbb{N}$ write $\mathbf{B}^n U(1)_{\mathrm{conn}}$ for the ∞-pullback

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1) & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1) \end{array}$$

in Smooth∞Grpd. The cocycle ∞-groupoid over some $X \in$ Smooth∞Grpd with coefficients in $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is the ∞-pullback

$$\mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \quad \simeq \quad \begin{array}{ccc} \mathbf{H}'_{\mathrm{diff}}(X, \mathbf{B}^n U(1)) & \xrightarrow{F} & \Omega^{n+1}_{\mathrm{cl}}(X) \\ \downarrow{\scriptstyle \mathbf{c}} & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n U(1)) & \xrightarrow{\mathrm{curv}} & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^{n+1} U(1)) \end{array} .$$

We call $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n U(1))$ and its primed version the cocycle ∞-groupoid for *ordinary smooth differential cohomology* in degree n .

Proposition 4.4.83. *For $n \geq 1$ and $X \in$ SmoothMfd, the abelian group $H'^n_{\mathrm{diff}}(X)$ sits in the following short exact sequences of abelian groups*

- *the* curvature exact sequence

$$0 \to H^n(X, U(1)_{\mathrm{disc}}) \to {H'}^n_{\mathrm{diff}}(X, U(1)) \xrightarrow{F} \Omega^{n+1}_{\mathrm{cl,int}}(X) \to 0$$

- *the* characteristic class exact sequence

$$0 \to \Omega^n_{\mathrm{cl}}/\Omega^n_{\mathrm{cl,int}}(X) \to {H'}^n_{\mathrm{diff}}(X, U(1)) \xrightarrow{\mathbf{c}} H^{n+1}(X, \mathbb{Z}) \to 0\,.$$

Here $\Omega^n_{\mathrm{cl,int}}$ denotes closed forms with integral periods.

Proof. For the curvature exact sequence we invoke prop. 3.9.40, which yields (for H_{diff} as for H'_{diff})

$$0 \to H^n_{\mathrm{flat}}(X, U(1)) \to {H'}^n_{\mathrm{diff}}(X, U(1)) \xrightarrow{F} \Omega^{n+1}_{\mathrm{cl,int}}(X) \to 0\,.$$

The claim then follows by using prop. 4.4.40 to get $H^n_{\mathrm{flat}}(X, U(1)) \simeq H^n(X, U(1)_{\mathrm{disc}})$.

For the characteristic class exact sequence, we have with 3.9.41 for the smaller group H^n_{diff} (the fiber over the vanishing curvature $(n+1)$-form $F = 0$) the sequence

$$0 \to H^n_{\mathrm{dR}}(X)/\Omega^n_{\mathrm{cl,int}}(X) \to {H'}^n_{\mathrm{diff}}(X, U(1)) \xrightarrow{c} H^{n+1}(X, \mathbb{Z}) \to 0$$

where we used prop. 4.4.47 to identify the de Rham cohomology on the left, and the fact that X is paracompact to identify the integral cohomology on the right. Since $\Omega^n_{\mathrm{cl,int}}(X)$ contains the exact forms (with all periods being $0 \in \mathbb{Z}$), the leftmost term is equivalently $\Omega^n_{\mathrm{cl}}(X)//\Omega^n_{\mathrm{cl,int}}(X)$. As we pass from H_{diff} to the bigger H'_{diff}, we get a copy of a torsor over this group, for each closed form F, trivial in de Rham cohomology, to a total of

$$\coprod_{F \in \Omega^{n+1}_{\mathrm{cl}}(X)} \{\omega | d\omega = F\}/\Omega^n_{\mathrm{cl,int}} \simeq \Omega^n(X)/\Omega^n_{\mathrm{cl,int}}(X)\,.$$

This yields the curvature exact sequence as claimed. □

If we invoke standard facts about Deligne cohomology, then prop. 4.4.83 is also implied by the following proposition, which asserts that in Smooth∞Grpd the groups ${H'}^\bullet_{\mathrm{diff}}$ not only share the above abstract properties of ordinary differential cohomology, but indeed coincide with it.

Theorem 4.4.84. *For $X \in$ SmoothMfd $\hookrightarrow$ Smooth∞Grpd a paracompact smooth manifold we have that the connected components of the object $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^n U(1))$ are given by*

$$H^n_{\mathrm{diff}}(X, U(1)) \simeq (\;\; H(X, \mathbb{Z}(n+1)^\infty_D) \;\;) \times_{\Omega^{n+1}_{\mathrm{cl}}(X)} H^{n+1}_{\mathrm{dR,int}}(X)\,.$$

Here on the right we have the subset of Deligne cocycles that picks for each integral de Rham cohomology class of X only one curvature form representative.

For the connected components of $\mathbf{H}'_{\mathrm{diff}}(X, \mathbf{B}^n U(1))$ we get the complete ordinary Deligne cohomology of X in degree $n+1$:

$${H'}^n_{\mathrm{diff}}(X, U(1)) \simeq \;\; H(X, \mathbb{Z}(n+1)^\infty_D)$$

Proof. Choose a differentiably good open cover, def. 4.4.2, $\{U_i \to X\}$ and let $C(\{U_i\}) \to X$ in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ be the corresponding Čech nerve projection, a cofibrant resolution of X.

Since the presentation of prop. 4.4.71 for the universal curvature class $\mathrm{curv}_{\mathrm{chn}} : \mathbf{B}^n U(1)_{\mathrm{diff,chn}} \to \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)_{\mathrm{chn}}$ is a global fibration and $C(\{U_i\})$ is cofibrant, also

$$[\mathrm{Cartp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}^n_{\mathrm{diff}} U(1)) \to [\mathrm{Cartp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \flat_{\mathrm{dR}} \mathbf{B}^n U(1))$$

is a Kan fibration by the fact that $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is an $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category. Therefore the homotopy pullback in question is computed as the ordinary pullback of this morphism.

By prop. 4.4.46 we can assume that the morphism $H^{n+1}_{\mathrm{dR}}(X) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \flat_{\mathrm{dR}}\mathbf{B}^{n+1})$ picks only cocycles represented by globally defined closed differential forms $F \in \Omega^{n+1}_{\mathrm{cl}}(X)$. We see that the elements in the fiber over such a globally defined $(n+1)$-form F are precisely the cocycles with values only in the upper row complex of $\mathbf{B}^n U(1)_{\mathrm{diff,chn}}$

$$C^\infty(-, U(1)) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^1(-) \stackrel{d_{\mathrm{dR}}}{\to} \cdots \stackrel{d_{\mathrm{dR}}}{\to} \Omega^n(-)\,,$$

such that F is the de Rham differential of the last term. This is the Deligne-Beilinson complex, def. 1.2.102, for Deligne cohomology in degree $(n+1)$. □

In terms of def. 3.9.42 we have the object $\mathbf{B}^n U(1)_{\mathrm{conn}}$ – the *moduli n-stack of circle n-bundles with connection* – which presents $\mathbf{H}'_{\mathrm{diff}}(-, \mathbf{B}^n U(1))$

$$\mathbf{H}'_{\mathrm{diff}}(-, \mathbf{B}^n U(1)) \simeq \mathbf{H}(-, \mathbf{B}^n U(1)_{\mathrm{conn}})\,.$$

4.4.16.2 The universal moduli n-stack of circle-principal n-connections

Definition 4.4.85. For $n \in \mathbb{N}$ and $k \leq n$ write

$$\Omega^{k \leq \bullet \leq n}_{\mathrm{cl}} := \mathrm{DK}\left(\Omega^k \xrightarrow{d_{\mathrm{dR}}} \Omega^{k+1} \longrightarrow \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^{n-1} \xrightarrow{d_{\mathrm{dR}}} \Omega^n_{\mathrm{cl}} \right)\,.$$

Write

$$\mathbf{B}^n U(1)_{\mathrm{conn}^k, \mathrm{chn}} := \mathrm{DK}\left(U(1)\Omega^1 \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^k \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \right)$$

for the simplicial presheaf which is the image under the Dold-Kan map of the chain complex concentrated in degrees n through $(n-k)$, as indicated. Notice that

$$\mathbf{B}^n U(1)_{\mathrm{conn}^0, \mathrm{chn}} = \mathbf{B}^n U(1)_{\mathrm{chn}}\,,$$

and we write

$$\mathbf{B}^n U(1)_{\mathrm{conn}, \mathrm{chn}} := \mathbf{B}^n U(1)_{\mathrm{conn}^n, \mathrm{chn}}\,.$$

Proposition 4.4.86. *The object $\mathbf{B}^n U(1)^k_{\mathrm{conn}} \in \mathrm{Smooth}\infty\mathrm{Grpd}$ is presented in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ by $\mathbf{B}^n U(1)_{\mathrm{conn}^k, \mathrm{chn}}$.*

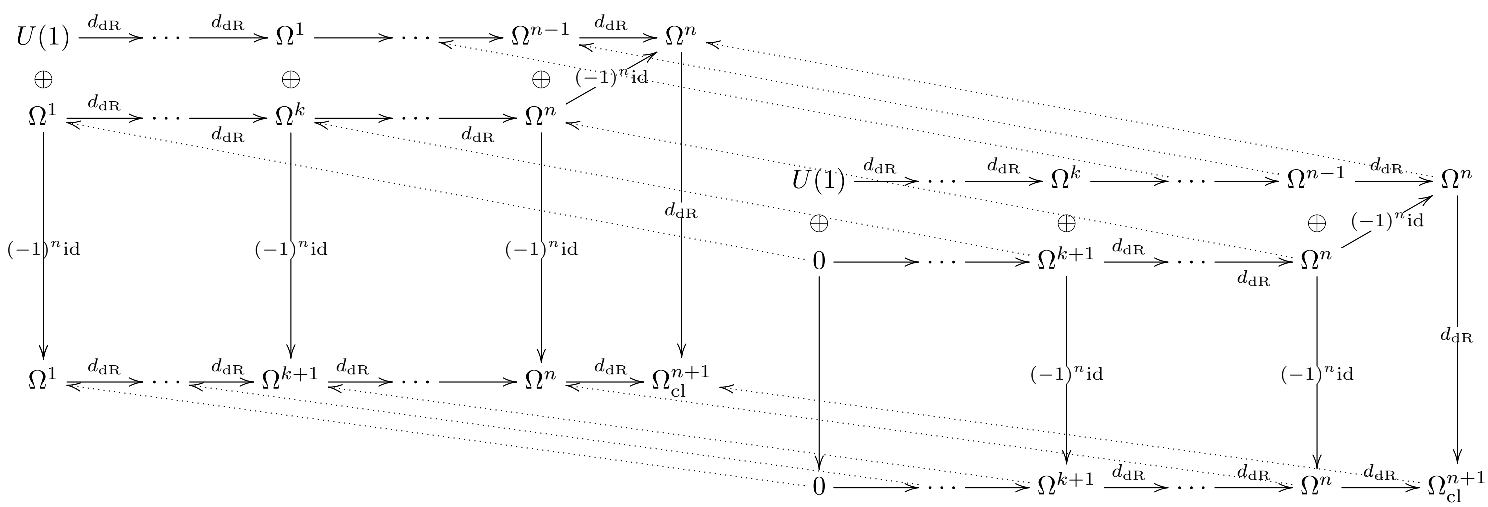
Proof. By prop. 4.4.71 the defining ∞-pullback

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}^k} & \xrightarrow{F_{(-)}} & \Omega^{k \leq \bullet \leq n}_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1) & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1) \end{array}$$

is presented by the homotopy pullback of presheaves of chain complexes

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{diff,chn}} & \longleftarrow & \mathbf{B}^n U(1)_{\mathrm{conn}^k, \mathrm{chn}} \\ \downarrow^{\mathrm{curv}_{\mathrm{chn}}} & & \downarrow \\ \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)_{\mathrm{chn}} & \longleftarrow & \Omega^{k \leq \bullet \leq n}_{\mathrm{cl}} \end{array}$$

(rotated here just for readability in the following) which in components is given as follows



This shows that $\mathbf{B}^n U(1)_{\mathrm{conn}^k}$ is presented by the chain complex appearing on the top right here. The canonical projection morphism from this pullback to $\mathbf{B}^n U(1)_{\mathrm{conn}^k,\mathrm{chn}}$ is clearly a weak equivalence. □

Remark 4.4.87. In particular this means that $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is presented by the Deligne complex

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \simeq \mathrm{DK}\left(U(1) \xrightarrow{d_{\mathrm{dR}}} \Omega^1 \xrightarrow{d_{\mathrm{dR}}} \cdots \longrightarrow \Omega^{n-1} \xrightarrow{d_{\mathrm{dR}}} \Omega^n \right)$$

The above proof of theorem 4.4.84 makes a statement not only about cohomology classes, but about the full moduli n-stacks:

Proposition 4.4.88. *The object $\mathbf{B}^n U(1)_{\mathrm{conn}} \in \mathbf{H}$ from def. 4.4.82 is presented by the simplicial presheaf which is the image under the Dold-Kan map Ξ, def. 2.2.31, of the Deligne complex in the corresponding degree.*

The canonical morphism $\mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ is similarly presented via Dold-Kan of the evident morphism of chain complexes of sheaves

$$\begin{array}{ccccccc} C^\infty(-,U(1)) & \xrightarrow{d_{\mathrm{dR}}\log} & \Omega^1(-) & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n(-) \\ \downarrow^{\mathrm{id}} & & \downarrow & & & & \downarrow \\ C^\infty(-,U(1)) & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \end{array} .$$

Proposition 4.4.89. *The moduli stack $\mathbf{B}U(1)_{\mathrm{conn}}$ of circle bundles (i.e. circle 1-bundles) with connection is 1-concrete, def. 3.7.7.*

Proof. Observing that the presentation by the Deligne complex under the Dold-Kan map is fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ and is the concrete sheaf presented by $U(1)$ in degree 1, this follows with prop. 3.7.8. □

4.4.16.3 The smooth moduli of connections over a given base

We discuss the *moduli stacks* of higher principal connections, over a fixed $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$, following the general abstract discussion in 3.9.6.4.

For $n \in \mathbb{N}$ and with $\mathbf{B}^n U(1)_{\mathrm{conn}} \in \mathrm{Smooth}\infty\mathrm{Grpd}$ the universal moduli stack for circle n-bundles with connection, def. 4.4.70, and for $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$, one may be tempted to regard the internal hom/mapping space $[X, \mathbf{B}^n U(1)_{\mathrm{conn}}]$ as the moduli stack of circle n-bundles with connection on X. However, for $U \in \mathrm{CartSp}$ an abstract coordinate system, U-plots and their k-morphisms in $[X, \mathbf{B}^n U(1)_{\mathrm{conn}}]$ are circle principal n-connections and their k-fold gauge transformations on $U \times X$, and this is not generally what one would want the U-plots of the moduli stack of such connections on X to be. Rather, that moduli stack should have

1. as U-plots smoothly U-parameterized collections $\{\nabla_u\}$ of n-connections on X;
2. as k-morphisms smoothly U-parameterized collections $\{\phi_u\}$ of gauge transformations between them.

The first item is equivalent to: a single n-connection on $U \times X$ such that its local connection n-forms have no legs along U. This is essentially the situation of moduli of differential forms which we have discussed above (...).

But the second item is different: a gauge transformation of a single n-connection ∇ on $U \times X$ needs to respect the curvature of the connection along U, but a family $\{\phi_u\}$ of gauge tranformations between the restrictions $\nabla|_u$ of ∇ to points of the coordinate patch U need not.

In order to capture this correctly, the concretification-process that yields the moduli spaces of differential forms is to be refined to a process that concretifies the higher stack $[X, \mathbf{B}^n U(1)_{\mathrm{conn}}]$ degreewise in stages.

Definition 4.4.90. For $n, k \in \mathbb{N}$ and $k \leq n$ write $\mathbf{B}^n U(1)_{\mathrm{conn}^k}$ for the ∞-pullback in

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}^k} & \longrightarrow & \Omega^{n+1 \leq \bullet \leq k}_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1) & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1) \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1) \end{array} .$$

By the universal property of the ∞-pullback, the canonical tower of morphisms

$$\Omega^{n+1}_{\mathrm{cl}} \longrightarrow \Omega^{n+1 \leq \bullet \leq n}_{\mathrm{cl}} \longrightarrow \cdots \longrightarrow \Omega^{n+1 \leq \bullet \leq 1}_{\mathrm{cl}} \xrightarrow{\simeq} \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)$$

induces a tower of morphisms

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \xrightarrow{\simeq} \mathbf{B}^n U(1)_{\mathrm{conn}^n} \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}^{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}^0} \xrightarrow{\simeq} \mathbf{B}^n U(1) \;.$$

Proposition 4.4.91. *We have*

$$\mathbf{B}^n U(1)_{\mathrm{conn}^k} \simeq \mathrm{DK} \left(U(1) \xrightarrow{d_{\mathrm{dR}}} \Omega^1 \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^k \xrightarrow{d_{\mathrm{dR}}} 0 \longrightarrow \cdots \longrightarrow 0 \right)$$

where the chain complex on the right is concentrated in degrees n through $n-k$. Under this equivalence the canonical morphism $\mathbf{B}^n U(1)_{\mathrm{conn}^{k+1}} \to \mathbf{B}^n U(1)_{\mathrm{conn}^k}$ is equivalent to the image under DK *to the chain map*

$$\begin{array}{ccccccccccccccc} U(1) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1 & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{k+1} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^k & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U(1) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1 & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{k+1} & \xrightarrow{d_{\mathrm{dR}}} & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \end{array}$$

Proof. By the presentation of curv as in prop. 4.4.71. □

Definition 4.4.92. For $X \in \mathbf{H}$ and $n \in \mathbb{N}$, $n \geq 1$, the *moduli of circle-principal n-connections* on X is the iterated ∞-fiber product

$$
\begin{aligned}
&(\mathbf{B}^{n-1}U(1))\mathbf{Conn}(X) \\
&:= \sharp_1[X, \mathbf{B}^n U(1)_{\mathrm{conn}^n}] \underset{\sharp_1[X,\mathbf{B}^n U(1)_{\mathrm{conn}^{n-1}}]}{\times} \sharp_2[X, \mathbf{B}^n U(1)_{\mathrm{conn}^{n-1}}] \underset{\sharp_2[X,\mathbf{B}^n U(1)_{\mathrm{conn}^{n-2}}]}{\times} \cdots \underset{\sharp_n[X,\mathbf{B}^n U(1)_{\mathrm{conn}^0}]}{\times} [X, \mathbf{B}^n U(1)_{\mathrm{conn}^0}],
\end{aligned}
$$

of the morphisms

$$\sharp_k[X, \mathbf{B}^n U(1)_{\mathrm{conn}^{n-k+1}}] \longrightarrow \sharp_k[X, \mathbf{B}^n U(1)_{\mathrm{conn}^{n-k}}]$$

which are the image under $\sharp_k$, def. 3.7.6, of the image under the internal hom $[X, -]$ of the canonical projections of prop. 4.4.90, and of the morphisms

$$\sharp_{k+1}[X, \mathbf{B}^n U(1)_{\mathrm{conn}^{n-k}}] \longrightarrow \sharp_k[X, \mathbf{B}^n U(1)_{\mathrm{conn}^{n-k}}]$$

of def. 3.7.6.

4.4.16.3.1 Moduli of smooth principal 1-connections

We discuss the general notion of moduli of G-principal connections, def. 3.9.50 for the special case that G is a 0-truncated group.

For $G = U(1)$ the circle group, the special case of def. 4.4.92 is the following.

Definition 4.4.93. For $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$, the *moduli of circle-principal connections* is given by the ∞-pullback

$$
\begin{array}{ccccc}
U(1)\mathbf{Conn}(X) & \longrightarrow & \sharp_2[X, \mathbf{B}U(1)] & \simeq & [X, \mathbf{B}U(1)] \,, \\
\downarrow & & \downarrow & & \\
\sharp_1[X, \mathbf{B}U(1)_{\mathrm{conn}}] & \xrightarrow{\sharp_1[X, U_{\mathbf{B}U(1)}]} & \sharp_1[X, \mathbf{B}U(1)] & &
\end{array}
$$

where $U_{\mathbf{B}U(1)} : \mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)$ is the canonical forgetful morphism.

Of course we have the analogous construction for G any Lie group:

Definition 4.4.94. For $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$, the *moduli of circle-principal connections* is given by the ∞-pullback

$$
\begin{array}{ccccc}
G\mathbf{Conn}(X) & \longrightarrow & \sharp_2[X, \mathbf{B}G] & \simeq & [X, \mathbf{B}G] \,, \\
\downarrow & & \downarrow & & \\
\sharp_1[X, \mathbf{B}G_{\mathrm{conn}}] & \xrightarrow{\sharp_1[X, U_{\mathbf{B}G}]} & \sharp_1[X, \mathbf{B}G] & &
\end{array}
$$

where $U_{\mathbf{B}G} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$ is the canonical forgetful morphism.

Proposition 4.4.95. *For $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$, the smooth groupoid $U(1)\mathbf{Conn}(X)$ of def. 4.4.93 is indeed the smooth moduli object/moduli stack of circle-principal connections on X; in that its U-plots of are smoothly U-parameterized collections of smooth circle-principal connections on X and its morphisms of U-plots are smoothly U-parameterized collections of smooth gauge transformation between these, on X.*

Proof.

By the discussion of n-image and using arguments as for the concretification of moduli of differential forms above, we have:

- $\sharp_1[X, \mathbf{B}U(1)_{\mathrm{conn}}]$ has as U-plots smoothly U-parameterized $U(1)$-principal connections on X that have a lift to a $U(1)$-principal connection on $U \times X$, and morphisms are discretely $\Gamma(U)$-parameterized collections of gauge transformations of these connections on X.

- $\sharp_1[X, \mathbf{B}U(1)]$ looks similarly, just without the connection information;

- $\sharp_1[X, U_{\mathbf{B}U(1)_{\mathrm{conn}}}]$ simply forgets the connection data on the collections of bundles-with-connection; the point to notice is that oveach each chart U it is a fibration(isofibration): given a $\Gamma(U)$-parameterized collection of gauge transformations out of a smoothly U-parameterized collection of bundles and then a smooth choice of smooth connections on these bundles, the $\Gamma(U)$ collection of gauge transformations of course also acts on these connections;

- $\sharp_2[X, \mathbf{B}U(1)] \simeq [X, \mathbf{B}U(1)]$ (because if two gauge transformations of bundles on $U \times X$ coincide on each point of U as gauge tranformations on X, then they were already equal).

From the third item it follows that we may compute equivalently simply the pullback in the 1-category of groupoid-valued presheaves on CartSp. This means that a U-plot of the pullback is a smoothly U-parameterized collection $\{\nabla_u\}$ of $U(1)$-principal connections on X which admits a lift to a $U(1)$-principal connection on $U \times X$, and that a morphism between such as a $\Gamma(U)$-parameterized collection of gauge transformations $\{\phi_u\}$ of connections, such that their underlying collection of gauge transformations of bundles is a smoothly U-parameterized family. But gauge transformations of 1-connections are entirely determined by the underlying gauge transformation of the underlying bundle, and so this just means that also the morphism of U-plots of the pullback are smoothly U-paramezerized collections of gauge transformations.

Consider then the functor from $U(1)\mathbf{Conn}(X)_U$ to this pullback which forgets the lift to a connection on $U \times X$. This is natural in U and hence to complete the proof we need to see that for each U it is an equivalence of groupoids. By the above it is clearly fully faithful, so it remains to see that it is essentially surjective, hence that every smoothly U-parameterized collection of connections on X comes from a single connection on $X \times U$. To this end, consider a smoothly U-parameterized collection $\{\nabla_u\}_{u \in U}$ of $U(1)$-principal connections on X. Choosing a differentiably good open cover $\{U_i \to X\}$ of X the collection of connections is equivalently given by a collection of cocycle data

$$\{g_{ij}^u \in C^\infty(U_i \cap U_j, U(1)), A_i^u \in \Omega^1(U_i)\}_{u \in U}$$

with $A_j^u = A_i^u + d_X \mathrm{log} g_{ij}^u$ on $U_i \cap U_j$ for all i, j in the index set and all $u \in U$. To see that this is the restriction of a single such cocycle datum on $\{U_i \times U \to X \times U\}$ we use the standard formula for the existence of connections on a given bundle represented by a given cocycle, but applied just to the U-factor. So let $\{\rho_i \in C^\infty(U_i \times U)\}$ be a partition of unity on $X \times U$ subordinate to the chosen cover and define $A_i \in \Omega^1(U_i \times U)$ by

$$A_i(u) := A_i^u + \sum_{i_0} \rho_{i_0} d_U \mathrm{log} g_{i_0 i}(u)$$

for each $u \in U$. This is clearly a lift on each patch, and it does constitute a cocycle for a connection on $X \times U$ since on each $U \times (U_i \cap U_j)$ we have:

$$\begin{aligned} A_j(u) - A_i(u) &= \sum_{i_0} \rho_{i_0} \left(A_j^u + d_U \mathrm{log} g_{i_0 j}(u) - A_i^u - d_U \mathrm{log} g_{i_0 i}(u) \right) \\ &= A_j^u - A_i^u + \sum_{i_0} \rho_{i_0} d_U \mathrm{log}(g_{i i_0}(u) g_{i_0 j}(u)) \\ &= d_X \mathrm{log} g_{ij}(u) + d_U \mathrm{log} g_{ji}(u) \\ &= d \mathrm{log} g_{ij}(u) \end{aligned} .$$

□

Proposition 4.4.96. *For $G \in$ Grp(Smth∞Grp) a 0-truncated group object and for $X \in$ Smth∞Grpd, we have an equivalence*

$$\Omega\left(G\mathbf{Conn}(X)\right) \simeq G$$

in Smooth∞Grpd*, between the loop space object of the moduli object of G def. 4.4.94, and G itself.*

Proof. For X a smooth manifold and G a Lie group, this is straightforward to check by inspection of the stack $\Omega\left(G\mathbf{Conn}(X)\right)$. Its U-plots are the smoothly U-parameterized collections of gauge transformations of the trivial G-principal connection on X. Any such is a constant G-valued function on X, hence an element of G, and so these form the set $C^\infty(U, G)$ of U-plots of G.

Generally, the statement follows abstractly from prop. 3.6.47. By that proposition and using that Ω commutes over ∞-fiber products (since both are ∞-limits) we have

$$\begin{aligned}
\Omega\left(G\mathbf{Conn}(X)\right) &\simeq \Omega\sharp_1[X, \mathbf{B}G_{\mathrm{conn}}] \underset{\Omega\sharp_1[X,\mathbf{B}G]}{\times} \Omega[X, \mathbf{B}G] \\
&\simeq \sharp\Omega[X, \mathbf{B}G_{\mathrm{conn}}] \underset{\sharp\Omega[X,\mathbf{B}G]}{\times} \Omega[X, \mathbf{B}G] \\
&\simeq \sharp[X, \Omega\mathbf{B}G_{\mathrm{conn}}] \underset{\sharp[X,\Omega\mathbf{B}G]}{\times} [X, \Omega\mathbf{B}G] \\
&\simeq \sharp[X, \flat G] \underset{\sharp[X,G]}{\times} [X, G] \\
&\simeq \sharp G \underset{\sharp[X,G]}{\times} [X, G]
\end{aligned} \quad .$$

This last ∞-fiber product is one of 0-truncated object hence is the ordinary fiber products of the corresponding sheaves. The U-plots of the left factor are discretely $\Gamma(U)$-parameterized collections of elements of G, the inclusion of these into $\sharp[X, G]$ is as $\Gamma(U)$-parameterized collections of constant G-valued functions on G, and the right factor picks out among these those that are smoothly parameterized over $X \times U$, hence over U. This is the statement to be shown. □

4.4.16.3.2 Moduli of smooth principal 2-connections

We discuss the general notion of moduli of G-principal connections, def. 3.9.50 for the special case that G is a 1-truncated group.

Proposition 4.4.97. *Given $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$, the moduli 2-stack $(\mathbf{B}U(1))\mathbf{Conn}(X)$ of circle 2-bundles with connection on X, given by the ∞-limit in*

$$\begin{array}{ccccc}
(\mathbf{B}U(1))\mathbf{Conn}(X) & \longrightarrow & & \longrightarrow & [X, \mathbf{B}^2U(1)] \\
\downarrow & & & & \downarrow \\
 & & \sharp_2[X, \mathbf{B}^2U(1)_{\mathrm{conn}^1}] & \longrightarrow & \sharp_2[X, \mathbf{B}^2U(1)] \\
\downarrow & & \downarrow & & \\
\sharp_1[X, \mathbf{B}^2U(1)_{\mathrm{conn}}] & \longrightarrow & \sharp_1[X, \mathbf{B}^2U(1)_{\mathrm{conn}^1}] & &
\end{array}$$

is equivalent to the 2-stack which assigns to any $U \in \mathrm{CartSp}$ the 2-groupoid whose objects, morphisms, and 2-morphisms are smoothly U-parameterized collections of circle-principal connections and their gauge transformations on X.

Proof. By a variant of the pasting law, we may compute the given ∞-limit as the pasting composite of three ∞-pullbacks:

$$\begin{array}{ccccc}
(\mathbf{B}U(1))\mathbf{Conn}(X) & \longrightarrow & \cdot & \longrightarrow & [X, \mathbf{B}^2U(1)] \\
\downarrow & & \downarrow & & \downarrow \\
\cdot & \longrightarrow & \sharp_2[X, \mathbf{B}^2U(1)_{\mathrm{conn}^1}] & \longrightarrow & \sharp_2[X, \mathbf{B}^2U(1)] \\
\downarrow & & \downarrow & & \\
\sharp_1[X, \mathbf{B}^2U(1)_{\mathrm{conn}}] & \longrightarrow & \sharp_1[X, \mathbf{B}^2U(1)_{\mathrm{conn}^1}] & &
\end{array} \quad .$$

Since this is a finite ∞-limit, we may compute it in ∞-presheaves over CartSp, hence as a homotopy pullback in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. For $\{U_i \to X\}_{i\in I}$ any choice of differentiable good open cover of X, our standard model for the mapping stacks appearing in the diagram are given by the Deligne complex, according to prop. 4.4.88. Since this takes values, under the Dold-Kan map, in strict ∞-groupoids, we find the $\sharp$-images by prop. 3.6.53. In this standard presentation all simplicial presheaves appearing in the diagram are fibrant and the two horizontal morphisms are fibrations. Therefore we conclude that the ∞-limit in question is in fact given by the pasting composite of three 1-categorical pullbacks of these presheaves of strict 2-groupoids. Using that pullbacks of presheaves of 2-groupoids are computed objectwise and degreewise, we find that the pullback presheaf is over $U \in \mathrm{CartSp}$ given by the following strict 2-groupoid:

- objects are smoothly U-parameterized collections of Deligne cocycles $\{B_i^u, A_{ij}^u, g_{ijk}^u\}_{i,j,k\in I, u\in\Gamma(U)}$ on X, such that there *exists* a lift to a single cocycle on $X \times U$ (this is the structure of the objects of $\sharp_1[X, \mathbf{B}^2 U(1)_{\mathrm{conn}}]$) and *equipped* with a choice of lift of the restricted $\mathbf{B}U(1)_{\mathrm{conn}^1}$-cocycle $\{0, A_{ij}^u, g_{ijk}^u\}_{u\in U}$ to a single restricted cocycle $\{0, A_{ij}, g_{ijk}\}$ on $U \times X$ (this is the structure of the objects of $\sharp_2[X, \mathbf{B}^2 U(1)_{\mathrm{conn}^1}]$);
- morphisms are smoothly U-parameterized collections of morphisms of cocycles on X such that there exists a lift to a morphism of restricted cocycles on $X \times U$;
- 2-morphisms are smoothly U-parameterized collections of 2-gauge transformations, hence 2-gauge transformations on $X \times U$.

This is almost verbatim the 2-groupoid claimed in the proposition, except for the appearance of the existence and choice of lifts. We need to show that up to equivalence these drop out.

Consider therefore the canonical 2-functor from the 2-groupoid thus described to the one consisting degreewise of genuine smoothly U-parameterized collections of cocycles and transformations, which forgets the lift and the existence of lifts. This 2-functor is clearly natural in U, hence is a morphism of simplicial presheaves. It is now sufficient to show that over each U this is an equivalence of 2-groupoids.

To see that this 2-functor is fully faithful, notice that by the strict abelian group structure on all objects we may restrict to considering the homotopy groups that are based at the 0-cocycle. But the automorphism groupoid of the trivial circle-principal 2-connection is that of flat circle-principal 1-connections. Hence fully faithfulness of this 2-functor amouts to the statement of prop. 4.4.95.

Therefore it remains to check essential surjectivity of the forgetful 2-functor. To this end, observe that the underlying circle-principal 2-bundles of a collection of 2-connections smoothly parameterized by a Cartesian (hence topologically contractible) space necessarily have the same class at all points $u \in U$ and so every object in the pullback 2-groupoid is equivalent to one for which $\{g_{ijk}^u\}$ is in fact independent of u. It is then sufficient to show that any such is in the image of the above forgetful 2-functor.

So consider a smoothly parameterized collection of Deligne cocycles on $\{U_i \to X\}_{i\in I}$ of the form $\{B_i^u, A_{ij}^u, g_{ijk}\}_{u\in U}$. Since now g_{ijk} is constant on U, we can obtain a lift of the 1-form part simply by defining for $i, j \in I$ a 1-form $A_{ij} \in \Omega^1(U \times (U_i \cap U_j))$ by declaring that at $u \in U$ it is given by

$$A_{ij}(u) := A_{ij}^u \,.$$

Next we need to similarly find a lift $\{B_i \in \Omega^2(U \times U_i)\}_{i\in I}$. For that, choose now a partition of unity $\{\rho_i \in C^\infty(U_i)\}_i$ of X, subordinate to the given cover and set

$$B_i(u) := B_i^u + \sum_{i_0} \rho_{i_0} d_U A_{i_0 i}(u) \,.$$

This is clearly patchwise a lift and we check that it satisfies the cocycle condition by computing for each

$i, j \in I$, $u \in U$:

$$\begin{aligned} B_j(u) - B_i(u) &= B_j^u - B_i^u + \sum_{i_0} \rho_{i_0} d_U (A_{i_0 j} - A_{i_0 i})(u) \\ &= d_X A_{ij}(u) + \sum_{i_0} \rho_{i_0} d_U (A_{ij} - d_X \log g_{i_0 i j})(u) \,, \\ &= d_{U \times X} A_{ij} \end{aligned}$$

where in the second but last step we used that at each u the A_{ij}^u satisfy their cocycle condition and where in the last step we used again that $g_{\dots}$ is constant on U on X.

So the 2-functor is also essentially surjective and this completes the proof. □

4.4.16.4 Equivariant circle n-bundles with connection We highlight some aspects of the *equivariant* version, def. 3.6.139, of smooth differential cohomology.

Observation 4.4.98. Let G be a Lie group acting on a smooth manifold X. Then the Deligne complex, def. 1.2.102, computes the correct G-equivariant differential cohomology on X if and only if the G-equivariant de Rham cohomology of X, prop. 4.4.51, coincides with the *G-invariant* Rham cohomology of X.

Proof. By prop. 4.4.51 we have that the G-equivariant de Rham cohomology of X is given for $n \geq 1$ by

$$H^{n+1}_{\mathrm{dR},G}(X) \simeq \pi_0 \mathbf{H}(X//G, \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}) \,.$$

Observe that $\pi_0 \mathbf{H}(X//G, \Omega^n_{\mathrm{cl}}(-))$ is set of *G-invariant* closed differential n-forms on X (which are in particular equivariant, but in general do not exhaust the equivariant cocycles). By prop. 4.4.84 the Deligne complex presents the homotopy pullback of $\Omega^{n+1}_{\mathrm{cl}}(-) \to \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}$ along the universal curvature map on $\mathbf{B}^n U(1)$. If therefore the inclusion $\pi_0 \mathbf{H}(X//G, \Omega^{n+1}_{\mathrm{cl}}(-)) \to \pi_0 \mathbf{H}(X//G, \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R})$ of invariant into equivariant de Rham cocycles is not surjective, then there are differential cocycles on $X//G$ not presented by the Deligne complex. □

In other words, if the G-invariant de Rham cocycles do not exhaust the equivariant cocycles, then $X//G$ is not *de Rham-projective*, and hence the representable variant, def. 3.9.42, of differential cohomology does not apply. The correct definition of differential cohomology in this case is the more general one from def. 3.9.37, which allows the curvature forms themselves to be in equivariant cohomology.

4.4.17 ∞-Chern-Weil homomorphism

We discuss the general abstract notion of Chern-Weil homomorphism, 3.9.7, realized in Smooth∞Grpd.

Recall that for $A \in$ Smooth∞Grpd a smooth ∞-groupoid regarded as a coefficient object for cohomology, for instance the delooping $A = \mathbf{B}G$ of an ∞-group G we have general abstractly that

- a characteristic class on A with coefficients in the circle Lie n-group, 4.4.21, is represented by a morphism
$$\mathbf{c} : A \to \mathbf{B}^n U(1) \,;$$
- the (unrefined) Chern-Weil homomorphism induced from this is the differential characteristic class given by the composite
$$\mathbf{c}_{\mathrm{dR}} : A \stackrel{\mathbf{c}}{\to} \mathbf{B}^n U(1) \stackrel{\mathrm{curv}}{\to} \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}$$
with the universal curvature characteristic, 3.9.5, on $\mathbf{B}^n U(1)$, or rather: is the morphism on cohomology
$$H^1_{\mathrm{Smooth}}(X, G) := \pi_0 \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}G) \stackrel{\pi_0((\mathbf{c}_{\mathrm{dR}})_*)}{\to} \pi_0 \mathrm{Smooth}\infty\mathrm{Grpd}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}) \simeq H^{n+1}_{\mathrm{dR}}(X)$$
induced by this.

By prop. 4.4.76 we have a presentation of the universal curvature class $\mathbf{B}^n\mathbb{R} \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}$ by a span

$$\begin{array}{ccc} \mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} & \xrightarrow{\mathrm{curv}_{\mathrm{smp}}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}} \\ \downarrow^{\simeq} & & \\ \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} & & \end{array}$$

in the model structure on simplicial presheaves $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, given by maps of smooth families of differential forms. We now insert this in the above general abstract definition of the ∞-Chern-Weil homomorphism to deduce a presentation of that in terms of smooth families L_∞-algebra valued differential forms.

The main step is the construction of a well-suited composite of two spans of morphisms of simplicial presheaves (of two ∞-anafunctors): we consider presentations of characteristic classes $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ in the image of the $\exp(-)$ map, def. 4.4.53, and presented by trunactions and quotients of morphisms of simplicial presheaves of the form

$$\exp(\mathfrak{g}) \stackrel{\exp(\mu)}{\to} \exp(b^{n-1}\mathbb{R})\,.$$

Then, using the above, the composite differential characteristic class $\mathbf{c}_{\mathrm{dR}}$ is presented by the zig-zag

$$\begin{array}{ccccc} & & \mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} & \xrightarrow{\mathrm{curv}_{\mathrm{smp}}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}} \\ & & \downarrow^{\simeq} & & \\ \exp(\mathfrak{g}) & \xrightarrow{\exp(\mu)} & \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} & & \end{array}$$

of simplicial presheaves. In order to efficiently compute which morphism in Smooth∞Grpd this presents we need to construct, preferably naturally in the L_∞-algebra $\mathfrak{g}$, a simplicial presheaf $\exp(\mathfrak{g})_{\mathrm{diff}}$ that fills this diagram as follows:

$$\begin{array}{ccccc} \exp(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{\exp(\mu,\mathrm{cs})} & \mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} & \xrightarrow{\mathrm{curv}_{\mathrm{smp}}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}} \\ \downarrow^{\simeq} & & \downarrow^{\simeq} & & \\ \exp(\mathfrak{g}) & \xrightarrow{\exp(\mu)} & \mathbf{B}^n\mathbb{R}_{\mathrm{smp}} & & \end{array} .$$

Given this, $\exp(\mathfrak{g})_{\mathrm{diff,smp}}$ serves as a new resolution of $\exp(\mathfrak{g})$ for which the composite differential characteristic class is presented by the ordinary composite of morphisms of simplicial presheaves $\mathrm{curv}_{\mathrm{smp}} \circ \exp(\mu, cs)$.

This object $\exp(\mathfrak{g})_{\mathrm{diff}}$ we shall see may be interpreted as the coefficient for *pseudo*-∞-connections with values in $\mathfrak{g}$.

There is however still room to adjust this presentation such as to yield in each cohomology class special nice cocycle representatives. This we will achieve by finding naturally a subobject $\exp(\mathfrak{g})_{\mathrm{conn}} \hookrightarrow \exp(\mathfrak{g})_{\mathrm{diff}}$ whose inclusion is an isomorphism on connected components and restricted to which the morphism $\mathrm{curv}_{\mathrm{smp}} \circ \exp(\mu, cs)$ yields nice representatives in the de Rham hypercohomology encoded by $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}_{\mathrm{smp}}$, namely globally defined differential forms. On this object the differential characteristic classes we will show factors naturally through the refinements to differential cohomology, and hence $\exp(\mathfrak{g})_{\mathrm{conn}}$ is finally identified as a presentation for the the coefficient object for ∞-connections with values in $\mathfrak{g}$.

Let $\mathfrak{g} \in L_\infty \stackrel{\mathrm{CE}}{\hookrightarrow} \mathrm{dgAlg}^{\mathrm{op}}$ be an L_∞-algebra, def. 1.2.114.

Definition 4.4.99. A L_∞-algebra cocycle on $\mathfrak{g}$ in degree n is a morphism

$$\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$$

to the line Lie n-algebra.

Observation 4.4.100. Dually this is equivalently a morphism of dg-algebras

$$\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{CE}(b^{n-1}\mathbb{R}) : \mu \,,$$

which we denote by the same letter, by slight abuse of notation. Such a morphism is naturally identified with its image of the single generator of $\mathrm{CE}(b^{n-1}\mathbb{R})$, which is a closed element

$$\mu \in \mathrm{CE}(\mathfrak{g})$$

in degree n, that we also denote by the same letter. Therefore L_∞-algebra cocycles are precisely the ordinary cocycles of the corresponding Chevalley-Eilenberg algebras.

Remark 4.4.101. After the injection of smooth ∞-groupoids into synthetic differential ∞-groupoids, discussed below in 4.5, there is an intrinsic abstract notion of cohomology of ∞-Lie algebras. Proposition 4.5.45 below asserts that the above definition is indeed a presentation of that abstract cohomological notion.

Definition 4.4.102. For $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ an L_∞-algebra cocycle with $n \geq 2$, write $\mathfrak{g}_\mu$ for the L_∞-algebra whose Chevalley-Eilenberg algebra is generated from the generators of $\mathrm{CE}(\mathfrak{g})$ and one single further generator b in degree $(n-1)$, with differential defined by

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)}|_{\mathfrak{g}^*} = d_{\mathrm{CE}(\mathfrak{g})} \,,$$

and

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)} : b \mapsto \mu \,,$$

where on the right we regard μ as an element of $\mathrm{CE}(\mathfrak{g})$, hence of $\mathrm{CE}(\mathfrak{g}_\mu)$, by observation 4.4.100.

Remark 4.4.103. Below in prop. 4.5.47 we show that, in the context of *synthetic differential cohesion* 4.5, $\mathfrak{g}_\mu$ is indeed the extension of $\mathfrak{g}$ classified by μ in the general sense of 3.6.14.

Definition 4.4.104. For $\mathfrak{g} \in L_\infty\mathrm{Alg}$ an L_∞-algebra, its *Weil algebra* $\mathrm{W}(\mathfrak{g}) \in \mathrm{dgAlg}$ is the unique representative of the free dg-algebra on the dual cochain complex underlying $\mathfrak{g}$ such that the canonical projection $\mathfrak{g}^*_\bullet[1] \oplus \mathfrak{g}^*_\bullet[2] \to \mathfrak{g}^*_\bullet[1]$ extends to a dg-algebra homomorphism

$$\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{W}(\mathfrak{g}) \,.$$

Since $\mathrm{W}(\mathfrak{g})$ is itself in $L_\infty\mathrm{Alg}^{\mathrm{op}} \hookrightarrow \mathrm{dgAlg}$ we can idenntify it with the Chevalley-Eilenberg algebra of an L_∞-algebra. That we write $\mathrm{inn}(\mathfrak{g})$ or $e\mathfrak{g}$:

$$\mathrm{W}(\mathfrak{g}) :=: \mathrm{CE}(e\mathfrak{g}) \,.$$

In terms of this the above canonical morphism reads

$$\mathfrak{g} \to e\mathfrak{g} \,.$$

Remark 4.4.105. This notation reflects the fact that $e\mathfrak{g}$ may be regarded as the infinitesimal groupal model of the universal $\mathfrak{g}$-principal ∞-bundle.

Proposition 4.4.106. *For $n \in \mathbb{N}$, $n \geq 2$ we have a pullback in* $L_\infty\mathrm{Alg}$

$$\begin{array}{ccc} b^{n-1}\mathbb{R} & \longrightarrow & e b^{n-1}\mathbb{R} \\ \downarrow & & \downarrow \\ * & \longrightarrow & b b^{n-1}\mathbb{R} \end{array} \,.$$

Proof. Dually this is the pushout diagram of dg-algebras that is free on the short exact sequence of cochain complexes concentrated in degrees n and $n+1$ as follows:

$$\left(\begin{array}{c} 0_{n+1} \\ {\scriptstyle d_{\mathrm{CE}(\mathfrak{b}^{n-1}\mathbb{R})}} \uparrow \\ \langle c \rangle_n \end{array} \right) \leftarrow \left(\begin{array}{c} \langle d \rangle_{n+1} \\ {\scriptstyle d_{\mathrm{CE}(\mathfrak{e}\mathfrak{b}^{n-1}\mathbb{R})}} \uparrow \simeq \\ \langle c \rangle_n \end{array} \right) \leftarrow \left(\begin{array}{c} \langle d \rangle_{n+1} \\ {\scriptstyle d_{\mathrm{CE}(\mathfrak{b}\mathfrak{b}^{n-1}\mathbb{R})}} \uparrow \\ 0_n \end{array} \right).$$

□

Proposition 4.4.107. *The L_∞-algebra $\mathfrak{g}_\mu$ from def. 4.4.102 fits into a pullback diagram in* $L_\infty\mathrm{Alg}$

$$\begin{array}{ccc} \mathfrak{g}_\mu & \longrightarrow & eb^{n-2}\mathbb{R} \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\mu} & bb^{n-2}\mathbb{R} \end{array}.$$

Proposition 4.4.108. *Let $\mu : \mathfrak{g} \to b^n\mathbb{R}$ be a degree-n cocycle on an L_∞-algebra and $\mathfrak{g}_\mu$ the L_∞-algebra from def. 4.4.102.*

We have that $\exp(\mathfrak{g}_\mu) \to \exp(\mathfrak{g})$ *presents the homotopy fiber of* $\exp(\mu) : \exp(\mathfrak{g}) \to \exp(b^{n-1}\mathbb{R})$ *in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Since $\exp(b^{n-1}\mathbb{R}) \simeq \mathbf{B}^n\mathbb{R}$ by prop. 4.4.61, this means that $\exp(\mathfrak{g}_\mu)$ is the $\mathbf{B}^{n-1}\mathbb{R}$-principal ∞-bundle classified by $\exp(\mu)$ in that we have an ∞-pullback

$$\begin{array}{ccc} \exp(\mathfrak{g}_\mu) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \exp(\mathfrak{g}) & \xrightarrow{\exp(\mu)} & \mathbf{B}^n\mathbb{R} \end{array}$$

in Smooth∞Grpd.

Proof. Since $\exp : L_\infty\mathrm{Alg} \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ preserves pullbacks (being given componentwise by a hom-functor) it follows from 4.4.107 that we have a pullback diagram

$$\begin{array}{ccc} \exp(\mathfrak{g}_\mu) & \longrightarrow & \exp(eb^{n-1}\mathbb{R}) \\ \downarrow & & \downarrow \\ \exp(\mathfrak{g}) & \xrightarrow{\exp(\mu)} & \exp(b^{n-1}\mathbb{R}) \end{array}.$$

The right vertical morphism is a fibration resolution of the point inclusion $* \to \exp(b^{n-1}\mathbb{R})$. Hence this is a homotopy pullback in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and the claim follows with prop. 2.3.13. □

We now come to the definition of differential refinements of exponentiated L_∞-algebras.

Definition 4.4.109. For $\mathfrak{g} \in L_\infty$ define the simplicial presheaf $\exp(\mathfrak{g})_{\mathrm{diff}} \in [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{smooth}}, \mathrm{sSet}]$ by

$$\exp(\mathfrak{g})_{\mathrm{diff}} : (U, [k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^k) & \longleftarrow & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^k) & \longleftarrow & \mathrm{W}(\mathfrak{g}) \end{array} \right\},$$

where on the left we have the set of commuting diagrams in dgAlg as indicated, with the vertical morphisms being the canonical projections.

Proposition 4.4.110. *The canonical projection*

$$\exp(\mathfrak{g})_{\mathrm{diff}} \to \exp(\mathfrak{g})$$

is a weak equivalence in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Moreover, for every L_∞-algebra cocycle it fits into a commuting diagram

$$\begin{array}{ccccc}
\exp(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{\exp(\mu)_{\mathrm{diff}}} & \exp(b^{n-1}\mathbb{R})_{\mathrm{diff}} & = & \mathbf{B}^n\mathbb{R}_{\mathrm{diff,smp}} \\
\downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{\simeq} \\
\exp(\mathfrak{g}) & \xrightarrow{\exp(\mu)} & \exp(b^{n-1}\mathbb{R}) & = & \mathbf{B}^n\mathbb{R}_{\mathrm{smp}}
\end{array}$$

for some morphism $\exp(\mu)_{\mathrm{diff}}$.

Proof. Use the contractibility of the Weil algebra. □

Definition 4.4.111. Let $G \in \mathrm{Smooth}\infty\mathrm{Grpd}$ be a smooth n-group given by Lie integration, 4.4.14, of an L_∞ algebra $\mathfrak{g}$, in that the delooping object $\mathbf{B}G$ is presented by the $(n+1)$-coskeleton simplicial presheaf $\mathbf{cosk}_{n+1}\exp(\mathfrak{g})$, def. 3.6.28.

Then for $X \in [\mathrm{CartSp}_{\mathrm{smooth}}, \mathrm{sSet}]_{\mathrm{proj}}$ any object and $\hat{X}$ a cofibrant resolution, we say that

$$[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}](\hat{X}, \mathbf{cosk}_{n+1}\exp(\mathfrak{g})_{\mathrm{diff}})$$

is the Kan complex of *pseudo-n-connections* on G-principal n-bundles.

We discuss now subobjects that pick out genuine ∞-connections.

Definition 4.4.112. An *invariant polynomial* on an L_∞-algebra $\mathfrak{g}$ is an element $\langle - \rangle \in \mathrm{W}(\mathfrak{g})$ in the Weil algebra, such that

1. $d_{\mathrm{W}(\mathfrak{g})}\langle -, - \rangle = 0$;
2. $\langle - \rangle \in \wedge^\bullet \mathfrak{g}^*[1] \hookrightarrow W(\mathfrak{g})$;

hence such that it is a closed element built only from shifted generators of $\mathrm{W}(\mathfrak{g})$.

Proposition 4.4.113. *For $\mathfrak{g}$ an ordinary Lie algebra, this definition of invariant polynomial is equivalent to the traditional one (for instance [AzIz95]).*

Proof. Let $\{t^a\}$ be a basis of $\mathfrak{g}^*$ and $\{r^a\}$ the corresponding basis of $\mathfrak{g}^*[1]$. Write $\{C^a{}_{bc}\}$ for the structure constants of the Lie bracket in this basis.

Then for $P = P_{(a_1, \cdots, a_k)} r^{a_1} \wedge \cdots \wedge r^{a_k} \in \wedge^r \mathfrak{g}^*[1]$ an element in the shifted generators, the condition that its image under $d_{\mathrm{W}(\mathfrak{g})}$ is in the shifted copy is equivalent to

$$C^b_{c(a_1} P_{b, \cdots, a_k)} t^c \wedge r^{a_1} \wedge \cdots \wedge r^{a_k} = 0\,,$$

where the parentheses around indices denotes symmetrization, so that this is equivalent to

$$\sum_i C^b_{c(a_i} P_{a_1 \cdots a_{i-1} b a_{i+1} \cdots, a_k)} = 0$$

for all choice of indices. This is the component-version of the defining invariance statement

$$\sum_i P(t_1, \cdots, t_{i-1}, [t_c, t_i], t_{i+1}, \cdots, t_k) = 0$$

for all $t_\bullet \in \mathfrak{g}$. □

Observation 4.4.114. For the line Lie n-algeba we have

$$\mathrm{inv}(b^{n-1}\mathbb{R}) \simeq \mathrm{CE}(b^n\mathbb{R})\,.$$

This allows us to identify an invariant polynomial $\langle - \rangle$ of degree $n+1$ with a morphism

$$\mathrm{inv}(\mathfrak{g}) \stackrel{\langle - \rangle}{\leftarrow} \mathrm{inv}(b^{n-1}\mathbb{R})$$

in dgAlg.

Remark 4.4.115. Write $\iota : \mathfrak{g} \to \mathrm{Der}_\bullet(\mathrm{W}(\mathfrak{g}))$ for the identification of elements of $\mathfrak{g}$ with inner graded derivations of the Weil-algebra, induced by contraction. For $v \in \mathfrak{g}$ write

$$\mathcal{L}_x := [d_{\mathrm{W}(\mathfrak{g})}, \iota_v] \in \mathrm{der}_\bullet(\mathrm{W}(\mathfrak{g}))$$

for the induced Lie derivative. Then the fist condition on an invariant polynomial $\langle - \rangle$ in def. 4.4.112 is equivalent to

$$\iota_v \langle - \rangle = 0 \quad \forall v \in \mathfrak{g}$$

and the second condition implies that

$$\mathcal{L}_v \langle - \rangle = 0 \quad \forall v \in \mathfrak{g}\,.$$

In Cartan calculus [Cart50a][Cart50b] elements satisfying these two conditions are called *basic elements* or *basic forms*. By prop. 4.4.113 on an ordinary Lie algebra the basic forms are precisely the invariant polynomials. But on a general L_∞-algebra there can be non-closed basic forms. Our definition of invariant polynomials hence picks the *closed basic forms* on an L_∞-algebra.

Definition 4.4.116. We say that an invariant polynomial $\langle - \rangle$ on $\mathfrak{g}$ is *in transgression* with an L_∞-algebra cocycle $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ if there is a morphism $\mathrm{cs} : \mathrm{W}(b^{n-1}\mathbb{R}) \to \mathrm{W}(\mathfrak{g})$ such that we have a commuting diagram

$$\begin{array}{ccccc} \mathrm{CE}(\mathfrak{g}) & \stackrel{\mu}{\longleftarrow} & \mathrm{CE}(b^{n-1}\mathbb{R}) & & \\ \uparrow & & \uparrow & & \\ \mathrm{W}(\mathfrak{g}) & \stackrel{\mathrm{cs}}{\longleftarrow} & \mathrm{W}(b^{n-1}\mathbb{R}) & & \\ \uparrow & & \uparrow & & \\ \mathrm{inv}(\mathfrak{g}) & \stackrel{\langle - \rangle}{\longleftarrow} & \mathrm{inv}(b^{n-1}\mathbb{R}) & = & \mathrm{CE}(b^n\mathbb{R}) \end{array}$$

hence such that

1. $d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs} = \langle - \rangle$;
2. $\mathrm{cs}|_{\mathrm{CE}(\mathfrak{g})} = \mu$.

We say that cs is a *Chern-Simons element* exhibiting the transgression between μ and $\langle - \rangle$.

We say that an L_∞-algebra cocycle is *transgressive* if it is in transgression with some invariant polynomial.

Observation 4.4.117. We have

1. There is a transgressive cocycle for every invariant polynomial.
2. Any two L_∞-algebra cocycles in transgression with the same invariant polynomial are cohomologous.
3. Every decomposable invariant polynomial (the wedge product of two non-vanishing invariant polynomials) transgresses to a cocycle cohomologous to 0.

Proof.

1. By the fact that the Weil algebra is free, its cochain cohomology vanishes and hence the definition property $d_{\mathrm{W}(\mathfrak{g})}\langle -\rangle = 0$ implies that there is some element $\mathrm{cs} \in W(\mathfrak{g})$ such that $d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs} = \langle -\rangle$. Then the image of cs along the canonical dg-algebra homomorphism $\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$ is $d_{\mathrm{CE}(\mathfrak{g})}$-closed hence is a cocycle on $\mathfrak{g}$. This is by construction in transgression with $\langle -\rangle$.

2. Let cs_1 and cs_2 be Chern-Simons elements for the to given L_∞-algebra cocycles. Then by assumption $d_{(\mathfrak{g})}(\mathrm{cs}_1 - \mathrm{cs}_2) = 0$. By the acyclicity of $\mathrm{W}(\mathfrak{g})$ there is then $\lambda \in \mathrm{W}(\mathfrak{g})$ such that $\mathrm{cs}_1 = \mathrm{cs}_2 + d_{\mathrm{W}(\mathfrak{g})}\lambda$. Since $\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$ is a dg-algebra homomorphism this implies that also $\mu_1 = \mu_2 + d_{\mathrm{CE}(\mathfrak{g})}\lambda|_{\mathrm{CE}(\mathfrak{g})}$.

3. Given two nontrivial invariant polynomials $\langle -\rangle_1$ and $\langle -\rangle_2$ let $\mathrm{cs}_1 \in \mathrm{W}(\mathfrak{g})$ be any element such that $d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs}_1 = \langle -\rangle_1$. Then $\mathrm{cs}_{1,2} := \mathrm{cs}_1 \wedge \langle -\rangle_2$ satisfies $d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs}_{1,2} = \langle -\rangle_1 \wedge \langle -\rangle_2$. By the first observation the restriction of $\mathrm{cs}_{1,2}$ to $\mathrm{CE}(\mathfrak{g})$ is therefore a cocycle in transgression with $\langle -\rangle_1 \wedge \langle -\rangle_2$. But by the definition of invariant polynomials the restriction of $\langle -\rangle_2$ vanishes, and hence so does that of $\mathrm{cs}_{1,2}$. The claim the follows with the second point above.

□

The following notion captures the equivalence relation induced by lifts of cocycles to Chern-Simons elements on invariant polynomials.

Definition 4.4.118. We say two invariant polynomials $\langle -\rangle_1, \langle -\rangle_2 \in \mathrm{W}(\mathfrak{g})$ are *horizontally equivalent* if there exists $\omega \in \ker(\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g}))$ such that

$$\langle -\rangle_1 = \langle -\rangle_2 + d_{\mathrm{W}(\mathfrak{g})}\omega \,.$$

Observation 4.4.119. Every decomposable invariant polynomial is horizontally equivalent to 0.

Proof. By the argument of prop. 4.4.117, item iii): for $\langle -\rangle = \langle -\rangle_1 \wedge \langle -\rangle_2$ let cs_1 be a Chern-Simons element for $\langle -\rangle_1$. Then $\mathrm{cs}_1 \wedge \langle -\rangle_2$ exhibits a horizontal equivalence $\langle -\rangle \sim 0$. □

Proposition 4.4.120. *For $\mathfrak{g}$ an L_∞-algebra, $\mu : \mathfrak{g} \to b^n\mathbb{R}$ a cocycle in transgression to an invariant polynomial $\langle\rangle$ on $\mathfrak{g}$ and $\mathfrak{g}_\mu$ the corresponding shifted central extension, 4.4.102, we have that*

1. *$\langle -\rangle$ defines an invariant polynomial also on $\mathfrak{g}_\mu$, by the defining identification of generators;*

2. *but on $\mathfrak{g}_\mu$ the invariant polynomial $\langle -\rangle$ is horizontally trivial.*

Proof. □

Definition 4.4.121. For $\mathfrak{g}$ an L_∞-algebra we write $\mathrm{inv}(\mathfrak{g})$ for the free graded algebra on horizontal equivalence classes of invariant polynomials. We regard this as a dg-algebra with trivial differential This comes with an inclusion of dg-algebras

$$\mathrm{inv}(\mathfrak{g}) \to W(\mathfrak{g})$$

given by a choice of representative for each class.

Observation 4.4.122. The algebra $\mathrm{inv}(\mathfrak{g})$ is generated from indecomposable invariant polynomials.

Proof. By observation 4.4.119. □

Definition 4.4.123. Define the simplicial presheaf $\exp(\mathfrak{g})_{\mathrm{ChW}} \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$ by the assignment

$$\exp(\mathfrak{g})_{\mathrm{ChW}} : (U, [k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U) & \xleftarrow{\langle F_A \rangle} & \mathrm{inv}(\mathfrak{g}) \end{array} \right\},$$

where on the right we have the set of horizontal morphisms in dgAlg making commuting diagrams with the canonical vertical morphisms as indicated.

We call $\langle F_A \rangle$ the *curvature characteristic forms* of A.

Let

$$\begin{array}{ccccc} \exp(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{(\exp(\mu_i, \mathrm{cs}_i))_i} & \prod_i \exp(b^{n_i - 1}\mathbb{R})_{\mathrm{diff}} & \xrightarrow{((\mathrm{curv}_i)_{\mathrm{smp}})} & \prod_i \flat_{\mathrm{dR}} \mathbf{B}^{n_i}_{\mathrm{smp}} \\ \downarrow \simeq & & & & \\ \exp(\mathfrak{g}) & & & & \end{array}$$

be the presentation, as above, of the product of all differental refinements of characteristic classes on $\exp(\mathfrak{g})$ induced from Lie integration of transgressive L_∞-algebra cocycles.

Proposition 4.4.124. *We have that* $\exp(\mathfrak{g})_{\mathrm{ChW}}$ *is the pullback in* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$ *of the globally defined closed forms along the curvature characteristics induced by all transgressive* L_∞*-algebra cocycles:*

$$\begin{array}{ccc} \exp(\mathfrak{g})_{\mathrm{ChW}} & \xrightarrow{\exp(\mu, cs)} & \prod_{n_i} \Omega^{n_i+1}_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \exp(\mathfrak{g})_{\mathrm{diff,smp}} & \xrightarrow{(\mathrm{curv}_i)_i} & \prod_i \flat_{\mathrm{dR}} \mathbf{B}^{n_i+1}\mathbb{R}_{\mathrm{smp}} \\ \downarrow \simeq & & \\ \exp(\mathfrak{g}) & & \end{array} .$$

Proof. By prop. 4.4.77 we have that the bottom horizontal morphims sends over each $(U, [k])$ and for each i an element

$$\begin{array}{ccc} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \end{array}$$

of $\exp(\mathfrak{g})(U)_k$ to the composite

$$\left(\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \stackrel{A}{\leftarrow} \mathrm{W}(\mathfrak{g}) \stackrel{\mathrm{cs}_i}{\leftarrow} \mathrm{W}(b^{n_i-1}\mathbb{R}) \leftarrow \mathrm{inv}(b^{n_i}\mathbb{R}) = \mathrm{CE}(b^{n_i}\mathbb{R})) \right)$$

$$= \left(\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \stackrel{\langle F_A \rangle_i}{\leftarrow} \mathrm{CE}(b^{n_i}\mathbb{R}) \right)$$

regarded as an element in $\flat_{\mathrm{dR}} \mathbf{B}^{n_i+1}_{\mathrm{smp}}(U)_k$. The right vertical morphism $\Omega^{n_i+1}(U) \to \flat_{\mathrm{dR}} \mathbf{B}^{n_i+1}\mathbb{R}_{\mathrm{smp}}(U)$ from the constant simplicial set of closed $(n_i + 1)$-forms on U picks precisely those of these elements for which

$\langle F_A \rangle$ is a basic form on the $U \times \Delta^k$-bundle in that it is in the image of the pullback $\Omega^\bullet(U) \to \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)$.
□

This way the abstract differential refinement recovers the notion of ∞-connections from Lie integration discussed before in 1.2.13.6.

4.4.18 Higher holonomy

We discuss the intrinsic notion of higher holonomy, 3.9.9, realized in Smooth∞Grpd.

Theorem 4.4.125. *If* $\Sigma \hookrightarrow \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ *is a closed manifold of dimension* $\dim\Sigma \leq n$ *then the intrinsic integration by truncation, def. 3.9.65, takes values in*

$$\tau_{\leq n-\dim\Sigma}\mathbf{H}(\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}}) \simeq B^{n-\dim\Sigma}U(1) \simeq K(U(1), n - \dim(\Sigma)) \qquad \in \infty\mathrm{Grpd}\,.$$

Moreover, in the case $\dim\Sigma = n$*, then the morphism*

$$\exp(iS_{\mathbf{c}}(-)) : \mathbf{H}(\Sigma, A_{\mathrm{conn}}) \to U(1)$$

is obtaind from the Lagrangian $\exp(iL_{\mathbf{c}}(-))$ *by forming the volume holonomy of circle n-bundles with connection (fiber integration in Deligne cohomology)*

$$S_{\mathbf{c}}(-) = \int_\Sigma L_{\mathbf{c}}(-)\,.$$

This is due to [FRS11b].

Proof. Since $\dim\Sigma \leq n$ we have by prop. 4.4.47 that $H(\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}) \simeq H^{n+1}_{\mathrm{dR}}(\Sigma) \simeq *$. It then follows by prop. 3.9.39 that we have an equivalence

$$\mathbf{H}_{\mathrm{diff}}(\Sigma, \mathbf{B}^n U(1)) \simeq \mathbf{H}_{\mathrm{flat}}(\Sigma, \mathbf{B}^n U(1)) =: \mathbf{H}(\mathbf{\Pi}(\Sigma), \mathbf{B}^n U(1))$$

with the flat differential cohomology on Σ, and by the $(\Pi \dashv \mathrm{Disc} \dashv \Gamma)$-adjunction it follows that this is equivalently

$$\begin{aligned} \cdots &\simeq \infty\mathrm{Grpd}(\Pi(\Sigma), \Gamma\mathbf{B}^n U(1)) \\ &\simeq \infty\mathrm{Grpd}(\Pi(\Sigma), B^n U(1)_{\mathrm{disc}}) \end{aligned},$$

where $B^n U(1)_{\mathrm{disc}}$ is an Eilenberg-MacLane space $\cdots \simeq K(U(1), n)$. By prop. 4.4.27 we have under $|-| : \infty\mathrm{Grpd} \simeq \mathrm{Top}$ a weak homotopy equivalence $|\Pi(\Sigma)| \simeq \Sigma$. Therefore the cocycle ∞-groupoid is that of ordinary cohomology

$$\cdots \simeq C^n(\Sigma, U(1))\,.$$

By general abstract reasoning it follows that we have for the homotopy groups an isomorphism

$$\pi_i \mathbf{H}_{\mathrm{diff}}(\Sigma, \mathbf{B}^n U(1)) \stackrel{\simeq}{\to} H^{n-i}(\Sigma, U(1))\,.$$

Now we invoke the universal coefficient theorem. This asserts that the morphism

$$\int_{(-)} (-) : H^{n-i}(\Sigma, U(1)) \to \mathrm{Hom}_{\mathrm{Ab}}(H_{n-i}(\Sigma, \mathbb{Z}), U(1))$$

which sends a cocycle ω in singular cohomology with coefficients in $U(1)$ to the pairing map

$$[c] \mapsto \int_{[c]} \omega$$

sits inside an exact sequence

$$0 \to \mathrm{Ext}^1(H_{n-i-1}(\Sigma, \mathbb{Z}), U(1)) \to H^{n-i}(\Sigma, U(1)) \to \mathrm{Hom}_{\mathrm{Ab}}(H_{n-i}(\Sigma, \mathbb{Z}), U(1)) \to 0\,,$$

But since $U(1)$ is an injective $\mathbb{Z}$-module we have

$$\mathrm{Ext}^1(-, U(1)) = 0\,.$$

This means that the integration/pairing map $\int_{(-)}(-)$ is an isomorphism

$$\int_{(-)}(-) : H^{n-i}(\Sigma, U(1)) \simeq \mathrm{Hom}_{\mathrm{Ab}}(H_{n-i}(\Sigma, \mathbb{Z}), U(1))\,.$$

For $i < (n - \mathrm{dim}\Sigma)$, the right hand is zero, so that

$$\pi_i \mathbf{H}_{\mathrm{diff}}(\Sigma, \mathbf{B}^n U(1)) = 0 \quad \text{for } i < (n - \mathrm{dim}\Sigma)\,.$$

For $i = (n - \mathrm{dim}\Sigma)$, instead, $H_{n-i}(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}$, since Σ is a closed $\mathrm{dim}\Sigma$-manifold and so

$$\pi_{(n-\mathrm{dim}\Sigma)} \mathbf{H}_{\mathrm{diff}}(\Sigma, \mathbf{B}^n U(1)) \simeq U(1)\,.$$

□

More generally, using fiber integration in Deligne hypercohomology as in [GoTe00], we get for compact oriented closed smooth manifolds Σ of dimension k a natural morphism

$$\exp(2\pi i \int_\sigma (-)) : [\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}}] \to \mathbf{B}^{n-k} U(1)_{\mathrm{conn}}\,.$$

4.4.19 Chern-Simons functionals

We discuss the realization of the intrinsic notion of Chern-Simons functionals, 3.9.11, in Smooth∞Grpd.

The proof of theorem 4.4.125 shows that for $\mathrm{dim}\Sigma = n$ and $\exp(iL) : A_{conn} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ an (Chern-Simons) Lagrangian, we may think of the composite

$$\exp(iS) : \mathbf{H}(\Sigma, A_{\mathrm{conn}}) \stackrel{\exp(iL)}{\to} \mathbf{H}(\Sigma, \mathbf{B}^n U(1)_{conn}) \stackrel{\int_{[\Sigma]}(-)}{\to} U(1)$$

as being indeed given by integrating the Lagrangian over Σ in order to obtain the action

$$S(-) = \int_\Sigma L(-)\,.$$

We consider precise versions of this statement in 5.7.

4.4.20 Geometric prequantization

We discuss the notion of cohesive prequantization, 3.9.13, realized in the model of smooth cohesion.

What is traditionally called *(geometric) prequantization* is the refinement of symplectic 2-forms to curvature 2-forms on line bundles with connection. Formally: for

$$H^2_{\mathrm{diff}}(X) \xrightarrow{\mathrm{curv}} \Omega^2_{\mathrm{int}}(X) \hookrightarrow \Omega^2_{\mathrm{cl}}(X)$$

the morphism that sends a class in degree-2 differential cohomology over a smooth manifold X to its curvature 2-form, geometric prequantization of some $\omega \in \Omega^2_{\mathrm{cl}}(X)$ is a choice of lift $\hat\omega \in H^2_{\mathrm{diff}}(X)$ through this morphism.

One says that $\hat{\omega}$ is (the class of) a *prequantum line bundle* or *quantization line bundle* with connection for ω. See for instance [WeXu91].

By the curvature exact sequence for differential cohomology, prop. 4.4.83, a lift $\hat{\omega}$ exists precisely if ω is an *integral* differential 2-form. This is called the *quantization condition* on ω. If it is fulfilled, the group of possible choices of lifts is the topological (for instance singular) cohomology group $H^1(X, U(1))$. Notice that the extra non-degeneracy condition that makes a closed 2-form a symplectic form does not appear in *pre*quantization.

The concept of geometric prequantization has an evident generalization to closed forms of degree $n+1$ for any $n \in \mathbb{N}$. For $\omega \in \Omega^{n+1}_{\mathrm{cl}}(X)$ a closed differential $(n+1)$-form on a manifold X, a geometric prequantization is a lift of ω through the canonical morphism

$$H^{n+1}_{\mathrm{diff}}(X) \xrightarrow{\mathrm{curv}} \Omega^{n+1}_{\mathrm{int}}(X) \hookrightarrow \Omega^{n+1}_{\mathrm{cl}}(X) \; .$$

Since the elements of the higher differential cohomology group $H^{n+1}_{\mathrm{diff}}(X)$ are classes of *circle n-bundles with connection* (equivalently *circle bundle $(n-1)$-gerbes with connection*) on X, we may speak of such a lift as a *prequantum circle n-bundle*. Again, the lift exists precisely if ω is integral and the group of possible choices is $H^n(X, U(1))$. Higher geometric prequantization for $n = 2$ has been considered in [Rog11]. By the discussion in 4.4.16 we may consider circle n-bundles with connection not just over smooth manifolds, but over any smooth ∞-groupoid (smooth ∞-stack) and hence consider, generally, geometric prequantization of higher forms on higher smooth stacks.

- 4.4.20.1 – Ordinary symplectic geometry and its prequantization;
- 4.4.20.2 – 2-Plectic geometry and its prequantization.

This section draws from joint work with Chris Rogers.

4.4.20.1 Ordinary symplectic geometry and its prequantization We discuss how the general abstract notion of higher geometric prequantization reduces to the traditional notion of geometric prequatization when interpreted in the smooth context and for $n = 1$.

The following is essentially a re-derivation of the discussion in section II.3 and II.4 of [Br93] (based on [Kos70]) from the abstract point of view of 3.9.13.

The traditional definition of Hamiltonian vector fields is the following.

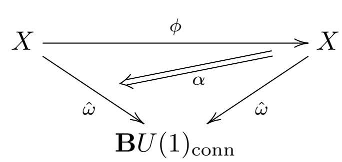
Definition 4.4.126. Let (X, ω) be a smooth symplectic manifold. A *Hamiltonian vector field* on X is a vector field $v \in \Gamma(TX)$ whose contraction with the symplectic form ω yields an exact form, hence such that

$$\exists\, h \in C^\infty(X) \; : \; \iota_v \omega = d_{\mathrm{dR}} h \, .$$

Here a choice of function h is called a *Hamiltonian* for v.

Proposition 4.4.127. *Let X be a smooth manifold which is simply connected, and let $\omega \in \Omega^2(X)_{\mathrm{int}}$ be an integral symplectic form on X. Then regarding (X, ω) as a symplectic 0-groupoid in* Smooth∞Grpd*, the general definition 3.9.94 reproduces the standard notion of Hamiltonian vector fields, def. 4.4.126 on the symplectic manifold (X, ω).*

Proof. A Hamiltonian symplectomorphism is an equivalence $\phi : X \to X$ that fits into a diagram



in Smooth∞Grpd. To compute the Lie algebra of the group of these diffeomorphisms, we need to consider smooth 1-parameter families of such and differentiate them.

Assume first that the connection 1-form in $\hat{\omega}$ is globally defined $A \in \Omega^1(X)$ with $dA = \omega$. Then the existence of the above diagram is equivalent to the condition

$$(\phi(t)^* A - A) = d\alpha(t)\,,$$

where $\alpha(t) \in C^\infty(X)$. Differentiating this at 0 yields the Lie derivative

$$\mathcal{L}_v A = d\alpha'\,,$$

where v is the vector field of which $t \mapsto \phi(t)$ is the flow and where $\alpha' := \frac{d}{dt}\alpha$. By Cartan calculus this is equivalently

$$d_{\mathrm{dR}} \iota_v A + \iota_v d_{dR} A = d\alpha'$$

and using that A is the connection on a prequantum circle bundle for ω

$$\iota_v \omega = d(\alpha' - \iota_v A)\,.$$

This says that for v to be Hamiltonian, its contraction with ω must be exact. This is precisely the definition of Hamiltonian vector fields. The corresponding Hamiltonian function h here is $\alpha' - \iota_v A$.

We now discuss the general case, where the prequantum bundle is not necessarily trivial. After a choice of cover that is compatible with the flows of vector fields, the argument proceeds by slight generalization of the previous argument.

We may assume without restriction of generality that X is connected. Choose then any base point $x_0 \in X$ and let

$$P_* X := [I, X] \times_X \{x_0\}$$

be the based smooth path space of X, regarded as a diffeological space, def. 4.4.14, where $I \subset \mathbb{R}$ is the standard closed interval. This comes equipped with the smooth endpoint evaluation map

$$p : P_* X \to X\,.$$

Pulled back along this map, every circle bundle has a trivialization, since $P_* X$ is topologically contractible. The corresponding Čech nerve $C(P_* X \to X)$ is the simplicial presheaf that starts out as

$$\cdots \rightrightarrows P_* X \times_X P_* X \underset{p_2}{\overset{p_1}{\rightrightarrows}} P_* X\ ,$$

where in first degree we have a certain smooth version of the based loop space of X. Any diffeomorphism $\phi = \exp(v) : X \to X$ lifts to an automorphism of the Čech nerve by letting

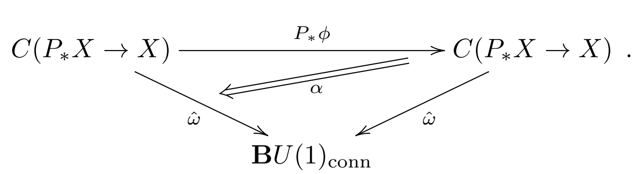
$$P_* \phi : P_* X \to P_* X$$

be given by

$$P_* \phi(\gamma) : (t \in [0,1]) \mapsto \exp(tv)(\gamma(t))$$

and similarly for $P_* \phi : P_* X \times_X P_* X \to P_* X \times_X P_* X$. If $\phi = \exp(tv)$ for v a vector field on X, we will write v also for the vector fields induced this way on the components of the Čech nerve.

With these preparations, every elements of the group in question is presented by a diagram of simplicial presheaves of the form



Here the vertical (diagonal) morphisms now exhibit Čech-Deligne cocycles with transition function

$$g \in C^\infty(P_*X \times_X P_*X)$$

and connection 1-form

$$A \in \Omega^1(P_*X)\,,$$

satisfiying

$$p_2^* A - p_1^* A = d_{\mathrm{dR}} \mathrm{log} g\,.$$

For $\phi(t) = \exp(tv)$ a 1-parameter family of diffeomorphisms, the homotopy in this diagram is a gauge transformation given by a function $\alpha(t) \in C^\infty(P_*X, U(1))$ such that

$$p_2^*\alpha(t) \cdot g \cdot p_1^*\alpha(t)^{-1} = \exp(tv)^* g$$

and

$$\exp(tv)^* A - A = d_{\mathrm{dR}} \mathrm{log}\alpha(t)\,.$$

Differentiating this at $t = 0$ and writing $\alpha' := \alpha'(0)$ as before, this yields

$$p_2^*\alpha' - p_1^*\alpha' = \mathcal{L}_v \mathrm{log} g$$

and

$$\mathcal{L}_v A = d_{\mathrm{dR}} \alpha'\,.$$

The latter formula says that on P_*X $\iota_v\omega$ is exact

$$\iota_v p^*\omega = d_{\mathrm{dR}}(\alpha' - \iota_v A)\,.$$

But in fact the function on the right descends down to X, because by the formulas above we have

$$\begin{aligned} p_2^*(\alpha' - \iota_v A) - p_1^*(\alpha' - \iota_v A) &= \mathcal{L}_v \mathrm{log} g - \iota_v(p_2^* A - p_1^* A) \\ &= 0\,. \end{aligned}.$$

Write therefore $h \in C^\infty(X)$ for the unique function such that $p^*h = \alpha' - \iota_v A$, then this satisfies

$$\iota_v \omega = dh$$

on X. □

The traditional definition of the Poisson-bracket Lie algebra associated with a symplectic manifold (X, ω) is the following.

Definition 4.4.128. Let (X, ω) be a smooth symplectic manifold. Then its *Poisson-bracket Lie algebra* is the Lie algebra whose underlying vector space is $C^\infty(X)$, the space of smooth function on X, and whose Lie bracket is given by

$$[h_1, h_2] := \iota_{v_2}\iota_{v_1}\omega$$

for all $h_1, h_2 \in C^\infty(X)$ and for v_1, v_2 the corresponding Hamiltonian vector fields, def. 4.4.126.

Proposition 4.4.129. *The general definition of Poisson ∞-Lie algebra, def. 3.9.94, applied to the symplectic manifold (X, ω) regarded as a symplectic smooth 0-groupoid, reproduces the traditional definition of the Lie algebra underlying the Poisson algebra of (X, ω).*

Proof. The smooth group $\mathbf{Aut}_{\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}}(\hat{\omega})$ is manifestly a subgroup of the semidirect product group $\mathrm{Diff}(X) \ltimes C^\infty(X)$, where the group structure on the second factor is given by addition, and the action of the first factor on the second is the canonical one by pullback. Accordingly, its Lie algebra may be identified with that of pairs (v, α) in $\Gamma(TX) \times C^\infty(X)$ such that, with the notation as in the proof of prop. 4.4.127, $\alpha - \iota_v A$ is a Hamiltonian for v; and the Lie bracket is given by

$$[(v_1, \alpha_1), (v_2, \alpha_2)] = ([v_1, v_2], \mathcal{L}_{v_2}\alpha_1 - \mathcal{L}_{v_1}\alpha_2).$$

Notice that these pairs are redundant in that v is entirely determined by α, we just use them to make explicit the embedding into the semidirect product.

It remains to check that with this bracket the map

$$\phi : \alpha \mapsto \alpha - \iota_v A$$

is a Lie algebra isomorphism to the Poisson-bracket Lie algebra, def. 4.4.128. For this first notice the equation

$$\begin{aligned} 2\iota_{v_2}\iota_{v_1}\omega &= \iota_{v_2} d_{\mathrm{dR}} h_1 - \iota_{v_1} d_{\mathrm{dR}} h_2 \\ &= \mathcal{L}_{v_2}(\alpha_1 - \iota_{v_1} A) - \mathcal{L}_{v_1}(\alpha_2 - \iota_{v_2} A) \\ &= \mathcal{L}_{v_2}\alpha_1 - \mathcal{L}_{v_1}\alpha_2 + \iota_{v_2}\iota_{v_1} d_{\mathrm{dR}} A - \iota_{[v_1, v_2]} A, \end{aligned}$$

where in the last step we used the identity

$$\iota_{v_2}\iota_{v_1} d_{\mathrm{dR}} A = \mathcal{L}_{v_1}\iota_{v_2} A - \mathcal{L}_{v_2}\iota_{v_1} A - \iota_{[v_1, v_2]} A.$$

Subtracting $\iota_{v_2}\iota_{v_1}\omega = \iota_{v_2}\iota_{v_1} d_{\mathrm{dR}} A$ on both sides yields

$$[h_1, h_2] = \mathcal{L}_{v_2}\alpha_1 - \mathcal{L}_{v_1}\alpha_2 - \iota_{[v_1, v_2]} A.$$

This is equivalently the equation

$$\begin{aligned} [\phi(\alpha_1), \phi(\alpha_2)] &= \mathcal{L}_{v_2}\alpha_1 - \mathcal{L}_{v_1}\alpha_2 - \iota_{[v_1, v_2]} A \\ &= \phi([\alpha_1, \alpha_2]) \end{aligned},$$

which exhibits ϕ as a Lie algebra homomorphism. □

We recover the following traditional facts from the general notions of 3.9.13.

Observation 4.4.130. The *Poisson-bracket group* of the symplectic manifold $(X, \hat{\omega})$ according to def. 3.9.94 is a central extension by $U(1)$ of the group of hamiltonian symplectomorphisms: we have a short exact sequence of smooth groups

$$U(1) \to \mathrm{Poisson}(X, \hat{\omega}) \to \mathrm{HamSympl}(X, \hat{\omega}).$$

On Lie algebras this exhibits the Poisson-bracket Lie algebra as a central extension of the Lie algebra of Hamiltonian vector fields.

$$\mathbb{R} \to \mathfrak{poisson}(X, \hat{\omega}) \to \mathcal{X}_{\mathrm{ham}}(X, \hat{\omega}).$$

If (X, ω) is a *symplectic vector space* in that X is a vector space and the symplectic differential form ω is constant with respect to (left or right) translation along X, then the *Heisenberg Lie algebra* is the sub Lie algebra

$$\mathfrak{heis}(X, \hat{\omega}) \hookrightarrow \mathfrak{poisson}(X, \hat{\omega})$$

on the constant and the linear functions, see remark 3.9.95.

Traditional literature knows different conventions about which Lie group to pick by default as the one integrating a Heisenberg Lie algebra (the unique simply-connected one or one of its discrete quotients). By remark 3.9.95 the inclusion

$$\mathrm{Heis}(X, \hat{\omega}) \hookrightarrow \mathrm{Poisson}(X, \hat{\omega})$$

picks the one where the central part is integrated to the circle group:

$$\mathrm{Heis}(X, \hat{\omega}) \simeq X \times U(1).$$

If in this decomposition we write the canonical generator in

$$\mathfrak{heis}(X, \hat{\omega}) \simeq X \oplus \mathfrak{u}(1)$$

of the summand $\mathfrak{u}(1) = \mathrm{Lie}(U(1))$ as "i" then the Lie bracket on $\mathfrak{heis}(X, \hat{\omega})$ is given on any two $f, g \in X$ by

$$[f, g] = \mathrm{i}\omega(f, g)\,.$$

Specifically for the special case $X = \mathbb{R}^2$ with canonical basis vectors denoted $\hat{q}$ and $\hat{p}$, and with ω the canonical symplectic form, the only nontrivial bracket in $\mathfrak{heis}(X, \hat{\omega})$ among these generators is

$$[\hat{q}, \hat{p}]_{\mathfrak{heis}} = \mathrm{i}\,.$$

The image of this equation under the map $\mathfrak{heis}(X, \hat{\omega}) \to \mathcal{X}_{\mathrm{Ham}}(X, \hat{\omega})$ is

$$[q, p]_{\mathcal{X}} = 0\,,$$

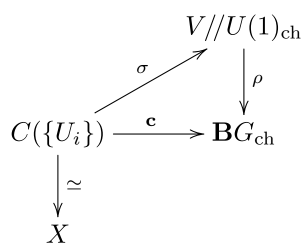
where now q, p denote the Hamiltonian vector fields associated with $\hat{q}$ and $\hat{p}$, respectively. The lift from the latter to the former equation is, historically, the archetypical hallmark of quantization.

Proposition 4.4.131. *For (X, ω) an ordinary prequantizable symplectic manifold and $\nabla : X \to \mathbf{B}^1 U(1)$ any choice of prequantum bundle, def. 3.9.77, let $V := \mathbb{C}$ and let ρ be the canonical representation of $U(1)$.*

Then def. 3.9.102 reduces to the traditional definition to prequantum operators in geometric quantization.

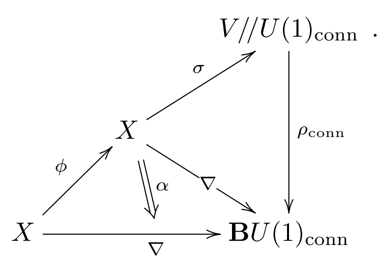
Proof. According to the discussion in 5.4.2 the space of sections $\Gamma_X(E)$ is that of the ordinary sections of the ordinary associated line bundle.

Notice that part of the statement there is that the standard presentation of $\rho : V//U(1) \to \mathbf{B}U(1)$ by a morphism of simplicial presheaves $V//U(1)_{\mathrm{ch}} \to \mathbf{B}U(1)_{\mathrm{ch}}$ is a fibration. In particular this means, as used there, that the ∞-groupoid of sections *up to homotopy* is presented already by the Kan complex (which here is just a set) of strict sections σ



and it is these that directly identify with the ordinary sections of the line bundle $E \to X$.

Now, a Hamiltonian diffeomorphism in the general sense of def. 3.9.102 takes such a section σ to the pasting composite



By the above, to identify this with a section of the line bundle in the ordinary sense, we need to find an equivalent homotopy-section whose homotopy is, however, trivial, hence a strict section which is equivalent to this as a homotopy section.

Inspection shows that there is a unique such equivalence whose underlying natural transformations has components induced by the inverse of α. Then for $h : X \to \mathbb{C}$ a given function and $t \mapsto (\phi(t), \alpha(t))$ the family of Hamiltonian diffeomorphism associated to it by prop. 4.4.127, the proof of that proposition shows that the infinitesimal difference between the original section σ and this new section is

$$i\nabla_{v_h}\sigma + h \cdot \sigma \,,$$

where v_h is the ordinary Hamiltonian vector field induced by h. This is the traditional formula for the action of the prequantum operator $\hat{h}$ on prequantum states. □

4.4.20.2 2-Plectic geometry and its prequantization We consider now the general notion of higher geometric prequantization, 3.9.13, specialized to the case of closed 3-forms on smooth manifolds, canonically regarded in Smooth∞Grpd. We show that this reproduces the *2-plectic geometry* and its prequantization studied in [Rog11].

The following two definitions are from [Rog11], def. 3.1, prop. 3.15.

Definition 4.4.132. A *2-plectic structure* on a smooth manifold X is a smooth closed differential 3-form $\omega \in \Omega^3_{\mathrm{cl}}(X)$, which is non-degenerate in that the induced morphism

$$\iota_{(-)}\omega : \Gamma(TX) \to \Omega^2(X)$$

has trivial kernel.

Definition 4.4.133. Let (X, ω) be a 2-plectic manifold. Then a 1-form $h \in \Omega^1(X)$ is called *Hamiltonian* if there exists a vector field $v \in \Gamma(TX)$ such that

$$d_{\mathrm{dR}} h = \iota_v \omega \,.$$

If this vector field exists, then it is unique and is called the *Hamiltonian vector field* corresponding to α. We write v_h to indicate this. We write

$$\Omega^1(X)_{\mathrm{Ham}} \hookrightarrow \Omega^1(X)$$

for the vector space of Hamiltonian 1-forms on (X, ω).

The *Lie 2-algebra of Hamiltonian vector fields* $L_\infty(X, \omega)$ is the (infinite-dimensional) L_∞-algebra, def. 1.2.114, whose underlying chain complex is

$$\cdots \longrightarrow 0 \longrightarrow C^\infty(X) \xrightarrow{d_{\mathrm{dR}}} \Omega^1_{\mathrm{Ham}}(X) \,,$$

whose non-trivial binary bracket is

$$[-,-] : (h_1, h_2) \mapsto \iota_{v_{h_2}} \iota_{v_{h_1}} \omega$$

and whose non-trivial trinary bracket is

$$[-,-,-] : (h_1, h_2, h_3) \mapsto \iota_{v_{h_1}} \iota_{v_{h_2}} \iota_{v_{h_3}} \omega \,.$$

Proposition 4.4.134. *Let (X, ω) be a 2-plectic smooth manifold, canonically regarded in* Smooth∞Grpd. *Then for $\hat{\omega} : X \to \mathbf{B}^2 U(1)_{\mathrm{conn}}$ any prequantum circle 2-bundle with connection (see 4.4.16) for ω, its Poisson Lie 2-algebra, def. 3.9.94, is equivalent to the Lie 2-algebra $L_\infty(X, \omega)$ from def. 4.4.133:*

$$\mathfrak{poisson}(X, \hat{\omega}) \simeq L_\infty(X, \omega) \,.$$

Proof. As in the proof of prop. 4.4.127, we first consider the case that ω is exact, so that there exists a globally defined 2-form $A \in \Omega^2(X)$ with $d_{\mathrm{dR}} A = \omega$. The general case follows from this by working on the path fibration surjective submersion, in straightforward generalization of the strategy in the proof of prop. 4.4.127.

By def. 3.9.94, an object of the smooth 2-group $\mathrm{Poisson}(X, \hat{\omega})$ is a diagram of smooth 2-groupoids

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ & {\scriptstyle A}\searrow \quad \Leftarrow_{\alpha} \quad \swarrow{\scriptstyle A} & \\ & \mathbf{B}^2 U(1)_{\mathrm{conn}} & \end{array} ,$$

such that map ϕ is a diffeomorphism. Given ϕ, such diagrams correspond to $\alpha \in \Omega^1(X)$ such that

$$(\phi^* A - A) = d_{\mathrm{dR}} \alpha \,. \tag{4.1}$$

Morphisms in the 2-group may go between two such objects $(f) : (\phi, \alpha_1) \to (\phi, \alpha_2)$ with the same ϕ and are given by $f \in C^\infty(X, U(1))$ such that

$$\alpha_2 = \alpha_1 + d_{\mathrm{dR}} \mathrm{log} f \,.$$

Under the 2-group product the objects (ϕ, α) form a genuine group with multiplication given by

$$(\phi_1, \alpha_1) \cdot (\phi_2, \alpha_2) = (\phi_2 \circ \phi_1, \alpha_1 + \phi_1^* \alpha_2) \,.$$

Similarly the group product on two morphisms $(f_1), (f_2) : (\phi, \alpha_1) \to (\phi, \alpha_2)$ is given by

$$(f_1) \cdot (f_2) = f_1 \cdot \phi^* f_2 \,.$$

Therefore this is a *strict* 2-group, def. 1.2.45, given by the subobject of the crossed module

$$C^\infty(X, U(1)) \xrightarrow{(0, d_{\mathrm{dR}} \mathrm{log})} \mathrm{Diff}(X) \ltimes \Omega^1(X)$$

on those pairs of vector fields and 1-forms that satisfy (4.1). Here $\mathrm{Diff}(X) \ltimes \Omega^1(X)$ is the semidirect product group induced by the pullback action on the additive group of 1-forms, and its action on $C^\infty(X, U(1))$ is again by the pullback action of the $\mathrm{Diff}(X)$-factor.

Therefore the L_∞-algebra $\mathfrak{poisson}(X, \hat{\omega})$ may be identified with the subobject of the corresponding strict Lie 2-algebra given by the differential crossed module, def. 1.2.46,

$$C^\infty(X) \xrightarrow{d_{\mathrm{dR}}} \Gamma(TX) \oplus \Omega^1(X)$$

on those pairs $(v, \alpha) \in \Gamma(TX) \times \Omega^1(X)$ for which

$$\mathcal{L}_v A = d_{\mathrm{dR}} \alpha \,,$$

hence, by Cartan's formula, for which

$$h := \alpha - \iota_v A$$

is a Hamiltonian 1-form for v, def. 4.4.133. Here $\Gamma(TX) \oplus \Omega^1(X)$ is the semidirect product Lie algebra with bracket

$$[(v_1, \alpha_1), (v_2, \alpha_2)] = ([v_1, v_2], \mathcal{L}_{v_2} \alpha_1 - \mathcal{L}_{v_1} \alpha_2)$$

and its action on $f \in C^\infty(X)$ is by Lie derivatives of the $\Gamma(TX)$-summand:

$$[(v, \alpha), f] = -\mathcal{L}_v f \,.$$

For emphasis, we write $\Omega^1_{\mathrm{Ham},p} \subset \Gamma(TX) \oplus \Omega^1(X)$ for the vector space of pairs (v, α) with $\alpha - \iota_v A$ Hamiltonian. The map $\phi : (\alpha, v) \mapsto \alpha - \iota_v A$ consistutes a vector space isomorphism

$$\phi : \Omega^1_{\mathrm{Ham},p} \stackrel{\simeq}{\to} \Omega^1_{\mathrm{Ham}}$$

and for the moment it is useful to keep this around explicitly. So $\mathfrak{poisson}(X, \hat{\omega})$ is given by the differential crossed module on the top of the diagram

$$\begin{array}{ccc} C^\infty(X) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1_{\mathrm{Ham},p}(X) \\ \downarrow{=} & & \downarrow \\ C^\infty(X) & \xrightarrow{d_{\mathrm{dR}}} & \Gamma(TX) \oplus \Omega^1(X) \end{array} ,$$

with brackets induced by this inclusion into the crossed module on the bottom.

We need to check that with these brackets the chain map

$$\begin{array}{ccc} C^\infty(X) & \xrightarrow{\mathrm{id}} & C^\infty(X) \\ \downarrow{d_{\mathrm{dR}}} & & \downarrow{d_{\mathrm{dR}}} \\ \Omega^1(X)_{\mathrm{Ham},p} & \xrightarrow{\phi} & \Omega^1(X)_{\mathrm{Ham}} \\ & & \\ [-,-] & & ([-,-]', J) \end{array}$$

is a Lie 2-algebra equivalence from the strict brackets $[-,-]$ to the brackets $([-,-]', [-,-,-]')$ of def. 4.4.133.

To that end, first notice the equation

$$\begin{aligned} 2\iota_{v_2}\iota_{v_1}\omega &= \iota_{v_2} d_{\mathrm{dR}} h_1 - \iota_{v_1} d_{\mathrm{dR}} h_2 \\ &= \mathcal{L}_{v_2}(\alpha_1 - \iota_{v_1} A) - \mathcal{L}_{v_1}(\alpha_2 - \iota_{v_2} A) + d_{\mathrm{dR}}(\iota_{v_1} h_2 - \iota_{v_2} h_1) \\ &= \mathcal{L}_{v_2}\alpha_1 - \mathcal{L}_{v_1}\alpha_2 + \iota_{v_2}\iota_{v_1} d_{\mathrm{dR}} A - \iota_{[v_1,v_2]} A + d_{\mathrm{dR}}(\iota_{v_1} h_2 - \iota_{v_2} h_1 - \iota_{v_2}\iota_{v_1} A) , \end{aligned}$$

where in the last step we used the identity

$$\iota_{v_2}\iota_{v_1} d_{\mathrm{dR}} A = \mathcal{L}_{v_1}\iota_{v_2} A - \mathcal{L}_{v_2}\iota_{v_1} A - \iota_{[v_1,v_2]} A + d_{\mathrm{dR}}\iota_{v_2}\iota_{v_1} A .$$

Subtracting $\iota_{v_2}\iota_{v_1}\omega = \iota_{v_2}\iota_{v_1} d_{\mathrm{dR}} A$ on both sides yields

$$\iota_{v_2}\iota_{v_1}\omega = \mathcal{L}_{v_2}\alpha_1 - \mathcal{L}_{v_1}\alpha_2 - \iota_{[v_1,v_2]} A + d_{\mathrm{dR}}(\iota_{v_1} h_2 - \iota_{v_2} h_1 - \iota_{v_2}\iota_{v_1} A) ,$$

Here on the left we have the bracket of h_1 with h_2 in def. 4.4.133, which we will write $[h_1, h_2]' := [\phi(v_1, \alpha_1), \phi(v_2, \alpha_2)]'$, whereas the first three terms on the right are the image under ϕ of the bracket from above, to be written $\phi[(v_1, \alpha_1), (v_2, \alpha_2)]$. Therefore this equation says that

$$[\phi(v_1, \alpha_1), \phi(v_2, \alpha_2)]' = \phi([(v_1, \alpha_1), (v_2, \alpha_2)]) + d_{\mathrm{dR}}(\iota_{v_1}\phi(v_2, \alpha_2) - \iota_{v_2}\phi(v_1, \alpha_1) - \iota_{v_2}\iota_{v_1} A) . \qquad (4.2)$$

In view of the exact term on the far right, this implies that the map

$$\Phi : \Omega^1(X)_{\mathrm{Ham},} \otimes \Omega^1(X)_{\mathrm{Ham},p} \to C^\infty(X)$$

given by

$$\Phi : (h_1 = \alpha_1 - \iota_{v_1} A, h_2 = \alpha_2 - \iota_{v_2} A) \mapsto \iota_{v_1} h_2 - \iota_{v_2} h_1 - \iota_{v_2}\iota_{v_1} A$$

should be a chain homotopy between the binary brackets

$$\begin{array}{ccc} (\Omega^1(X)_{\mathrm{Ham},p} \otimes C^\infty(X)) \oplus (C^\infty(X) \otimes \Omega^1(X)_{\mathrm{Ham},p}) & \xrightarrow{[-,-]'-[-,-]} & C^\infty(X) \\ \downarrow{\scriptstyle (\mathrm{id}\otimes d_{\mathrm{dR}})\oplus(d_{\mathrm{dR}}\otimes \mathrm{id})} & \nearrow{\scriptstyle \Phi} & \downarrow{\scriptstyle d_{\mathrm{dR}}} \\ \Omega^1(X)_{\mathrm{Ham},p} \otimes \Omega^1(X)_{\mathrm{Ham},p} & \xrightarrow[{[\phi(-),\phi(-)]'-\phi([-,-])}]{} & \Omega^1_{\mathrm{Ham}}(X) \end{array} .$$

Indeed, the bottom right triangle commutes manifestly, by equation (4.2). For the top left triangle notice that $[-,-]'$ vanishes here, by definition, and $[-,-]$ is given by

$$[(v,\alpha), f] = -\mathcal{L}_v f \,.$$

On the other hand, since the Hamiltonian vector field of $d_{\mathrm{dR}} f$ vanishes, we also have

$$\begin{aligned} \Phi((v,\alpha),(0,d_{\mathrm{dR}}f)) &= \iota_v d_{\mathrm{dR}} f \\ &= \mathcal{L}_v f \end{aligned} .$$

It remains to check that Φ respects the Jacobiator, sending the trivial one on $\Omega^1(X)_{\mathrm{Ham},p}$ to the nontrivial one of def. 4.4.133. From now on we leave the isomorphism $\phi : \Omega^1(X)_{\mathrm{Ham},p} \xrightarrow{\simeq} \Omega^1(X)_{\mathrm{Ham}}$ implicit, regarding $[-,-]'$ and $[-,-]$ as two different brackets on the same vector space.

Observe that generally, with a chain homotopy of binary brackets Φ given as above, setting

$$J(h_1,h_2,h_3) := \Phi(h_1,[h_2,h_3]) + \mathrm{cyc}$$

for all h_1,h_2,h_3 makes the collection of brackets $([-,-]',J)$ (extended by 0 to $C^\infty(X)$) a Lie 2-algebra structure on $C^\infty(X) \to \Omega^1(X)_{\mathrm{Ham}}$ such that (ϕ,Φ) a Lie 2-algebra equivalence. Notice that we may equivalently write

$$J(h_1,h_2,h_3) = -\Phi(D(h_1 \vee h_2 \vee h_3))\,,$$

where $(\vee^\bullet \Omega^1(X)_{\mathrm{Ham}}, D)$ is the differential coalgebra incarnation of the Lie algebra $[-,-]$.

Indeed, J vanishes on the image of d_{dR}, because

$$\begin{aligned} \Phi(d_{\mathrm{dR}}f,[h_2,h_3]) + \Phi(h_2,[h_3,d_{\mathrm{dR}}f]) + \Phi(h_3,[d_{\mathrm{dR}}f,h_2]) &= -d_{\mathrm{dR}}\left([f,[h_2,h_3 + [h_2,[h_3,f + [h_3,[f,h_2)\right. \\ &= 0 \end{aligned} ,$$

where we used the chain homotopy property of ϕ and the identities of the differential crossed module $[-,-]$.

Using this, the coherence law of the Jacobiator, which a priori involves $[-,-]'$, is equivalently formulated in terms of $[-,-]$ (because the two differ by something in the image of d_{dR}), where it then reads

$$J(D(h_1 \vee h_2 \vee h_3 \vee h_4)) = 0\,,$$

with $(\vee^\bullet \Omega^1(X)_{\mathrm{Ham}}, D)$ as before. This equation follows now due to $D^2 = 0$.

Finally, to see that J as above indeed is a Jacobiator for $[-,-]'$ we compute

$$\begin{aligned} [h_1,[h_2,h_3]']' + \mathrm{cyc} &= [h_1,[h_2,h_3 + d_{\mathrm{dR}}\Phi(h_2,h_3)]' + \mathrm{cyc} \\ &= [h_1,[h_2,h_3 + [h_1,d_{\mathrm{dR}}\Phi(h_2,h_3)] + d_{\mathrm{dR}}\Phi(h_1,[h_2,h_3] + d_{\mathrm{dR}}\Phi(h_2,h_3)) + \mathrm{cyc}\,, \\ &= d_{\mathrm{dR}}\Phi(h_1,[h_2,h_3]) + \mathrm{cyc} \end{aligned}$$

where in the last step the first summand disappears due to the Jacobi identity satisfied by $[-,-]$, and where we used the chain homotopy propoerty of Φ to cancel two terms.

This way we have produced an equivalence of Lie 2-algebras

$$(\phi,\Phi) : \mathfrak{poisson}(X,\hat{\omega}) \to ((C^\infty(X) \to \Omega^1(X)_{\mathrm{Ham}}), [-,-]', J)\,,$$

where on the right the binary bracket is that of def. 4.4.133. The last thing to check is that the Jacobiator J is indeed that of def. 4.4.133. But since the differential in the Lie 2-algebra is d_{dR}, any two Jacobiators for the same binary bracket must differ by a constant function on X. Since at the same time the Jacobiators are linear, that constant must be 0, and hence the two Jacobiators must coincide. □

4.5 Synthetic differential ∞-groupoids

We discuss ∞-groupoids equipped with *synthetic differential cohesion*, a version of smooth cohesion in which an explicit notion of smooth *infinitesimal* spaces exists.

Notice that the category $\mathrm{CartSp}_{\mathrm{smooth}}$, def. 4.4.4, is (the syntactic category of) a finitary algebraic theory: a *Lawvere theory* (see chapter 3, volume 2 of [Borc94]).

Definition 4.5.1. Write

$$\mathrm{SmoothAlg} := \mathrm{Alg}(\mathrm{CartSp}_{\mathrm{smooth}})$$

for the category of algebras over the algebraic theory $\mathrm{CartSp}_{\mathrm{smooth}}$: the category of product-preserving functors $\mathrm{CartSp}_{\mathrm{smooth}} \to \mathrm{Set}$.

These algebras are traditionally known as C^∞*-rings* or C^∞*-algebras* [KaKrMi87].

Proposition 4.5.2. *The map that sends a smooth manifold X to the product-preserving functor*

$$C^\infty(X) : \mathbb{R}^k \mapsto \mathrm{SmoothMfd}(X, \mathbb{R}^k)$$

extends to a full and faithful embedding

$$\mathrm{SmoothMfd} \hookrightarrow \mathrm{SmoothAlg}^{\mathrm{op}} .$$

Proposition 4.5.3. *Let A be an ordinary (associative) $\mathbb{R}$-algebra that as an $\mathbb{R}$-vector space splits as $\mathbb{R} \oplus V$ with V finite dimensional as an $\mathbb{R}$-vector space and nilpotent with respect to the algebra structure: $(v \in V \hookrightarrow A) \Rightarrow (v^2 = 0)$.*

There is a unique lift of A through the forgetful functor $\mathrm{SmoothAlg} \to \mathrm{Alg}_{\mathbb{R}}$.

Proof. Use Hadamard's lemma. □

Remark 4.5.4. In the context of synthetic differential geometry the algebras of prop. 4.5.3 are usually called *Weil algebras*. In other contexts however the underlying rings are known as *Artin rings*, see for instance [LurieFormalGeometry].

Definition 4.5.5. Write

$$\mathrm{InfSmoothLoc} \hookrightarrow \mathrm{SmoothAlg}^{\mathrm{op}}$$

for the full subcategory of the opposite of smooth algebras on those of the form of prop. 4.5.3. We call this the category of *infinitesimal smooth loci* or of *infinitesimally thickened points.*

Write

$$\mathrm{CartSp}_{\mathrm{synthdiff}} := \mathrm{CartSp}_{\mathrm{smooth}} \ltimes \mathrm{InfSmoothLoc} \hookrightarrow \mathrm{SmoothAlg}^{\mathrm{op}}$$

for the full subcategory of the opposite of smooth algebras on those that are products

$$X \simeq U \times D$$

in $\mathrm{SmoothAlg}^{\mathrm{op}}$ of an object U in the image of $\mathrm{CartSp}_{\mathrm{smooth}} \hookrightarrow \mathrm{SmoothMfd} \hookrightarrow \mathrm{SmoothAlg}^{\mathrm{op}}$ and an object D in the image of $\mathrm{InfSmoothLoc} \hookrightarrow \mathrm{SmoothAlg}^{\mathrm{op}}$.

Define a coverage on $\mathrm{CartSp}_{\mathrm{synthdiff}}$ whose covering families are precisely those of the form $\{U_i \times D \stackrel{(f_i,\mathrm{id})}{\to} U \times D\}$ for $\{U_i \stackrel{f_i}{\to} U\}$ a covering family in $\mathrm{CartSp}_{\mathrm{smooth}}$.

Remark 4.5.6. This definition appears in [Kock86], following [Dub79b]. The sheaf topos $\mathrm{Sh}(\mathrm{CartSp}_{\mathrm{synthdiff}}) \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$ over this site is equivalent to the *Cahiers topos* [Dub79b] which is a model of some set of axioms of *synthetic differential geometry* (see [Lawv97] for the abstract idea, where also the relation to the axiomatics of cohesion is vaguely indicated). Therefore the following definition may be thought of as describing the ∞-*Cahiers topos* providing a higher geometry version of this model of synthetic differential smooth geometry.

Definition 4.5.7. The ∞-topos of *synthetic differential smooth ∞-groupoids* is

$$\mathrm{SynthDiff}\infty\mathrm{Grpd} := \mathrm{Sh}_{(\infty,1)}(\mathrm{CartSp}_{\mathrm{synthdiff}})\,.$$

Proposition 4.5.8. SynthDiff∞Grpd *is a cohesive ∞-topos.*

Proof. Using that the covering families of $\mathrm{CartSp}_{\mathrm{synthdiff}}$ do by definition not depend on the infinitesimal smooth loci D and that these each have a single point, one finds that $\mathrm{CartSp}_{\mathrm{synthdiff}}$ is an ∞-cohesive site, def. 3.4.8, by reducing to the argument as for $\mathrm{CartSp}_{\mathrm{top}}$, prop. 4.3.2. The claim then follows with prop. 3.4.9. □

Definition 4.5.9. Write FSmoothMfd for the category of *formal smooth manifolds* – manifolds modeled on $\mathrm{CartSp}_{\mathrm{synthdiff}}$, equipped with the induced site structure.

Proposition 4.5.10. *We have an equivalence of ∞-categories*

$$\mathrm{SynthDiff}\infty\mathrm{Grpd} \simeq \hat{\mathrm{Sh}}_{(\infty,1)}(\mathrm{FSmoothMfd})$$

with the hypercomplete ∞-topos over formal smooth manifolds.

Proof. By definition $\mathrm{CartSp}_{\mathrm{synthdiff}}$ is a dense sub-site of FSmoothMfd. The statement then follows as in prop. 4.3.7. □

Write $i : \mathrm{CartSp}_{\mathrm{smooth}} \hookrightarrow \mathrm{CartSp}_{\mathrm{synthdiff}}$ for the canonical embedding.

Proposition 4.5.11. *The functor i^* given by restriction along i exhibits* SynthDiff∞Grpd *as an infinitesimal cohesive neighbourhood, def. 3.5.1, of* Smooth∞Grpd, *in that we have a quadruple of adjoint ∞-functors*

$$(i_! \dashv i^* \dashv i_* \dashv i^!) : \mathrm{Smooth}\infty\mathrm{Grpd} \to \mathrm{SynthDiff}\infty\mathrm{Grpd}\,,$$

such that $i_!$ is full and faithful and preserves the terminal object.

Proof. We observe that $\mathrm{CartSp}_{\mathrm{smooth}} \hookrightarrow \mathrm{CartSp}_{\mathrm{synthdiff}}$ is an infinitesimal neighbourhood of sites, according to def. 3.5.5. The claim then follows with prop. 3.5.6. □

We now discuss the general abstract structures in cohesive ∞-toposes, 3.9 and 3.5, realized in SynthDiff∞Grpd

- 4.5.1 – ∞-Lie algebroids;
- 4.5.6 – Formally smooth/étale/unramified morphisms;
- 4.5.7 – Formally étale groupoids;
- 4.5.2 – Manifolds
- 4.5.3 – Cohomology;
- 4.5.5 – Paths and geometric Postnikov towers;
- 4.5.8 – Chern-Weil theory.

4.5.1 ∞-Lie algebroids

We discuss explicit presentations for first order formal cohesive ∞-groupoids, 3.10.9, realized in SynthDiff∞Grpd. We call these L_∞*-algebroids*, subsuming the traditional notion of L_∞*-algebras* [LaMa95].

In the standard presentation of SynthDiff∞Grpd by simplicial presheaves over formal smooth manifolds these L_∞-algebroids are presheaves in the image of the *monoidal Dold-Kan correspondence* [CaCo04] of semi-free differential graded algebras. This construction amounts to identifying the traditional description of Lie algebras, Lie algebroids and L_∞-algebras by their Chevalley-Eilenberg algebras, def. 1.2.114, as a convenient characterization of the corresponding cosimplicial algebras whose formal dual simplicial presheaves are manifest presentations of infinitesimal smooth ∞-groupoids.

- 4.5.1.1 – L_∞-Algebroids and smooth commutative dg-algebras;
- 4.5.1.2 – Infinitesimal smooth ∞-groupoids;
- 4.5.1.3 – Lie 1-algebroids as infinitesimal simplicial presheaves

4.5.1.1 L_∞-Algebroids and smooth commutative dg-algebras Recall the characterization of L_∞-algebra structures in terms of dg-algebras from prop. 1.2.116.

Definition 4.5.12. Let

$$\mathrm{CE} : L_\infty \mathrm{Algd} \hookrightarrow \mathrm{cdgAlg}_{\mathbb{R}}^{\mathrm{op}}$$

be the full subcategory on the opposite category of cochain dg-algebras over $\mathbb{R}$ on those dg-algebras that are

- graded-commutative;
- concentrated in non-negative degree (the differential being of degree $+1$);
- in degree 0 of the form $C^\infty(X)$ for $X \in \mathrm{SmoothMfd}$;
- semifree: their underlying graded algebra is isomorphic to an exterior algebra on an $\mathbb{N}$-graded locally free projective $C^\infty(X)$-module;
- of finite type;

We call this the category of L_∞*-algebroids* over smooth manifolds.

More in detail, an object $\mathfrak{a} \in L_\infty\mathrm{Algd}$ may be identified (non-canonically) with a pair $(\mathrm{CE}(\mathfrak{a}), X)$, where

- $X \in \mathrm{SmoothMfd}$ is a smooth manifold – called the *base space* of the L_∞-algebroid ;
- $\mathfrak{a}$ is the module of smooth sections of an $\mathbb{N}$-graded vector bundle of degreewise finite rank;

- $\mathrm{CE}(\mathfrak{a}) = (\wedge^\bullet_{C^\infty(X)} \mathfrak{a}^*, d_{\mathfrak{a}})$ is a semifree dg-algebra on $\mathfrak{a}^*$ – a Chevalley-Eilenberg algebra – where

$$\wedge^\bullet_{C^\infty(X)} \mathfrak{a}^* = C^\infty(X) \;\oplus\; \mathfrak{a}_0^* \;\oplus\; \left((\mathfrak{a}_0^* \wedge_{C^\infty(X)} \mathfrak{a}_0^*) \oplus \mathfrak{a}_1^* \right) \;\oplus\; \cdots$$

with the kth summand on the right being in degree k.

Definition 4.5.13. An L_∞-algebroid with base space $X = *$ the point is an L_∞-algebra $\mathfrak{g}$, def. 1.2.114, or rather is the pointed delooping of an L_∞-algebra. We write $b\mathfrak{g}$ for L_∞-algebroids over the point. They form the full subcategory

$$L_\infty \mathrm{Alg} \hookrightarrow L_\infty \mathrm{Algd}\,.$$

The following fact is standard and straightforward to check.

Proposition 4.5.14. *1. The full subcategory* $L_\infty\mathrm{Alg} \hookrightarrow L_\infty\mathrm{Algd}$ *from def. 4.5.12 is equivalent to the traditional definition of the category of L_∞-algebras and "weak morphisms" / "sh-maps" between them.*

2. *The full subcategory* $\mathrm{LieAlgd} \hookrightarrow L_\infty\mathrm{Algd}$ *on the 1-truncated objects is equivalent to the traditional category of Lie algebroids (over smooth manifolds).*

3. *In particular the joint intersection* $\mathrm{LieAlg} \hookrightarrow L_\infty\mathrm{Alg}$ *on the 1-truncated L_∞-algebras is equivalent to the category of ordinary Lie algebras.*

We now construct an embedding of $L_\infty\mathrm{Algd}$ into $\mathrm{SynthDiff}\infty\mathrm{Grpd}$. Below in 4.5.1.2 we show that this embedding exhibitsthe above algebraic data as a presentation of synthetic differential ∞-groupoids which are infinitesimal objects in the abstract intrinsic sense of 4.5.1.2.

Remark 4.5.15. The functor

$$\Xi : \mathrm{Ch}^\bullet_+(\mathbb{R}) \to \mathrm{Vect}^\Delta_{\mathbb{R}}$$

of the Dold-Kan correspondence, prop. 2.2.31, from non-negatively graded cochain complexes of vector spaces to cosimplicial vector spaces is a lax monoidal functor and hence induces a functor (which we will denote by the same symbol)

$$\Xi : \mathrm{dgAlg}^+_{\mathbb{R}} \to \mathrm{Alg}^\Delta_{\mathbb{R}}$$

from non-negatively graded commutative cochain dg-algebras to cosimplicial commutative algebras (over $\mathbb{R}$).

Definition 4.5.16. Write

$$\Xi\mathrm{CE} : L_\infty\mathrm{Algd} \to (\mathrm{CAlg}^\Delta_{\mathbb{R}})^{\mathrm{op}}$$

for the restriction of the functor Ξ from remark 4.5.15 along the defining inclusion $\mathrm{CE} : L_\infty\mathrm{Algd} \hookrightarrow \mathrm{dgAlg}^{\mathrm{op}}_{\mathbb{R}}$.

There are several different ways to present $\Xi\mathrm{CE}$ explicitly in components. Below we make use of the following fact, pointed out in [CaCo04] (see the discussion around equations (26) and (49) there).

Proposition 4.5.17. *The functor* $\Xi\mathrm{CE}$ *from def. 4.5.16 is given as follows.*
For $\mathfrak{a} \in L_\infty\mathrm{Algd}$*, the underlying cosimplicial vector space of* $\Xi\mathrm{CE}(\mathfrak{a})$ *is*

$$\Xi\mathrm{CE}(\mathfrak{a}) : [n] \mapsto \bigoplus_{i=0}^{n} \mathrm{CE}(\mathfrak{a})_i \otimes \wedge^i \mathbb{R}^n\,.$$

The product of the $\mathbb{R}$*-algebra structure on this space in degree n is given on homogeneous elements* $(\omega, x), (\lambda, y) \in \mathrm{CE}(\mathfrak{a})_i \otimes \wedge^i \mathbb{R}^n$ *in the tensor product by*

$$(\omega, x) \cdot (\lambda, y) = (\omega \wedge \lambda, x \wedge y)\,.$$

(Notice that $\Xi\mathfrak{a}$ *is indeed a* commutative *cosimplicial algebra, since* ω *and* x *in* (ω, x) *are by definition in the same degree.)*

To define the cosimplicial structure, let $\{v_j\}_{j=1}^n$ be the canonical basis of $\mathbb{R}^n$ and consider and set $v_0 := 0$ to obtain a set of vectors $\{v_j\}_{j=0}^n$. Then for $\alpha : [k] \to [l]$ a morphism in the simplex category, set

$$\alpha : v_j \mapsto v_{\alpha(j)} - v_{\alpha(0)}$$

and extend this skew-multilinearly to a map $\alpha : \wedge^\bullet \mathbb{R}^k \to \wedge^\bullet \mathbb{R}^l$. In terms of all this the action of α on homogeneous elements (ω, x) in the cosimplicial algebra is defined by

$$\alpha : (\omega, x) \mapsto (\omega, \alpha x) + (d_{\mathfrak{a}}\omega, v_{\alpha(0)} \wedge \alpha(x))$$

Remark 4.5.18. The commutative algebras appearing here may be understood geometrically as being algebras of functions on spaces of infinitesimal based simplices. This we discuss in more detail in 4.5.1.3 below, see prop. 4.5.29 there.

We now refine the image of Ξ to cosimplicial *smooth* algebras, def. 4.5.1. Notice that there is a canonical forgetful functor

$$U : \mathrm{SmoothAlg} \to \mathrm{CAlg}_{\mathbb{R}}$$

from the category of smooth algebras to the category of commutative associative algebras over the real numbers.

Proposition 4.5.19. *There is a unique factorization of the functor $\Xi\mathrm{CE} : L_\infty\mathrm{Algd} \to (\mathrm{CAlg}_{\mathbb{R}}^\Delta)^{\mathrm{op}}$ from def. 4.5.16 through the forgetful functor $(\mathrm{SmoothAlg}_{\mathbb{R}}^\Delta)^{\mathrm{op}} \to (\mathrm{CAlg}_{\mathbb{R}}^\Delta)^{\mathrm{op}}$ such that for any $\mathfrak{a}$ over base space X the degree-0 algebra of smooth functions $C^\infty(X)$ lifts to its canonical structure as a smooth algebra*

$$\begin{array}{ccc} & & (\mathrm{SmoothAlg}^\Delta)^{\mathrm{op}} \\ & \overset{\Xi\mathrm{CE}}{\nearrow} & \downarrow U \\ L_\infty\mathrm{Algd} & \longrightarrow & (\mathrm{CAlg}_{\mathbb{R}}^\Delta)^{\mathrm{op}} \end{array} .$$

Proof. Observe that for each n the algebra $(\Xi\mathrm{CE}(\mathfrak{a}))_n$ is a finite nilpotent extension of $C^\infty(X)$. The claim then follows with the fact that $C^\infty : \mathrm{SmoothMfd} \to \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}$ is faithful and using Hadamard's lemma for the nilpotent part. □

Proposition 4.5.20. *The functor $\Xi\mathrm{CE}$ preserves limits of L_∞-algebras. It preserves pullbacks of L_∞-algebroids if the two morphisms in degree 0 are transveral maps of smooth manifolds.*

Proof. The functor $\Xi : \mathrm{cdgAlg}_{\mathbb{R}}^+ \to \mathrm{CAlg}_{\mathbb{R}}^\Delta$ evidently preserves colimits. This gives the first statement. The second follows by observing that the functor from smooth manifolds to the opposite of smooth algebras preserves transversal pullbacks. □

4.5.1.2 Infinitesimal smooth groupoids

We discuss how the L_∞-algebroids from def. 4.5.12 serve to present the intrinsically defined infinitesimal smooth ∞-groupoids from 3.10.9.

Definition 4.5.21. Write $i : L_\infty\mathrm{Algd} \to \mathrm{SynthDiff}\infty\mathrm{Grpd}$ for the composite ∞-functor

$$L_\infty\mathrm{Algd} \xrightarrow{\Xi\mathrm{CE}} (\mathrm{SmoothAlg}^\Delta)^{\mathrm{op}} \xrightarrow{j} [\mathrm{CartSp}_{\mathrm{synthdiff}}^{\mathrm{op}}, \mathrm{sSet}] \xrightarrow{PQ} ([\mathrm{CartSp}_{\mathrm{synthdiff}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{loc}})^\circ \xrightarrow{\simeq} \mathrm{SynthDiff}\infty\mathrm{Grpd} ,$$

where the first morphism is the monoidal Dold-Kan correspondence as in prop. 4.5.19, the second is degreewise the external Yoneda embedding

$$\mathrm{SmoothAlg}^{\mathrm{op}} \to [\mathrm{CartSp}_{\mathrm{synthdiff}}, \mathrm{Set}] ,$$

and PQ is any fibrant-cofibrant resolution functor in the local model structure on simplicial presheaves.

We discuss now that $L_\infty\mathrm{Algd}$ is indeed a presentation for objects in $\mathrm{SynthDiff}\infty\mathrm{Grpd}$ satisfying the abstract axioms from 3.10.9.

Lemma 4.5.22. *For* $\mathfrak{a} \in L_\infty\mathrm{Algd}$ *and* $i(\mathfrak{a}) \in [\mathrm{FSmoothMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *its image in the presentation for* $\mathrm{SynthDiff}\infty\mathrm{Grpd}$*, we have that*

$$\left(\int^{[k]\in\Delta} \mathbf{\Delta}[k]\cdot i(\mathfrak{a})_k\right) \stackrel{\simeq}{\to} i(\mathfrak{a})$$

is a cofibrant resolution, where $\mathbf{\Delta} : \Delta \to sSet$ *is the* fat simplex*.*

Proof. The coend over the tensoring

$$\int^{[k]\in\Delta}(-)\cdot(-) : [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}} \times [\Delta^{\mathrm{op}}, [\mathrm{FSmoothMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{inj}} \to [\mathrm{FSmoothMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

for the projective and injective global model structure on functors on the simplex category and its opposite is a left Quillen bifunctor, prop. 2.3.17. We have moreover

1. The fat simplex is cofibrant in $[\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$, prop. 2.3.19.
2. The object $i(\mathfrak{a})_\bullet \in [\Delta^{\mathrm{op}}, [\mathrm{FSmoothMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{inj}}$ is cofibrant, because every representable $\mathrm{FSmoothMfd} \hookrightarrow [\mathrm{FSmoothMfd}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ is cofibrant.

□

Proposition 4.5.23. *Let* $\mathfrak{g}$ *be an* L_∞*-algebra, regarded as an* L_∞*-algebroid* $b\mathfrak{g} \in L_\infty\mathrm{Algd}$ *over the point by the embedding of def. 4.5.12. Then* $i(b\mathfrak{g}) \in \mathrm{SynthDiff}\infty\mathrm{Grpd}$ *is an infinitesimal object, def. 3.10.49, in that it is geometrically contractible*

$$\Pi b\mathfrak{g} \simeq *$$

and has as underlying discrete ∞*-groupoid the point*

$$\Gamma b\mathfrak{g} \simeq * \,.$$

Proof. We present now $\mathrm{SynthDiff}\infty\mathrm{Grpd}$ by $[\mathrm{CartSp}_{\mathrm{synthdiff}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Since $\mathrm{CartSp}_{\mathrm{synthdiff}}$ is an ∞-cohesive site by prop. 4.5.8, we have by the proof of prop. 3.4.9 that Π is presented by the left derived functor $\mathbb{L}\lim_\to$ of the degreewise colimit and Γ is presented by the left derived functor of evaluation on the point.

With lemma 4.5.22 we can evaluate

$$\begin{aligned}(\mathbb{L}\lim_{\to})i(b\mathfrak{g}) &\simeq \lim_{\to}\int^{[k]\in\Delta}\mathbf{\Delta}[k]\cdot(b\mathfrak{g})_k \\ &\simeq \int^{[k]\in\Delta}\mathbf{\Delta}[k]\cdot\lim_{\to}(b\mathfrak{g})_k \;, \\ &= \int^{[k]\in\Delta}\mathbf{\Delta}[k]\cdot *\end{aligned}$$

because each $(b\mathfrak{g})_n \in \mathrm{InfPoint} \hookrightarrow \mathrm{CartSp}_{\mathrm{smooth}}$ is an infinitesimally thickened point, hence representable and hence sent to the point by the colimit functor.

That this is equivalent to the point follows from the fact that $\emptyset \to \mathbf{\Delta}$ is an acylic cofibration in $[\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$, and that

$$\int^{[k]\in\Delta}(-)\times(-) : [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}} \times [\Delta^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Qillen}}]_{\mathrm{inj}} \to \mathrm{sSet}_{\mathrm{Quillen}}$$

is a Quillen bifunctor, using that $* \in [\Delta^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{inj}}$ is cofibrant.

Similarly, we have degreewise that

$$\mathrm{Hom}(*, (b\mathfrak{g})_n) = *$$

by the fact that an infinitesimally thickened point has a single global point. Therefore the claim for Γ follows analogously. □

Proposition 4.5.24. *Let* $\mathfrak{a} \in L_\infty\mathrm{Algd} \hookrightarrow [\mathrm{CartSp}_{\mathrm{synthdiff}}, \mathrm{sSet}]$ *be an* L_∞*-algebroid, def. 4.5.12, over a smooth manifold* X*, regarded as a simplicial presheaf and hence as a presentation for an object in* SynthDiff∞Grpd *according to def. 4.5.21.*

We have an equivalence

$$\mathbf{\Pi}_{\mathrm{inf}}(\mathfrak{a}) \simeq \mathbf{\Pi}_{\mathrm{inf}}(X)\,.$$

Proof. Let first $X = U \in \mathrm{CartSp}_{\mathrm{synthdiff}}$ be a representable. Then according to prop. 4.5.22 we have that

$$\hat{\mathfrak{a}} := \left(\int^{k \in \Delta} \mathbf{\Delta}[k] \cdot \mathfrak{a}_k\right) \simeq \mathfrak{a}$$

is cofibrant in $[\mathrm{CartSp}_{\mathrm{synthdiff}}^{op}, \mathrm{sSet}]_{\mathrm{proj}}$. Therefore, by prop. 3.5.6, we compute the derived functor

$$\begin{aligned} \mathbf{\Pi}_{\mathrm{inf}}(\mathfrak{a}) &\simeq i_* i^* \mathfrak{a} \\ &\simeq \mathbb{L}((-) \circ p)\mathbb{L}((-) \circ i)\mathfrak{a} \\ &\simeq ((-) \circ ip)\hat{\mathfrak{a}} \end{aligned}$$

with the notation as used there. In view of def. 4.5.16 we have for all $k \in \mathbb{N}$ that $\mathfrak{a}_k = X \times D$ where D is an infinitesimally thickened point. Therefore $((-)\circ ip)\mathfrak{a}_k = ((-)\circ ip)X$ for all k and hence $((-)\circ ip)\hat{\mathfrak{a}} \simeq \mathbf{\Pi}_{\mathrm{inf}}(X)$.

For general X choose first a cofibrant resolution by a split hypercover that is degreewise a coproduct of representables (which always exists, by the cofibrant replacement theorem of [Dugg01]), then pull back the above discussion to these covers. □

Corollary 4.5.25. *Every* L_∞*-algebroid in the sense of def. 4.5.12 under the embedding of def. 4.5.21 is indeed a formal cohesive* ∞*-groupoid in the sense of def. 3.10.49.*

4.5.1.3 Lie 1-algebroids as infinitesimal simplicial presheaves We characterize Lie 1-algebroids $(E \to X, \rho, [-,-])$ as precisely those synthetic differential ∞-groupoids that under the presentation of def. 4.5.21 are locally, on any chart $U \to X$ of their base space, given by simplicial smooth loci of the form

$$\cdots\cdots U \times \tilde{D}(k,2) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; U \times \tilde{D}(k,1) \;\substack{\longrightarrow\\ \longrightarrow}\; U$$

where $k = \mathrm{rank}(E)$ is the dimension of the fibers of the Lie algebroid and where $\tilde{D}(k,n)$ is the smooth locus of *infinitesimal* k*-simplices* based at the origin in $\mathbb{R}^n$. (These smooth loci have been highlighted in section 1.2 of [Kock10]).

The following definition may be either taken as an informal but instructive definition – in which case the next definition 4.5.27 is to be taken as the precise one – or in fact it may be already itself be taken as the fully formal and precise definition if one reads it in the internal logic of any smooth topos with line object R – which for the present purpose is the *Cahiers topos* [Dub79b] $\mathrm{Sh}(\mathrm{CartSp}_{\mathrm{synthdiff}})$ with line object R, remark 4.5.6.

Definition 4.5.26. For $k, n \in \mathbb{N}$, an *infinitesimal* k*-simplex* in R^n based at the origin is a collection $(\vec{\epsilon}_a \in R^n)_{a=1}^k$ of points in R^n, such that each is an infinitesimal neighbour of the origin

$$\forall a: \;\; \vec{\epsilon}_a \sim 0$$

and such that all are infinitesimal neighbours of each other

$$\forall a, a' : \quad (\vec{\epsilon}_a - \vec{\epsilon}_{a'}) \sim 0\,.$$

Write $\tilde{D}(k, n) \subset R^{k \cdot n}$ for the space of all such infinitesimal k-simplices in R^n.

Equivalently:

Definition 4.5.27. For $k, n \in \mathbb{N}$, the smooth algebra

$$C^\infty(\tilde{D}(k, n)) \in \mathrm{SmoothAlg}$$

is the unique lift through the forgetful functor $U : \mathrm{SmoothAlg} \to \mathrm{CAlg}_{\mathbb{R}}$ of the commutative $\mathbb{R}$-algebra generated from $k \times n$ many generators

$$(\epsilon_a^j)_{1 \leq j \leq n, 1 \leq a \leq k}$$

subject to the relations

$$\forall a, j, j' : \quad \epsilon_a^j \epsilon_a^{j'} = 0$$

and

$$\forall a, a', j, j' : \quad (\epsilon_a^j - \epsilon_{a'}^j)(\epsilon_a^{j'} - \epsilon_{a'}^{j'}) = 0\,.$$

Remark 4.5.28. In the above form these relations are the manifest analogs of the conditions $\vec{\epsilon}_a \sim 0$ and $(\vec{\epsilon}_a - \vec{\epsilon}_{a'}) \sim 0$. But by multiplying out the latter set of relations and using the former, we find that jointly they are equivalent to the single set of relations

$$\forall a, a', j, j' : \quad \epsilon_a^j \epsilon_{a'}^{j'} + \epsilon_{a'}^j \epsilon_a^{j'} = 0\,,$$

which of course is equivalent to

$$\forall a, a', j, j' : \quad \epsilon_a^j \epsilon_{a'}^{j'} + \epsilon_a^{j'} \epsilon_{a'}^j = 0\,.$$

In this expression the roles of the two sets of indices is manifestly symmetric. Hence another equivalent way to state the relations is to say that

$$\forall a, a', j : \quad \epsilon_a^j \epsilon_{a'}^j = 0$$

and

$$\forall a, a', j, j' : \quad (\epsilon_a^j - \epsilon_a^{j'})(\epsilon_{a'}^j - \epsilon_{a'}^{j'}) = 0$$

This appears around (1.2.1) in [Kock10].

The following proposition identifies these algebras of functions on spaces of infinitesimal based simplices with the algebras that appear in the component expression of the monoidal Dold-Kan correspondence, as displayed in prop. 4.5.17.

Proposition 4.5.29. *For all $k, n \in \mathbb{N}$ we have a natural isomorphism of real commutative and hence of smooth algebras*

$$\phi : \; C^\infty(\tilde{D}(k, n)) \xrightarrow{\simeq} \oplus_{i=0}^n (\wedge^i \mathbb{R}^k) \otimes (\wedge^i \mathbb{R}^n) \;\;,$$

where on the right we have the algebras that appear degreewise in def. 4.5.16, where the product is given on homogeneous elements by

$$(\omega, x) \cdot (\lambda, y) = (\omega \wedge \lambda, x \wedge y)\,.$$

Proof. Let $\{t_a\}$ be the canonical basis for $\mathbb{R}^k$ and $\{e^i\}$ the canonical basis for $\mathbb{R}^n$. We claim that an isomorphism is given by the assignment which on generators is

$$\phi : \epsilon^i_a \mapsto (t_a, e^i)\,.$$

To see that this defines indeed an algebra homomorphism we need to check that it respects the relations on the generators. By remark 4.5.28 for this it is sufficient to observe that for all pairs of pairs of indices we have

$$\begin{aligned} \phi(\epsilon^i_a \epsilon^{i'}_{a'}) &= (t_a \wedge t_{a'}, e^i \wedge e^{i'}) \\ &= -(t_{a'} \wedge t_a, e^i \wedge e^{i'})\,. \\ &= -\phi(\epsilon^i_{a'} \epsilon^{i'}_a) \end{aligned}$$

□

Remark 4.5.30. The proof of prop. 4.5.29 together with remark 4.5.28 may be interpreted as showing how the skew-linearity which is the hallmark of traditional Lie theory arises in the synthetic differential geometry of infinitesimal simplices. In the context of the tangent Lie algebroid, discussed as example 4.5.33 below, this pleasant aspect of Kock's "combinatorial differential forms" had been amplified in [BM00]. See also [Stel10].

Proposition 4.5.31. *For $\mathfrak{a} \in L_\infty\mathrm{Alg}$ a 1-truncated object, hence an ordinary Lie algebroid of rank k over a base manifold X, its image under the map $i : L_\infty\mathrm{Alg} \to (\mathrm{SmoothAlg}^\Delta)^{op}$, def. 4.5.21, is such that its restriction to any chart $U \to X$ is, up to isomorphism, of the form*

$$i(\mathfrak{a})|_U : [n] \mapsto U \times \tilde{D}(k,n)\,.$$

Proof. Apply prop. 4.5.29 in def. 4.5.16, using that by definition $\mathrm{CE}(\mathfrak{a})$ is given by the exterior algebra on locally free $C^\infty(X)$ modules, so that

$$\begin{aligned} \mathrm{CE}(\mathfrak{a}|_U) &\simeq (\wedge^\bullet_{C^\infty(U)} \Gamma(U \times \mathbb{R}^k)^*, d_{\mathfrak{a}|_U}) \\ &\simeq (C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^k, d_{\mathfrak{a}|_U}) \end{aligned}\,.$$

□

Example 4.5.32 (Lie algebra as infinitesimal simplicial complex)**.** For G a Lie group, consider the simplicial manifold

$$\mathbf{B}G_{\mathrm{ch}} = \left(\cdots\cdots G \times G \xrightarrow{\to\to\to} G \xrightarrow{\to\to} * \right) \in \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{synthdiff}}, \mathrm{sSet}]$$

which presents the internal delooping $\mathbf{B}G$ by prop. 4.3.21. Consider then the subobject (as simplicial formal manifolds)

$$\begin{array}{ccc} \vdots & & \vdots \\ \tilde{D}(k,2) & \overset{i_2}{\hookrightarrow} & G \times G \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\ \tilde{D}(k,1) & \overset{i_1}{\hookrightarrow} & G \\ \downarrow\downarrow & & \downarrow\downarrow \\ * & \xrightarrow{i_0} & * \\ \\ (\mathbf{B}\mathfrak{g})_{\mathrm{ch}} & \hookrightarrow & (\mathbf{B}G)_{\mathrm{ch}} \end{array},$$

where $k = \dim(G)$, defined as follows:

1. i_1 includes the first order infinitesimal neighbourhood of the neutral element of G, hence synthetically $\{g \in G | g \sim_1 0\}$.
2. i_2 includes the space of pairs of points in G which are first order neighbours of the neutral element and of each other: $\{(g_1, g_2) \in G \times G | g_1 \sim_1 e, g_2 \sim_1 e, g_1 \sim g_2\}$.

This is implicitly the inclusion that is used in [Kock10] in the discussion of Lie algebras in synthetic differential geometry. By the above discussion the above identifies $\tilde{D}(k,1) \simeq \mathfrak{g} = T_e(G)$ as the Lie algebra of G and $\tilde{D}(k,2) \simeq \mathfrak{g} \wedge \mathfrak{g}$. Then formula 6.8.2 in [Kock10]together with theorem 6.6.1 there show how the group product on the right turns into the Lie bracket on the left.

More in detail, formula 6.8.2 in [Kock10] says that for $g_1, g_2 \sim_1 e$ and $g_1 \sim_1 g_2$ we have

$$g_1 \cdot g_2 = g_1 + g_2 + \frac{1}{2}\{g_1, g_2\} - \frac{3}{2}e\,,$$

where $\{g_1, g_2\} = g_1 g_2 g_1^{-1} g_2^{-1}$ is the group commutator. Theorem 6.6.1 in [Kock10] identifies this on the given elements infinitesimally close to e with the Lie bracket on these elements.

Example 4.5.33 (tangent Lie algebroid as infinitesimal simplicial complex)**.** For X a smooth manifold and TX its tangent Lie algebroid, its incarnation as a simplicial smooth locus via def. 4.5.21, prop. 4.5.31 is the simplicial complex of *infinitesimal simplices* $\{(x_0, \cdots, x_n) \in X^n | \forall i,j : x_i \sim x_j\}$ in X. The normalized cosimplicial function algebra of this complex is called the algebra of *combinatorial differential forms* in [Kock10]. The corresponding normalized chain dg-algebra is observed there to be isomorphic to the de Rham complex of X, which here is a direct consequence of the monoidal Dold-Kan correspondence. This is made explicit in [Stel10].

Notice that accordingly for $\mathfrak{g}$ any L_∞-algebra, flat $\mathfrak{g}$-valued differential forms are equivalently morphisms of dg-algebras

$$\Omega^\bullet(X) \leftarrow \mathrm{CE}(\mathfrak{g}) : A$$

as well as ("synthetically") morphisms

$$TX \to \mathfrak{g}$$

of simplicial objects in the Cahiers topos $\mathrm{Sh}(\mathrm{CartSp}_{\mathrm{synthdiff}})$.

4.5.1.4 ∞-Lie differentiation

Definition 4.5.34. Write

$$\mathrm{Inf}\infty\mathrm{Grpd} := \mathrm{PSh}_\infty(\mathrm{InfSmothLoc})$$

for the ∞-category of ∞-presheaves on the site of infinitesimal smooth loci of def. 4.5.5 (formal duals of Weil algebras/Artin algebras). Write

$$\mathrm{Inf}\infty\mathrm{Grpd}_1 \hookrightarrow \mathrm{Inf}\infty\mathrm{Grpd}$$

for the reflective localization at the effective epimorphisms in InfSmoothLoc.

Proposition 4.5.35. *We have an ∞-pushout diagram of ∞-toposes of the form*

$$\begin{array}{ccc} \mathrm{Smooth}\infty\mathrm{Grpd} & \longrightarrow & \mathrm{SynthDiff}\infty\mathrm{Grpd} \\ \downarrow & & \downarrow \\ \infty\mathrm{Grpd} & \longrightarrow & \mathrm{Inf}\infty\mathrm{Grpd} \end{array}\,.$$

Proof. By prop. 6.3.2.3 of [LuHTT] ∞-pushouts of ∞-toposes are computed as ∞-limits of ∞-categories with respect to the corresponding inverse image functors. Therefore the statement is that Inf∞Grpd is the kernel of $i^* : \mathrm{SynthDiff}\infty\mathrm{Grpd} \to \mathrm{Smooth}\infty\mathrm{Grpd}$. Since inverse images preserve ∞-colimits in the ∞-topos, we may compute this kernel on generators, hence on the site. The statement then follows by observing the evident pullback diagram

$$\begin{array}{ccc} \mathrm{CartSp}_{\mathrm{smooth}} & \xleftarrow{p} & \mathrm{CartSp}_{\mathrm{synthdiff}} \\ \uparrow & & \uparrow^{j} \\ * & \longleftarrow & \mathrm{InfSmoothLoc} \end{array} .$$

□

Proposition 4.5.36. *Write*

$$L\infty\mathrm{Alg} \hookrightarrow \mathrm{Inf}\infty\mathrm{Grpd}_1$$

for the full sub-∞-category on those objects which are sent by Γ to the point. This is the ∞-category of L_∞-algebras.

Proof. By the central result of [LurieFormalGeometry]. □

Definition 4.5.37. The functor

$$\mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd}) \simeq \mathrm{Smooth}\infty\mathrm{Grpd}^{*/}_{\geq 1} \xrightarrow{i_!^{*/}} \mathrm{Smooth}\infty\mathrm{Grpd}^{*/}_{\geq 1} \xrightarrow{j^*} L_\infty\mathrm{Alg}$$

is ∞-*Lie differentiation*.

(...)

4.5.2 Manifolds

We discuss the general abstract notion of *separated manifolds*, 3.10.6, realized in the model of synthetic differential cohesion.

Let $\mathbb{A}^1 := \mathbb{R}^1$ be the standard line object of Smooth∞Grpd exhibiting its cohesion, by prop. 4.3.33.

Proposition 4.5.38. *The full subcategory of* Smooth∞Grpd *on the separated* $\mathbb{R}$*-manifolds, def. 3.10.36 is equivalently that of smooth Hausdorff paracompact manifolds*

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}\,.$$

4.5.3 Cohomology

We discuss aspects of the intrinsic cohomology, 3.6.9, in SynthDiff∞Grpd.

- 4.5.3.1 – Cohomology localization;
- 4.5.3.2 – Lie group cohomology
- 4.5.3.3 – ∞-Lie algebroid cohomology
- 4.5.3.2 – Lie group cohomology;
- 4.5.3.3 – L_∞-algebroid cohomology;
- 4.5.4 – Infinitesimal principal ∞-bundles / extensions of L_∞-algebroids

4.5.3.1 Cohomology localization

Observation 4.5.39. The canonical line object of the Lawvere theory $\mathrm{CartSp}_{\mathrm{smooth}}$ (the free algebra on the singleton) is the real line

$$\mathbb{A}^1_{\mathrm{CartSp}_{\mathrm{smooth}}} = \mathbb{R}\,.$$

We shall write $\mathbb{R}$ also for the underlying additive group

$$\mathbb{G}_a = \mathbb{R}$$

regarded canonically as an abelian ∞-group object in $\mathrm{SynthDiff}\infty\mathrm{Grpd}$. For $n \in \mathbb{N}$ write $\mathbf{B}^n\mathbb{R} \in \mathrm{SynthDiff}\infty\mathrm{Grpd}$ for its n-fold delooping. For $n \in \mathbb{N}$ and $X \in \mathrm{SynthDiff}\infty\mathrm{Grpd}$ write

$$H^n_{\mathrm{shdiff}}(X, \mathbb{R}) := \pi_0 \mathrm{SynthDiff}\infty\mathrm{Grpd}(X, \mathbf{B}^n\mathbb{R})$$

for the cohomology group of X with coefficients in the canonical line object in degree n.

Definition 4.5.40. Write

$$\mathbf{L}_{\mathrm{sdiff}} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$$

for the cohomology localization of $\mathrm{SynthDiff}\infty\mathrm{Grpd}$ at $\mathbb{R}$-cohomology: the full sub-∞-category on the W-local objects with respect to the class W of morphisms that induce isomorphisms in all $\mathbb{R}$-cohomology groups.

Proposition 4.5.41. *Let $\mathrm{Ab}^{\Delta}_{\mathrm{proj}}$ be the model structure on cosimplicial abelian groups, whose fibrations are the degreewise surjections and whose weak equivalences the quasi-isomorphisms under the normalized cochain functor.*

The transferred model structure along the forgetful functor

$$U : \mathrm{SmothAlg}^{\Delta} \to \mathrm{Ab}^{\Delta}$$

exists and yields a cofibrantly generated simplicial model category structure on cosimplicial smooth algebras (cosimplicial C^∞-rings).

See [Stel10] for an account.

Proposition 4.5.42. *Let $j : (\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}} \to [\mathrm{CartSp}_{\mathrm{synthdiff}}, \mathrm{sSet}]$ be the prolonged external Yoneda embedding.*

1. *This constitutes the right adjoint of a simplicial Quillen adjunction*

$$(\mathcal{O} \dashv j) : \; (\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}} \underset{j}{\overset{\mathcal{O}}{\rightleftarrows}} [\mathrm{CartSp}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj,loc}} \;,$$

where the left adjoint $\mathcal{O}(-) = C^\infty(-, \mathbb{R})$ degreewise forms the algebra of functions obtained by homming presheaves into the line object $\mathbb{R}$.

2. *Restricted to simplicial formal smooth manifolds of finite truncation along*

$$\mathrm{FSmoothMfd}^{\Delta^{\mathrm{op}}}_{\mathrm{fintr}} \hookrightarrow (\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}}$$

the right derived functor of j is a full and faithful ∞-functor that factors through the cohomology localization and thus identifies a full reflective sub-∞-category

$$(\mathrm{FSmoothMfd}^{\Delta^{op}})^{\circ}_{\mathrm{fintr}} \hookrightarrow \mathbf{L}_{\mathrm{sdiff}} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}\,.$$

3. *The intrinsic* $\mathbb{R}$*-cohomology of any object* $X \in \mathrm{SynthDiff}\infty\mathrm{Grpd}$ *is computed by the ordinary cochain cohomology of the Moore cochain complex underlying the cosimplicial abelian group of the image of the left derived functor* $(\mathbb{L}\mathcal{O})(X)$ *under the Dold-Kan correspondence:*

$$H^n_{\mathrm{SynthDiff}}(X, \mathbb{R}) \simeq H^n_{\mathrm{cochain}}(N^\bullet(\mathbb{L}\mathcal{O})(X)) .$$

Proof. By prop. 4.5.10 we may equivalently work over the site FSmoothMfd. The proof there is given in [Stel10], following [Toën06]. □

4.5.3.2 Lie group cohomology

Proposition 4.5.43. *Let* $G \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$ *be a Lie group.*

Then the intrinsic group cohomology in $\mathrm{Smooth}\infty\mathrm{Grpd}$ *and in* $\mathrm{SynthDiff}\infty\mathrm{Grpd}$ *of* G *with coefficients in*

1. *discrete abelian groups* A*;*

2. *the additive Lie group* $A = \mathbb{R}$

coincides with Segal's refined Lie group cohomology [Sega70], [Bry00].

$$H^n_{\mathrm{Smooth}}(\mathbf{B}G, A) \simeq H^n_{\mathrm{SynthDiff}}(\mathbf{B}G, A) \simeq H^n_{\mathrm{Segal}}(G, A) .$$

Proof. For discrete coefficients this is shown in theorem 4.4.33 for H_{Smooth}, which by the full and faithful embedding then also holds in SynthDiff∞Grpd.

Here we demonstrate the equivalence for $A = \mathbb{R}$ by obtaining a presentation for $H^n_{\mathrm{SynthDiff}}(\mathbf{B}G, \mathbb{R})$ that coincides explicitly with a formula for Segal's cohomology observed in [Bry00].

Let therefore $\mathbf{B}G_{\mathrm{ch}} \in [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{Set}$ be the standard presentation of $\mathbf{B}G \in \mathrm{SynthDiff}\infty\mathrm{Grpd}$ by the nerve of the Lie groupoid $(G \rightrightarrows *)$ as discussed in 4.4.2. We may write this as

$$\mathbf{B}G_{\mathrm{ch}} = \int^{[k]\in\Delta} \Delta[k] \cdot G^{\times_k} .$$

By prop. 4.5.42 the intrinsic $\mathbb{R}$-cohomology of $\mathbf{B}G$ is computed by the cochain cohomology of the cochain complex of the underlying simplicial abelian group of the value $(\mathbb{L}\mathcal{O})\mathbf{B}G_{\mathrm{ch}}$ of the left derived functor of $\mathcal{O}$.

In order to compute this we shall build and compare various resolutions, as in prop. 4.3.16, moving back and forth through the Quillen equivalences

$$[\Delta^{\mathrm{op}}, D]_{\mathrm{inj}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} [\Delta^{\mathrm{op}}, D]_{\mathrm{Reedy}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} [\Delta^{\mathrm{op}}, D]_{\mathrm{proj}}$$

between injective, projective and Reedy model structures on functors with values in a combinatorial model category D, with D either $\mathrm{sSet}_{\mathrm{Quillen}}$ or with D itself the injective or projective model structure on simplicial presheaves over $\mathrm{CartSp}_{\mathrm{synthdiff}}$.

To begin with, let $(\,Q\mathbf{B}G_{\mathrm{ch}})_\bullet \xrightarrow{\simeq} (G^{\times\bullet}\,) \in [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{Reedy}}$ be a Reedy-cofibrant resolution of the simplicial presheaf $\mathbf{B}G_{\mathrm{ch}}$ with respect to the projective model structure. This is in particular degreewise a weak equivalence of simplicial presheaves, hence

$$\int^{[k]\in\Delta} \Delta[k] \cdot (Q\mathbf{B}G_{\mathrm{ch}})_k \overset{\simeq}{\to} \int^{[k]\in\Delta} \Delta[k] \cdot G^{\times_k} = \mathbf{B}G_c$$

exists and is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{inj}}$, hence in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj}}$, hence in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, because

1. $\Delta \in [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$ is cofibrant in the Reedy model structure;
2. every simplicial presheaf X is Reedy cofibrant when regarded as an object $X_\bullet \in [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{Reedy}}$;
3. the coend over the tensoring

$$\int^{\Delta} : [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}} \times [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{Reedy}} \to [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{inj}}$$

is a left Quillen bifunctor ([LuHTT], prop. A.2.9.26), hence in particular a left Quillen functor in one argument when the other argument is fixed on a cofibrant object, hence preserves weak equivalences between cofibrant objects in that case.

To make this a projective cofibrant resolution we further pull back along the Bousfield-Kan fat simplex projection $\mathbf{\Delta} \to \Delta$ with $\mathbf{\Delta} := N(\Delta/(-))$ to obtain

$$\int^{[k]\in\Delta} \mathbf{\Delta}[k] \cdot (Q\mathbf{B}G_{\mathrm{ch}})_k \xrightarrow{\simeq} \int^{[k]\in\Delta} \Delta[k] \cdot (Q\mathbf{B}G_{\mathrm{ch}})_k \xrightarrow{\simeq} \mathbf{B}G_{\mathrm{ch}} \,,$$

which is a weak equivalence again due to the left Quillen bifunctor property of $\int^{\Delta}(-)\cdot(-)$, now applied with the second argument fixed, and the fact that $\mathbf{\Delta} \to \Delta$ is a weak equivalence between cofibrant objects in $[\Delta, \mathrm{sSet}]_{\mathrm{Reedy}}$. (This is the *Bousfield-Kan map*). Finally, that this is indeed cofibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ follows from

1. the fact that the Reedy cofibrant $(Q\mathbf{B}G_{\mathrm{ch}})_\bullet$ is also cofibrant in $[\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{inj}}$;
2. the left Quillen bifunctor property of

$$\int^{\Delta} : [\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}} \times [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{inj}} \to [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj}} \,;$$

3. the fact that the fat simplex is cofibrant in $[\Delta, \mathrm{sSet}]_{\mathrm{proj}}$.

The central point so far is that in order to obtain a projective cofibrant resolution of $\mathbf{B}G_{\mathrm{ch}}$ we may form a compatible degreewise projective cofibrant resolution but then need to form not just the naive diagonal $\int^{\Delta} \Delta[-]\cdot(-)$ but the fattened diagonal $\int^{\Delta} \mathbf{\Delta}[-]\cdot(-)$. In the remainder of the proof we observe that for computing the left derived functor of $\mathcal{O}$, the fattened diagonal is not necessary after all.

For that observe that the functor

$$[\Delta^{\mathrm{op}}, \mathcal{O}] : [\Delta^{\mathrm{op}}, [\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj,loc}}] \to [\Delta^{\mathrm{op}}, (\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}}]$$

preserves Reedy cofibrant objects, because the left Quillen functor $\mathcal{O}$ preserves colimits and cofibrations and hence the property that the morphisms $L_k X \to X_k$ out of latching objects $\varinjlim_{s \xrightarrow{+} k} X_s$ are cofibrations. Therefore we may again apply the Bousfield-Kan map after application of $\mathcal{O}$ to find that there is a weak equivalence

$$(\mathbb{L}\mathcal{O})(\mathbf{B}G_{\mathrm{ch}}) \simeq \int^{[k]\in\Delta} \mathbf{\Delta}[k]\cdot \mathcal{O}((Q\mathbf{B}G_{\mathrm{ch}})_k) \simeq \int^{[k]\in\Delta} \Delta[k]\cdot \mathcal{O}((Q\mathbf{B}G_{\mathrm{ch}})_k)$$

in $(\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}}$ to the object where the fat simplex is replaced back with the ordinary simplex. Therefore by prop. 4.5.42 the $\mathbb{R}$-cohomology that we are after is equivalently computed as the cochain cohomology of the image under the left adjoint

$$(N^\bullet)^{\mathrm{op}} U^{\mathrm{op}} : (\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}} \to (\mathrm{Ch}^\bullet)^{\mathrm{op}}$$

(where $U : \mathrm{SmoothAlg}^\Delta \to \mathrm{Ab}^\Delta$ is the forgetful functor) of

$$\int^{[k]\in\Delta} \Delta[k]\cdot \mathcal{O}(Q\mathbf{B}G_{\mathrm{ch}})_k \in (\mathrm{SmoothAlg}^\Delta)^{\mathrm{op}}\,,$$

which is

$$(N^\bullet)^{\mathrm{op}} \int^{[k]\in\Delta} \Delta[k]\cdot U^{\mathrm{op}}\mathcal{O}((Q\mathbf{B}G_{\mathrm{ch}})_k) \in (\mathrm{Ch}^\bullet)^{\mathrm{op}}\,,$$

Notice that

1. for $S_{\bullet,\bullet}$ a bisimplicial abelian group we have that the coend $\int^{[k]\in\Delta}\Delta[k]\cdot S_{\bullet,k} \in (\mathrm{Ab}^\Delta)^{\mathrm{op}}$ is isomorphic to the diagonal simplicial abelian group and that forming diagonals of bisimplicial abelian groups sends degreewise weak equivalences to weak equivalences;
2. the Eilenberg-Zilber theorem asserts that the cochain complex of the diagonal is the total complex of the cochain bicomplex: $N^\bullet \mathrm{diag} S_{\bullet,\bullet} \simeq \mathrm{tot} C^\bullet(S_{\bullet,\bullet})$;
3. the complex $N^\bullet\mathcal{O}(Q\mathbf{B}G_{\mathrm{ch}})_k)$ – being the correct derived hom-space between $G^{\times_k}$ and $\mathbb{R}$ – is related by a zig-zag of weak equivalences to $\Gamma(G^{\times_k}, I_{(k)})$, where $I_{(k)}$ is an injective resolution of the sheaf of abelian groups $\mathbb{R}$

Therefore finally we have

$$H^n_{\mathrm{SynthDiff}}(G,\mathbb{R}) \simeq H^n_{\mathrm{cochain}} \mathrm{Tot}\Gamma(G^{\times_\bullet}, I^\bullet_\bullet)\,.$$

On the right this is manifestly $H^n_{\mathrm{Segal}}(G,\mathbb{R})$, as observed in [Bry00]. □

Corollary 4.5.44. *For G a compact Lie group we have for $n \geq 1$ that*

$$H^n_{\mathrm{SynthDiff}\infty\mathrm{Grpd}}(G,U(1)) \simeq H^n_{\mathrm{Smooth}\infty\mathrm{Grpd}}(G,U(1)) \simeq H^{n+1}_{\mathrm{Top}}(BG,\mathbb{Z})\,.$$

Proof. For G compact we have, by [Blan85], that $H^n_{\mathrm{Segal}}(G,\mathbb{R}) \simeq 0$. The claim then follows with prop. 4.5.43 and theorem 4.4.33 applied to the long exact sequence in cohomology induced by the short exact sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = U(1)$. □

4.5.3.3 ∞-Lie algebroid cohomology

We discuss the intrinsic cohomology, 3.6.9, of ∞-Lie algebroids, 4.5.1, in SynthDiff∞Grpd.

Proposition 4.5.45. *Let $\mathfrak{a} \in L_\infty\mathrm{Algd}$ be an L_∞-algebroid. Then its intrinsic real cohomoloogy in* SynthDiff∞Grpd

$$H^n(\mathfrak{a},\mathbb{R}) := \pi_0 \mathrm{SynthDiff}\infty\mathrm{Grpd}(\mathfrak{a}, \mathbf{B}^n\mathbb{R})$$

coincides with its ordinary L_∞-algebroid cohomology: the cochain cohomology of its Chevalley-Eilenberg algebra

$$H^n(\mathfrak{a},\mathbb{R}) \simeq H^n(\mathrm{CE}(\mathfrak{a}))\,.$$

Proof. By prop. 4.5.42 we have that

$$H^n(\mathfrak{a},\mathbb{R}) \simeq H^n N^\bullet(\mathbb{L}\mathcal{O})(i(\mathfrak{a}))\,.$$

By lemma 4.5.22 this is

$$\cdots \simeq H^n N^\bullet \left(\int^{[k]\in\Delta} \mathbf{\Delta}[k]\cdot \mathcal{O}(i(\mathfrak{a})_k) \right)\,.$$

Observe that $\mathcal{O}(\mathfrak{a})_\bullet$ is cofibrant in the Reedy model structure $[\Delta^{\mathrm{op}}, (\mathrm{SmoothAlg}^{\Delta}_{\mathrm{proj}})^{\mathrm{op}}]_{\mathrm{Reedy}}$ relative to the opposite of the projective model structure on cosimplicial algebras: the map from the latching object in degree n in $\mathrm{SmoothAlg}^{\Delta})^{\mathrm{op}}$ is dually in $\mathrm{SmoothAlg} \hookrightarrow \mathrm{SmoothAlg}^{\Delta}$ the projection

$$\oplus_{i=0}^{n} \mathrm{CE}(\mathfrak{a})_i \otimes \wedge^i \mathbb{R}^n \to \oplus_{i=0}^{n-1} \mathrm{CE}(\mathfrak{a})_i \otimes \wedge^i \mathbb{R}^n$$

hence is a surjection, hence a fibration in $\mathrm{SmoothAlg}^{\Delta}_{\mathrm{proj}}$ and therefore indeed a cofibration in $(\mathrm{SmoothAlg}^{\Delta}_{\mathrm{proj}})^{\mathrm{op}}$.

Therefore using the Quillen bifunctor property of the coend over the tensoring in reverse to lemma 4.5.22 the above is equivalent to

$$\cdots \simeq H^n N^\bullet \left(\int^{[k]\in\Delta} \Delta[k] \cdot \mathcal{O}(i(\mathfrak{a})_k) \right)$$

with the fat simplex replaced again by the ordinary simplex. But in brackets this is now by definition the image under the monoidal Dold-Kan correspondence of the Chevalley-Eilenberg algebra

$$\cdots \simeq H^n(N^\bullet \Xi \mathrm{CE}(\mathfrak{a}))\,.$$

By the Dold-Kan correspondence we have hence

$$\cdots \simeq H^n(\mathrm{CE}(\mathfrak{a}))\,.$$

□

Remark 4.5.46. It follows that an intrinsically defined degree-n $\mathbb{R}$-cocycle on $\mathfrak{a}$ is indeed presented by a morphism in $L_\infty\mathrm{Algd}$

$$\mu : \mathfrak{a} \to b^n\mathbb{R}\,,$$

as in def. 4.4.99. Notice that if $\mathfrak{a} = b\mathfrak{g}$ is the delooping of an L_∞- algebra $\mathfrak{g}$ this is equivalently a morphism of L_∞-algebras

$$\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}\,.$$

4.5.4 Extensions of L_∞-algebroids

We discuss the general notion of extensions of cohesive ∞-groups, 3.6.14, for infinitesimal objects in SynthDiff∞Grpd: extensions of L_∞-algebras, def. 4.5.12.

Proposition 4.5.47. *Let $\mu : b\mathfrak{g} \to b^{n+1}\mathbb{R}$ be an $(n+1)$-cocycle on an L_∞-algebra $\mathfrak{g}$. Then under the embedding of def. 4.5.21 the L_∞-algebra $\mathfrak{g}_\mu$ of def. 4.4.102 is the extension classified by μ, according to the general definition 3.6.245.*

Proof. We need to show that

$$b\mathfrak{g}_\mu \to \mathfrak{g} \xrightarrow{\mu} b^{n+1}\mathbb{R}$$

is a fiber sequence in SynthDiff∞Grpd. By prop. 4.4.107 this sits in a pullback diagram of L_∞-algebras (connected L_∞-algebroids)

$$\begin{array}{ccc} b\mathfrak{g}_\mu & \longrightarrow & eb^n\mathbb{R} \\ \downarrow & & \downarrow \\ b\mathfrak{g} & \xrightarrow{\mu} & b^{n+1}\mathbb{R} \end{array}\,.$$

By prop. 4.5.20 this pullback is preserved by the embedding into $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj}}$. Here the right vertical morphism is found to be a fibration replacement of the point inclusion $* \to b^{n+1}\mathbb{R}$. By the discussion in 2.3.2.1 this identifies $b\mathfrak{g}_\mu$ as the homotopy fiber of μ. □

4.5.5 Infinitesimal path groupoid and de Rham spaces

We discuss the intrinsic notion of infinitesimal geometric paths in objects in a ∞-topos of infinitesimal cohesion, 3.10.1, realized in SynthDiff∞Grpd.

Observation 4.5.48. For $U \times D \in \mathrm{CartSp}_{\mathrm{smooth}} \ltimes \mathrm{InfinSmoothLoc} = \mathrm{CartSp}_{\mathrm{synthdiff}} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$ we have that

$$\mathbf{Red}(U \times D) \simeq U$$

is the *reduced smooth locus*: the formal dual of the smooth algebra obtained by quotienting out all nilpotent elements in the smooth algebra $C^\infty(K \times D) \simeq C^\infty(K) \otimes C^\infty(D)$.

Proof. By the model category presentation of $\mathbf{Red} = \mathbb{L}\mathrm{Lan}_i \circ \mathbb{R}i^*$ of the proof of prop. 4.5.11 and using that every representable is cofibrant and fibrant in the local projective model structure on simplicial presheaves we have

$$\begin{aligned} \mathbf{Red}(U \times D) &\simeq (\mathbb{L}\mathrm{Lan}_i)(\mathbb{R}i^*)(U \times D) \\ &\simeq (\mathbb{L}\mathrm{Lan}_i) i^*(U \times D) \\ &\simeq (\mathbb{L}\mathrm{Lan}_i) U \\ &\simeq \mathrm{Lan}_i U \\ &\simeq U \end{aligned} ,$$

where we are using again that i is a full and faithful functor. □

Corollary 4.5.49. *For $X \in \mathrm{SmoothAlg}^{\mathrm{op}} \to \mathrm{SynthDiff}\infty\mathrm{Grpd}$ a smooth locus, we have that $\mathbf{\Pi}_{\mathrm{inf}}(X)$ is the corresponding* de Rham space*, the object characterized by*

$$\mathrm{SynthDiff}\infty\mathrm{Grpd}(U \times D, \mathbf{\Pi}_{\mathrm{inf}}(X)) \simeq \mathrm{SmoothAlg}^{\mathrm{op}}(U, X) .$$

Proof. By the $(\mathbf{Red} \dashv \mathbf{\Pi}_{\mathrm{inf}})$-adjunction relation we have

$$\begin{aligned} \mathrm{SynthDiff}\infty\mathrm{Grpd}(U \times D, \mathbf{\Pi}_{\mathrm{inf}}(X)) &\simeq \mathrm{SynthDiff}\infty\mathrm{Grpd}(\mathbf{Red}(U \times D), X) \\ &\simeq \mathrm{SynthDiff}\infty\mathrm{Grpd}(U, X) \end{aligned} .$$

□

4.5.6 Formally smooth/étale/unramified morphisms

We discuss the general notion of formally smooth/étale/unramified morphisms, 3.10.4, realized in the differential ∞-topos $i : \mathrm{Smooth}\infty\mathrm{Grpd} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$. given by prop. 4.5.11.

Proposition 4.5.50. *A morphism $f : X \to Y$ in SynthDiff∞Grpd is formally étale in the general sense of def. 3.10.19 precisely if for all infinitesimall thickened points $D \in \mathrm{InfSmoothLoc} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$ the canonical diagrams*

$$\begin{array}{ccc} X^D & \xrightarrow{f^D} & Y^D \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

(where the vertical morphism are induced by the unique point inclusion $* \to D$) are ∞-pullbacks under i^*.

Proof. We will write $i : \mathbf{H} \hookrightarrow \mathbf{H}_{\mathrm{th}}$ as shorthand for $i : \mathrm{Smooth}\infty\mathrm{Grpd} \hookrightarrow \mathrm{SynthDiff}\infty\mathrm{Grpd}$. The defining ∞-pullback diagram of def. 3.10.19 induces and is detected by ∞-pullback diagrams for all $U \times D \in \mathrm{CartSp}_{\mathrm{synthdiff}}$ of the form

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{th}}(U \times D, X) & \longrightarrow & \mathbf{H}_{\mathrm{th}}(U \times D, Y) \\ \downarrow & & \downarrow \\ \mathbf{H}_{\mathrm{th}}(U \times D, i_* i^* X) & \longrightarrow & \mathbf{H}_{\mathrm{th}}(U \times D, i_* i^* Y) \end{array} .$$

By the ∞-Yoneda lemma, the $(i^* \dashv i_*)$-adjunction, the definition of i and the formula for the internal hom, this is equivalent to the diagram

$$\begin{array}{ccc} \mathbf{H}(U, i^* X^D) & \longrightarrow & \mathbf{H}(U, i^* Y^D) \\ \downarrow & & \downarrow \\ \mathbf{H}(U, i^* X) & \longrightarrow & \mathbf{H}(U, i^* Y) \end{array}$$

being an ∞-pullback for all $U \in \mathrm{CartSp}$. By one more application of the ∞-Yoneda lemma this is the statement to be proven. □

Remark 4.5.51. Since i^* is right adjoint and hence preserves ∞-pullbacks, it is sufficient for a morphism $f \in \mathrm{SynthDiff}\infty\mathrm{Grpd}$ to be formally étale that

$$\begin{array}{ccc} X^D & \xrightarrow{f^D} & Y^D \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is an ∞-pullback in $\mathrm{SynthDiff}\infty\mathrm{Grpd}$. In this form, when restricted to 0-truncated objects, formally étale morphisms are axiomatized in [Kock06], around p. 82, in a topos for synthetic differential geometry, such as the Cahier topos $\tau_{\leq 0}\mathrm{SynthDiff}\infty\mathrm{Grpd} \simeq \mathrm{Sh}(\mathrm{CartSp})$ considered here.

We now discuss in more detail the special case of formally étale maps between objects that are presented by simplicial smooth manifolds.

Proposition 4.5.52. *Let $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ be presented by a simplicial smooth manifold under the canonical inclusion $X_\bullet \in \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]$. Then $i_! X$ is presented by the same simplicial smooth manifold, under the canonical inclusion*

$$X_\bullet \in \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}} \hookrightarrow [\mathrm{CartSp}_{\mathrm{synthdiff}}^{\mathrm{op}}, \mathrm{sSet}] .$$

Proposition 4.5.53. *Let $f : X \to Y$ be a morphism in* SmthMfd*, a smooth function between finite dimensional paracompact smooth manifolds, regarded, by cor. 4.4.10, as a morphism in* $\mathrm{Smooth}\infty\mathrm{Grpd}$*. Then*

- *f is a submersion $\Leftrightarrow$ f is formally i-smooth;*
- *f is a local diffeomorphism $\Leftrightarrow$ f is formally i-étale;*
- *f is an immersion $\Leftrightarrow$ f is formally i-unramified;*

where on the left we have the traditional notions, and on the right those of def. 3.10.16.

Proof. By lemma 4.5.52 the canonical diagram

$$\begin{array}{ccc} i_! X & \xrightarrow{i_! f} & i_! Y \\ \downarrow & & \downarrow \\ i_* X & \xrightarrow{i_* f} & i_* Y \end{array}$$

in SynthDiff∞Grpd is presented in $[\mathrm{CartSp}^{\mathrm{op}}_{\mathrm{synthdiff}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ by the diagram of presheaves

$$U \times D \;\; \mapsto \;\; \begin{array}{ccc} \mathrm{FSmthMfd}(U \times D, X) & \xrightarrow{\mathrm{FSmthMfd}(U \times D, f)} & \mathrm{FSmthMfd}(U \times D, Y) \\ \downarrow & & \downarrow \\ \mathrm{FSmthMfd}(U, X) & \xrightarrow{\mathrm{FSmthMfd}(U, f)} & \mathrm{FSmthMfd}(U, Y) \end{array} ,$$

where FSmthMfd is the category of formal smooth manifolds from def. 4.5.9, U is an ordinary smooth manifold and D an infinitesimal smooth loci, def. 4.5.5.

Consider this first for the case that $D := \mathbb{D} \hookrightarrow \mathbb{R}$ is the first order infinitesimal neighbourhood of the origin in the real line. Restricted to this case the above diagram of presheaves is that represented on SmthMfd by the diagram of smooth manifolds

$$\begin{array}{ccc} TX & \xrightarrow{df} & TY \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} ,$$

where on the top we have the tangent bundles of X and Y and the differential of f mapping between them.

Since pullbacks of presheaves are computed objectwise, f being formally smooth/étale/unramified implies that the canonical morphism

$$TX \to X \times_Y TY = f^* TY$$

is an epi-/iso-/mono-morphism, respectively. This by definition means that f is a submersion/local diffeomorphism/immersion, respectively.

Conversely, by standard facts of differential geometry, f being a submersion means that it is locally a projection, f being a local isomorphism means that it is in particular étale, and f being an immersion means that it is locally an embedding. This implies that also for D any other infinitesimal smooth locus, so that X^D, Y^D are bundles of possibly higher order formal curves, the morphism

$$X^D \to X \times_Y Y^D$$

is an epi-/iso-/mono-morphism, respectively. □

4.5.7 Formally étale groupoids

We discuss the general notion of formally étale groupoids in a differential ∞-topos, 3.10.5, realized in $\mathrm{Smooth}\infty\mathrm{Grpd} \overset{i}{\hookrightarrow} \mathrm{SynthDiff}\infty\mathrm{Grpd}$.

Definition 4.5.54. Call a simplicial smooth manifold $X \in \mathrm{SmoothMfd}^{\Delta^{\mathrm{op}}}$ an *étale simplicial smooth manifold* if it is fibrant as an object of $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and if moreover all face and degeneracy morphisms are étale morphisms.

Example 4.5.55. The nerve of an étale Lie groupoid in the traditional sense is an étale simplicial smooth manifold.

Proposition 4.5.56. *Let* $X \in \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}}$ *be an étale simplicial manifold, def. 4.5.54. Then equipped with its canonical atlas, observation 2.3.29, it presents a formally étale groupoid object in* $\mathrm{Smooth}\infty\mathrm{Grpd} \stackrel{i}{\hookrightarrow} \mathrm{SynthDiff}\infty\mathrm{Grpd}$*, according to def. 3.10.33.*

Proof. We need to check that $i_! X_0$ is the ∞-pullback $i_* X_0 \times_{i_* X} i_! X$. By prop. 2.3.13, lemma 4.5.52 and prop. 2.3.33 it is sufficient to show for the décalage replacement $\mathrm{Dec}_0 X \to X$ of the atlas, that $i_! \mathrm{Dec}_0 X$ is the ordinary pullback of simplicial presheaves $(i_* \mathrm{Dec}_0 X) \times_{i_* X} i_! X$. Since pullbacks of simplicial presheaves are computed degreewise, this is the case by prop. 4.5.53 if for all $n \in \mathbb{N}$ the morphism $(\mathrm{Dec}_0 X)_n \to X_n$ is an étale morphism of smooth manifolds, in the traditional sense. By prop. 2.3.32 this morphism is the face map d_{n+1} of X. This is indeed étale by the very assumption that X is an étale simplicial smooth manifold. □

4.5.8 Chern-Weil theory

We discuss the notion of ∞-connections, 4.4.17, in the context SynthDiff∞Grpd.

4.5.8.1 ∞-Cartan connections A *Cartan connection* on a smooth manifold is a principal connection subject to an extra constraint that identifies a component of the connection at each point with the tangent space of the base manifold at that point. The archetypical application of this notion is to the formulation of the field theory of *gravity*, 5.3.1.

We indicate a notion of Cartan ∞-connections.

The following notion is classical, see for instance section 5.1 of [Sha97].

Definition 4.5.57. Let $(H \hookrightarrow G)$ be an inclusion of Lie groups with Lie algebras $(\mathfrak{h} \hookrightarrow \mathfrak{g})$. A $(H \to G)$-*Cartan connection* on a smooth manifold X is

1. a G-principal bundle $P \to X$ equipped with a connection ∇;
2. such that
   (a) the structure group of P reduces to H, hence the classifying morphism factors as $X \to \mathbf{B}H \to \mathbf{B}G$;
   (b) for each point $x \in X$ and any local trivialization of (P, ∇) in some neighbourhood of X, the canonical linear map
   $$T_x X \xrightarrow{\nabla} \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$$
   is an isomorphism,

Here $(\mathfrak{h} \to \mathfrak{g})$ are the Lie algebras of the given Lie groups and $\mathfrak{g}/\mathfrak{h}$ is the quotient of the underlying vector spaces.

4.6 Supergeometric ∞-groupoids

We discuss ∞-groupoids equipped with *discrete super cohesion*, with *smooth super cohesion* and *synthetic differential super cohesion*, where "super" is in the sense of *superalgebra* and *supergeometry* (see for instance [DelMor99] for a review of traditional superalgebra and supergeometry).

We first introduce *discrete super ∞-groupoids* which have super-grading but no smooth structure. This is the canonical context in which (higher) *superalgebra* takes place: an $\mathbb{R}$-module internal to super ∞-groupoids is externally a chain complex of *super vector spaces* and an $\mathbb{R}$-algebra internal to super ∞-groupoids is externally a real *superalgebra*. Then we add smooth structure by passing further to *smooth super ∞-groupoids*. This is the canonical context for supergeometry. Notably the traditional category of smooth supermanifolds faithfully embeds into smooth super ∞-groupoids. Finally we further refine to *synthetic differential super ∞-groupoids* where the smooth structure is refined by explicit commutative infinitesimals in addition to the super/graded infinitesimals of supergeometry. In summary, this yields a super-refinement of three cohesive structures discussed before:

supergeometric refinement	differential geometry	discussed in section
Super∞Grpd	Disc∞Grpd	4.1
SmoothSuper∞Grpd	Smooth∞Grpd	4.4
SynthDiffSuper∞Grpd	SynthDiff∞Grpd	4.5

Accordingly, the canonical site of definition of the most inclusive of these cohesive ∞-toposes, which is SynthDiffSuper∞Grpd, contains objects denoted $\mathbb{R}^{p\oplus k|q}$ – *synthetic differential super Cartesian space* – that have three gradings:

- an ordinary dimension p;
- an order k of their infinitesimal thickening;
- a super dimension q.

And of course ∞-groupoids over this site have furthermore their homotopy theoretic degree.

In terms of the formally dual function algebras $C^\infty(\mathbb{R}^{p\oplus k|l})$ on these objects, k is the number of *commuting* nilpotent generators, while q is the number of *graded-commuting* nilpotent generators. In this sense supergeometry may be understood as a $\mathbb{Z}_2$-graded variant of synthetic differential geometry. This is a perspective that had been explored in [Yet88] and more recently in [CarRoy12].

On the other hand, the role played by supergeometry in applications is well reflected by the perspective where smooth/synthetic differential supergeometry is regarded as ordinary smooth/synthetic differential geometry but *internal* to the "bare super context", which is the context parameterized over just the *superpoints* $\mathbb{R}^{0|q}$. This perspective on supergeometry had been proposed independently in 1984 in [Schw84], [Molo84] and [Vor84]. A review is in the appendix of [KonSch00], whose main part discusses aspects of those synthetic differential superspaces in this language.

In terms of (higher) topos theory this perspective means that passing from higher differential geometry to higher supergeometry means to change the *base ∞-topos* from Disc∞Grpd $\simeq$ ∞Grpd to Super∞Grp. We find below the ∞-toposes for differential-, synthetic differential- and supergeometry to arrange in a diagram of geometric morphisms of the form

$$
\begin{array}{ccccc}
\mathrm{SmoothSuper}\infty\mathrm{Grpd} & \hookrightarrow & \mathrm{SynthDiffSuper}\infty\mathrm{Grpd} & \longrightarrow & \infty\mathrm{SuperGrpd} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Smooth}\infty\mathrm{Grpd} & \hookrightarrow & \mathrm{SynthDiff}\infty\mathrm{Grpd} & \longrightarrow & \infty\mathrm{Grpd}
\end{array}
.
$$

Here the bottom line is the differential cohesion over the base of discrete ∞-groupoids discussed in 4.5. The top line is the super-refinement exhibited by differential cohesion, but now over the base Super∞Grpd of discrete but "super" ∞-groupoids. This diagram of ∞-toposes we present by a diagram of sites which, with the above notation for synthetic differential super Cartesian spaces, looks as follows.

$$\begin{array}{ccccc} \{\mathbb{R}^{p|q}\}_{p,q} & \hookrightarrow & \{\mathbb{R}^{p\oplus k|q}\}_{p,k,q} & \longrightarrow & \{\mathbb{R}^{0|q}\}_q \\ \downarrow && \downarrow && \downarrow \\ \{\mathbb{R}^p\}_p & \hookrightarrow & \{\mathbb{R}^{p\oplus k}\}_{p,k} & \longrightarrow & \{*\} \end{array} .$$

Definition 4.6.1. Let $\mathrm{GrassmannAlg}_{\mathbb{R}}$ be the category whose objects are finite dimensional free $\mathbb{Z}_2$-graded commutative $\mathbb{R}$-algebras (Grassmann algebras). Write

$$\mathrm{SuperPoint} := \mathrm{GrassmannAlg}_{\mathbb{R}}^{\mathrm{op}}$$

for its opposite category. For $q \in \mathbb{N}$ we write $\mathbb{R}^{0|q} \in \mathrm{SuperPoint}$ for the object corresponding to the free $\mathbb{Z}_2$-graded commutative algebra on q generators and speak of the *superpoint* of order q.

We think of SuperPoint as a site by equipping it with the trivial coverage.

Definition 4.6.2. Write

$$\mathrm{SuperSet} := \mathrm{Sh}(\mathrm{SuperPoint}) \simeq \mathrm{PSh}(\mathrm{SuperPoint})$$

for the topos of presheaves over SuperPoint.

Definition 4.6.3. Write

$$\mathrm{Super}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{SuperPoint}) \simeq \mathrm{PSh}_\infty(\mathrm{SuperPoint})$$

for the ∞-topos of ∞-sheaves over SuperPoint. We say an object $X \in \mathrm{Super}\infty\mathrm{Grpd}$ is a *super ∞-groupoid.*

We shall conceive of higher superalgebra and higher supergeometry as being the higher algebra and geometry *over the base ∞-topos* ([John03], chapter B3) Super∞Grpd instead of over the canonical base ∞-topos ∞Grpd. Except for the topos-theoretic rephrasing, this perspective has originally been suggested in [Schw84] and [Molo84].

Proposition 4.6.4. *The ∞-topos* Super∞Grpd *is cohesive, def. 3.4.1.*

$$\mathrm{Super}\infty\mathrm{Grpd} \underset{\underset{\mathrm{coDisc}}{\longleftarrow}}{\overset{\overset{\Pi}{\longrightarrow}}{\underset{\Gamma}{\overset{\mathrm{Disc}}{\leftrightarrows}}}} \infty\mathrm{Grpd} .$$

Proof. The site SuperPoint is ∞-cohesive, according to def. 3.4.8. Hence the claim follows by prop. 3.4.9. □

Proposition 4.6.5. *The inclusion* $\mathrm{Disc} : \infty\mathrm{Grpd} \hookrightarrow \mathrm{Super}\infty\mathrm{Grpd}$ *exhibits the collection of super ∞-groupoids as forming an infinitesimal cohesive neighbourhood, def. 3.5.1, of the discrete ∞-groupoids, 4.1.*

Proof. Observe that the point inclusion $i : \mathrm{Point} := * \hookrightarrow \mathrm{SuperPoint}$ is both left and right adjoint to the unique projection $p : \mathrm{SuperPoint} \to \mathrm{Point}$. Therefore we have even a periodic sequence of adjunctions

$$(\cdots \dashv i^* \dashv p^* \dashv i^* \dashv p^* \dashv \cdots) : \mathrm{Super}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}\,,$$

and $p^* \simeq \mathrm{Disc} \simeq \mathrm{coDisc}$ is full and faithful. □

Definition 4.6.6. Write $\mathbb{R} \in \mathrm{Super}\infty\mathrm{Grpd}$ for the presheaf $\mathrm{SuperPoint}^{\mathrm{op}} \to \mathrm{Set} \hookrightarrow \infty\mathrm{Grpd}$ given by

$$\mathbb{R} : \mathbb{R}^{0|q} \mapsto C^\infty(\mathbb{R}^{0|q}) := (\Lambda_q)_{\mathrm{even}} \,,$$

which sends the order-q superpoint to the underlying set of the even subalgebra of the Grassmann algebra on q generators.

Remark 4.6.7. The object $\mathbb{R} \in \mathrm{Super}\infty\mathrm{Grpd}$ is canonically equipped with the structure of an internal ring object. Morever, under both Π and Γ it maps to the ordinary real line $\mathbb{R} \in \mathrm{Set} \hookrightarrow \infty\mathrm{Grpd}$ while respecting the ring structures on both sides.

Proposition 4.6.8. *The theory of ordinary (linear) $\mathbb{R}$-algebra internal to the 1-topos* $\mathrm{SuperSet} = \mathrm{Super}0\mathrm{Grpd} \hookrightarrow \mathrm{Super}\infty\mathrm{Grpd}$ *is equivalent to the theory of $\mathbb{R}$-superalgebra in* Set*.*

This is due to [Molo84].

In view prop. 4.6.8 we may define *smooth* super ∞-groupoids exactly as we defined ordinary smooth ∞-groupoids in 4.4, but working over the base ∞-topos $\mathrm{Super}\infty\mathrm{Grpd}$ instead of over the canonical base ∞-topos $\infty\mathrm{Grpd}$.

Definition 4.6.9. Write $\mathrm{CartSp}_{\mathrm{super}}$ for the internal site ([John03], section C2.4) in $\mathrm{SuperSet} \hookrightarrow \mathrm{Super}\infty\mathrm{Grpd}$, whose objects are the natural numbers, whose morphisms are smooth morphisms $\mathbb{R}^k \to \mathbb{R}^l$ in SuperSet, and whose covers ar given by differentiably good open covers.

According to prop. C2.5.4 of [John03] for every internal site there is an external site such that the internal sheaves on the former are equivalen to the external sheavs on the latter.

Proposition 4.6.10. *The external site corresponding to def. 4.6.9 is the cartesian product site* $\mathrm{CartSp}_{\mathrm{smooth}} \times \mathrm{SuperPoint}$ *(the first factor from def. 4.4.4, the second from def. 4.6.1), hence the subsite of that of smooth supermanifolds on those of the form $\mathbb{R}^{p|q}$ for $p, q \in \mathbb{N}$*

Remark 4.6.11. A morphism $\mathbb{R}^{p_1|q_1} \to \mathbb{R}^{p_2|q_2}$ in $\mathrm{CartSp}_{\mathrm{super}}$ is equivalently a tuple consisting of p_2 even elements and q_2 odd elements of the superalgebra $C^\infty(\mathbb{R}^{p_1|q_1})$. In particular, under the restricted Yoneda embedding the line of def. 4.6.6 is $\mathbb{R} \simeq \mathbb{R}^{1|0}$.

Definition 4.6.12. Write

$$\mathrm{SmoothSuper}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp}_{\mathrm{smooth}} \times \mathrm{SuperPoint}) \,.$$

An object in this ∞-topos we call a *smooth super ∞-groupoid*.

Proposition 4.6.13. *We have a commuting diagram of cohesive ∞-toposes*

$$\begin{array}{ccc}
\mathrm{SmoothSuper}\infty\mathrm{Grpd} & \begin{array}{c} \xrightarrow{\Pi_{\mathrm{super}}} \\ \xleftarrow{\mathrm{Disc}_{\mathrm{super}}} \\ \xrightarrow{\Gamma_{\mathrm{super}}} \\ \xleftarrow{\mathrm{coDisc}_{\mathrm{super}}} \end{array} & \mathrm{Super}\infty\mathrm{Grpd} \\
\downarrow\uparrow & & \downarrow\uparrow \\
\mathrm{Smooth}\infty\mathrm{Grpd} & \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\mathrm{coDisc}} \end{array} & \infty\mathrm{Grpd}
\end{array} \,.$$

For emphasis we shall refer to the objects of $\mathrm{Super}\infty\mathrm{Grp}$ as *discrete super ∞-groupoids*: these refine discrete ∞-groupoids, 4.1 with super-cohesion and are themselves further refined by smooth super ∞-groupoids with smooth cohesion.

We now discuss the various general abstract structures in a cohesive ∞-topos, 3.9, realized in Super∞Grpd and SmoothSuper∞Grpd.

- 4.6.1 – Associated bundles
- 4.6.2 – Exponentiated ∞-Lie algebras

4.6.1 Associated bundles

We discuss aspects of the general notion of associated fiber ∞-bundles, 3.6.11, realized in the context of supergeometric cohesion.

In 4.4.10 above we discussed the 2-stack $2\mathbf{Line}_{\mathbb{C}}$ of smooth complex line 2-bundles. Since the B-field that the bosonic string is charged under has moduli in the differential refinement $\mathbf{B}^2\mathbb{C}^\times_{\mathrm{conn}}$, we may hence say that it is given by 2-connections on complex *2-line bundles*. However, a careful analysis (due [DiFrMo11] and made more explicit in [?]) shows that for the superstring the background B-field is more refined. Expressed in the language of higher stacks the statement is that it is a connection on a complex *super*-2-line bundle. Precisely, in the language of stacks for supergeometry we are to pass to the higher topos SmoothSuper∞Grpd $\simeq$ Sh_∞(SuperMfd) on the site of smooth supermanifolds (section 4.6 of [?]). Internal to that the term *algebra* now means *superalgebra* and hence the 2-stack

$$2\mathbf{sLine}_{\mathbb{C}} \in \mathrm{SmoothSuper}\infty\mathrm{Grpd}$$

now has global points that are identified with complex Azumaya *super*algebras. Of these it turns out there is, up to equivalence, not just one, but two: the canonical super 2-line and its "superpartner". Moreover, there are now, up to equivalence, two different invertible 2-linear maps from each of these super-lines to itself. In summary, the homotopy sheaves of the super 2-stack of super line 2-bundles are

- $\pi_0(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{Z}_2$,
- $\pi_1(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{Z}_2$,
- $\pi_2(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{C}^\times \in \mathrm{Sh}(\mathrm{SuperMfd})$.

(where in the last line we emphasize that the *homotopy sheaf* is that represented by $\mathbb{C}^\times$ as a smooth (super-)manifold). With the discussion in 3.8.1 it follows that the geometric realization of this 2-stack has homotopy groups

- $\pi_0(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}_2$,
- $\pi_1(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}_2$,
- $\pi_2(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq 0$,
- $\pi_3(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}$.

These are precisely the correct coefficients for the twists of complex K-theory, witnessing the fact that the B-field background of the superstring twists the Chan-Paton bundles on the D-branes.

The braided monoidal structure of $2\mathrm{sVect}_{\mathbb{C}}$ induces on $2\mathbf{sLine}_{\mathbb{C}}$ the structure of a *braided 3-group*. Therefore the above general abstract definition of universal moduli for differential cocycles/higher connections produces a moduli 3-stack $\mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}$ which is the supergeometric refinement of the coefficient object $\mathbf{B}^3\mathbb{C}^\times_{\mathrm{conn}}$ for the extended Lagrangian of bosonic 3-dimensional Chern-Simons theory. Therefore for G a super-Lie group a super-Chern-Simons theory that induces the super-WZW action functional on G is given by an extended Lagragian which is a map of higher moduli stacks of the form

$$\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}} .$$

By the canonical inclusion $\mathbf{B}^3\mathbb{C}^\times_{\mathrm{conn}} \to \mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}$ every bosonic extended Lagrangian of 3-d Chern-Simons type induces such a supergeometric theory with trivial super-grading part.

4.6.2 Exponentiated ∞-Lie algebras

According to prop. 4.6.8 the following definition is justified.

Definition 4.6.14. A *super L_∞-algebra* is an L_∞-algebra, def. 1.2.114, internal to the topos SuperSet, def. 4.6.2, over the ring object $\mathbb{R}$ from def. 4.6.6.

Observation 4.6.15. The Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$, def. 1.2.117, of a super L_∞-algebra $\mathfrak{g}$ is externally

- a graded-commutative algebra over $\mathbb{R}$ on generators of bigree in $(\mathbb{N}_+, \mathbb{Z}_2)$ – the *homotopical degree* $\deg_h$ and the *super degree* $\deg_s$;
- such that for any two generators a, b the product satisfies

$$ab = (-1)^{\mathrm{def}_h(a)\deg_h(b) + \mathrm{def}_s(a)\deg_s(b)}\, ba\,;$$

- and equipped with a differential d_{CE} of bidegree $(1, \mathrm{even})$ such that $d_{\mathrm{CE}}^2 = 0$.

Examples 4.6.16.
- Every ordinary L_∞-algebra is canonically a super L_∞-algebra where all element are of even superdegree.
- Ordinary super Lie algebras are canonically identified with precisely the super Lie 1-algebras.
- For every $n \in \mathbb{N}$ there is the *super line super Lie $(n+1)$-algebra* $b^n\mathbb{R}^{0|1}$ characterized by the fact that its Chevalley-Eilenberg algebra has trivial differential and a single generator in bidegree (n, odd).
- For $\mathfrak{g}$ any super L_∞-algebra and $\mu : \mathfrak{g} \to b^n\mathbb{R}$ a cocycle, its homotopy fiber is the super L_∞-algebra extension of $\mathfrak{g}$, as in def. 4.4.102.

  Below in 5.3.2 we discuss in detail a class of super L_∞-algebras that arise by higher extensions from a super Poincaré Lie algebra.

Observation 4.6.17. The Lie integration

$$\exp(\mathfrak{g}) \in [\mathrm{CartSp}_{\mathrm{smooth}} \times \mathrm{SuperPoint}, \mathrm{sSet}] = [\mathrm{SuperPoint}, [\mathrm{CartSp}_{\mathrm{smooth}}, \mathrm{sSet}$$

of a super L_∞-algebra $\mathfrak{g}$ according to 4.4.14 is a system of Lie integrated ordinary L_∞-algebras

$$\exp(\mathfrak{g}) : \mathbb{R}^{0|q} \mapsto \exp((\mathfrak{g} \otimes_{\mathbb{R}} \Lambda_q)_{\mathrm{even}})\,,$$

where $\Lambda_q = C^\infty(\mathbb{R}^{0|q})$ is the Grassmann algebra on q generators.

Over each $U \in \mathrm{CartSp}$ this is the discrete super ∞-groupoid given by

$$\exp(\mathfrak{g})_U : \mathbb{R}^{0|q} \mapsto \mathrm{Hom}_{\mathrm{dgsAlg}}(\mathrm{CE}(\mathfrak{g} \otimes \Lambda_q)_{\mathrm{even}}, \Omega^\bullet_{\mathrm{vert}}(U \times \mathbb{R}^{0|q} \times \Delta^\bullet))\,,$$

where on the right we have super differential forms vertical with respect to the projection $U \times \mathbb{R}^{0|q} \times \Delta^n \to U \times \mathbb{R}^{0|q}$ of supermanifolds.

Proof. The first statement holds by the proof of prop. 4.6.8. The second statement is an example of a stadard mechanism in superalgebra: Using that the category sVect of finite-dimensional super vector space is a compact closed category, we compute

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{dgsAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{vert}}(U \times \mathbb{R}^{0|q} \times \Delta^n)) &\simeq \mathrm{Hom}_{\mathrm{dgsAlg}}(\mathrm{CE}(\mathfrak{g}), C^\infty(\mathbb{R}^{0|q}) \otimes \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^n)) \\
&\simeq \mathrm{Hom}_{\mathrm{dgsAlg}}(\mathrm{CE}(\mathfrak{g}), \Lambda_q \otimes \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^n)) \\
&\subset \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathrm{sVect})}(\mathfrak{g}^*[1], \Lambda_q \otimes \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^n)) \\
&\simeq \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathrm{sVect})}(\mathfrak{g}^*[1] \otimes (\Lambda^q)^*, \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^n)) \quad . \\
&\simeq \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathrm{sVect})}((\mathfrak{g} \otimes \Lambda_q)^*[1], \Omega^\bullet_{\mathrm{vert}}(\Delta^n)) \\
&\simeq \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathrm{sVect})}((\mathfrak{g} \otimes \Lambda_q)^*[1]_{\mathrm{even}}, \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^n)) \\
&\supset \mathrm{Hom}_{\mathrm{dgsAlg}}(\mathrm{CE}((\mathfrak{g} \otimes_k \Lambda_q)_{\mathrm{even}}), \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^n))
\end{aligned}$$

Here in the third step we used that the underlying dg-super-algebra of $\mathrm{CE}(\mathfrak{g})$ is free to find the space of morphisms of dg-algebras inside that of super-vector spaces (of generators) as indicated. Since the differential on both sides is Λ_q-linear, the claim follows. $\square$

5 Applications

We study aspects of the realization of the general abstract Chern-Weil theory in a cohesive ∞-topos, 3.9.7, in the model Smooth∞Grpd, 4.4. The generalization of ordinary Chern-Weil theory in ordinary differential geometry obtained this way comes from two directions:

1. The ∞-Chern-Weil homomorphism applies to G-principal ∞-bundles for G more general than a Lie group.

   - In the simplest case G may be a higher connected cover of a Lie group, realized as a smooth n-group for some $n > 1$. Applied to these, the ∞-Chern-Weil homomorphism sees fractional refinements of the ordinary differential characteristic classes as seen by the ordinary Chern-Weil homomorphism. This we discuss in 5.1.
   - More generally, G may be any mooth ∞-groupoid, for instance obtained from a general ∞-Lie algebra or ∞-Lie algebroid by Lie integration. In 5.5 we observe that symplectic forms in *higher symplectic geometry* may be understood as examples of ∞-Chern-Weil homomorphisms. In 5.7 we discuss a list of examples for which the higher parallel transport of the circle n-bundles with connection in the image of the ∞-Chern-Weil homomorphism reproduces action functionals of various σ-model/Chern-Simons-like field theories.

2. The ∞-Chern-Weil homomorphism is not just a function on cohomology sets, but an ∞-functor on the full cocycle ∞-groupoids. This allows to access the homotopy fibers of this ∞-functor. Over the trivial cocycle these encode the differential refinement of the obstruction theory associated to the underlying bare cocycle. Over nontrivial cocycles they encode the corresponding twisted cohomology. We formalize this in terms of *twisted differential* **c**-*structures* in 3.9.8. A central class of examples are *higher differential Spin structures*, 5.4.7.3, induced from the Whitehead tower of the orthogonal group. These appear in various guises in string background gauge fields. But also *differential T-duality pairs* are an example, as we discuss in 5.4.9.

Finally, we observe that the ∞-Chern-Weil homomorphism may be understood as providing the Lagrangian of higher analogs of Chern-Simons theory, in that its intrinsic integration, 3.9.11, yields a functional on the ∞-groupoid of ∞-connections that generalizes the action functional of Chern-Simons theory from ordinary semisimple Lie algebras and their Killing form to arbitrary ∞-Lie algebroids and arbitrary invariant polynomials on them. We conclude in 5.7 by a discussion of a list of field theories obtained this way.

5.1 Higher Spin-structures

For any $n \in \mathbb{N}$, the Lie group $\mathrm{Spin}(n)$ is the universal simply connected cover of the special orthogonal group $\mathrm{SO}(n)$. Since $\pi_1 \mathrm{SO}(n) \simeq \mathbb{Z}_2$, it is an extension of Lie groups of the form

$$\mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n)\,.$$

The lift of an $\mathrm{SO}(n)$-principal bundle through this extension to a $\mathrm{Spin}(n)$-principal bundle is a called a choice of *spin structure*. A classical textbook on the geometry of spin structures is [LaMi89].

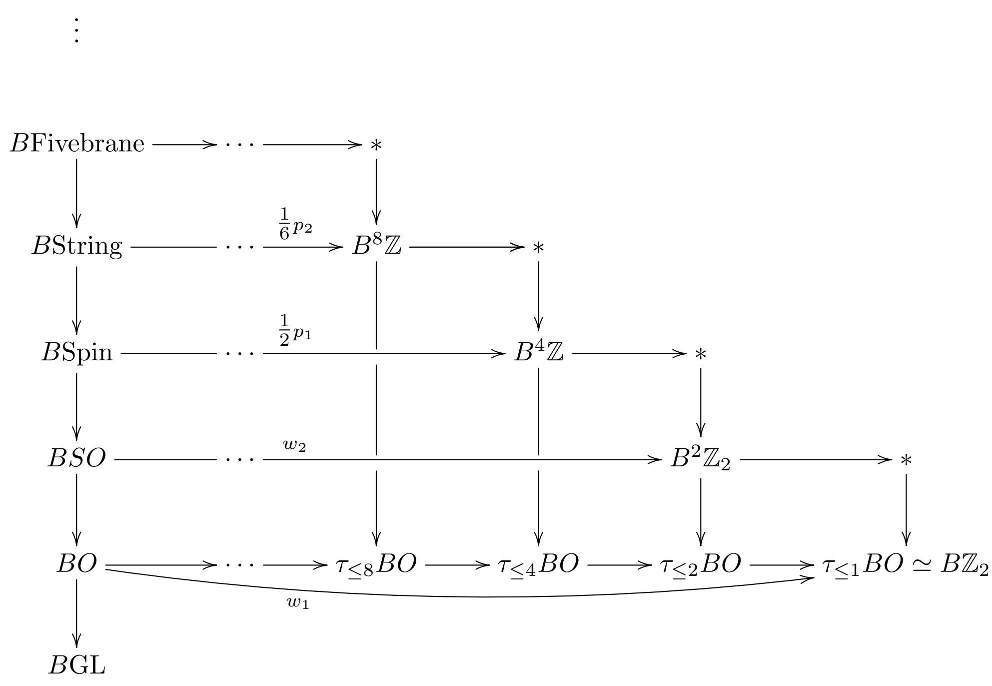
We discuss how this construction is only one step in a whole tower of analogous constructions involving smooth n-groups for various n. These are higher smooth analogs of the Spin-group and define higher analogs of smooth spin structures.

The Spin-group carries its name due to the central role that it plays in the description of the physics of quantum *spinning particles*. In 1.1.4 we indicated how the higher spin structures to be discussed here are similarly related to spinning quantum strings and 5-branes. More in detail, this requires *twisted* higher spin structures, which we turn to below in 3.9.8.

5.1.1 Overview: the smooth and differential Whitehead tower of BO

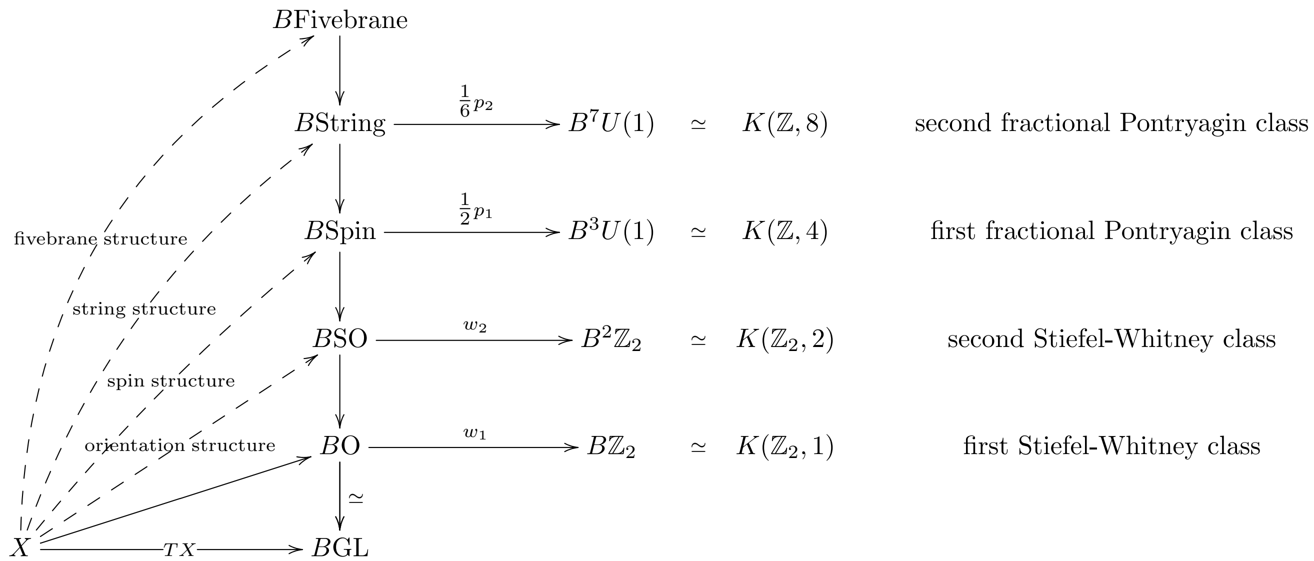
We survey the constructions and results about the smooth and differential refinement of the Whitehead tower of BO, to be discussed in the following.

By definition 3.8.10 applied in $\infty\mathrm{Grpd} \simeq \mathrm{Top}$, the first stages of the Whitehead tower of the classifying space BO of the orthogonal group, together with the corresponding obstruction classes is constructed by iterated pasting of homotopy pullbacks as in the following diagram:



Here the bottom horizontal tower is the Postnikov tower, def. 3.6.25, of BO and all rectangles are homotopy pullbacks.

For X a smooth manifold, there is a canonically given map $X \to B\mathrm{GL}$, which classifies the tangent bundle TX. The lifts of this classifying map through the above Whitehead tower correspond to structures on X as indicated in the following diagram:



Here the horizontal morphisms denote representatives of universal characteristic classes, such that each sub-diagram of the shape

$$\begin{array}{ccc} B\hat{G} & & \\ \downarrow & & \\ BG & \xrightarrow{c} & B^n K \end{array}$$

is a fiber sequence, def. 3.6.141.

The lifting problem presented by each of these steps is exemplified in terms of a smooth manifold X, which comes with a canonical map $X \to B\mathrm{GL}$ that classifies the tangent bundle TX of X.

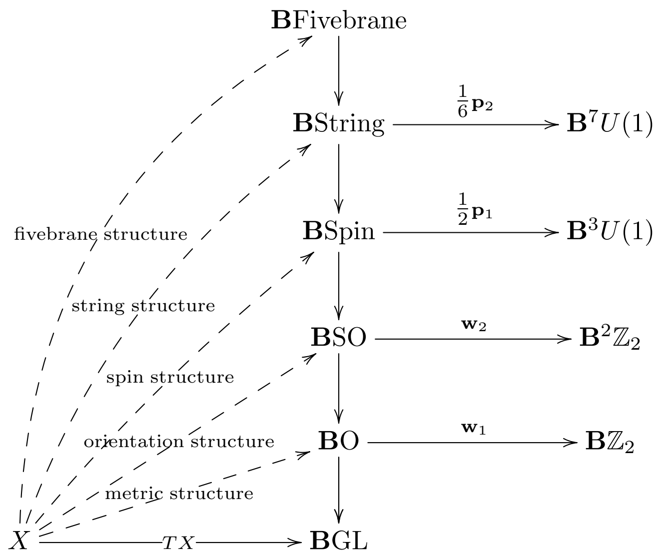
In the first step, since the $B\mathrm{O} \to B\mathrm{GL}$ is a weak equivalence in $\mathrm{Top} \simeq \infty\mathrm{Grpd}$, we may always factor $X \to B\mathrm{GL}$, up to homotopy, through $B\mathrm{O}$. The homotopy class of the resulting composite $X \to B\mathrm{O} \stackrel{w_1}{\to} B\mathbb{Z}_2$ is the first Stiefel-Whitney class of the manifold. The fact that $B\mathrm{SO}$ is the homotopy fiber of w_1 means, by the universal property of the homotopy pullback, that the further lift to a map $X \to B\mathrm{SO}$ exists precisely if the first Stiefel-Whitney class vanishes. While this is a classical fact, it is useful to make its relation to homotopy pullbacks explicit here, since this illuminates the following steps in this tower as well as all the steps in the smooth and differential refinements to follow.

Next, if the first Stiefel-Whitney class of X vanishes, then any *choice* of orientation, hence any choice of lift $X \to B\mathrm{SO}$ induces the composite map $X \to B\mathrm{SO} \stackrel{w_2}{\to} B^2\mathbb{Z}_2$, whose homotopy class is the second Stiefel-Whitney class of X equipped with that orientation. If that class vanishes, there exists a choice of lift $X \to B\mathrm{Spin}$, which is a choice of spin structure on X. The resulting composite $X \to B\mathrm{Spin} \stackrel{\frac{1}{2}p_1}{\to} B^3U(1)$ is a representative of the *first fractional Pontryagin class*. If this vanishes, there exists a choice of lift $X \to B\mathrm{String}$, which equips X with a *string structure*. The induced composite $X \to B\mathrm{String} \stackrel{\frac{1}{6}p_2}{\to} B^7U(1)$ is a representative of the second fractional Pontryagin class of X. If that vanishes, there exists a choice of lift $X \to B\mathrm{Fivebrane}$, which is a choice of *fivebrane srructure* on X.

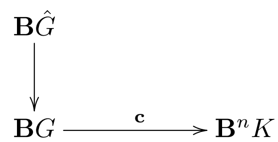
In this or slightly different terminology, this is a classical construction in homotopy theory. We show in the following that this tower has a *smooth lift* from topological spaces through the geometric realization functor, 4.4.4,

$$\mathrm{Smooth}\infty\mathrm{Grpd} \xrightarrow{\Pi} \infty\mathrm{Grpd} \xrightarrow[\simeq]{|-|} \mathrm{Top}$$

to smooth ∞-groupoids, of the form



Here $\mathbf{B}^n U(1)$ is the smooth circle $(n+1)$-group, def. 4.4.21, the smooth classifying n-stack of smooth circle n-bundles. This is such that still all diagrams of the form



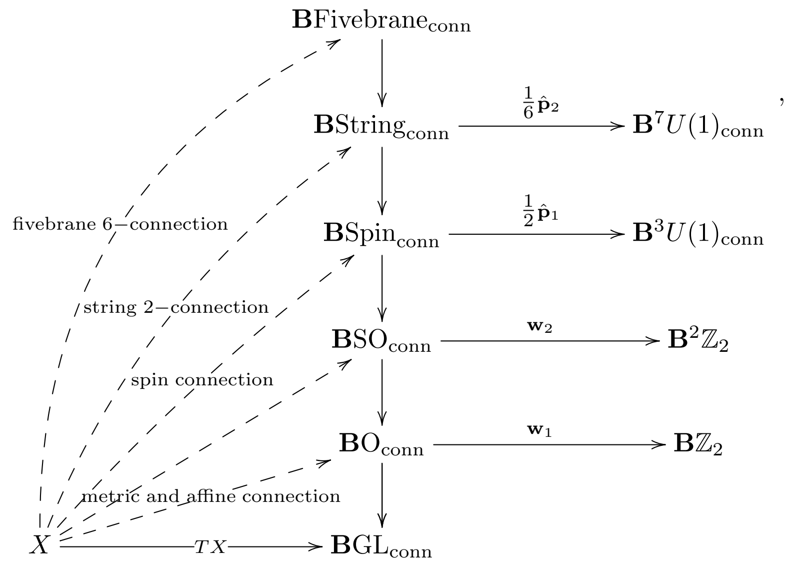
are fiber sequences, now in the cohesive ∞-topos Smooth∞Grpd, exhibiting the smooth moduli ∞-stack $\mathbf{B}\hat{G}$ as the homotopy fiber of the smooth universal characteristic map $\mathbf{c}$ which is a smooth refinement of the corresponding ordinary characteristic map c.

The corresponding choices of lifts now are more refined than before, as they correspond to *smooth structures*. In the first step, the choice of lift from a morphism $X \to \mathbf{B}\mathrm{GL}$ to a morphism $X \to \mathbf{B}\mathrm{SO}$ encodes now genuine information, namely a choice of *Riemannian metric* on X. This is discussed in 5.4.4.1 below.

Further up, a choice of lift $X \to \mathbf{B}\mathrm{Spin}$ is a choice of smooth Spin-principal bundle on X. Next, the object denoted String is a smooth 2-group, and a lift $X \to \mathbf{B}\mathrm{String}$ is a choice of smooth String-principal 2-bundle on X. The object denoted Fivebrane is a smooth 6-group and a choice of lift $X \to \mathbf{B}\mathrm{Fivebrane}$ is a choice of smooth Fivebrane-principal 6-bundle.

One consequence of the smooth refinement, which is important for the *twisted* such structures discussed below in 3.9.8, is that the spaces of choices of lifts are much more refined than those of the ordinary non-smooth case. Another consequence is that it allows to proceed and next consider a *differential* refinement, def. 3.9.59:

we show that the above smooth Whitehead tower further lifts to a *differential Whitehead tower* of the form



where $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is the moduli n-stack of circle n-bundles with connection, according to 4.4.16. Still, all diagrams of the form

$$\begin{array}{ccc} \mathbf{B}\hat{G}_{\mathrm{conn}} & & \\ \downarrow & & \\ \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^n K_{\mathrm{conn}} \end{array}$$

are fiber sequences in Smooth∞Grpd, exhibiting the smooth moduli ∞-stack $\mathbf{B}\hat{G}_{\mathrm{conn}}$, def. 3.9.59, of higher $\hat{G}$-connections as the homotopy fiber of the differential refinement $\hat{\mathbf{c}}$ of the given characteristic map c. Choices of lifts through this tower correspond to choices of smooth higher connections on smooth higher bundles.

5.1.2 Orienation structure

Before going to higher degree beyond the Spin-group, it is instructive to first consider a *lower* degree. The special orthogonal Lie group itself is a kind of extension of the orthogonal Lie group. To see this clearly, consider the smooth delooping $\mathbf{B}\mathrm{SO}(n) \in \mathrm{Smooth}\infty\mathrm{Grpd}$ according to 4.4.2.

Proposition 5.1.1. *The canonical morphism* $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$ *induces a long fiber sequence in* Smooth∞Grpd *of the form*

$$\mathbb{Z}_2 \to \mathbf{B}\mathrm{SO}(n) \to \mathbf{B}O(n) \stackrel{\mathbf{w}_1}{\to} \mathbf{B}\mathbb{Z}_2 \,,$$

where $\mathbf{w}_1$ *is the universal smooth first Stiefel-Whitney class from example 1.2.109.*

Proof. It is sufficient to show that the homotopy fiber of $\mathbf{w}_1$ is $\mathbf{B}\mathrm{SO}(n)$. This implies the rest of the statement by prop. 3.6.142.

To see this, notice that by the discussion in 3.6.9 we are to compute the $\mathbb{Z}_2$-principal bundle over the Lie groupoid $\mathbf{B}\mathrm{SO}(n)$ that is classified by the above injection. By observation 3.6.182 this is accomplished by forming a 1-categorical pullback of Lie groupoids

$$\begin{array}{ccc} \mathbb{Z}_2//\mathrm{O}(n) & \longrightarrow & \mathbb{Z}_2//\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ *//\mathrm{O}(n) & \longrightarrow & *//\mathbb{Z}_2 \end{array} .$$

One sees that the canonical projection

$$\mathbb{Z}_2//\mathrm{O}(n) \stackrel{\simeq}{\to} *//\mathrm{SO}(n)$$

is a weak equivalence (it is an essentially surjective and full and faithful functor of groupoids). $\square$

Definition 5.1.2. For $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ any object equipped with a morphism $r_X : X \to \mathbf{B}O(n)$, we say a lift o_X of r through the above extension

$$\begin{array}{ccc} & & \mathbf{B}\mathrm{SO}(n) \\ & {}^{o_X}\nearrow & \downarrow \\ X & \xrightarrow{r} & \mathbf{B}O(n) \end{array}$$

is an *orientation structure* on (X, r_X).

5.1.3 Spin structure

Proposition 5.1.3. *The classical sequence of Lie groups* $\mathbb{Z}_2 \to \mathrm{Spin} \to \mathrm{SO}$ *induces a long fiber sequence in* $\mathrm{Smooth}\infty\mathrm{Grpd}$ *of the form*

$$\mathbb{Z}_2 \to \mathrm{Spin} \to \mathrm{SO} \to \mathbf{B}\mathbb{Z}_2 \to \mathbf{B}\mathrm{Spin} \to \mathbf{B}\mathrm{SO} \stackrel{\mathbf{w}_2}{\to} \mathbf{B}^2\mathbb{Z}_2\,,$$

where $\mathbf{w}_2$ *is the universal smooth second Stiefel-Whitney class from example 1.2.110.*

Proof. It is sufficient to show that the homotopy fiber of $\mathbf{w}_2$ is $\mathbf{B}\mathrm{Spin}(n)$. This implies the rest of the statement by prop. 3.6.142.

To see this notice that the top morphism in the stanard anafunctor that presents $\mathbf{w}_2$

$$\begin{array}{ccc} \mathbf{B}(\mathbb{Z}_2 \to \mathrm{O}(n))_{\mathrm{ch}} & \longrightarrow & \mathbf{B}(\mathbb{Z}_2 \to 1)_{\mathrm{ch}} \qquad \mathbf{B}^2\mathbb{Z}_2 \\ \downarrow{\simeq} & & \\ \mathbf{B}\mathrm{SO}(n) & & \end{array}$$

is a fibration in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. By proposition 2.3.13 this means that the homotopy fiber is given by the 1-categorical pullback of simplicial presheaves

$$\begin{array}{ccc} \mathbf{B}(\mathbb{Z}_2 \to \mathrm{O}(n))_{\mathrm{ch}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathbb{Z}_2 \to \mathrm{O}(n))_{\mathrm{ch}} & \xrightarrow{\mathbf{w}_2} & \mathbf{B}(\mathbb{Z}_2 \to 1)_{\mathrm{ch}} \end{array} .$$

The canonical projection

$$\mathbf{B}(\mathbb{Z}_2 \to \mathrm{O}(n))_{\mathrm{ch}} \stackrel{\simeq}{\to} \mathbf{B}\mathrm{SO}(n)_{\mathrm{ch}}$$

is seen to be a weak equivalence. $\square$

Definition 5.1.4. For $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ an object equipped with orientation structure $o_X : X \to \mathbf{B}\mathrm{SO}(n)$, def. 5.1.2, we say a choice of lift $\hat{o}_X$ in

$$\begin{array}{ccc} & & \mathbf{B}\mathrm{Spin} \\ & {}^{\hat{o}_X}\nearrow & \downarrow \\ X & \xrightarrow{o_X} & \mathbf{B}\mathrm{SO}(n) \end{array}$$

equips (X, o_X) with *spin structure*.

5.1.4 Smooth string structure and the String-2-group

The sequence of Lie groupoids

$$\cdots \to \mathbf{B}\mathrm{Spin}(n) \to \mathbf{B}\mathrm{SO}(n) \to \mathbf{B}\mathrm{O}(n)$$

discussed in 5.1.2 and 5.1.3 is a smooth refinement of the first two steps of the *Whitehead tower* of $BO(n)$. We discuss now the next step. This is no longer presented by Lie groupoids, but by smooth 2-groupoids.

Write $\mathfrak{so}(n)$ for the special orthogonal Lie algebra in dimension n. We shall in the following notationally suppress the dimension and just write $\mathfrak{so}$. The simply connected Lie group integrating $\mathfrak{so}$ is the Spin-group .

Proposition 5.1.5. *Pulled back to* $B\mathrm{Spin}$ *the universal first Pontryagin class* $p_1 : BO \to B^4\mathbb{Z}$ *is 2 times a generator* $\frac{1}{2}p_1$ *of* $H^4(B\mathrm{Spin}, \mathbb{Z})$

$$\begin{array}{ccc} B\mathrm{Spin} & \xrightarrow{\frac{1}{2}p_1} & B^4\mathbb{Z} \\ \downarrow & & \downarrow^{\cdot 2} \\ BO & \xrightarrow{p_1} & B^4\mathbb{Z} \end{array} .$$

We call $\frac{1}{2}p_1$ *the* first fractional Pontryagin class *.*

This is due to [Bott58]. See [SSS09b] for a review.

Definition 5.1.6. Write $B\mathrm{String}$ for the homotopy fiber in Top $\simeq \infty$Grpd of the first fractional Pontryagin class

$$\begin{array}{ccc} B\mathrm{String} & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathrm{Spin} & \xrightarrow{\frac{1}{2}p_1} & B^4\mathbb{Z} \end{array} .$$

Its loop space is the *string group*

$$\mathrm{String} := O\langle 7\rangle := \Omega B\mathrm{String} .$$

This is defined up to equivalence as an ∞-group object, but standard methods give a presentation by a genuine topological group and often the term *string group* is implicitly reserved for such a topological group model. See also the review in [Scho10].

We now discuss smooth refinements of $\frac{1}{2}p_1$ and of String as lifts through the intrinsic geometric realization, def. 3.8.2, $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}$ in Smooth∞Grpd, 4.4.

Proposition 5.1.7. *We have a weak equivalence*

$$\mathbf{cosk}_3(\exp(\mathfrak{so})) \xrightarrow{\simeq} \mathbf{B}\mathrm{Spin}_c$$

in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$*, between the Lie integration, 4.4.14, of* $\mathfrak{so}$ *and the standard presentation, 4.4.2, of* $\mathbf{B}\mathrm{Spin}$*.*

Proof. By prop. 4.4.57. ☐

Corollary 5.1.8. *The image of* $\mathbf{B}\mathrm{Spin} \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *under the fundamental* ∞*-groupoid/geometric realization functor* Π*, 4.3.4, is the classifying space* $B\mathrm{Spin}$ *of the topological* Spin*-group*

$$|\Pi \mathbf{B}\mathrm{Spin}| \simeq B\mathrm{Spin} .$$

Proof. By prop. 4.3.30 applied to prop. 4.4.19. ☐

Theorem 5.1.9. *The image under Lie integration, 4.4.14, of the canonical Lie algebra 3-cocycle*

$$\mu = \langle -, [-,-]\rangle : \mathfrak{so} \to b^2\mathbb{R}$$

on the semisimple Lie algebra $\mathfrak{so}$ *of the* Spin *group is a morphism in* Smooth∞Grpd *of the form*

$$\frac{1}{2}\mathbf{p}_1 := \exp(\mu) : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3 U(1)$$

whose image under the the fundamental ∞-groupoid ∞-functor/ geometric realization, 4.3.4, $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}$ *is the ordinary fractional Pontryagin class* $\frac{1}{2}p_1 : B\mathrm{Spin} \to B^4\mathbb{Z}$ *in* Top, *and up to equivalence* $\exp(\mu)$ *is the unique lift of* $\frac{1}{2}p_1$ *from* Top *to* Smooth∞Grpd *with codomain* $\mathbf{B}^3U(1)$. *We write* $\frac{1}{2}\mathbf{p}_1 := \exp(\mu)$ *and call it the* smooth first fractional Pontryagin class*.*

Moreover, the corresponding refined differential characteristic class, 4.4.17,

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{H}_{\mathrm{conn}}(-, \mathbf{B}\mathrm{Spin}) \to \mathbf{H}_{\mathrm{diff}}(-, \mathbf{B}^3 U(1))\,,$$

wich we call the fractional Pontryagin class*, is in cohomology the corresponding ordinary refined Chern-Weil homomorphism [HoSi05]*

$$[\frac{1}{2}\hat{\mathbf{p}}_1] : H^1_{\mathrm{Smooth}}(X, \mathrm{Spin}) \to H^4_{\mathrm{diff}}(X)$$

with values in ordinary differential cohomology that corresponds to the Killing form invariant polynomial $\langle -,-\rangle$ *on* $\mathfrak{so}$*.*

Proof. This is shown in [FSS10].

Using corollary. 5.1.7 and unwinding all the definitions and using the characterization of smooth de Rham coefficient objects, 4.4.13, and smooth differential coefficient objects, 4.4.16, one finds that the post-composition with $\exp(\mu, \mathrm{cs})_{\mathrm{diff}}$ induces on Čech cocycles precisely the operation considered in [BrMc96b], and hence the conclusion follows essentially as by the reasoning there: one reads off the 4-curvature of the circle 3-bundle assigned to a Spin bundle with connection ∇ to be $\propto \langle F_\nabla \wedge F_\nabla\rangle$, with the normalization such that this is the image in de Rham cohomology of the generator of $H^4(B\mathrm{Spin}) \simeq \mathbb{Z} \simeq \langle \frac{1}{2}p_1\rangle$.

Finally that $\frac{1}{2}\mathbf{p}_1$ is the unique smooth lift of $\frac{1}{2}p_1$ follows from theorem 4.4.33. □

By the unique smooth refinement of the first fractional Pontryagin class, 5.1.9, we obtain a smooth refinement of the String-group, def. 5.1.6.

Definition 5.1.10. Write $\mathbf{B}$String for the homotopy fiber in Smooth∞Grpd of the smooth refinement of the first fractional Pontryagin class from prop. 5.1.9:

$$\begin{array}{ccc} \mathbf{B}\mathrm{String} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3 U(1) \end{array} .$$

We say its loop space object is the *smooth string 2-group*

$$\mathrm{String}_{\mathrm{smooth}} := \Omega \mathbf{B}\mathrm{String}\,.$$

We speak of a smooth *2-group* because $\mathrm{String}_{\mathrm{smooth}}$ is a categorical homotopy 1-type in Smooth∞Grpd, being an extension

$$\mathbf{B}U(1) \to \mathrm{String}_{\mathrm{smooth}} \to \mathrm{Spin}$$

of the categorical 0-type Spin by the categorical 1-type $\mathbf{B}U(1)$ in Smooth∞Grp.

Proposition 5.1.11. *The categorical homotopy groups of the smooth String 2-group, $\pi_n(\mathbf{B}\mathrm{String}) \in \mathrm{Sh}(\mathrm{CartSp})$, are*

$$\pi_1(\mathbf{B}\mathrm{String}) \simeq \mathrm{Spin}$$

and

$$\pi_2(\mathbf{B}\mathrm{String}) \simeq U(1)\,.$$

All other categorical homotopy groups are trivial.

Proof. Notice that by construction the non-trivial categorical homotopy groups of $\mathbf{B}\mathrm{Spin}$ and $\mathbf{B}^3U(1)$ are $\pi_1\mathbf{B}\mathrm{Spin} = \mathrm{Spin}$ and $\pi_3\mathbf{B}^3U(1) = U(1)$, respectively. Using the long exact sequence of homotopy sheaves (use [LuHTT] remark 6.5.1.5,with $X = *$ the base point) applied to def. 5.1.10, we obtain the long exact sequence of pointed objects in Sh(CartSp)

$$\cdots \to \pi_{n+1}(\mathbf{B}^3U(1)) \to \pi_n(\mathbf{B}\mathrm{String}) \to \pi_n(\mathbf{B}\mathrm{Spin}) \to \pi_n(\mathbf{B}^3U(1)) \to \pi_{n-1}(\mathbf{B}\mathrm{String}) \to \cdots$$

this yields for $n = 0$

$$0 \to \pi_1(\mathbf{B}\mathrm{String}) \to \mathrm{Spin} \to 0$$

and for $n = 2$

$$0 \to U(1) \to \pi_2(\mathbf{B}\mathrm{String}) \to 0$$

and for $n \geq 3$

$$0 \to \pi_n(\mathbf{B}\mathrm{String}) \to 0\,.$$

□

However the *geometric* homotopy type, 3.8.1, of $\mathbf{B}\mathrm{String}$ is not bounded, in fact it coincides with that of the topological string group:

Proposition 5.1.12. *Under intrinsic geometric realization, 4.4.4, $|-| : \mathrm{Smooth}\infty\mathrm{Grpd} \stackrel{\Pi}{\to} \infty\mathrm{Grp} \stackrel{|-|}{\to} \mathrm{Top}$ the smooth string 2-group maps to the topological string group*

$$|\mathrm{String}_{\mathrm{smooth}}| \simeq \mathrm{String}\,.$$

Proof. Since $\mathbf{B}^3U(1)$ has a presentation by a simplicial object in SmoothMfd, prop. 4.4.29 asserts that

$$|\mathrm{String}_{\mathrm{smooth}}| \simeq \mathrm{hofib}|\frac{1}{2}\mathbf{p}_1|\,.$$

The claim then follows with prop. 5.1.9

$$\cdots \simeq \mathrm{hofib}\frac{1}{2}p_1$$

and def. 5.1.6

$$\cdots \simeq \mathrm{String}\,.$$

□

Notice the following important subtlety:

Proposition 5.1.13. *There exists an infinite-dimensional Lie group $\mathrm{String}_{1\mathrm{smooth}}$ whose underlying topological group is a model for the String group in* Top*, def. 5.1.6.*

This is due to [NSW11], by a refinement of a construction in [Stol96].

Remark 5.1.14. However, $\mathbf{B}\mathrm{String}_{1\mathrm{smooth}}$ itself is not a model for def. 5.1.10, because it is an internal 1-type in Smooth∞Grpd, hence because $\pi_2\mathbf{B}\mathrm{String}_{\mathrm{smooth}} = 0$. In [NSW11] a smooth 2-group with the correct internal homotopy groups based on $\mathrm{String}_{1\mathrm{smooth}}$ is given, but it is not clear yet whether or not this is a model for def. 5.1.10.

We proceed by discussing concrete presentations of the smooth string 2-group.

Definition 5.1.15. Write

$$\mathfrak{string} := \mathfrak{so}_\mu$$

for the L_∞-algebra extension of $\mathfrak{so}$ induced by μ according to def 4.4.102.

We call this the *string Lie 2-algebra*

Observation 5.1.16. The indecomposable invariant polynomials on $\mathfrak{string}$ are those of $\mathfrak{so}$ except for the Killing form:

$$\mathrm{inv}(\mathfrak{string}) = \mathrm{inv}(\mathfrak{so})/(\langle -, - \rangle)\,.$$

Proof. As a special case of prop. 4.4.120. □

Proposition 5.1.17. *The smooth ∞-groupoid that is the Lie integration of $\mathfrak{so}_\mu$ is a model for the smooth string 2-group*

$$\mathbf{B}\mathrm{String} \simeq \mathbf{cosk}_3 \exp(\mathfrak{so}_\mu)\,.$$

Notice that this statement is similar to, but different from, the statement about the untruncated exponentiated L_∞-algebras in prop. 4.4.108.

Proof. By prop. 5.1.9 an explicit presentation for $\mathbf{B}\mathrm{String}$ is given by the pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}_c & \longrightarrow & \mathbf{E}\mathbf{B}^2 U(1)_c \\ \downarrow & & \downarrow \\ \mathbf{cosk}_3 \exp(\mathfrak{so}) & \xrightarrow{\int_{\Delta^\bullet} \exp(\mu)} & \mathbf{B}^3 U(1)_c \end{array}$$

in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$, where $\mathbf{B}^3 U(1)_c$ is the simplicial presheaf whose 3-cells form the space $U(1)$, and where $\mathbf{E}B^2 U(1)$ is the simplicial presheaf whose 2-cells form $U(1)$ and whose 3-cells form the space of arbitrary quadruples of elements in $U(1)$. The right vertical morphism forms the oriented sum of these quadruples.

Since all objects are 3-truncated, it is sufficient to consider the pullback of the simplices in degrees 0 to 3. In degrees 0 to 1 the morphism $\mathbf{E}\mathbf{B}^2 U(1) \to \mathbf{B}^3 U(1)_c$ is the identity, hence in these degrees $\mathbf{B}\mathrm{String}_c$ coincides with $\mathbf{cosk}_3 \exp(\mathfrak{so})$. In degree 2 the pullback is the product of $\mathbf{cosk}_3(\mathfrak{so})_2$ with $U(1)$, hence the 2-cells of $\mathbf{B}\mathrm{String}_c$ are pairs (f, c) consisting of a smooth map $f : \Delta^2 \to \mathrm{Spin}$ (with sitting instants) and an elemement $c \in U(1)$. Finally a 3-cell in $\mathbf{B}\mathrm{String}_c$ is a pair $(\sigma, \{c_i\})$ of a smooth map $\sigma : \Delta^3 \to \mathrm{Spin}$ and four labels $c_i \in U(1)$, subject to the condition that the sum of the labels is the integral of the cocycle μ over σ:

$$c_4 c_2 c_1^{-1} c_3^{-1} = \int_{\Delta^3} \sigma^* \mu(\theta) \bmod \mathbb{Z}\,,$$

(with θ the Maurer-Cartan form on Spin).

The description of the cells in $\mathbf{cosk}_3 \exp(\mathfrak{g}_\mu)$ is similar: a 2-cells is a pair (f, B) consisting of a smooth function $f : \Delta^2 \to \mathrm{Spin}$ and a smooth 2-form $B \in \Omega^2(\Delta^2)$ (both with sitting instants), and a 3-cell is a pair consisting of a smooth function $\sigma : \Delta^3 \to \mathrm{Spin}$ and a 2-form $\hat{B} \in \Omega^2(\Delta^3)$ such that $d\hat{B} = \sigma^* \mu(\theta)$.

There is an evident morphism

$$p : \int_{\Delta^\bullet} : \mathbf{cosk}_3(\mathfrak{so}_\mu) \to \mathbf{B}\mathrm{String}_c$$

that is the identity on the smooth maps from simplices into the Spin-group and which sends the 2-form labels to their integral over the 2-faces

$$p_2 : (f, B) \mapsto (f, (\int_{\Delta^2} B) \mathrm{mod} \mathbb{Z})\,.$$

We claim that this is a weak equivalence. The first simplicial homotopy group on both sides is Spin itself (meaning: the presheaf on CartSp represented by Spin). The nontrivial simplicial homotopy group to check is the second. Since $\pi_2(\mathrm{Spin}) = 0$ every pair (f, B) on $\partial\Delta^3$ is homotopic to one where f is constant. It follows from prop. 4.4.61 that the homotopy classes of such pairs where also the homotopy involves a constant map $\partial\Delta^3 \times \Delta^1 \to \mathrm{Spin}$ are given by $\mathbb{R}$, being the integral of the 2-forms. But then moreover there are the non-constant homotopies in Spin from the constant 2-sphere to itself. Since $\pi_3(\mathrm{Spin}) = \mathbb{Z}$ and $\mu(\theta)$ is an integral form, this reduces the homotopy classes to $U(1) = \mathbb{R}/\mathbb{Z}$. This are the same as in $\mathbf{B}\mathrm{String}_c$ and the integration map that sends the 2-forms to elements in $U(1)$ is an isomorphism on these homotopy classes. □

Remark 5.1.18. Propositions 5.1.17 and 5.1.12 together imply that the geometric realization $|\mathbf{cosk}_3 \exp(\mathfrak{so}_\mu)|$ is a model for BString in Top

$$|\exp(\mathfrak{so}_\mu)| \simeq B\mathrm{String}.$$

With slight differences in the technical realization of $\exp(\mathfrak{g}_m u)$ this was originally shown in [Henr08], theorem 8.4. For the following discussion however the above perspective, realizing $\mathbf{cosk}_3 \exp(\mathfrak{so}_\mu)$ as a presentation of the homotopy fiber of the smooth first fractional Pontryagin class, def 5.1.10, is crucial.

We now discuss three equivalent but different models of the smooth String 2-group by diffeological *strict* 2-groups, hence by crossed modules of diffeological groups. See [BCSS07] for the general notion of strict Fréchet-Lie 2-groups and for discussion of one of the following models.

Definition 5.1.19. For $(G_1 \to G_0)$ a crossed module of diffeological groups (groups of concrete sheaves on CartSp) write

$$\Xi(G_1 \to G_0) \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$$

for the corresponding presheaf of simplicial groups.

There is an evident strictification of $\mathbf{B}\mathrm{String}_c$ from the proof of prop 5.1.17 given by the following definition. For the notion of thin homotopy classes of paths and disks see [ScWaII].

Definition 5.1.20. Write

$$\hat{\Omega}_{\mathrm{th}}\mathrm{Spin} \to P_{\mathrm{th}}\mathrm{Spin}\,,$$

for the crossed module where

- P_{th}Spin is the group whose elements are *thin-homotopy* classes of based smooth paths in G and whose product is obtained by rigidly translating one path so that its basepoint matches the other path's endpoint and then concatenating;

- $\hat{\Omega}_{\mathrm{th}}$Spin is the group whose elements are equivalence classes of pairs (d, x) consisting of *thin homotopy* classes of disks $d : D^2 \to G$ in G with sitting instant at a chosen point on the boundary, together with an element $x \in \mathbb{R}/\mathbb{Z}$. Two such pairs are taken to be equivalent if the boundary of the disks has the same thin homotopy classes and if the labels x and x' differ, in $\mathbb{R}/\mathbb{Z}$, by the integral $\int_{D^3} f^*\mu(\theta)$ over any 3-ball $f : D^3 \to G$ cobounding the two disks. The product is given by translating and then *gluing* of disks at their basepoint (so that their boundary paths are being concatenated, hence multiplied in P_{th}Spin) and adding the labels in $\mathbb{R}/\mathbb{Z}$.

  The map from $\hat{\Omega}_{\mathrm{th}}$Spin to P_{th}Spin is given by sending a disk to its boundary path.

  The action of P_{th}Spin on Ω_{th}Spin is given by whiskering a disk by a path and its reverse path.

Proposition 5.1.21. *Let*

$$\mathbf{B}\mathrm{String}_c \to \mathbf{B}\Xi(\hat{\Omega}_{\mathrm{th}}\mathrm{Spin} \to P_{\mathrm{th}}\mathrm{Spin})$$

be the morphism that sends maps to Spin *to their thin-homotopy class. This is a weak equivalence in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

We produce now two equivalent crossed modules that are both obtained as central extensions of path groups. This is joint with Danny Stevenson, based on results in [MuSt03].

The following proposition is standard.

Proposition 5.1.22. *Let $H \subset G$ be a normal subgroup of some group G and lat $\hat{H} \to H$ be a central extension of groups such that the conjugation action of G on H lifts to an automorphism action $\alpha : G \to \mathrm{Aut}(\hat{H})$ on the central extension. Then $(\hat{H} \to G)$ with this α is a crossed module.*

We construct classes of examples of this type from central extensions of path groups.

Proposition 5.1.23. *Let $G \subset \Gamma$ be a simply connected normal Lie subgroup of a Lie group Γ. Write PG for the based path group of G whose elements are smooth maps $[0,1] \to G$ starting at the neutral element and whose product is given by the pointwise product in G. Consider the complex with differential $d \pm \delta$ of simplicial forms on $\mathbf{B}G_{\mathrm{ch}}$. Let (F, a, β) be a triple where*
i. $a \in \Omega^1(G \times G)$ such that $\delta a = 0$;
ii. F is a closed integral 2-form on G such that $\delta F = da$;
iii. $\beta : \Gamma \to \Omega^1(G)$ such that, for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$,

- *$\gamma^* F = F + d\beta_\gamma$;*
- *$(\gamma_1)^* \beta_{\gamma_2} - \beta_{\gamma_1 \gamma_2} + \beta_{\gamma_1} = 0$;*
- *$a = \gamma^* a + \delta(\beta_\gamma)$;*
- *for all based paths $f : [0,1] \to G$, $f^* \beta_\gamma = (f, \gamma^{-1})^* a + (\gamma, f\gamma^{-1})^* a$.*

1. *Then the map $c : PG \times PG \to U(1)$ given by $c : (f,g) \mapsto c_{f,g} := \exp\left(2\pi i \int\limits_{0,1} (f,g)^* a\right)$ is a group 2-cocycle leading to a central extension $\widehat{P}G = PG \ltimes U(1)$ with product $(\gamma_1, x_1) \cdot (\gamma_2, x_2) = (\gamma_1 \cdot \gamma_2, x_1 x_2 c_{\gamma_1, \gamma_2})$.*
2. *Since G is simply connected every loop in G bounds a disk D. There is a normal subgroup $N \subset \widehat{P}G$ consisting of pairs (γ, x) with $\gamma(1) = e$ and $x = \exp(2\pi i \int_D F)$ for any disk D in G such that $\partial D = \gamma$.*
3. *Finally, $\tilde{G} := \widehat{PG}/N$ is a central extension of G by $U(1)$ and the conjugation action of Γ on G lifts to $\tilde{G}$ by setting $\alpha(\gamma)(f,x) := (\alpha(\gamma)(f),\ x \exp(\in_f \beta_\gamma))$ such that $\mathrm{Cent}(G, \Gamma, F, a, \beta) := (\tilde{G} \to \Gamma)$ is a Lie crossed module and hence a strict Lie 2-group of the type in prop. 5.1.22.*

Proof. All statements about the central extension $\hat{G}$ can be found in [MuSt03]. It remains to check that the action $\alpha : \Gamma \to \mathrm{Aut}(\tilde{G})$ satisfies the required axioms of a crossed module, in particular the condition $\alpha(t(h))(h') = hh'h^{-1}$. For this we have to show that

$$\alpha(h(1))([f,z]) = [h,1][f,z]\left[h^{-1}, \exp(-\int_{(h,h^{-1})} a)\right] ,$$

where h denotes a based path in $P\mathcal{G}$, so that $[h,1]$ represents an element of $\tilde{\mathcal{G}}$. By definition of the product in $\tilde{\mathcal{G}}$, the right hand side is equal to

$$\left[hfh^{-1}, z \exp\left(\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a\right)\right] .$$

This is not exactly in the form we want, since the left hand side is equal to $\left[h(1)fh(1)^{-1}, z\exp(\int_f \beta_h)\right]$. Therefore, we want to replace hfh^{-1} with the homotopic path $h(1)fh(1)^{-1}$. An explicit homotopy between these two paths is given by $H(s,t) = h((1-s)t+s)f(t)h((1-s)t+s)^{-1}$. Therefore, we have the equality

$$\left[hfh^{-1}, z\exp\left(\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a\right)\right]$$
$$= \left[h(1)fh(1)^{-1}, z\exp\left(\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a + \int H^*F\right)\right] .$$

Using the relation $\delta(F) = da$ and the fact that the pullback of F along the maps $[0,1]\times[0,1] \to G$, $(s,t)\mapsto h((1-s)t+s)$ vanish, we see that

$$\int H^*F = \int_{(f,h(1)^{-1})} a - \int_{(f,h^{-1})} a + \int_{(h,h^{-1})} a + \int_{(h(1),fh(1)^{-1})} a - \int_{(h,fh^{-1})} a .$$

Therefore the sum of integrals

$$\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a + \int H^*F$$

can be written as

$$\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a + \int_{(f,h(1)^{-1})} a - \int_{(f,h^{-1})} a + \int_{(h,h^{-1})} a + \int_{(h(1),fh(1)^{-1})} a - \int_{(h,fh^{-1})} a .$$

Using the condition $\delta(a) = 0$, we see that this simplifies down to $\int_{(f,h(1)^{-1})} a + \int_{(h(1),fh(1)^{-1})} a$. Therefore, a sufficient condition to have a crossed module is the equation $f^*\beta_h = (f,h(1))^*a + (h(1),fh(1)^{-1})^*a$. □

Proposition 5.1.24. *Given triples (F,a,β) and (F',a',β') as above and given $b\in\Omega^1(G)$ such that*

$$F' = F + db\,, \tag{5.1}$$

$$a' = a + \delta(b) \tag{5.2}$$

and for all $\gamma\in\Gamma$

$$\beta_\gamma + \gamma^* b = b + \beta'_\gamma\ , \tag{5.3}$$

then there is an isomorphism $\mathrm{Cent}(G,\Gamma,F,a,\beta) \simeq \mathrm{Cent}(G,\Gamma,F',a',\beta')$.

In [BCSS07] the following special case of this general construction was considered.

Definition 5.1.25. Let G be a compact, simple and simply-connected Lie group with Lie algebra $\mathfrak{g}$. Let $\langle\cdot,\cdot\rangle$ be the Killing form invariant polynomial on $\mathfrak{g}$, normalized such that the Lie algebra 3-cocycle $\mu := \langle\cdot,[\cdot,\cdot]\rangle$ extends left invariantly to a 3-form on G which is the image in deRham cohomology of one of the two generators of $H^3(G,\mathbb{Z}) = \mathbb{Z}$. Let ΩG be the based loop group of G whose elements are smooth maps $\gamma : [0,1]\to G$ with $\gamma(0) = \gamma(1) = e$ and whose product is by pointwise multiplication of such maps. Define $F\in\Omega^2(\Omega G)$, $a\in\Omega^1(\Omega G\times\Omega G)$ and $\beta : \Gamma\to\Omega^1(\Omega G)$

$$\begin{aligned} F(\gamma,X,Y) &:= \int_0^{2\pi} \langle X, Y'\rangle dt \\ a(\gamma_1,\gamma_2,X_1,X_2) &:= \int_0^{2\pi} \langle X_1, \dot\gamma_2\gamma_2^{-1}\rangle dt \\ \beta(p)(\gamma,X) &:= \int_0^{2\pi} \langle p^{-1}\dot p, X\rangle dt \end{aligned}$$

This satisfies the axioms of prop. 5.1.23 and we write

$$\mathrm{String}_{\mathrm{BCSS}}(G) := \Xi\mathrm{Cent}(\Omega G, PG, F, \alpha, \beta)$$

for the corresponding diffeological strict 2-group. If $G = \mathrm{Spin}$ we write just $\mathrm{String}_{\mathrm{BCS}}$ for this.

There is a variant of this example, using another cocycle on loop groups that was given in [Mick87].

Definition 5.1.26. With all assumptions as in definition 5.1.25 define now

$$\begin{aligned} F(\gamma, X, Y) &:= \frac{1}{2}\int_0^{2\pi} \langle \gamma^{-1}\dot\gamma, [X,Y]\rangle dt \\ a(\gamma_1,\gamma_2,X_1,X_2) &:= \frac{1}{2}\int_0^{2\pi} \left(\langle X_1, \dot\gamma_2\gamma_2^{-1}\rangle - \langle \gamma_1^{-1}\dot\gamma_1, \gamma_2 X_2 \gamma_2^{-1}\rangle\right) dt \\ \beta(p)(\gamma, X) &:= \frac{1}{2}\int_0^{2\pi} \langle \gamma^{-1}p^{-1}\dot p\gamma + p^{-1}\dot p, X\rangle dt \end{aligned}$$

This satisfies the axioms of proposition 5.1.23 and we write

$$\mathrm{String}_{\mathrm{Mick}}(G) := \Xi\mathrm{Cent}(\Omega G, PG, F, \alpha, \beta)$$

for the corresponding 2-group. If $G = \mathrm{Spin}$ we write just $\mathrm{String}_{\mathrm{Mick}}$ for this.

Proposition 5.1.27. *There is an isomorphism of 2-groups* $\mathrm{String}_{\mathrm{BCSS}}(G) \xrightarrow{\simeq} \mathrm{String}_{\mathrm{Mick}}(G)$.

Proof. We show that $b \in \Omega^1(\Omega G)$ defined by $b(\gamma, X) := \frac{1}{4\pi}\int_0^{2\pi}\langle \gamma^{-1}\dot\gamma, X\rangle dt$ satisfies the conditions of prop. 5.1.24 and hence defines the desired isomorphism.

- Proof of equation 5.1: We calculate the exterior derivative db. To do this we first calculate the derivative $Xb(y)$: if $\gamma_t = \gamma e^{tX}$ then to first order in t, $\gamma_t^{-1}\dot\gamma_t$ is equal to $\gamma^{-1}\dot\gamma + t[\gamma^{-1}\dot\gamma, X] + tX'$. Therefore

  $$Xb(Y) = \frac{1}{2}\int_0^{2\pi} \left(\langle \gamma^{-1}\dot\gamma, [X,Y]\rangle + \langle X', Y\rangle\right) dt \; .$$

  Hence db is equal to

  $$\frac{1}{2}\int_0^{2\pi} \left(\langle \gamma^{-1}\dot\gamma, [X,Y]\rangle + \langle X', Y\rangle + \langle \gamma^{-1}\dot c, [X,Y]\rangle - \langle Y', X\rangle - \langle \gamma^{-1}\dot\gamma, [X,Y]\rangle\right) \; ,$$

  which is easily seen to simplify down to

  $$-\int_0^{2\pi} \langle X, Y\rangle dt + \frac{1}{2}\int_0^{2\pi}\langle \gamma^{-1}\dot\gamma, [X,Y]\rangle dt \, .$$

- Proof of equation 5.2: We get

  $$\begin{aligned} \frac{1}{2}\int_0^{2\pi} \big\{ &\langle \gamma_2\dot\gamma_2^{-1}, X_2\rangle - \langle \gamma_2^{-1}\gamma_1^{-1}\dot\gamma_1\gamma_2, \gamma_2^{-1}X_1\gamma_2\rangle - \langle \gamma_2^{-1}\gamma_1^{-1}\dot\gamma_1\gamma_2, X_2\rangle \\ &-\langle \gamma_2^{-1}\dot\gamma_2, \gamma_2^{-1}X_1\gamma_2\rangle - \langle \gamma_2^{-1}\dot\gamma_2, X_2\rangle + \langle \gamma_1^{-1}\dot\gamma_1, X_1\rangle \big\}\, dt \; , \end{aligned}$$

  which is equal to

  $$\frac{1}{2}\int_0^{2\pi} \left\{ -\langle \gamma_1^{-1}\dot\gamma_1, \gamma_2 X_2\gamma_2^{-1}\rangle - \langle \dot\gamma_2\gamma_2^{-1}, X_1\rangle \right\} dt \; ,$$

  which in turn equals

  $$\frac{1}{2}\int_0^{2\pi} \left\{ \langle X_1, \dot\gamma_2\gamma_2^{-1}\rangle - \langle \gamma_1^{-1}\dot\gamma_1, \gamma_2 X_2\gamma_2^{-1}\rangle \right\} dt - \frac{1}{2\pi}\int_0^{2\pi} \langle X_1, \dot\gamma_2\gamma_2^{-1}\rangle dt \; .$$

- Proof of equation 5.3: we get

$$
\begin{aligned}
p^*b(\gamma;\gamma X) &= b(p\gamma p^{-1}; p\gamma p^{-1}(pXp^{-1})) \\
&= \frac{1}{2}\int_0^{2\pi} \langle p\gamma p^{-1}(\dot p\gamma p^{-1} + p\dot\gamma p^{-1} - p\gamma p^{-1}\dot p p^{-1}, pXp^{-1}\rangle dt \\
&= \frac{1}{2}\int_0^{2\pi} \langle p\gamma^{-1}p^{-1}\dot p\gamma p^{-1} + p\gamma^{-1}\dot\gamma p^{-1} - \dot p p^{-1}, pXp^{-1}\rangle dt \\
&= \frac{1}{2}\int_0^{2\pi} \langle \gamma^{-1}p^{-1}\dot p\gamma + \gamma^{-1}\dot\gamma - p^{-1}\dot p, X\rangle dt \\
&= b(\gamma,\gamma X) + \frac{1}{2}\int_0^{2\pi} \langle \gamma^{-1}p^{-1}\dot p\gamma - p^{-1}\dot p, X\rangle dt \\
&= b(\gamma,\gamma X) + \frac{1}{2}\int_0^{2\pi} \langle \gamma^{-1}p^{-1}\dot p\gamma + p^{-1}\dot p, X\rangle dt - \frac{1}{2\pi}\int_0^{2\pi}\langle p^{-1}\dot p, X\rangle dt
\end{aligned}
$$

The three conditions in proposition 5.1.24 are satisfied and, therefore, the desired isomorphism is established.
□

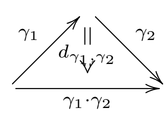
Proposition 5.1.28. *The strict 2-group* $\mathrm{String}_{\mathrm{Mick}}$ *from definition 5.1.26 is equivalent to the model* $\Xi(\hat\Omega_{\mathrm{th}}\mathrm{Spin} \to P_{\mathrm{th}})\mathrm{Spin}$ *from def. 5.1.20.*

Proof. We define a morphism $F : \mathbf{B}\mathrm{String}_{\mathrm{Mick}} \to \mathbf{B}\Xi(\hat\Omega_{\mathrm{th}}\mathrm{Spin} \to P_{\mathrm{th}})\mathrm{Spin}$. Its action on 1- and 2-morphisms is obvious: it sends parameterized paths $\gamma : [0,1] \to G = \mathrm{Spin}$. to their thin-homotopy equivalence class

$$F : \gamma \mapsto [\gamma]$$

and similarly for parameterized disks. On the $\mathbb{R}/\mathbb{Z}$-labels of these disks it acts as the identity.

The subtle part is the compositor measuring the coherent failure of this assignment to respect composition: Define the components of this compositor for any two parameterized based paths $\gamma_1, \gamma_2 : [0,1] \to G$ with pointwise product $(\gamma_1\cdot\gamma_2) : [0,1] \to G$ and images $[\gamma_1], [\gamma_2], [\gamma_1\cdot\gamma_2]$ in thin homotopy classes to be represented by a parameterized disk in G



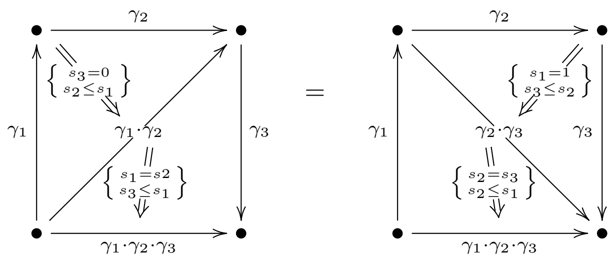
equipped with a label $x_{\gamma_1,\gamma_2} \in \mathbb{R}/\mathbb{Z}$ to be determined. Notice that this triangle is a diagram in $\Xi(\hat\Omega_{\mathrm{th}}\mathrm{Spin} \to P_{\mathrm{th}})\mathrm{Spin}$, so that composition of 1-morphisms is concatenation $\gamma_1 \circ \gamma_2$ of paths. A suitable disk in G is obtained via the map

$$D^2 \xrightarrow{a} [0,1]^2 \xrightarrow{(s_1,s_2)\mapsto \gamma_1(s_1)\cdot\gamma_2(s_2)} G \ ,$$

where a is a smooth surjection onto the triangle $\{(s_1,s_2)|s_2 \leq s_1\} \subset [0,1]^2$ such that the lower semi-circle of $\partial D^2 = S^1$ maps to the hypotenuse of this triangle. The coherence law for this compositor for all triples of parameterized paths $\gamma_1,\gamma_2,\gamma_3 : [0,1] \to G$ amounts to the following: consider the map

$$D^3 \xrightarrow{a} [0,1]^3 \xrightarrow{(s_1,s_2,s_3)\mapsto \gamma_1(s_1)\cdot\gamma_2(s_2)\cdot\gamma_3(s_3)} G \ ,$$

where the map a is a smooth surjection onto the tetrahedron $\{(s_3 \leq s_2 \leq s_1)\} \subset [0,1]^3$. Then the coherence condition



requires that the integral of the canonical 3-form on G pulled back to the 3-ball along these maps accounts for the difference in the chosen labels of the disks involved:

$$\int_{D^3}(b\circ a)^*\mu = \int_{s_3\leq s_2\leq s_1}(\gamma_1\cdot\gamma_2\cdot\gamma_3)^*\mu = x_{\gamma_1,\gamma_2} + x_{\gamma_1\cdot\gamma_2,\gamma_3} - x_{\gamma_1,\gamma_2\cdot\gamma_3} - x_{\gamma_2,\gamma_3} \quad \in \mathbb{R}/\mathbb{Z}\,.$$

(Notice that there is no further twist on the right hand side because whiskering in $\mathbf{B}\Xi(\hat{\Omega}_{\mathrm{th}}G \to P_{\mathrm{th}}G)$ does not affect the labels of the disks.) To solve this condition, we need a 2-form to integrate over the triangles. This is provided by the degree 2 component of the simplicial realization $(\mu,\nu)\in\Omega^3(G)\times\Omega^2(G\times G)$ of the first Pontryagin form as a simplicial form on $\mathbf{B}G_{\mathrm{ch}}$:

for $\mathfrak{g}$ a semisimple Lie algebra, the image of the normalized invariant bilinear polynomial $\langle\cdot,\cdot\rangle$ under the Chern-Weil map is $(\mu,\nu)\in\Omega^3(G)\times\Omega^2(G\times G)$ with

$$\mu := \langle\theta\wedge[\theta\wedge\theta]\rangle$$

and

$$\nu := \langle\theta_1\wedge\bar{\theta}_2\rangle\,,$$

where θ is the left-invariant canonical $\mathfrak{g}$-valued 1-form on G and $\bar{\theta}$ the right-invariant one.

So, define the label assigned by our compositor to the disks considered above by

$$x_{\gamma_1,\gamma_2} := \int_{s_2\leq s_1}(\gamma_1,\gamma_2)^*\nu\,.$$

To show that this assignment satisfies the above condition, use the closedness of (μ,ν) in the complex of simplicial forms on $\mathbf{B}G_{\mathrm{ch}}$: $\delta\mu = d\nu$ and $\delta\nu = 0$. From this one obtains

$$(\gamma_1\cdot\gamma_2\cdot\gamma_3)^*\mu = -d(\gamma_1\cdot\gamma_2,\gamma_3)^*\nu = -d(\gamma_1,\gamma_2\cdot\gamma_3)^*\nu$$

and

$$(\gamma_1,\gamma_2\cdot\gamma_3)^*\nu = (\gamma_1\cdot\gamma_2,\gamma_3)^*\nu + (\gamma_1,\gamma_2)^*\nu - (\gamma_2,\gamma_3)^*\nu\,.$$

Now we compute as follows: Stokes' theorem gives

$$\int\limits_{s_3\leq s_2\leq s1}(\gamma_1\cdot\gamma_2\cdot\gamma_3)^*\mu = \left(\int\limits_{s_3=0,s_2\leq s_1} + \int\limits_{s_1=s_2,s_3\leq s_1} - \int\limits_{s_1=1,s_3\leq s_2} - \int\limits_{s_2=s_3,s_2\leq s_1}\right)(\gamma_1,\gamma_2\cdot\gamma_3)^*\nu\,.$$

The first integral is manifestly equal to x_{γ_1,γ_2}. The last integral is manifestly equal to $-x_{\gamma_1,\gamma_2\cdot\gamma_3}$. For the remaining two integrals we rewrite

$$\cdots = x_{\gamma_1,\gamma_2} - x_{\gamma_1,\gamma_2\cdot\gamma_3} + \left(\int\limits_{s_1=s_2,s_3\leq s_1} - \int\limits_{s_1=1,s_3\leq s_2}\right)\left((\gamma_1\cdot\gamma_2,\gamma_3)^*\nu + (\gamma_1,\gamma_2)^*\nu - (\gamma_2,\gamma_3)^*\nu\right)\,.$$

The first term in the integrand now manifestly yields $x_{\gamma_1\cdot\gamma_2,\gamma_3} - x_{\gamma_2,\gamma_3}$. The second integrand vanishes on the integration domain. The third integrand, finally, gives the same contribution under both integrals and thus drops out due to the relative sign. So in total what remains is indeed

$$\cdots = x_{\gamma_1,\gamma_2} - x_{\gamma_1,\gamma_2\cdot\gamma_3} + x_{\gamma_1\cdot\gamma_2,\gamma_3} - x_{\gamma_2,\gamma_3}\,.$$

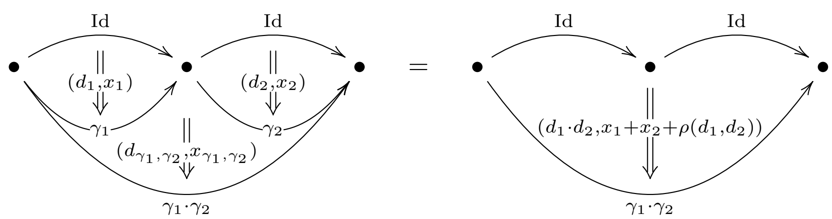
This establishes the coherence condition for the compositor.

Finally we need to show that the compositor is compatible with the horizontal composition of 2-morphisms. We consider this in two steps, first for the horizontal composition of two 2-morphisms both starting at the identity 1-morphism in $\mathbf{B}\mathrm{String}_{\mathrm{Mick}}(G)$ – this is the product in the loop group $\hat{\Omega}G$ centrally extended using Mickelsson's cocycle – then for the horizontal composition of an identity 2-morphism in $\mathbf{B}\mathrm{String}_{\mathrm{Mick}}(G)$ with a 2-morphism starting at the identity 1-morphisms – this is the action of PG on $\hat{\Omega}G$. These two cases then imply the general case.

- Let (d_1, x_1) and (d_2, x_2) represent two 2-morphisms in $\mathbf{B}\mathrm{String}_{\mathrm{Mick}}$ starting at the identity 1-morphisms. So

  $$d_i : [0,1] \to \Omega G$$

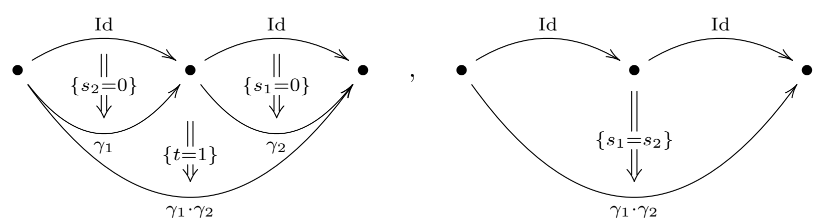
  is a based path in loops in G and $x_i \in U(1)$. We need to show that

  as a pasting diagram equation in $\mathbf{B}\Xi(\hat{\Omega}_{\mathrm{th}}G \to P_{\mathrm{th}}G)$. Here on the left we have gluing of disks in G along their boundaries and addition of their labels, while on the right we have the pointwise product from definition 5.1.26 of labeled disks as representing the product of elements $\hat{\Omega}G$.

  There is an obvious 3-ball interpolating between the disk on the left and on the right of the above equation:

  $$(\{s_2 \leq s_1\} \subset [0,1]^3) \to G$$
  $$(s_1, s_2, t) \mapsto (d_1(t, s_1) \cdot d_2(t, s_2))$$

  The compositor property demands that the integral of the canonical 3-form over this ball accounts for the difference between x_{γ_1,γ_2} and $\rho(\gamma_1,\gamma_2)$

  $$\rho(d_1, d_2) = \int\limits_{\substack{s_2\leq s_1\\ 0\leq t\leq 1}} (d_1\cdot d_2)^*\mu + \int\limits_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu\,.$$

  Now use again the relation between μ and $d\nu$ to rewrite this as

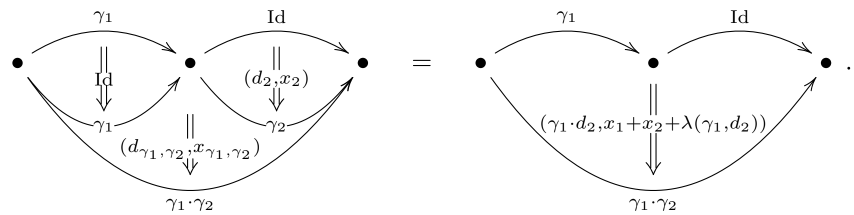
  $$\cdots = \int\limits_{\substack{s_2\leq s_1\\ 0\leq t\leq 1}} ((d_1)^*\mu + (d_2)^*\mu - d(d_1,d_2)^*\nu) + \int\limits_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu\,.$$

The first two integrands vanish. The third one leads to boundary integrals

$$\cdots = -\left(\int_{s_2=0} + \int_{s_1=0}\right)(d_1,d_2)^*\nu - \int_{\substack{t=1\\ s_2\leq s_1}} (d_1,d_2)^*\nu + \int_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu + \int_{\substack{0\leq t\leq 1\\ s_1=s_2}} (d_1,d_2)^*\nu\,.$$

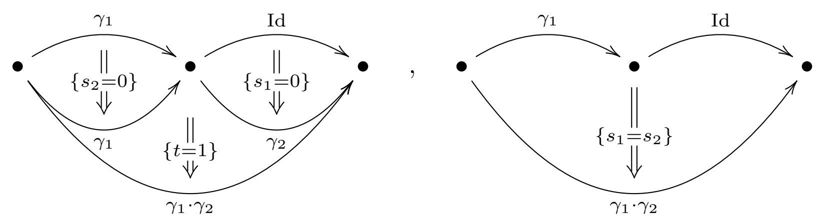
The first two integrands vanish on their integration domain. The third integral cancels with the fourth one. The remaining fifth one is indeed the 2-cocycle on $P\Omega G$ from definition 5.1.26.

- The second case is entirely analogous: for γ_1 a path and (d_2,x_2) a centrally extended loop we need to show that



There is an obvious 3-ball interpolating between the disk on the left and on the right of the above equation:

$$(\{s_2\leq s_1\}\subset[0,1]^3)\to G$$
$$(s_1,s_2,t)\mapsto(\gamma_1(s_1)\cdot d_2(t,s_2))$$



The compositor property demands that the integral of the canonical 3-form over this ball accounts for the difference between x_{γ_1,γ_2} and $\lambda(\gamma_1,\gamma_2)$

$$\lambda(\gamma_1,d_2) = \int_{\substack{s_2\leq s_1\\ 0\leq t\leq 1}} (d_1\cdot d_2)^*\mu + \int_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu\,.$$

This is essentially the same computation as before, so that the result is

$$\cdots = \int_{\substack{0\leq t\leq 1\\ s_1=s_2}} (\gamma_1,d_2)^*\nu\,.$$

This is indeed the quantity from definition 5.1.26.

□

Applied to the case $G=\mathrm{Spin}$ in summary this shows that all these strict smooth 2-groups are indeed presentations of the abstractly defined smooth String 2-group from def. 5.1.10.

Theorem 5.1.29. *We have equivalences of smooth 2-groups*

$$\mathrm{String}\simeq\Omega\mathbf{cosk}_3\exp(\mathfrak{so}_\mu)\simeq\mathrm{String}_{\mathrm{BCSS}}\simeq\mathrm{String}_{\mathrm{Mick}}\,.$$

Notice that all the models on the right are degreewise diffeological and in fact Fréchet, but not degreewise finite dimensional. This means that neither of these models is a differentiable stack or Lie groupoid in the traditional sense, even though they are perfectly good models for objects in Smooth∞Grpd. Some authors found this to be a deficiency. Motivated by this it has been shown in [Scho10] that there exist finite dimensional models of the smooth String-group. Observe however the following:

1. If one allows arbitrary disjoint unions of finite dimensional manifolds, then by prop. 2.2.18 *every* object in Smooth∞Grpd has a presentation by a simplicial object that is degreewise of this form, even a presentation which is degreewise a union of just Cartesian spaces.
2. Contrary to what one might expect, it is not the degreewise finite dimensional models that seem to lend themselves most directly to differential refinements and differential geometric computations with objects in Smooth∞Grpd, but the models of the form $\mathbf{cosk}_n \exp(\mathfrak{g})$. See also the discussion in 5.4.7.3 below.

5.1.5 Smooth fivebrane structure and the Fivebrane-6-group

We now climb up one more step in the smooth Whitehead tower of the orthogonal group, to find a smooth and differential refinement of the *Fivebrane group*.

Proposition 5.1.30. *Pulled back along $B\mathrm{String} \to BO$ the second Pontryagin class is 6 times a generator $\frac{1}{6}p_2$ of $H^8(B\mathrm{String}, \mathbb{Z}) \simeq \mathbb{Z}$:*

$$\begin{array}{ccc} B\mathrm{String} & \xrightarrow{\frac{1}{6}p_2} & B^8\mathbb{Z} \\ \downarrow & & \downarrow^{\cdot 6} \\ B\mathrm{Spin} & \xrightarrow{p_2} & B^8\mathbb{Z} \end{array} .$$

This is due to [Bott58]. We call $\frac{1}{6}p_2$ the *second fractional Pontryagin class* .

Definition 5.1.31. Write BFivebrane for the homotopy fiber of the second fractional Pontryagin class in Top $\simeq$ ∞Grpd

$$\begin{array}{ccc} B\mathrm{Fivebrane} & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathrm{String} & \xrightarrow{\frac{1}{6}p_2} & B^8\mathbb{Z} \end{array} .$$

Write

$$\mathrm{Fivebrane} := \Omega B\mathrm{Fivebrane}$$

for its loop space, the topological *fivebrane ∞-group*.

This is the next step in the topological Whitehead tower of O after String, often denoted $O\langle 7\rangle$. For a discussion of its role in the physics of super-Fivebranes that gives it its name here in analogy to String $= O\langle 3\rangle$ see [SSS09b]. See also [DoHeHi10], around remark 2.8. We now construct smooth and then differential refinements of this object.

Theorem 5.1.32. *The image under Lie integration, 4.4.14, of the canonical Lie algebra 7-cocycle*

$$\mu_7 = \langle -, [-,-], [-,-], [-,-]\rangle : \mathfrak{so}_{\mu_3} \to b^6\mathbb{R}$$

on the string Lie 2-algebra $\mathfrak{so}_{\mu_3}$, def. 5.1.15, is a morphism in Smooth∞Grpd *of the form*

$$\frac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{String} \to \mathbf{B}^7 U(1)$$

whose image under the the fundamental ∞*-groupoid* ∞*-functor/ geometric realization, 4.3.4,* $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}$ *is the ordinary second fractional Pontryagin class* $\frac{1}{6}p_2 : B\mathrm{String} \to B^8\mathbb{Z}$ *in* Top. *We call* $\frac{1}{6}\hat{\mathbf{p}}_2 := \exp(\mu_7)$ *the* second smooth fractional Pontryagin class

Moreover, the corresponding refined differential characteristic cocycle, 4.4.17,

$$\frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{H}_{\mathrm{conn}}(-, \mathbf{B}\mathrm{Spin}) \to \mathbf{H}_{\mathrm{diff}}(-, \mathbf{B}^7U(1)),$$

induces in cohomology the ordinary refined Chern-Weil homomorphism [HoSi05]

$$[\frac{1}{6}\hat{\mathbf{p}}_2] : H^1_{\mathrm{Smooth}}(X, \mathrm{String}) \to H^4_{\mathrm{diff}}(X)$$

of $\langle -, -, -, - \rangle$ *restricted to those* Spin*-principal bundles* P *that have* String*-lifts*

$$[P] \in H^1_{\mathrm{smooth}}(X, \mathrm{String}) \hookrightarrow H^1_{\mathrm{smooth}}(X, \mathrm{Spin}).$$

Proof. This is shown in [FSS10]. The proof is analogous to that of prop. 5.1.9. □

Definition 5.1.33. Write $\mathbf{B}$Fivebrane for the homotopy fiber in Smooth∞Grpd of the smooth refinement of the second fractional Pontryagin class, prop. 5.1.32:

$$\begin{array}{ccc} \mathbf{B}\mathrm{Fivebrane} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{String} & \xrightarrow{\frac{1}{6}\mathbf{p}_2} & \mathbf{B}^7U(1) \end{array} .$$

We say its loop space object is the *smooth fivebrane 6-group*

$$\mathrm{Fivebrane}_{\mathrm{smooth}} := \Omega\mathbf{B}\mathrm{Fivebrane}.$$

This has been considered in [SSS09c]. Similar discussion as for the smooth String 2-group applies.

5.2 Higher Spin^c-structures

In 5.1 we saw that the classical extension

$$\mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

is only the first step in a tower of *smooth* higher spin groups.

There is another classical extension of $\mathrm{SO}(n)$, not by $\mathbb{Z}_2$ but by the circle group [LaMi89]:

$$U(1) \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n)\,.$$

Here we discuss higher smooth analogs of this construction.

This section draws form [FiSaScIII].

We find below that Spin^c is a special case of the following simple general notion, that turns out to be useful to identify and equip with a name.

Definition 5.2.1. Let $\mathbf{H}$ be an ∞-topos, $G \in \infty\mathrm{Grp}(\mathbf{H})$ an ∞-group object, let A be an abelian group object and let

$$\mathbf{p} : \mathbf{B}G \to \mathbf{B}^{n+1}A$$

be a characteristic map. Write $\hat{G} \to G$ for the extension classified by $\mathbf{p}$, exhibited by a fiber sequence

$$\mathbf{B}^n A \to \hat{G} \to G$$

in $\mathbf{H}$. Then for $H \in \infty\mathrm{Grp}(\mathbf{H})$, any other ∞-group with characteristic map of the same form

$$\mathbf{c} : \mathbf{B}H \to \mathbf{B}^{n+1}A$$

we write

$$\hat{G}^{\mathbf{c}} := \Omega\left(\mathbf{B}G_{\mathbf{p}} \times_{\mathbf{c}} \mathbf{B}H\right) \in \infty\mathrm{Grp}(\mathbf{H})$$

for the loop space object of the ∞-pullback

$$\begin{array}{ccc} \mathbf{B}\hat{G}^{\mathbf{c}} & \longrightarrow & \mathbf{B}H \\ \downarrow & & \downarrow^{\mathbf{c}} \\ \mathbf{B}G & \xrightarrow{\mathbf{p}} & \mathbf{B}^{n+1}A \end{array} .$$

Remark 5.2.2. Since the Eilenberg-MacLane object $\mathbf{B}^{n+1}A$ is tself an ∞-group object, by the Mayer-Vietoris fiber sequence in $\mathbf{H}$, prop. 3.6.145, the object $\mathbf{B}\hat{G}^{\mathbf{c}}$ is equivalently the homotopy fiber of the difference $(\mathbf{p} - \mathbf{c})$ of the two characteristic maps

$$\begin{array}{ccc} \mathbf{B}\hat{G}^{\mathbf{c}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}G \times \mathbf{B}H & \xrightarrow{\mathbf{p}-\mathbf{c}} & \mathbf{B}^n A \end{array} .$$

5.2.1 Spin^c as a homotopy fiber product in $\mathrm{Smooth}\infty\mathrm{Grpd}$

A classical definition of Spin^c is the following (for instance [LaMi89]).

Definition 5.2.3. For each $n \in \mathbb{N}$ the Lie group $\mathrm{Spin}^c(n)$ is the fiber product of Lie groups

$$\begin{aligned} \mathrm{Spin}^c(n) &:= \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1) \\ &= (\mathrm{Spin}(n) \times U(1))/\mathbb{Z}_2\,, \end{aligned}$$

where the quotient is by the canonical subgroup embeddings.

We observe now that in the context of Smooth∞Grpd this Lie group has the following intrinsic characterization.

Proposition 5.2.4. *In* Smooth∞Grpd *we have an* ∞*-pullback diagram of the form*

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin}^c & \longrightarrow & \mathbf{B}U(1) \\ \downarrow & & \downarrow^{\mathbf{c}_1 \mathrm{mod} 2} \\ \mathbf{B}\mathrm{SO} & \xrightarrow{\mathbf{w}_2} & \mathbf{B}^2\mathbb{Z}_2 \end{array} ,$$

where the right morphism is the smooth universal first Chern class, example 1.2.106, composed with the mod-2 reduction $\mathbf{B}\mathbb{Z} \to \mathbf{B}\mathbb{Z}_2$*, and where* $\mathbf{w}_2$ *is the smooth universal second Stiefel-Whiteney class, example 1.2.110.*

Proof. By the discussion at these examples, these universal smooth classes are represented by spans of simplicial presheaves

$$\begin{array}{ccccc} \mathbf{B}(\mathbb{Z} \to \mathbb{R})_{\mathrm{ch}} & \xrightarrow{\mathbf{c}_1} & \mathbf{B}(\mathbb{Z} \to 1)_{\mathrm{ch}} & = & \mathbf{B}^2\mathbb{Z} \\ \downarrow^{\simeq} & & & & \\ \mathbf{B}U(1)_{\mathrm{ch}} & & & & \end{array}$$

and

$$\begin{array}{ccccc} \mathbf{B}(\mathbb{Z}_2 \to \mathrm{Spin})_{\mathrm{ch}} & \longrightarrow & \mathbf{B}(\mathbb{Z}_2 \to 1)_{\mathrm{ch}} & = & \mathbf{B}^2(\mathbb{Z}_2)_{\mathrm{ch}} \\ \downarrow^{\simeq} & & & & \\ \mathbf{B}\mathrm{SO}_{\mathrm{ch}} & & & & \end{array} .$$

Here both horizontal morphism are fibrations in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Therefore by prop. 2.3.13 the ∞-pullback in question is given by the ordinary fiber product of these two morphisms. This is

$$\begin{array}{ccc} \mathbf{B}(\mathbb{Z} \to \mathrm{Spin} \times \mathbb{R})_{\mathrm{ch}} & \longrightarrow & \mathbf{B}(\mathbb{Z} \to \mathbb{R})_{\mathrm{ch}} \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathbb{Z} \stackrel{\mathrm{mod} 2}{\to} \mathrm{Spin})_{\mathrm{ch}} & \longrightarrow & \mathbf{B}(\mathbb{Z} \to 1)_{\mathrm{ch}} \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathbb{Z}_2 \to \mathrm{Spin})_{\mathrm{ch}} & \longrightarrow & \mathbf{B}(\mathbb{Z}_2 \to 1)_{\mathrm{ch}} \end{array} ,$$

where the crossed module $(\mathbb{Z} \stackrel{\partial}{\to} \mathrm{Spin} \times \mathbb{R})$ is given by

$$\partial : n \mapsto (n \bmod 2, n) .$$

Since this is a monomorphism, including (over the neutral element) the fiber of a locally trivial bundle we have an equivalence

$$\mathbf{B}(\mathbb{Z} \to \mathrm{Spin} \times \mathbb{R}) \stackrel{\simeq}{\to} \mathbf{B}(\mathbb{Z}_2 \to \mathrm{Spin} \times U(1)) \stackrel{\simeq}{\to} \mathbf{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} U(1))$$

in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. On the right is, by def. 5.2.3, the delooping of Spin^c. □

Remark 5.2.5. Therefore by def. 5.2.1 we have

$$\mathrm{Spin}^c \simeq \mathrm{Spin}^{\mathbf{c}_1 \mathrm{mod} 2} ,$$

which is the very motivation for the notation in that definition.

Remark 5.2.6. From prop. 5.2.4 we obtain the following characterization of Spin^c-structures in $\mathbf{H} = \mathrm{Smoth}\infty\mathrm{Grpd}$ over a smooth manifold expressed in terms of traditional Čech cohomology, 4.3.8.1.

For $X \in \mathrm{SmthMfd}$, the fact that $\mathbf{H}(X, -)$ preserves ∞-limits implies from prop. 5.2.4 that we have an ∞-pullback of cocycle ∞-groupoids

$$\begin{array}{ccc} \mathbf{H}(X, B\mathrm{Spin}^c) & \longrightarrow & \mathbf{H}(X, \mathbf{B}U(1)) \\ \downarrow & & \downarrow^{\mathbf{c}_1 \mathrm{mod} 2} \\ \mathbf{H}(X, \mathbf{B}\mathrm{SO}) & \xrightarrow{\mathbf{w}_2} & \mathbf{H}(X, \mathbf{B}^2\mathbb{Z}_2) \end{array} .$$

Picking any choice of differentially good open cover $\{U_i \to X\}$ of X and using the standard presentation of the coeffcient moduli stacks appearing here by sheaves of groupoids as discussed in 4.4.2, each of the four ∞-groupoids appearing here is canonically identified with the groupoid (or 2-groupoid in the bottom right) of Čech cocycles and Čech coboundaries with respect to the given cover and with coefficients in the given group. Moreover, in this presentation the right vertical morphism of the above diagram is clearly a fibration, and so by prop. 2.3.8 the ordinary pullback of these groupoids is already the correct ∞-pullback, hence is the groupoid $\mathbf{H}(X, \mathbf{B}\mathrm{Spin}^c)$ of Spin^c-structure on X. So we read off from the diagram and the construction in the above proof: given a Čech 1-cocycle for an SO-structure on X the corresponding Spin^c-structure is a lift to a $(\mathbb{Z} \to \mathbb{R})$-valued Čech cocycle of the $\mathbb{Z}_2$-valued Čech 2-cocycle that represents the second Stiefel-Whitney class, as described in 1.2.110, through the evident projetion $(\mathbb{Z} \to \mathbb{R}) \to (\mathbb{Z}_2 \to *)$ that by example. 1.2.106 presents the universal first Chern class.

5.2.2 Smooth $\mathrm{String}^{\mathbf{c}_2}$

We consider smooth 2-groups of the form $\mathrm{String}^{\mathbf{c}}$, according to def. 5.2.1, where $\mathbf{B}U(1) \to \mathrm{String} \to \mathrm{Spin}$ in $\mathrm{Smooth}\infty\mathrm{Grpd}$ is the smooth String-2-group extension of the Spin-group from def. 5.1.10.

In [Sa10b] the following notion is introduced.

Definition 5.2.7. Let

$$p_1^c : B\mathrm{Spin}^c \to B\mathrm{Spin} \stackrel{\frac{1}{2}p_1}{\to} K(\mathbb{Z}, 4)$$

in $\mathrm{Top} \simeq \infty\mathrm{Grpd}$, where the first map is induced on classifying spaces by the defining projection, def. 5.2.3, and where the second represents the fractional first Pontryagin class from prop. 5.1.5.

Then write String^c for the topological group, well defined up to weak homotopy equivalence, that models the loop space of the homotopy pullback

$$\begin{array}{ccc} B\mathrm{String}^c & \longrightarrow & (BU(1)) \times (BU(1)) \\ \downarrow & & \downarrow^{c_1 \cup c_1} \\ B\mathrm{Spin}^c & \xrightarrow{p_1^c} & K(\mathbb{Z}, 4) \end{array}$$

in Top.

This construction, and the role it plays in [Sa10b], is evidently an example of general structure of def. 5.2.1, the notation of which is motivated from this example. We consider now smooth and differential refinements of such objects.

To that end, recall from theorem. 5.1.9 the smooth refinement of the first fractional Pontryagin class

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3U(1)$$

and from def. 5.1.10 the defining fiber sequence

$$\mathbf{B}\mathrm{String} \longrightarrow \mathbf{B}\mathrm{Spin} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3U(1) \ .$$

The proof of theorem 5.1.9 rests only on the fact that Spin is a compact and simply connected simple Lie group. The same is true for the special unitary group SU and the exceptional Lie group E_8.

Proposition 5.2.8. *The first two non-vanishing homotopy groups of E_8 are*

$$\pi_3(E_8) \simeq \mathbb{Z}$$

and

$$\pi_{15}(E_8) \simeq \mathbb{Z}\,.$$

This is a classical fact[BoSa58]. It follows with the Hurewicz theorem that

$$H^4(BE_8, \mathbb{Z}) \simeq \mathbb{Z}\,.$$

Therefore the generator of this group is, up to sign, a canonical characteristic class, which we write

$$[a] \in H^4(BE_8, \mathbb{Z})$$

corresponding to a characeristic map $a : BE_8 \to K(\mathbb{Z}, 4)$. Hence we obtain analogously the following statements.

Corollary 5.2.9. *The second Chern-class*

$$c_2 : B\mathrm{SU} \to K(\mathbb{Z}, 4)$$

has an essentially unique lift through $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd} \simeq \mathrm{Top}$ *to a morphism of the form*

$$\mathbf{c}_2 : \mathbf{B}\mathrm{SU} \to \mathbf{B}^3 U(1)$$

and a representative is provided by the Lie integration $\exp(\mu_3^{\mathfrak{su}})$ *of the canonical Lie algebra 3-cocycle* $\mu_3^{\mathfrak{su}} : \mathfrak{su} \to b^2\mathbb{R}$

$$\mathbf{c}_2 \simeq \exp(\mu_3^{\mathfrak{su}})\,.$$

Similarly the characteristic map

$$a : BE_8 \to \mathbb{K}(\mathbb{Z}, 4)$$

has an essentially unique lift through $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd} \simeq \mathrm{Top}$ *to a morphism of the form*

$$\mathbf{a} : \mathbf{B}E_8 \to \mathbf{B}^3 U(1)$$

and a representative is provided by the Lie integration $\exp(\mu_3^{\mathfrak{e}_8})$ *of the canonical Lie algebra 3-cocycle* $\mu_3^{\mathfrak{e}_8} : \mathfrak{e}_8 \to b^2\mathbb{R}$

$$\mathbf{a} \simeq \exp(\mu_3^{\mathfrak{e}_8})\,.$$

Therefore we are entitled to the following special case of def. 5.2.1.

Definition 5.2.10. The smooth 2-group

$$\mathrm{String}^{\mathbf{c}_2} \in \infty\mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$$

is the loop space object of the ∞-pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}^{\mathbf{c}_2} & \longrightarrow & \mathbf{B}\mathrm{SU} \\ \downarrow & & \downarrow^{\mathbf{c}_2} \\ \mathbf{B}\mathrm{Spin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3 U(1) \end{array}\,.$$

Analogously, the smooth 2-group

$$\mathrm{String}^{\mathbf{a}} \in \infty\mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$$

is the loop space object of the ∞-pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}^{\mathbf{a}} & \longrightarrow & \mathbf{B}E_8 \\ \downarrow & & \downarrow^{\mathbf{a}} \\ \mathbf{B}\mathrm{Spin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \end{array} .$$

Remark 5.2.11. By prop. 3.6.145, $\mathrm{String}^{\mathbf{a}}$ is equivalently is the homotopy fiber of the difference $\frac{1}{2}\mathbf{p}_1 - \mathbf{a}$

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}^{\mathbf{a}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathrm{Spin} \times E_8) & \xrightarrow{\frac{1}{2}\mathbf{p}_1 - \mathbf{a}} & \mathbf{B}^3U(1) \end{array} .$$

We consider now a presentation of $\mathrm{String}^{\mathbf{a}}$ by Lie integration, as in 4.4.14.

Definition 5.2.12. Let

$$(\mathfrak{so} \otimes \mathfrak{e}_8)_{\mu_3^{\mathfrak{so}} - \mu_3^{\mathfrak{e}_8}} \in L_\infty\mathrm{Alg}$$

be the L_∞-algebra extension, according to def. 4.4.102, of the tensorproduct Lie algebra $\mathfrak{so} \otimes \mathfrak{e}_8$ by the difference of the canonical 3-cocycles on the two factors.

Proposition 5.2.13. *The Lie integration, def. 4.4.53, of the Lie 2-algebra $(\mathfrak{so} \otimes \mathfrak{e}_8)_{\mu_3^{\mathfrak{so}} - \mu_3^{\mathfrak{e}_8}}$ is a presentation of* $\mathrm{String}^{\mathbf{a}}$*:*

$$\mathrm{String}^{\mathbf{a}} \simeq \tau_2 \exp\left(\mathfrak{so} \otimes \mathfrak{e}_8)_{\mu_3^{\mathfrak{so}} - \mu_3^{\mathfrak{e}_8}} \right)$$

Proof. With remark 5.2.11 this is directly analogous to prop. 5.1.17. □

Remark 5.2.14. Therefore a 2-connection on a $\mathrm{String}^{\mathbf{a}}$-principal 2-bundle is locally given by

- an $\mathfrak{so}$-valued 1-form ω;
- an $\mathfrak{e}_8$-valued 1-form A;
- a 2-form B;

such that the 3-form curvature of B is, locally, the sum of the de Rham differential of B with the difference of the Chern-Simons forms of ω and A, respectively:

$$H_3 = dB + \mathrm{cs}(\omega) - \mathrm{cs}(A) \,.$$

We discuss the role of such 2-connections in string theory below in 5.4.7.3.2 and 5.7.9.3.

5.3 Classical supergravity

Action functionals of *supergravity* are extensions to super-geometry, 4.6, of the *Einstein-Hilbert action functional* that models the physics of *gravity*. While these action functionals are not themselves, generally, of higher Chern-Simons type, 3.9.11, or of higher Wess-Zumino-Witten type, 3.9.12, some of them are low-energy effective actions of *super string field theory* action functionals, that are of this type, as we discuss below in 5.7.10. Accordingly, supergravity action functionals typically exhibit rich Chern-Simons-like sub-structures.

A traditional introduction to the general topic can be found in [DEFJKMMW]. A textbook that aims for a more systematic formalization is [CaDAFr91]. Below in 5.3.3 we observe that the discussion of supergravity there is secretly in terms of ∞-connections, 1.2.13.6, with values in super L_∞-algebras, 4.6.2.

- 5.3.1 – First-order/gauge theory formulation of gravity
- 5.3.2 – Higher extensions of the super Poincaré Lie algebra;
- 5.3.3 – Supergravity fields are super L_∞-connections

5.3.1 First-order/gauge theory formulation of gravity

The field theory of gravity ("general relativity") has a natural *first order formulation* where a field configuration over a given $(d+1)$-dimensional manifold X is given by a $\mathfrak{iso}(d,1)$-valued Cartan connection, def. 4.5.57. The following statements briefly review this and related facts (see for instance also the review in the introduction of [Zane05]).

Definition 5.3.1. For $d \in \mathbb{N}$, the *Poincaré group* $\mathrm{ISO}(d,1)$ is the group of auto-isometries of the Minkowski space $\mathbb{R}^{d,1}$ equipped with its canonical pseudo-Riemannian metric η.

This is naturally a Lie group. Its Lie algebra is the *Poincaré Lie algebra* $\mathfrak{iso}(d,1)$.

We recall some standard facts about the Poincaré group.

Observation 5.3.2. The Poncaré group is the semidirect product

$$\mathrm{ISO}(d,1) \simeq \mathrm{O}(d,1) \ltimes \mathbb{R}^{d+1}$$

of the *Lorentz group* $\mathrm{O}(d,1)$ of *linear* auto-isometries of $\mathbb{R}^{d,1}$, and the abelian translation group in $(d+1)$ dimensions, with respect to the defining action of $\mathrm{O}(d,1)$ on $\mathbb{R}^{d,1}$. Accordingly there is a canonical embedding of Lie groups

$$O(d,1) \hookrightarrow \mathrm{ISO}(d,1)$$

and the corresponding coset space is Minkowski space

$$\mathrm{ISO}(d,1)/\mathrm{O}(d,1) \simeq \mathbb{R}^{d,1}\,.,$$

Analogously the Poincaré Lie algebra is the semidirect product

$$\mathfrak{iso}(d,1) \simeq \mathfrak{so}(d,1) \ltimes \mathbb{R}^{d,1}\,,$$

Accordingly there is a canonical embedding of Lie algebras

$$\mathfrak{so}(d,1) \hookrightarrow \mathfrak{iso}(d,1)$$

and the corresponding quotient of vector spaces is Minkowski space

$$\mathfrak{iso}(d,1)/\mathfrak{so}(d,1) \simeq \mathbb{R}^{d,1}\,.$$

Minkowski space $\mathbb{R}^{d,1}$ is the local model for *Lorentzian manifolds*.

Definition 5.3.3. A *Lorentzian manifold* is a pseudo-Riemannian manifold (X, g) such that each tangent space $(T_x X, g_x)$ for any $x \in X$ is isometric to a Minkowski space $(\mathbb{R}^{d,1}, \eta)$.

Proposition 5.3.4. *Equivalence classes of* $(O(d,1) \hookrightarrow \mathrm{ISO}(d,1))$*-valued Cartan connections, def. 4.5.57, on a smooth manifold* X *are in canonical bijection with Lorentzian manifold structures on* X*.*

This follows from the following observations.

Observation 5.3.5. Locally over a patch $U \to X$ a $\mathfrak{iso}(d,1)$ connection is given by a 1-form

$$A = (E, \Omega) \in \Omega^1(U, \mathfrak{iso}(d,1))$$

with a component

$$E \in \Omega^1(U, \mathbb{R}^{d+1})$$

and a component

$$\Omega \in \Omega^1(U, \mathfrak{so}(d,1)) .$$

If this comes from a $(\mathrm{O}(d,1) \to \mathrm{ISO}(d,1))$-*Cartan connection* then E is non-degenerate in that for all $x \in X$ the induced linear map

$$E : T_x X \to \mathbb{R}^{d+1}$$

is a linear isomorphism. In this case X is equipped with the Lorentzian metric

$$g := E^* \eta$$

and Ω is naturally identified with a compatible metric connection on TX. Then curvature 2-form of the connection

$$F_A = (F_\Omega, F_E) \in \Omega^2(U, \mathfrak{iso}(d,1))$$

has as components the *Riemann curvature*

$$F_\Omega = d\Omega + \frac{1}{2}[\Omega \wedge \Omega] \in \Omega^2(U, \mathfrak{so}(d,1))$$

of the metric connection, as well as the *torsion*

$$F_E = dE + [\Omega \wedge E] \in \Omega^2(U, \mathbb{R}^{d,1}) .$$

Therefore precisely if in addition the torsion vanishes is Ω uniquely fixed to be the Levi-Civita connection on (X, g).

Therefore the configuration space of gravity on a smooth manifold X may be identified with the moduli space of $\mathfrak{iso}(d,1)$-valued Cartan connections on X. The field content of *supergravity* is obtained from this by passing from the to Poincaré Lie algebra to one of its *super Lie algeba extensions*, a *super Poincaré* Lie algebra.

There are different such extensions. All involve some spinor representation of the Lorentz Lie algebra $\mathfrak{so}(d,1)$ as odd-degree elements in the super Lie algebra The choice of number N of irreps in this representation. But there are in general more choices, given by certain exceptional *polyvector extensions* of such super-Poincaré-Lie algebras which contain also new even-graded elements.

Below we show that these Lie superalgebra polyvector extensions , in turn, are induced from canonical *super* L_∞*-algebra extensions* given by exceptional super Lie algebra cocycles, and that the configuration spaces of higher dimensional supergravity may be identified with moduli spaces of ∞-connections, 1.2.13, withvalues in a super L_∞-algebra, def. 4.6.14. that arise as higher central extensions, def. 4.4.102, of a super Poincaré Lie algebra.

5.3.2 L_∞-extensions of the super Poincaré Lie algebra

The super-Poincaré Lie algebra is the local gauge algebra of supergravity. It inherits the cohomology of the special orthogonal or Lorentz Lie algebra $\mathfrak{so}(d,1)$, but crucially it exhibits a finite number of exceptional $\mathfrak{so}(d,1)$-invariant cocycles on its super-translation algebra. The super L_∞-algebra extensions induced by these cocycles control the structure of higher dimensional supergravity fields as well as of super-p-brane σ-models that propagate in a supergravity background.

- 5.3.2.1 – The super Poincaré Lie algebra;
- 5.3.2.2 – M2-brane Lie 3-algebra and the M-theory Lie algebra;
- 5.3.2.3 – Exceptional cocycles and the brane scan.

5.3.2.1 The super Poincaré Lie algebra

Definition 5.3.6. For $n \in \mathbb{N}$ and S a spinor representation of $\mathfrak{so}(n,1)$, the corresponding *super Poincaré Lie algebra* $\mathfrak{sIso}(n,1)$ is the super Lie algebra whose Chevalley-Eilenberg algebra CE($\mathfrak{sIso}(10,1)$ is generated from

1. generators $\{\omega^{ab}\}$ in degree $(1,\mathrm{even})$ dual to the standard basis of $\mathfrak{so}(n,1)$,
2. generators $\{e^a\}$ in degree $(1,\mathrm{even})$
3. and generators $\{\psi^\alpha\}$ in degree $(1,\mathrm{odd})$, dual to the spinor representation S

with differential defined by

$$d_{\mathrm{CE}}\omega^a{}_b = \omega^a{}_c \wedge \omega^c{}_d$$
$$d_{\mathrm{CE}}e^a = \omega^a{}_b \wedge e^b + \frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi$$
$$d_{\mathrm{CE}}\psi = \frac{1}{4}\omega^{ab}\Gamma_{ab}\psi\,,$$

where $\{\Gamma^a\}$ is the corresponding representation of the Clifford algebra $\mathrm{Cl}_{n,1}$ on S, and here and in the following $\Gamma^{a_1\cdots a_k}$ is shorthand for the skew-symmetrization of the matrix product $\Gamma^{a_1}\cdot\cdots\cdot\Gamma^{a_k}$ in the k indices.

5.3.2.2 M2-brane Lie 3-algebra and the M-theory Lie algebra We discuss an exceptional extension of the super Poincaré Lie algebra in 11-dimensions by a super Lie 3-algebra and further by super Lie 6-algebra. We show that the corresponding automorphism L_∞-algebra contains the polyvector extension called the *M-theory super Lie algebra*.

Proposition 5.3.7. *For* $(n,1) = (10,1)$ *and* S *the canonical spinor representation, we have an exceptional super Lie algebra cohomology class in degree 4*

$$[\mu_4] \in H^{2,2}(\mathfrak{sIso}(10,1))$$

with a representative given by

$$\mu_4 := \frac{1}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge e_a\wedge e_b\,.$$

This is due to [DAFr82].

Definition 5.3.8. The *M2-brane super Lie 3-algebra* $\mathfrak{m2brane}_{\mathrm{gs}}$ is the $b\mathbb{R}$-extension of $\mathfrak{sIso}(10,1)$ classified by μ_4, according to prop. 4.4.107

$$b^2\mathbb{R} \to \mathfrak{m2brane}_{\mathrm{gs}} \to \mathfrak{siso}(10,1)\,.$$

In terms of its Chevalley-Eilenberg algebra this extension was first considered in [DAFr82].

Definition 5.3.9. The *polyvector extension* [ACDP03] of $\mathfrak{sIso}(10,1)$ – called the *M-theory Lie algebra* – is the super Lie algebra obtained by adjoining to $\mathfrak{sIso}(10,1)$ generators $\{Q_\alpha, Z^{ab}\}$ that transform as spinors with respect to the existing generators, and whose non-vanishing brackets among themselves are

$$[Q_\alpha, Q_\beta] = i(C\Gamma^a)_{\alpha\beta} P_a + (C\Gamma_{ab}) Z^{ab}$$
$$[Q_\alpha, Z^{ab}] = 2i(C\Gamma^{[a})_{\alpha\beta} Q^{b]\beta} \,.$$

Proposition 5.3.10. *The automorphism super L_∞-algebra $\mathfrak{der}(\mathfrak{m2brane}_{\mathrm{gs}})$, def. 1.2.120, contains the polyvector extension of the 11d-super Poinceré algebra, def. 5.3.9 precisely as its graded Lie algebra of exact elements.*

Proof. One can see that this is secretly what [Ca95] shows. □

Proposition 5.3.11. *There is a nontrivial degree-7 class $[\mu_7] \in H^{5,2}(\mathfrak{m2brane}_{\mathrm{gs}})$ in the super-L_∞-algebra cohomology of the M2-brane Lie 3-algebra, a cocycle representative of which is*

$$\mu_7 := -\frac{1}{2}\bar\psi \wedge \Gamma^{a_1 \cdots a_5}\psi \wedge e_{a_1} \wedge \cdots \wedge e_{a_5} - \frac{13}{2}\bar\psi \wedge \Gamma^{a_1 a_2}\psi \wedge e_{a_1} \wedge e_{s_2} \wedge c_3 \,,$$

where c_3 is the extra generator of degree 3 in $\mathrm{CE}(\mathfrak{m2brane}_{\mathrm{gs}})$.

This is due to [DAFr82].

Definition 5.3.12. The *M5-brane Lie 6-algebra* $\mathfrak{m5brane}_{\mathrm{gs}}$ is the $b^5\mathbb{R}$-extension of $\mathfrak{m2brane}_{\mathrm{gs}}$ classified by μ_7, according to prop. 4.4.107

$$b^5\mathbb{R} \to \mathfrak{m5brane}_{\mathrm{gs}} \to \mathfrak{m2brane}_{\mathrm{gs}} \,.$$

5.3.2.3 Exceptional cocycles and the brane scan The exceptional cocycles discussed above are part of a pattern which traditionally goes by the name *brane scan* [Duff87].

Proposition 5.3.13. *For $d, p \in \mathbb{N}$, let $\mathfrak{sIso}(d,1)$ be the super Poincaré Lie algebra, def. 5.3.6, and consider the element*

$$\bar\psi\Gamma_{a_0,\cdots,a_{p+1}} \wedge \psi \wedge e^{a_0} \wedge \cdots \wedge e^{a_{p+1}} \in \mathrm{CE}(\mathfrak{sIso}(d,1))$$

in degree $p+2$ of the Chevalley-Eilenberg algebra. This is closed, hence is a cocycle, for the combinations of $D := d+1$ and $p \geq 1$ precisely where there are non-empty and non-parenthesis entries in the following table.

	$p=1$	2	3	4	5
$D=11$		$\mathfrak{m2brane}_{\mathrm{gs}}$			($\mathfrak{m5brane}_{\mathrm{gs}}$)
10	$\mathfrak{string}_{\mathrm{gs}}$				$\mathfrak{ns5brane}_{\mathrm{gs}}$
9				*	
8			*		
7		*			
6	*		*		
5		*			
4	*	*			
3	*				

The entries in the top two rows are labeled by the name of the extension of $\mathfrak{sIso}(d,1)$ that the corresponding cocycle classifies. By prop. 5.3.8 the 7-cocycle that defines $\mathfrak{m5brane}_{\mathrm{gs}}$ does not live on the Lie algebra $\mathfrak{sIso}(10,1)$, but only on its Lie 3-algebra extension $\mathfrak{m2brane}_{\mathrm{gs}}$. This is why in the context of the brane scan it does not appear in the classical literature, which does not know about higher Lie algebras.

An explicitly Lie-theoretic discussion of these cocycles is in chapter 8 of [AzIz95]. The extension

$$b\mathbb{R} \to \mathfrak{string}_{\mathrm{gs}} \to \mathfrak{sIso}(9,1)$$

and its Lie integration has been considered in [Huer11].

5.3.3 Supergravity fields are super L_∞-connections

Among the varied literature in theoretical physics on the topic of *supergravity* the book [CaDAFr91] and the research program that it summarizes, starting with [DAFr82], stands out as an attempt to identify and make use of a systematic mathematical structure controlling the general theory. By careful comparison one can see that the notions considered in that book may be translated into notions considered here under the following dictionary

- "FDA": the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ of a super L_∞-algebra $\mathfrak{g}$ (def. 4.6.14), def. 4.5.12;
- "soft group manifold": the Weil algebra $\mathrm{W}(\mathfrak{g})$ of $\mathfrak{g}$, def. 4.4.104
- "field configuration": $\mathfrak{g}$-valued ∞-connection, def. 1.2.13.6
- "field strength": curvature of $\mathfrak{g}$-valued ∞-connection, def. 1.2.140
- "horizontality condition": second ∞-Ehresmann condition, remark 1.2.149
- "cosmo-cocycle condition": characterization of $\mathfrak{g}$-Chern-Simons elements, def. 4.4.116, to first order in the curvatures;

All the super L_∞-algebras $\mathfrak{g}$ appearing in [CaDAFr91] are higher shifted central extensions, in the sense of prop. 4.4.107, of the super-Poincaré Lie algebra.

5.3.3.1 The graviton and the gravitino

Example 5.3.14. For X a supermanifold and $\mathfrak{g} = \mathfrak{sIso}(n,1)$ the super Poincaré Lie algebra from def. 5.3.6, $\mathfrak{g}$-valued differential form data

$$A : TX \to \mathfrak{siso}(n,1)$$

consists of

1. an $\mathbb{R}^{n+1}$-valued even 1-form $E \in \Omega^1(X, \mathbb{R}^{n+1})$ – the *vielbein*, identified as the propagating part of the *graviton* field;
2. an $\mathfrak{so}(n,1)$-valued even 1-form $\Omega \in \Omega^1(X, \mathfrak{so}(n,1))$ – the *spin connection*, identified as the non-propagating auxiliary part of the graviton field;
3. a spin-representaton -valued odd 1-form $\Psi \in \Omega^1(X, S)$ – identified as the *gravitino field*.

5.3.3.2 The 11d supergravity C_3-field

Example 5.3.15. For $\mathfrak{g} = \mathfrak{m2brane}_{\mathrm{gs}}$ the Lie 3-algebra from def. 5.3.8, a $\mathfrak{g}$-valued form

$$A : TX \to \mathfrak{sugra}_3(10,1)$$

consists in addition to the field content of a $\mathfrak{siso}(10,1)$-connection from example 5.3.14 of

- a 3-form $C_3 \in \Omega^3(X)$.

This 3-form field is the local incarnation of what is called the *supergravity C_3-field*. The global nature of this field is discussed in 5.4.8.

5.3.3.3 The magnetic dual 11d supergravity C_6-field

Example 5.3.16. For $\mathfrak{g} = \mathfrak{m5brane}_{\mathrm{gs}}$ the 11d-supergravity Lie 6-algebra, def. 5.3.12, a $\mathfrak{g}$-valued form

$$A : TX \to \mathfrak{sugra}_6(10,1)$$

consists in addition to the field content of a $\mathfrak{sugra}_3(10,1)$-connection given in remark 5.3.15 of

- a 6-form $C_6 \in \Omega^3(X)$ – the dual *supergravity C-field*.

The identification of this field content is also due to the analysis of [DAFr82].

5.4 Twisted ∞-bundles / twisted differential structures

We discuss various examples of *twisted ∞-bundles*, 3.6.14, and the corresponding *twisted differential structures*, 3.9.8.

Most of these appear in various guises in string theory, which we survey in

- 5.4.6 – Twisted topological c-structures in String theory.

Below we discuss the following differential refinements and applications.

- 5.4.1 – Definition and overview
- 5.4.4 – Reduction of structure groups
  - 5.4.4.1 – Orthogonal/Riemannian structure
  - 5.4.4.2 – Type II generalized geometry
  - 5.4.4.3 – U-duality geometry / exceptional generalized geometry
- 5.4.5 –Orientifolds and higher orientifolds
- 5.4.6 – Twisted topological structures in quantum anomaly cancellation
- 5.4.7 – Tisted differential structures in quantum anomaly cancellation
  - 5.4.7.1 – Twisted differential $\mathbf{c}_1$-structures
  - 5.4.7.2 – Twisted differential spin^c-structures
  - 5.4.7.3 – Higher differential spin structures: string and fivebrane structures
- 5.4.8 – The supergravity C-field
- 5.4.9 – Differential T-duality

The discussion in this section draws from [FiSaScI], which in turn draws from the examples discussed in [SSS09c], [FiSaScIII].

5.4.1 Overview

The following table lists some of main (classes of) examples. The left column displays a given extension of smooth ∞-groups, to be regarded as a bundle of coefficients with typical ∞-fiber shown on the far left. The middle column names the principal ∞-bundles, or equivalently the nonabelian cohomology classes, that are classified by the base of these extensions. These are to be thought of as twisting cocycles. The right column names the corresponding twisted ∞-bundles, or eqivalently the corresponding twisted cohomology classes.

extension / ∞-bundle of coefficients	twisting ∞-bundle / twisting cohomology	twisted ∞-bundle / twisted cohomology
$V \longrightarrow V//G \xrightarrow{\rho} \mathbf{B}G$	ρ-associated V-∞-bundle	section
$\mathrm{GL}(d)/O(d) \longrightarrow \mathbf{B}O(d) \longrightarrow \mathbf{B}\mathrm{GL}(d)$	tangent bundle	orthogonal structure / Riemannian geometry
$O(d)\backslash O(d,d)/O(d) \to \mathbf{B}(O(d)\times O(d)) \longrightarrow \mathbf{B}O(d,d)$	generalized tangent bundle	generalized (type II) Riemannian geometry
$\mathbf{B}U(n) \longrightarrow \mathbf{B}\mathrm{PU}(n) \xrightarrow{\mathbf{dd}} \mathbf{B}^2U(1)$	circle 2-bundle / bundle gerbe	twisted vector bundle / bundle gerbe module
$\mathbf{B}^2U(1) \longrightarrow \mathbf{B}\mathrm{Aut}(\mathbf{B}U(1)) \longrightarrow \mathbf{B}\mathbb{Z}_2$	double cover	orientifold structure / Jandl bundle gerbe
$\mathbf{B}^2\ker(G) \longrightarrow \mathbf{B}\mathrm{Aut}(\mathbf{B}G) \longrightarrow \mathbf{B}\mathrm{Out}(G)$	band (*lien*)	nonabelian (Giraud-Breen) G-∞-gerbe
$\mathbf{B}\mathrm{String} \longrightarrow \mathbf{B}\mathrm{Spin} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3U(1)$	circle 3-bundle / bundle 2-gerbe	twisted String 2-bundle
$Q \longrightarrow \mathbf{B}(\mathbb{T}\times\mathbb{T}^*) \xrightarrow{\langle \mathbf{c}_1 \cup \mathbf{c}_1\rangle} \mathbf{B}^3U(1)$	circle 3-bundle / bundle 2-gerbe	twisted T-duality structure
$\mathbf{B}\mathrm{Fivebrane} \longrightarrow \mathbf{B}\mathrm{String} \xrightarrow{\frac{1}{6}\mathbf{p}_2} \mathbf{B}^7U(1)$	circle 7-bundle	twisted Fivebrane 6-bundle
$\flat\mathbf{B}^nU(1) \longrightarrow \mathbf{B}^nU(1) \xrightarrow{\mathrm{curv}} \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$	curvature $(n+1)$-form	circle n-bundle with connection

The following table lists smooth twisting ∞-bundles $\mathbf{c}$ that become *identities under geometric realization*, def. 4.3.24, (the last one on 15-coskeleta). This means that the twists are purely geometric, the underlying topological structure being untwisted.

universal twisting ∞-bundle	twisted cohomology	relative twisted cohomology
$\mathbf{B}O(d) \to \mathbf{B}\mathrm{GL}(d)$	Riemannian geometry, orthogonal structure	
$\mathbf{B}O(d)\times O(d) \to \mathbf{B}O(d,d)$	type II NS-NS generalized geometry	
$\mathbf{B}H_n \to \mathbf{B}E_{n(n)}$	U-duality geometry, exceptional generalized geometry	
$\mathbf{B}\mathrm{PU}(\mathcal{H}) \xrightarrow{\mathbf{dd}} \mathbf{B}^2U(1)$	twisted $U(n)$-principal bundles	Freed-Witten anomaly cancellation on Spin^c-branes: *B*-field with twisted gauge bundles on D-branes
$\mathbf{B}E_8 \xrightarrow{\mathbf{2a}} \mathbf{B}^3U(1)$	twisted String(E_8)-principal 2-bundles	M5-brane anomaly cancellation: *C*-field with twisted gauge 2-bundles on M5-branes

The following table lists smooth twisted ∞-bundles that control various quantum anomaly cancellations in string theory.

universal twisting ∞-bundle	twisted cohomology	relative twisted cohomology
$\mathbf{B}\mathrm{SO} \xrightarrow{\mathbf{W}_3} \mathbf{B}^2U(1)$	twisted Spin^c-structure	
$\mathbf{B}\mathrm{PU}(\mathcal{H})\times\mathrm{SO} \xrightarrow{\mathbf{dd}-\mathbf{W}_3} \mathbf{B}^2U(1)$		general Freed-Witten anomaly cancellation: B-field with twisted gauge bundles on D-branes
$\mathbf{B}\mathrm{Spin} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3U(1)$	twisted String-2-bundles; heterotic Green-Schwarz anomaly cancellation	
$\mathbf{B}\mathrm{String} \xrightarrow{\frac{1}{6}\mathbf{p}_2} \mathbf{B}^7U(1)$	twisted Fivebrane-7-bundles; dual heterotic Green-Schwarz anomaly cancellation	

The following table lists twisting ∞-bundles that encode geometric structure preserving higher supersymmetry.

universal twisting ∞-bundle	twisted cohomology	relative twisted cohomology
$\mathbf{B}\mathrm{U}(d,d) \to \mathbf{B}\mathrm{O}(2d,2d)$	generalized complex geometry	
$\mathbf{B}\mathrm{SU}(3)\times \mathrm{SU}(3) \to \mathbf{B}\mathrm{O}(6,6)$	$d=6$, $N=2$ type II compactification	
$\mathbf{B}\mathrm{SU}(7) \to \mathbf{B}E_{7(7)}$	$d=7$, $N=1$ 11d sugra compactification	

5.4.2 Sections of vector bundles – twisted 0-bundles

We discuss here for illustration purposes twisted ∞-bundles in *lower* degree than traditionally considered, namely *twisted 0-bundles.* This degenerate case is in itself simple, but all the more does it serve to illustrate by familiar example the general notions of twisted ∞-bundles.

So we consider coefficient ∞-bundles such as

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C}//U(1) \\ & & \downarrow \\ & & \mathbf{B}U(1) \end{array} ,$$

where

- $\mathbf{B}U(1)$ is the smooth moduli stack of smooth circle bundles;
- $\mathbb{C}$ is the complex plane, regarded as a smooth manifold.

By 3.6.13 this corresponds equivalently to a representation of the Lie group $U(1)$ on $\mathbb{C}$, and this we take to be the canonical such representation. Accordingly, the above bundle is indeed the *universal complex line bundle* over the base space of the universal $U(1)$-principal bundle.

It will be meaningful and useful to think of $\mathbb{C}$ itself as a moduli ∞-stack: it is the smooth *moduli 0-stack of complex 0-vector bundles*, where, therefore, a complex 0-vector bundle on a smooth space X is simply a smooth function $\in C^\infty(X, \mathbb{C})$. Accordingly, we should find that such 0-vector bundles can be twisted by a principal $U(1)$-bundle and indeed, by feeding the above coefficient ∞-bundle through the definition of twisted ∞-bundles in 3.6.14, one finds, as we discuss below, that a *twisted 0-bundle* is a smooth section of the *associated line bundle*, hence, by local triviality of the line bundle, locally a complex-valued function, but globally twisted by the twisting circle bundle.

Let G be a Lie group, V a vector space and $\rho : V \times G \to V$ a smooth representation of G on V in the traditional sense. We discuss how this is an ∞-group representation in the sense of def. 3.6.152.

Definition 5.4.1. Write

$$V//G := V \times G \underset{\rho}{\overset{p_1}{\rightrightarrows}} V$$

for the *action groupoid* of ρ, the weak quotient of V by G, regarded as a smooth ∞-groupoid $V//G \in$ Smooth∞Grpd.

Notice that this is equipped with a canonical morphism $V//G \to \mathbf{B}G$ and a canonical inclusion $V \to V//G$.

Proposition 5.4.2. *We have a fiber sequence*

$$V \to V//G \to \mathbf{B}G$$

in Smooth∞Grpd.

Proof. One finds that in the canonical presentation by simplicial presheaves as in 4.4.2, the morphism $V//G_{\mathrm{ch}} \to \mathbf{B}G_{\mathrm{ch}}$ is a fibration in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Therefore by prop. 2.3.13 the homotopy fiber is given by the ordinary fiber of this presentation. This ordinary fibe is V. □

Remark 5.4.3. By remark 3.6.209 we may think of the fiber sequence

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

as the vector bundle over the classifying stack $\mathbf{B}G$ which is ρ-associated to the universal G-principal bundle.

More formally, the next proposition shows that the ρ-associated bundles according to def. 3.6.209 are the ordinary associated vector bundles.

Proposition 5.4.4. *Let X be a smooth manifold and $P \to X$ be a smooth G-principal bundle. If $g : X \to \mathbf{B}G$ is a cocycle for P as in 4.4.7, then the ρ-associated vector bundle $P \times_G V \to X$ is equivalent to the homotopy pullback of $V//G \to \mathbf{B}G$ along G:*

$$\begin{array}{ccc} P \times_G V & \longrightarrow & V//G \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} .$$

Proof. By the discussion in 4.4.7 we may present g by a morphism in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ of the form

$$\begin{array}{ccc} C(\{U_i\}) & \xrightarrow{g} & \mathbf{B}G_{\mathrm{ch}} \\ \downarrow{\simeq} & & \\ X & & \end{array} ,$$

where $C(\{U_i\})$ is the Čech nerve of a good open cover of X. Since $V//G_{\mathrm{ch}} \to \mathbf{B}G_{\mathrm{ch}}$ is a fibration in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, by prop. 2.3.13 its ordinary pullback of simplicial presheaves along g presents the homotopy pullback in question. By inspection one finds that this is the Lie groupoid whose space of objects is $\coprod_i U_i \times V$ and which has a unique morphism from $(x \in U_i, \sigma_i(x) \in V)$ to $(x \in U_j, \sigma_j(x))$ if $\sigma_j(x) = \rho(g_{ij}(x))(\sigma_i(x))$.

Due to the uniqueness of morphisms, the evident projection from this Lie groupoid to the smooth manifold $P \times_G V$ which is the total space of the V-bundle ρ-accociated to P is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, hence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. So $P \times_G V$ is indeed (one representative of) the homotopy pullback in question. □

Since therefore all the information about ρ is encoded in the bundle $V \hookrightarrow V//G \to \mathbf{B}G$, we may identify that bundle with the action. Accordingly we write

$$\rho : V//G \to \mathbf{B}G .$$

Regarding ρ then as a universal local coefficient bundle, we obtain the corresponding twisted cohomology, 3.6.12, and twisted ∞-bundles, 3.6.14. We show now that the general statement of prop. 3.6.220 on twisted cohomology in terms of sections of associated ∞-bundles reduces for twists relative to ρ to the standard notion of spaces of sections.

Proposition 5.4.5. *Let $P \to X$ be a G-principal bundle over a smooth manifold X. Then the ∞-groupoid of P-twisted cocycles relative to ρ, equivalently the ∞-groupoid of P-twisted V-0-bundles is equivalent to the ordinary set of sections of the vector bundle $E \to X$ which is ρ-associated to P:*

$$\Gamma_X(E) \simeq \mathbf{H}_{/\mathbf{B}G}(g, \rho) .$$

Here $g : X \to \mathbf{B}G$ is the morphism classifying P.

Proof. The hom ∞-groupoid of the slice ∞-topos over $\mathbf{B}G$ is the ∞-pullback

$$\begin{array}{ccc} \mathbf{H}_{/\mathbf{B}G}(g, \rho) & \longrightarrow & \mathbf{H}(X, V//G) \\ \downarrow & & \downarrow \\ * & \xrightarrow{[g]} & \mathbf{H}(X, \mathbf{B}G) \end{array} .$$

Since the Čech nerve $C(\{U_i\})$ of the good cover $\{U_i \to X\}$ is a cofibrant representative of X in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, and since $\mathbf{B}G_{\mathrm{ch}}$ and $V//G_{\mathrm{ch}}$ from above are fibrant representatives of $\mathbf{B}G$ and $V//G$, respectively, by the properties of simplicial model categories the right vertical morphism here is presented by the morphism of Kan complexes.

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), V//G_{\mathrm{ch}}) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}G_{\mathrm{ch}})\,.$$

Moreover, since this is the simplicial hom out of a cofibrant object into a fibration, the properties of simplicial model categories imply that this morphism is indeed a Kan fibration. It follows with prop. 2.3.8 that the ordinary fiber of this morphism over $[g]$ is a Kan complex that presents the twisted cocycle ∞-groupoid in question.

Since $V//G_{\mathrm{ch}} \to \mathbf{B}G_{\mathrm{ch}}$ is a faithful functor of groupoids, this fiber is a set, meaning a constant simplicial set. A $V//G_{\mathrm{ch}}$-valued cocycle is a collection of smooth functions $\{\sigma_i : U_i \to V\}_i$ and smooth functions $\{g_{ij} : U_{i,j} \to G\}_{i,j}$, satisfying the condition that on all U_{ij} we have $\sigma_j = \rho(g_{ij})(\sigma_i)$. This is a vertex in the fiber precisely if the second set of functions is that given by the cocycle g which classifies P. In this case this condition is precisely that which identifies the $\{\sigma_i\}_i$ as a section of the associated vector bundle, expressed in terms of the local trivialization that corresponds to g.

In conclusion, this shows that $\mathbf{H}_{/\mathbf{B}G}(g, \rho)$ is an ∞-groupoid equivalent to set of sections of the vector bundle ρ-associated to P. □

5.4.3 Sections of 2-bundles – twisted vector bundles and twisted K-classes

We construct now a coefficient ∞-bundle of the form

$$\begin{array}{ccc} \mathbf{B}U & \longrightarrow & (\mathbf{B}U)//\mathbf{B}U(1) \\ & & \downarrow{\scriptstyle \mathbf{dd}} \\ & & \mathbf{B}^2 U(1) \end{array}\,,$$

where

- $\mathbf{B}^2 U(1)$ is the smooth moduli 2-stack for smooth circle 2-bundles / bundle gerbes;
- $\mathbf{B}U = \varinjlim_n \mathbf{B}U(n)$ is the inductive ∞-limit over the smooth moduli stacks of smooth unitary rank-n vector bundles (equivalently: $U(n)$-principal bundles).

Equivalently, this is a smooth ∞-action of the smooth circle 2-group $\mathbf{B}U(1)$ on the smooth ∞-stack $\mathbf{B}U$.

This may be thought of as the canonical 2-representation of the circle 2-group $\mathbf{B}U(1)$, def. 4.3.48, being the higher analogue to the canonical representation of the circle group $U(1)$ on the complex plane $\mathbb{C}$, discussed above in 5.4.2.

We show that the notion of twisted cohomology induced by this local coefficient bundle according to 3.6.12 is reduced *twisted K-theory* and that the notion of twisted ∞-bundles induced by it according to 3.6.14 are ordinary *twisted vector bundles* also known as *bundle gerbe modules*. (See for instance chapter 24 of [May] for basics of K-theory that we need here, and see for instance [CBMMS02] for a discussion of twisted K-theory in terms of twisted bundles.)

This not only shows how the traditional notion of twisted K-theory is reproduced from the perspective of cohomology in an ∞-topos. It also refines the traditional constructions to the smooth context. Notice that there is a slight clash of terminology, as traditionally the term *smooth K-theory* is often used synonymously with *differential K-theory*. However, there is a geometric refinement in between bare (twisted) K-classes and differential (twisted) K-classes, namely smooth cocycle spaces of smooth (twisted) vector bundles and *smooth* gauge transformations between them. This is the smooth refinement of the situation that we find here, by regarding (twisted) K-theory as (twisted) cohomology internal to the ∞-topos Smooth∞Grpd.

The construction of the traditional topological classifying space for reduced K^0 proceeds as follows. For $n \in \mathbb{N}$, let $BU(n)$ be the classifying space of the unitary group in complex dimension n. The inclusion of groups $U(n) \to U(n+1)$ induced by the inclusion $\mathbb{C}^n \to \mathbb{C}^{n+1}$ by extension by 0 in the, say, last coordinate gives an inductive system of topological spaces

$$* \longrightarrow \cdots BU(n) \longrightarrow BU(n+1) \longrightarrow \cdots .$$

Definition 5.4.6. Write

$$BU := \varinjlim_n BU(n)$$

for the homotopy colimit in $\mathrm{Top}_{\mathrm{Quillen}}$.

Notice that by prop. 4.4.19 and prop. 4.3.30 we have, for each $n \in \mathbb{N}$, a smooth refinement of $BU(n) \in \mathrm{Top} \simeq \infty\mathrm{Grpd}$ to a smooth moduli stack $\mathbf{B}U(n) \in \mathrm{Smooth}\infty\mathrm{Grpd}$. This refines the set $[X, BU(n)]$ of equivalences classes of rank-n unitary vector bundles to the groupoid $\mathbf{H}(X, \mathbf{B}U(n))$ of unitary bundles and smooth gauge transformations between them.

We therefore consider now similarly a smooth refinement to moduli ∞-stacks of the inductive limit BU.

Definition 5.4.7. Write

$$\mathbf{B}U := \varinjlim_n \mathbf{B}U(n)$$

for the ∞-colimit in $\mathrm{Smooth}\infty\mathrm{Grpd}$ over the smooth moduli stacks of smooth $U(n)$-principal bundles.

Proposition 5.4.8. *The canonical morphism*

$$\varinjlim_n \mathbf{B}U(n) \to \mathbf{B} \varinjlim_n U(n)$$

is an equivalence in $\mathrm{Smooth}\infty\mathrm{Grpd}$.

Proof. Write $\mathbf{B}U(n)_{\mathrm{ch}} := N(\, U(n) \rightrightarrows * \,) \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ for the standard presentation of the delooping, prop. 4.4.19. Observe then that the diagram $n \mapsto \mathbf{B}U(n)_{\mathrm{ch}}$ is cofibrant when regarded as an object of $[(\mathbb{N}, \leq), [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}]_{\mathrm{proj}}$, because, by example 2.3.16, a cotower is projectively cofibrant if it consists of monomorphisms and if the first object, and hence all objects, are cofibrant. Therefore the ∞-colimit is presented by the ordinary colimit over this diagram. Since this is a filtered colimit, it commutes with finite limits of simplicial presheaves:

$$\begin{aligned} \varinjlim_n \mathbf{B}U(n)_{\mathrm{ch}} &= \varinjlim_n N(\, U(n) \rightrightarrows * \,) \\ &= N(\, \varinjlim_n U(n) \rightrightarrows * \,) \\ &= (\mathbf{B} \varinjlim_n U(n))_{\mathrm{ch}} . \end{aligned}$$

□

Proposition 5.4.9. *The smooth object* $\mathbf{B}U$ *is a smooth refinement of the topological space* BU *in that it reproduces the latter under geometric realization, 4.3.4.1:*

$$|\mathbf{B}U| \simeq BU .$$

Proof. By prop. 4.3.29 for every $n \in \mathbb{N}$ we have

$$|\mathbf{B}U(n)| \simeq BU(n) .$$

Moreover, by the discussion at 4.3.4.1, up to the equivalence $\mathrm{Top} \simeq \infty\mathrm{Grpd}$ the geometric realization is given by applying the functor $\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}$. That is a left ∞-adjoint and hence preserves

∞-colimits:

$$
\begin{aligned}
|\mathbf{B}U| &\simeq |\varinjlim_n \mathbf{B}U(n)| \\
&\simeq \varinjlim_n |\mathbf{B}U(n)| \\
&\simeq \varinjlim_n BU(n) \\
&\simeq BU\,.
\end{aligned}
$$

□

Corollary 5.4.10. *For $X \in$ SmthMfd $\hookrightarrow$ Smooth∞Grpd, the intrinsic cohomology of X with coefficients in the smooth stack $\mathbf{B}U$ is the reduced K-theory $\tilde{K}(X)$:*

$$
H^1_{\mathrm{smooth}}(X, U) := \pi_0 \mathbf{H}(X, \mathbf{B}U) \simeq \tilde{K}(X)\,.
$$

Proof. By prop. 4.3.39 the set $\pi_0\mathbf{H}(X,\mathbf{B}U)$ is the Čech cohomology of X with coefficients in the stable unitary group U. By classification theory (as discussed in [RoSt12]) this is isomorphic to the set of homotopy classes of maps $\pi_0\mathrm{Top}(X, BU)$ into the classifying space BU for reduced K-theory. □

Proposition 5.4.11. *Let X be a compact smooth manifold. Then*

$$
\mathbf{H}(X, \mathbf{B}U) \simeq \varinjlim_n \mathbf{H}(X, \mathbf{B}U(n))
$$

and

$$
\mathbf{H}(X, \mathbf{B}PU) \simeq \varinjlim_n \mathbf{H}(X, \mathbf{B}PU(n))\,.
$$

Proof. That X is a compact manifold means by def. 3.6.57 that it is a *representably compact object* in the site SmthMfd. Since X is in particular paracompact, prop. 3.6.63 says that it is also a *representably paracompact object* in the site, def. 3.6.62. With this the statement is given by prop. 3.6.64. □

We now discuss twisted bundles induced by the local coefficient bundles $\mathbf{dd}_n : \mathbf{B}PU(n) \to \mathbf{B}^2U(1)$ for every $n \in \mathbb{N}$. This is immediately generalized to general central extensions.

So let $U(1) \to \hat{G} \to G$ be any $U(1)$-central extension of a Lie group G and let $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^2U(1)$ the classifying morphism of moduli 2-stacks, according to prop. 3.6.151, sitting in the fiber sequence

$$
\begin{array}{ccc}
\mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\
 & & \downarrow{\scriptstyle \mathbf{c}} \\
 & & \mathbf{B}^2U(1)
\end{array}
\quad .
$$

Proposition 5.4.12. *Let $U(1) \to \hat{G} \to G$ be a group extension of Lie groups. Let $X \in$ SmoothMfd $\hookrightarrow$ Smooth∞Grpd be a smooth manifold with differentiably good open cover $\{U_i \to X\}$.*

1. *Relative to this data every twisting cocycle $[\alpha] \in H^2_{\mathrm{Smooth}}(X, U(1))$ is a Čech-cohomology representative given by a collection of functions*

$$
\{\alpha_{ijk} : U_i \cap U_j \cap U_k \to U(1)\}
$$

satisfying on every quadruple intersection the equation

$$
\alpha_{ijk}\alpha_{ikl} = \alpha_{jkl}\alpha_{ijl}\,.
$$

2. *In terms of this cocycle data, the twisted cohomology $H^1_{[\alpha]}(X, \hat{G})$ is given by equivalence classes of cocycles consisting of*

   (a) *collections of functions*
   $$\{g_{ij} : U_i \cap U_j \to \hat{G}\}$$
   subject to the condition that on each triple overlap the equation
   $$g_{ij}\dot{g}_{jk} = g_{ik} \cdot \alpha_{ijk}$$
   holds, where on the right we are injecting α_{ijk} via $U(1) \to \hat{G}$ into $\hat{G}$ and then form the product there;

   (b) *subject to the equivalence relation that identifies two such collections of cocycle data $\{g_{ij}\}$ and $\{g'_{ij}\}$ if there exists functions*
   $$\{h_i : U_i \to \hat{G}\}$$
   and
   $$\{\beta_{ij} : U_i \cap U_j \to \hat{U}(1)\}$$
   such that
   $$\beta_{ij}\beta_{jk} = \beta_{ik}$$
   and
   $$g'_{ij} = h_i^{-1} \cdot g_{ij} \cdot h_j \cdot \beta_{ij}\,.$$

Proof. We pass to the standard presentation of Smooth∞Grpd by the projective local model structure on simplicial presheaves over the site $\mathrm{CartSp}_{\mathrm{smooth}}$. There we compute the defining ∞-pullback by a homotopy pullback, according to remark 2.3.14.

Write $\mathbf{B}\hat{G}_{\mathrm{ch}}, \mathbf{B}^2U(1)_{\mathrm{ch}} \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ etc. for the standard models of the abstract objects of these names by simplicial presheaves, as discussed in 4.4.2. Write accordingly $\mathbf{B}(U(1) \to \hat{G})_{\mathrm{ch}}$ for the delooping of the crossed module 2-group associated to the central extension $\hat{G} \to G$.

By prop. 3.6.151, in terms of this the characteristic class $\mathbf{c}$ is represented by the ∞-anafunctor

$$\begin{array}{ccc} \mathbf{B}(U(1) \to \hat{G})_{\mathrm{ch}} & \stackrel{\mathbf{c}}{\longrightarrow} & \mathbf{B}(U(1) \to 1)_{\mathrm{ch}} = \mathbf{B}^2U(1)_{\mathrm{ch}} \\ \downarrow{\simeq} & & \\ \mathbf{B}G_{\mathrm{ch}} & & \end{array} ,$$

where the top horizontal morphism is the evident projection onto the $U(1)$-labels. Moreover, the Čech nerve of the good open cover $\{U_i \to X\}$ forms a cofibrant resolution

$$\emptyset \hookrightarrow C(\{U_i\}) \stackrel{\simeq}{\twoheadrightarrow} X$$

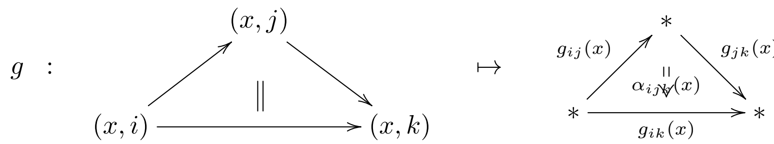
and so α is presented by an ∞-anafunctor

$$\begin{array}{ccc} C(\{U_i\}) & \stackrel{\alpha}{\longrightarrow} & \mathbf{B}^2U(1)_c \\ \downarrow{\simeq} & & \\ X & & \end{array} .$$

Using that $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is a simplicial model category this means in conclusion that the homotopy pullback in question is given by the ordinary pullback of simplicial sets

$$\begin{array}{ccc} \mathbf{H}^1_{[\alpha]}(X, \hat{G}) & \longrightarrow & * \\ \downarrow & & \downarrow{\alpha} \\ [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}(U(1) \to \hat{G})_c) & \stackrel{\mathbf{c}_*}{\longrightarrow} & [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}^2U(1)_c) \end{array} .$$

An object of the resulting simplicial set is then seen to be a simplicial map $g : C(\{U_i\}) \to \mathbf{B}(U(1) \to \hat{G})_c$ that assigns



such that projection out along $\mathbf{B}(U(1) \to \hat{G})_c \to \mathbf{B}(U(1) \to 1)_c = \mathbf{B}^2 U(1)_c$ produces α.

Similarily for the morphisms. Writing out what these diagrams in $\mathbf{B}(U(1) \to \hat{G})_c$ mean in equations, one finds the formulas claimed above. □

5.4.4 Reduction of structure groups

We discuss the traditional notion of *reduction* of a structure group in terms of twisted differential nonabelian cohomology. This perspective turns out to embed this standard notion seamlessly into more general notion of twisted differential structures, def. 3.9.61. Conversely, this prespective shows that the general notion of twisted differential structures may be thought of as a generalization of the classical notion of reduction of structure groups from principal bundles to principal ∞-bundles.

Let G be a Lie group and let $K \hookrightarrow G$ be a Lie subgroup. Write

$$\mathbf{c} : \mathbf{B}K \to \mathbf{B}G$$

for the induced morphism of smooth moduli stacks of smooth principal bundles, according to prop. 4.4.19.

Observation 5.4.13. The action groupoid $G//K$, def. 1.2.41, is 0-truncated, hence the canonical morphism to the smooth manifold quotient

$$G//K \xrightarrow{\simeq} G/K$$

is an equivalence in Smooth∞Grpd.

We have a fiber sequence of smooth stacks

$$G/K \to \mathbf{B}K \to \mathbf{B}G\,.$$

This is presented by the evident sequence of simplicial presheaves

$$G//K \to *//K \to *//G\,.$$

Proof. The equivalence follows because the action of a subgroup is free. The fiber sequence may be computed for instance with the factorization lemma, prop. 2.3.9. □

In applications, an important class of examples is the following.

Observation 5.4.14. For G a conneced Lie group, let $K \hookrightarrow G$ be the inclusion of its maximal compact subgroup. Then $\mathbf{c} : \mathbf{B}K \to \mathbf{B}G$ is a Π-equivalence, def. 3.8.23 (hence becomes an equivalence under geometric realization, def. 3.8.2). Therefore, while the groupoids of K, G-principal bundles are different and

$$\mathbf{H}(X, \mathbf{B}K) \to \mathbf{H}(X, \mathbf{B}G)$$

is not an equivalence, unless G is itself already compact, it does induce an isomorphism on connected components (nonabelian cohomology sets)

$$H^1(X, K) \xrightarrow{\simeq} H^1(X, G)\,.$$

In the following discussion this difference between the classifying spaces $BG \simeq \Pi(\mathbf{B}G) \simeq \Pi(\mathbf{B}K) \simeq BK$ and their smooth refinements is crucial.

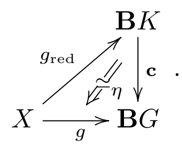
Theorem 4.3.47 in the present case says that $\Pi(G/K) \simeq *$ contractible. This recovers the classical statement that, as a topological space, G is a product of its maximal compact subgroup with a contractible space.

Proof. It is a classical fact that the maximal compact subgroup inclusion $K \hookrightarrow G$ is a homotopy equivalence on the underlying topological spaces. The statement then follows by prop. 4.3.31. $\square$

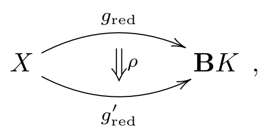
Given a subgroup inclusion $K \hookrightarrow G$ and a G-principal bundle P, a standard question is whether the structure group of P may be reduced to K.

Definition 5.4.15. Let $K \hookrightarrow G$ be an inclusion of Lie groups and let $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ be any object (for instance a smoot manifold). Let $g : X \to \mathbf{B}G$ be a smooth classifying morphism for a G-principal bundle $P \to X$.

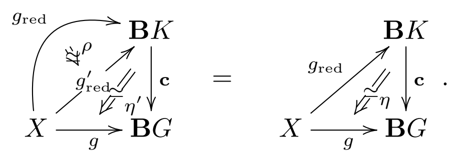
A choice of *reduction of the structure group* of G along $K \hookrightarrow G$ (or *K-reduction* for short) is a choice of lift g_{red} and a choice of homotopy (gauge transformation) η of smooth stacks in the diagram



For (g_{red}, η) and $(g'_{\mathrm{red}}, \eta')$ two K-reductions of P, an *isomorphism* of K-reductions from the first to the second is a natural transformation of morphisms of smooth stacks



hence a choice of gauge transformation between the corresponding K-principal bundles, such that



With the obvious notion of composition of such isomorphisms, this defines a *groupoid of K-reductions* of P.

Remark 5.4.16. The crucial information is in the *choice* of the smooth transformation η. Notably in the case that $K \hookrightarrow G$ is the inclusion of a maximal compact subgroup as in observation 5.4.14 the underlying reduction problem after geometric realization in the homotopy theory of topological spaces is trivial: all bundles involved in the above are equivalent. The important information in η is about *how* they are chosen to be equivalent, and smoothly so.

Below in 5.4.4.1 we see that in the case that $P = TX$ is the tangent bundle of a manifold, η is identified with a choice of *vielbein* or *soldering form*.

Comparison with the discussion in 3.6.12 reveals that therefore structure group reduction is a topic in *twisted nonabelian cohomology*. In particular, we may apply def. 3.9.61 to form the groupoid of all choices of reductions.

Proposition 5.4.17. *For $g : X \to \mathbf{B}G$ (the cocycle for) a G-principal bundle $P \to X$, the* groupoid of K-reductions *of P according to def. 5.4.15 is the groupoid of $[g]$-twisted $\mathbf{c}$-structures, def. 3.9.61, hence the homotopy pullback* $\mathbf{c}\mathrm{Struc}_{[g]}(X)$ *in*

$$
\begin{array}{ccc}
\mathbf{c}\mathrm{Struc}_{[g]}(X) & \longrightarrow & * \\
\downarrow & & \downarrow^{g} \\
\mathbf{H}(X, \mathbf{B}K) & \xrightarrow{\mathbf{H}(X,\mathbf{c})} & \mathbf{H}(X, \mathbf{B}G)
\end{array} ,
$$

where

$$\mathbf{c} : \mathbf{B}K \to \mathbf{B}G$$

is the induced morphism of smooth moduli stacks.

Proof. Using that $\mathbf{B}K$ and $\mathbf{B}G$ are 1-truncated objects in $\mathbf{H} := \mathrm{Smooth}\infty\mathrm{Grpd}$, by construction, one sees that the groupoid defined in def. 5.4.15 is equivalently the hom-groupoid $\mathbf{H}_{/\mathbf{B}G}(g, \mathbf{c})$ in the slice ∞-topos $\mathbf{H}_{/\mathbf{B}G}$. Using this, the statement is a special case of prop. 3.6.221. □

Remark 5.4.18. By observation 5.4.13 we may equivalently speak of $\mathbf{c}\mathrm{Struc}_g(X)$ as the *groupoid of twisted $G//K$-structures* on X (where the latter is given by a corresponding groupoid-principal bundle).

If we think, according to remark 5.4.16, of a choice of K-reduction as a choice of *vielbein* or *soldering form*, then this says that *locally* their moduli space is the cose G/K (while globally there may be a twist).

The morphism $\mathbf{c}$ as above always has a canonical differential refinement

$$\hat{\mathbf{c}} : \mathbf{B}K_{\mathrm{conn}} \to \mathbf{B}G_{\mathrm{conn}}$$

given by prop. 1.2.78. Accordingly, we may also apply def. 3.9.62 to the case of structure group reduction.

Definition 5.4.19. For $K \to G$ a Lie subgroup inclusion, and for $\nabla : X \to \mathbf{B}G_{\mathrm{conn}}$ (a cocycle for) a G-principal bundle with conneciton on X, we say the *groupoid of K-reductions* of ∇ is the groupoid $\hat{\mathbf{c}}\mathrm{Struc}_{[\nabla]}(X)$ of *twisted differential $\hat{\mathbf{c}}$-structures*, given as the homotopy pullback

$$\begin{array}{ccc} \hat{\mathbf{c}}\mathrm{Struc}_{[\nabla]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \nabla} \\ \mathbf{H}(X, \mathbf{B}K_{\mathrm{conn}}) & \xrightarrow{\mathbf{H}(X,\hat{\mathbf{c}})} & \mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}}) \end{array} .$$

However, here the differential refinement does not change the homotopy type of the twisted cohomology

Proposition 5.4.20. *For P a G-principal bundle with connection ∇ the groupoid of K-reductions of ∇ is equivalent to the groupoid of K-reductions of just P*

$$\hat{\mathbf{c}}\mathrm{Struc}_{[\nabla]}(X) \simeq \mathbf{c}\mathrm{Struc}_{[P]}(X)\,.$$

Remark 5.4.21. This degeneracy of notions does not hold for twisted structures controled by higher groups. That it holds in the special case of ordinary K-reductions is an incarnation of a classical fact in differential geometry: as we will see in 5.4.4.1 below, for reductions of tangent bundle structure it comes down to the fact that for every choice of Riemannian metric and torsion there is a unique metric-compatible connection with that torsion. Prop. 5.4.20 may be understood as stating this in the fullest generality of G-principal bundles for G a Lie group.

5.4.4.1 Orthogonal/Riemannian structure

For X a smooth manifold, we discuss the traditional notion of *Riemannian* structure or equivalently of *orthogonal structure* on X as a special case of $\mathbf{c}$-twisted cohomology for suitable $\mathbf{c}$. This perspective on ordinary Riemannian geometry proves to be a useful starting point for generalizations.

Let X be a smooth manifold of dimension d. Its tangent bundle TX is associated to an essentially canonical $\mathrm{GL}(d)$-principal bundle. We write

$$TX : X \to \mathbf{B}\mathrm{GL}(d)$$

for the corresponding classifying morphism, where $\mathbf{B}\mathrm{GL}(d)$ is the smooth moduli stack of smooth $\mathrm{GL}(d)$-principal bundles.

Consider the defining inclusion of Lie groups

$$\mathrm{O}(d) \hookrightarrow \mathrm{GL}(d)$$

and the induced morphism of the corresponding moduli stacks

$$\mathbf{orth} : \mathbf{B}\mathrm{O}(d) \to \mathbf{B}\mathrm{GL}(d)\,.$$

The general observation 5.4.13 here reads

Observation 5.4.22. The homotopy fiber of **orth** is the quotient manifold $\mathrm{GL}(d)/\mathrm{O(d)}$. We have a fiber sequence of smooth stacks

$$\mathrm{GL}(d)/\mathrm{O(d)} \longrightarrow \mathbf{B}\mathrm{O}(d) \xrightarrow{\mathbf{orth}} \mathbf{B}\mathrm{GL}(d)\ .$$

Notice that $\mathrm{O}(d) \hookrightarrow \mathrm{GL}(d)$ is a maximal compact subgroup inclusion, so that observation 5.4.14 applies.

Definition 5.4.17 now becomes

Definition 5.4.23. Write $\mathbf{orth}\mathrm{Struc}_{TX}$ for the groupoid of TX-twisted **orth**-structures on X, hence the homotopy pullback in

$$\begin{array}{ccc} \mathbf{orth}\mathrm{Struc}(X) & \longrightarrow & * \\ \downarrow & \swArrow_{\simeq} & \downarrow^{TX} \\ \mathbf{H}(X, \mathbf{B}\mathrm{O}(d)) & \xrightarrow[\mathbf{H}(X,\mathbf{orth})]{} & \mathbf{H}(X, \mathbf{B}\mathrm{GL}(d)) \end{array} .$$

Proposition 5.4.24. *The groupoid* $\mathbf{orth}\mathrm{Struc}_{TX}(X)$ *is naturally identified with the groupoid of choices of vielbein fields (soldering forms) on* TX*.*

Proof. Let $\{U_i \to X\}$ be any good open cover of X by coordinate patches $\mathbb{R}^d \simeq U_i$. Let $C(\{U_i\})$ be the corresponding Čech groupoid. There is then a canonical span of simplicial presheaves

$$\begin{array}{ccc} C(\{U_i\}) & \xrightarrow{TX_{\mathrm{ch}}} & \mathbf{B}\mathrm{GL}(d)_{\mathrm{ch}} \\ \downarrow^{\simeq} & & \\ X & & \end{array} .$$

presenting TX. Moreover, every morphism $g : X \to \mathbf{B}\mathrm{O}(d)$ has a presentation by a similar span g_{ch} with values in $\mathbf{B}\mathrm{O}(d)$.

An object in $\mathbf{orth}\mathrm{Struc}_{TX}(X)$ is

1. a cocycle g_{ch} for an $\mathrm{O}(d)$-principal bundle as above;
2. over each U_i an element $e|_{U_i} \in C^\infty(U_i, \mathrm{GL}(d))$

such that e is compatible, on double overlaps, with the left $\mathrm{O}(d)$-action by the transition functions g_{ch} and the right $\mathrm{GL}(d)$-action by the transitiuon functions TX_{ch}.

A morphism $e \to e'$ in $\mathbf{orth}\mathrm{Struc}_{TX}(X)$ is a gauge transformation $g_{\mathrm{ch}} \to g'_{\mathrm{ch}}$ of $\mathrm{O}(d)$-principal bundles whose left action takes e to e'.

From this it is clear that

$$e = \{e^a{}_\mu\}_{a,\mu \in \{1,\cdots,d\}}$$

is a choice of vielbein. □

There is an evident differential refinement of **orth**

$$\hat{\mathbf{orth}} : \mathbf{B}\mathrm{O}(d)_{\mathrm{conn}} \to \mathbf{B}\mathrm{GL}(d)_{\mathrm{conn}}\,.$$

Definition 5.4.25. Let $\mathrm{Conn}TX \to \mathbf{H}(X, \mathbf{B}\mathrm{GL}(d)_{\mathrm{conn}})$ be the left vertical morphism in the homotopy pullback

$$\begin{array}{ccc} \mathrm{Conn}TX & \longrightarrow & * \\ \downarrow & & \downarrow^{TX} \\ \mathbf{H}(X, \mathbf{B}\mathrm{GL}(d)_{\mathrm{conn}}) & \longrightarrow & \mathbf{H}(X, \mathbf{B}\mathrm{GL}(d)) \end{array},$$

where the bottom map is the morphism that forgets the connection.

This morphism may be thought of as the inclusion of connections on the tangent bundle into the groupoid of all $\mathrm{GL}(d)$-principal connections.

Proposition 5.4.26. *The homotopy pullback in*

$$\begin{array}{ccc} \hat{\mathbf{orth}}\mathrm{Struc}_{TX,\mathrm{conn}}(X) & \longrightarrow & \mathrm{Conn}TX \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}\mathrm{O}(d)_{\mathrm{conn}}) & \xrightarrow{\mathbf{H}(X,\hat{\mathbf{orth}})} & \mathbf{H}(X, \mathbf{B}\mathrm{GL}(d)_{\mathrm{conn}}) \end{array}$$

or equivalently that in

$$\begin{array}{ccc} \hat{\mathbf{orth}}\mathrm{Struc}_{TX,\mathrm{conn}}(X) & \longrightarrow & * \\ \downarrow & & \downarrow^{TX} \\ \mathbf{H}(X, \mathbf{B}\mathrm{O}(d)_{\mathrm{conn}}) & \longrightarrow & \mathbf{H}(X, \mathbf{B}\mathrm{GL}(d)) \end{array}$$

is equivalent to the set of pairs of Riemannian metrics on X and correspondingly metric-compatible connections on TX.

Proof. The two pullbacks are equivalent by def. 5.4.25 and the pasting law, prop. 2.3.2.

Consider the first version. As in the proof of prop. 5.4.24 an object in the groupoid has an underlying choice of vielbein e. This now being a morphism of bundles with connection, it related, locally on each U_i, the goven connection form Γ on TX with a connection form ω on the $\mathrm{O}(d)$-principal bundle, via

$$\omega^a{}_b = e^a{}_\alpha \Gamma^\alpha{}_\beta (e^{-1})^b{}_\beta + e^a{}_\alpha d_{\mathrm{dR}} (e^{-1})^b{}_\beta \,.$$

But since ω is by definition an orthogonal connection, by this isomorphism Γ is a metric-compatible connection. $\square$

5.4.4.2 Type II NS-NS generalized geometry

The target space geometry for type II superstrings in the NS-NS sector is naturally encoded by a variant of "generalized complex geometry" with metric structure, discussed for instance in [GMPW08]. We discuss here how this *type II NS-NS generalized geometry* is a special case of twisted **c**-structures as in 5.4.4.

Definition 5.4.27. Consider the Lie group inclusion

$$\mathrm{O}(d) \times \mathrm{O}(d) \to \mathrm{O}(d, d)$$

of those orthogonal transformations, that preserve the positive definite part or the negative definite part of the bilinear form of signature (d, d), respectively.

If $\mathrm{O}(d, d)$ is presented as the group of $2d \times 2d$-matrices that preserve the bilinear form given by the $2d \times 2d$-matrix

$$\eta := \begin{pmatrix} 0 & \mathrm{id}_d \\ \mathrm{id}_d & 0 \end{pmatrix}$$

then this inclusion sends a pair (A_+, A_-) of orthogonal $n \times n$-matrices to the matrix

$$(A_+, A_-) \mapsto \frac{1}{\sqrt{2}} \left(\begin{array}{cc} A_+ + A_- & A_+ - A_- \\ A_+ - A_- & A_+ + A_- \end{array} \right) .$$

The inclusion of Lie groups induces the corresponding morphism of smooth moduli stacks of principal bundles

$$\mathbf{TypeII} : \mathbf{B}(\mathrm{O}(d) \times \mathrm{O}(d)) \to \mathbf{B}\mathrm{O}(d,d) .$$

Observation 5.4.13 here becomes

Observation 5.4.28. There is a fiber sequence of smooth stacks

$$\mathrm{O}(d,d)/(\mathrm{O}(d) \times \mathrm{O}(d)) \longrightarrow \mathbf{B}(\mathrm{O}(d) \times \mathrm{O}(d)) \xrightarrow{\mathbf{TypeII}} \mathbf{B}\mathrm{O}(d,d) \;\; .$$

Definition 5.4.29. There is a canonical embedding

$$\mathrm{GL}(d) \hookrightarrow \mathrm{O}(d,d) .$$

In the above matrix presentation this is given by sending

$$a \mapsto \left(\begin{array}{cc} a & 0 \\ 0 & a^{-T} \end{array} \right) ,$$

where in the bottom right corner we have the transpose of the inverse matrix of the invertble matrix a.

Observation 5.4.30. We have a homotopy pullback of smooth stacks

$$\begin{array}{ccc} \mathrm{GL}(d)\backslash\backslash \mathrm{O}(d,d)//(\mathrm{O}(d) \times \mathrm{O}(d)) & \longrightarrow & \mathbf{B}\mathrm{GL}(d) \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathrm{O}(d) \times \mathrm{O}(d)) & \longrightarrow & \mathbf{B}\mathrm{O}(d,d) \end{array} .$$

Definition 5.4.31. Under inclusion def. 5.4.27 the tangent bundle of a d-dimensional manifold X defines an $\mathrm{O}(d,d)$-cocycle

$$TX \otimes T^*X : \; X \xrightarrow{TX} \mathbf{B}\mathrm{GL}(d) \longrightarrow \mathbf{B}\mathrm{O}(d,d) \;\; .$$

The vector bundle canonically associated to this composite cocycles may canonically be identified with the tensor product vector bundle $TX \otimes T^*X$, and so we will refer to this cocycle by these symbols, as indicated.

Therefore we may canonically consider the groupoid of $TX \otimes T^*X$-twisted **TypeII**-structures, according to def. 5.4.17:

Definition 5.4.32. Write $\mathbf{TypeII}\mathrm{Struc}_{TX \otimes T^*X}(X)$ for the homotopy pullback

$$\begin{array}{ccc} \mathbf{TypeII}\mathrm{Struc}_{TX\otimes T^*X}(X) & \longrightarrow & * \\ \downarrow & & \downarrow^{TX \otimes T^*X} \\ \mathbf{H}(X, \mathbf{B}(O(d) \times O(d))) & \xrightarrow{\mathbf{H}(X, \mathbf{TypeII})} & \mathbf{H}(X, \mathbf{B}O(d,d)) \end{array} .$$

Proposition 5.4.33. *The groupoid* $\mathbf{TypeII}\mathrm{Struc}_{TX\otimes T^*X}(X)$ *is that of "generalized vielbein fields" on* X*, as considered for instance around equation (2.24) of [GMPW08] (there only locally, but the globalization is evident).*

In particular, its set of equivalence classes is the set of type-II generalized geometry structures on X*.*

Proof. This is directly analogous to the proof of prop. 5.4.24. □

Over a local patch $\mathbb{R}^d \simeq U_i \hookrightarrow X$, the most general such generalized vielbein (hence the most general $\mathrm{O}(d,d)$-valued function) may be parameterized as

$$E = \frac{1}{2}\begin{pmatrix} (e_+ + e_-) + (e_+^{-T} - e_-^{-T})B & (e_+^{-T} - e_-^{-T}) \\ (e_+ - e_-) - (e_+^{-T} + e_-^{-T})B & (e_+^{-T} + e_-^{-T}) \end{pmatrix},$$

where $e_+, e_- \in C^\infty(U_i, \mathrm{O}(d))$ are thought of as two ordinary vielbein fields, and where B is any smooth skew-symmetric $n \times n$-matrix valued function on $\mathbb{R}^d \simeq U_i$.

By an $\mathrm{O}(d) \times \mathrm{O}(d)$-transformation this can always be brought into a form where $e_+ = e_- =: \frac{1}{2}e$ such that

$$E = \begin{pmatrix} e & 0 \\ -e^{-T}B & e^{-T} \end{pmatrix}.$$

The corresponding "generalized metric" over U_i is

$$E^T E = \begin{pmatrix} e^T & Be^{-1} \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} e & 0 \\ -e^{-T}B & e^{-T} \end{pmatrix} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix},$$

where

$$g := e^T e$$

is the metric (over $\mathbb{R}^q \simeq U_i$ a smooth function with values in symmetric $n \times n$-matrices) given by the ordinary vielbein e.

5.4.4.3 U-duality geometry / exceptional generalized geometry The scalar and bosonic fields of 11-dimensional supergravity compactified on tori to dimension d *locally* have moduli spaces identified with the quotients $E_{n(n)}/H_n$ of the split real form $E_{n(n)}$ in the E-series of exceptional Lie groups by their maximal compact subgroups H_n, where $n = 11 - d$. The canonical action of $E_{n(n)}$ on this coset space – or of a certain discrete subgroup $E_{n(n)}(\mathbb{Z}) \hookrightarrow E_{n(n)}$ – is called the *U-duality* global symmetry of the supergravity, or of its string UV-completion, respectively [HT94].

In [Hull07] it was pointed out that therefore the geometry of the field content of compactfied supergravity should be encoded by a *exceptional generalized geometry* which in direct analogy to the variant of *generalized complex geometry* that controls the NS-NS sector of type II strings, as discussed above in 5.4.4.2, is encoded by vielbein fields that exhibit reduction of a structure group along the inclusion $H_n \hookrightarrow E_{n(n)}$.

By the general discussion in 5.4.4, we have that all these geometries are encoded by twisted differential **c**-structures, where

$$\mathbf{c} : \mathbf{B}H_n \to \mathbf{B}E_{n(n)}$$

is the induced morphism of smooth moduli stacks.

5.4.5 Orientifolds and higher orientifolds

We discuss the notion of circle n-bundles with connection over double covering spaces with *orientifold* structure (see [SSW05] and [DiFrMo11] for the notion of orientifolds for 2-bundles).

Proposition 5.4.34. *The smooth automorphism 2-group of the circle group $U(1)$ is that corresponding to the smooth crossed module (as discussed in 2.2.6)*

$$\mathrm{AUT}(U(1)) \simeq [U(1) \to \mathbb{Z}_2],$$

where the differential $U(1) \to \mathbb{Z}_2$ is trivial and where the action of $\mathbb{Z}_2$ on $U(1)$ is given under the identification of $U(1)$ with the unit circle in the plane by reversal of the sign of the angle.

This is an extension of smooth ∞-groups, def. 3.6.245, of $\mathbb{Z}_2$ by the circle 2-group $\mathbf{B}U(1)$:

$$\mathbf{B}U(1) \to \mathrm{AUT}(U(1)) \to \mathbb{Z}_2.$$

Proof. The nature of $\mathrm{AUT}(U(1))$ is clear by definition. Let $\mathbf{B}U(1) \to \mathrm{AUT}(U(1))$ be the evident inclusion. We have to show that its delooping is the homotopy fiber of $\mathbf{B}\mathrm{AUT}(U(1)) \to \mathbf{B}\mathbb{Z}_2$.

Passing to the presentation of Smooth∞Grpd by the model structure on simplicial presheaves $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ and using prop. 2.3.13, it is sufficient to show that the simplicial presheaf $\mathbf{B}^2U(1)_c$ from 4.4.2 is equivalent to the ordinary pullback of simplicial presheaves $\mathbf{B}\mathrm{AUT}(U(1))_c \times_{\mathbf{B}\mathbb{Z}_2} \mathbf{E}\mathbb{Z}_2$ of the $\mathbb{Z}_2$-universal principal bundle, as discussed in 1.2.5.

This pullback is the 2-groupoid whose

- objects are elements of $\mathbb{Z}_2$;
- morphisms $\sigma_1 \to \sigma_2$ are labeled by $\sigma \in \mathbb{Z}_2$ such that $\sigma_2 = \sigma\sigma_1$;
- all 2-morphisms are endomorphisms, labeled by $c \in U(1)$;
- vertical composition of 2-morphisms is given by the group operation in $U(1)$,
- horizontal composition of 1-morphisms with 1-morphisms is given by the group operation in $\mathbb{Z}_2$
- horizontal composition of 1-morphisms with 2-morphisms (*whiskering*) is given by the action of $\mathbb{Z}_2$ on $U(1)$.

Over each $U \in \mathrm{CartSp}$ this 2-groupoid has vanishing π_1, and $\pi_2 = U(1)$. The inclusion of $\mathbf{B}^2U(1)$ into this pullback is given by the evident inclusion of elements in $U(1)$ as endomorphisms of the neutral element in $\mathbb{Z}_2$. This is manifestly an isomorphism on π_2 and trivially an isomorphism on all other homotopy groups. Therefore it is a weak equivalenc. □

Observation 5.4.35. A $U(1)$-gerbe in the full sense Giraud (see [LuHTT], section 7.2.2) as opposed to a $U(1)$-bundle gerbe / circle 2-bundle is equivalent to an $\mathrm{AUT}(U(1))$-principal 2-bundle, not in general to a circle 2-bundle, which is only a special case.

More generally we have:

Proposition 5.4.36. *For every $n \in \mathbb{N}$ the automorphism $(n+1)$-group of $\mathbf{B}^nU(1)$ is given by the crossed complex (as discussed in 2.2.6)*

$$\mathrm{AUT}(\mathbf{B}^nU(1)) \simeq [U(1) \to 0 \to \cdots \to 0 \to \mathbb{Z}_2]$$

with $U(1)$ in degree $n+1$ and $\mathbb{Z}_2$ acting by automorphisms. This is an extension of smooth ∞-groups

$$\mathbf{B}^{n+1}U(1) \longrightarrow \mathrm{AUT}(\mathbf{B}^nU(1)) \longrightarrow \mathbb{Z}_2 \ .$$

With slight abuse of notation we also write

$$\mathbf{B}^nU(1)//\mathbb{Z}_2 := \mathbf{B}\mathrm{AUT}(\mathbf{B}^{n-1}U(1))\,.$$

Definition 5.4.37. Write

$$\mathbf{J}_n : \mathbf{B}^{n+1}U(1)//\mathbb{Z}_2 \to \mathbf{B}\mathbb{Z}_2$$

for the corresponding universal characteristic map.

Definition 5.4.38. For $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$, a *double cover* $\hat{X} \to X$ is a $\mathbb{Z}_2$-principal bundle.

For $n \in \mathbb{N}$, $n \geq 1$, an *orientifold circle n-bundle (with connection)* is an $\mathrm{AUT}(\mathbf{B}^{n-1}U(1))$-principal ∞-bundle (with ∞-connection) on X that extends $\hat{X} \to X$ (by def. 3.6.245) with respect to the extension of $\mathbb{Z}_2$ by $\mathrm{AUT}(\mathbf{B}^nU(1))$, prop. 5.4.36.

This means that relative to a cocycle $g : X \to \mathbf{B}\mathbb{Z}^2$ for a double cover $\hat{X}$, the structure of an orientifold circle n-bundle is a factorization of this cocycle as

$$g : X \xrightarrow{\hat{g}} \mathbf{B}\mathrm{AUT}(\mathbf{B}^{n-1}U(1)) \to \mathbf{B}\mathbb{Z}^2$$

where $\hat{g}$ is the cocycle for the corresponding $\mathrm{AUT}(\mathbf{B}^n U(1))$-principal ∞-bundle.

Proposition 5.4.39. *Every orientifold circle n-bundle (with connection) on X induces an ordinary circle n-bundle (with connection) $\hat{P} \to \hat{X}$ on the given double cover $\hat{X}$ such that restricted to any fiber of $\hat{X}$ this is equivalent to* $\mathrm{AUT}(\mathbf{B}^{n-1}U(1)) \to \mathbb{Z}_2$.

Proof. There is a pasting diagram of ∞-pullbacks of the form

$$\begin{array}{ccccccc}
(U(1) \to \cdots \to \mathbb{Z}_2)^\rho & \longrightarrow & P & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z}_2 & \longrightarrow & \hat{X} & \longrightarrow & \mathbf{B}^n U(1) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \xrightarrow{x} & X & \xrightarrow{g} & \mathbf{B}^n U(1)//\mathbb{Z}_2 & \xrightarrow{\mathbf{J}_{n-1}} & \mathbf{B}\mathbb{Z}_2
\end{array}$$

□

Proposition 5.4.40. *Orientifold circle 2-bundles over a smooth manifold are equivalent to the* Jandl gerbes *introduced in [SSW05].*

Proof. By prop. 4.3.39 we have that $[U(1) \to \mathbb{Z}_2]$-principal ∞-bundles on X are given by Čech cocycles relative to any good open cover of X with coefficients in the sheaf of 2-groupoids $\mathbf{B}[U(1) \to \mathbb{Z}_2]$. Writing this out in components it is straightforward to check that this coincides with the data of a Jandl gerbe (with connection) over this cover. □

Remark 5.4.41. Orientifold circle n-bundles are not $\mathbb{Z}_2$-equivariant circle n-bundles: in the latter case the orientation reversal acts by an equivalence between the bundle and its pullback along the orientation reversal, whereas for an orientifold circle n-bundle the orientation reversal acts by an equivalence to the *dual* of the pulled-back bundle.

Proposition 5.4.42. *The geometric realization, def. 3.8.2,*

$$\tilde{R} := |\mathbf{B}[U(1) \to \mathbb{Z}_2]|$$

of $\mathbf{B}[U(1) \to \mathbb{Z}]$ *is the homotopy 3-type with homotopy groups*

$$\pi_0(\tilde{R}) = 0\,;$$

$$\pi_1(\tilde{R}) = \mathbb{Z}_2\,;$$

$$\pi_2(\tilde{R}) = 0\,;$$

$$\pi_3(R') = \mathbb{Z}$$

and nontrivial action of π_1 on π_3.

Proof. By prop. 4.4.27 and the results of 4.3.8 we have

1. specifically

(a) $|\mathbf{B}\mathbb{Z}_2| \simeq B\mathbb{Z}_2$;

(b) $|\mathbf{B}^2 U(1)| \simeq B^2 U(1) \simeq K(\mathbb{Z}; 3)$;

where on the right we have the ordinary classifying spaces going by these names;

2. generally geometric realization preserves fiber sequences of nice enough objects, such as those under consideration, so that we have a fiber sequence

$$K(\mathbb{Z}, 3) \to \tilde{R} \to B\mathbb{Z}_2$$

in Top.

Since $\pi_3(K(\mathbb{Z}), 3) \simeq \mathbb{Z}$ and $\pi_1(B\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and all other homotopy groups of these two spaces are trivial, the homotopy groups of $\tilde{R}$ follow by the long exact sequence of homotopy groups associated to our fiber sequence.

Finally, since the action of $\mathbb{Z}_2$ in the crossed module is nontrivial, $\pi_1(\tilde{R})$ must act notriviall on $\pi_3(\mathbb{Z})$. It can only act nontrivial in a single way, up to homotopy. □

The space

$$R := \mathbb{Z}_2 \times \tilde{R}$$

is taken to be the coefficient object for orientifold (differential) cohomology as appearing in string theory in [DiFrMo11].

The following definition gives the differential refinement of $\mathbf{B}\mathrm{AUT}(\mathbf{B}^{n-1}U(1))$. With slight abuse of notation we will also write

$$\mathbf{B}^n U(1)//\mathbb{Z}_2 := \mathbf{B}\mathrm{AUT}(\mathbf{B}^{n-1}U(1)) .$$

Definition 5.4.43. For $n \geq 2$ write $\mathbf{B}^n U(1)_{\mathrm{conn}}//\mathbb{Z}_2$ for the smooth n-stack presented by the presheaf of n-groupoids which is given by the presheaf of crossed complexes of groupoids

$$\Omega^n(-) \times C^\infty(-, U(1)) \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}}\log)} \Omega^n(-) \times \Omega^1(-) \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})} \cdots \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})} \Omega^n(-) \times \Omega^{n-2}(-) \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})}$$

$$\xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})} \Omega^n(-) \times \Omega^{n-1}(-) \times \mathbb{Z}_2 \rightrightarrows \Omega^n(-) \;,$$

where

1. the groupoid on the right has as morphisms $(A, \sigma) : B \to B'$ between two n-forms B, B' pairs consisting of an $(n-1)$-form A and an element $\sigma \in \mathbb{Z}_2$, such that $(-1)^\sigma B' = B + dA$;

2. the bundles of groups on the left are all trivial as bundles;

3. the $\Omega^1(-) \times \mathbb{Z}_2$-action is by the $\mathbb{Z}_2$-factor only and on forms given by multiplication by ± 1 and on $U(1)$-valued functions by complex conjugation (regarding $U(1)$ as the unit circle in the complex plane).

Remark 5.4.44. A detailed discussion of $\mathbf{B}^2 U(1)_{\mathrm{conn}}//\mathbb{Z}_2$ is in [ScWaII] and [ScWaIII].

We now discuss differential cocycles with coefficients in $\mathbf{B}^n U(1)_{\mathrm{conn}}//\mathbb{Z}_2$ over $\mathbb{Z}_2$-quotient stacks / orbifolds. Let Y be a smooth manifold equipped with a smoth $\mathbb{Z}_2$-action ρ. Write $Y//\mathbb{Z}_2$ for the corresponding global orbifold and $\rho : Y//\mathbb{Z}_2 \to \mathbf{B}\mathbb{Z}_2$ for its classifying morphism, hence for the morphism that fits into a fiber sequence of smooth stacks

$$Y \longrightarrow Y//\mathbb{Z}_2 \longrightarrow \mathbf{B}\mathbb{Z}_2 \;.$$

Definition 5.4.45. An *n-orientifold* structure $\hat{G}_\rho$ on (Y, ρ) is a *ρ-twisted $\hat{\mathbf{J}}_n$-structure* on $Y//\mathbb{Z}_2$, def. 3.9.61, hence a dashed morphism in the diagram

$$\begin{array}{ccc} & & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}//\mathbb{Z}_2 \\ & \overset{\hat{G}_\rho}{\nearrow} & \downarrow{\scriptstyle \hat{\mathbf{J}}_n} \\ Y//\mathbb{Z}_2 & \xrightarrow{\rho} & \mathbf{B}\mathbb{Z}_2 \end{array} .$$

Observation 5.4.46. By corollary 5.4.39, an n-orientifold structure decomposes into an ordinary $(n+1)$-form connection $\hat{G}$ on a circle $(n+1)$-bundle over Y, subject to a $\mathbb{Z}_2$-twisted $\mathbb{Z}_2$-equivariance condition

$$\begin{array}{ccccc} Y & \xrightarrow{\hat{G}} & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ Y//\mathbb{Z}_2 & \xrightarrow{\hat{G}_\rho} & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}//\mathbb{Z}_2 & \xrightarrow{\hat{\mathbf{J}}} & \mathbf{B}\mathbb{Z}_2 \end{array} \quad (\text{bottom composite: } \rho: Y//\mathbb{Z}_2 \to \mathbf{B}\mathbb{Z}_2).$$

For $n = 1$ this reproduces, via observation 5.4.40, the *Jandl gerbes with connection* from [SSW05], hence ordinary string orientifold backgrounds, as discussed there. For $n = 2$ this reproduces background structures for membranes as discussed below in 5.4.8.7.

5.4.6 Twisted topological structures in quantum anomaly cancellation

We discuss here cohomological conditions arising from anomaly cancellation in string theory, for various σ-models. In each case we introduce a corresponding notion of topological *twisted structures* and interpret the anomaly cancellation condition in terms of these. This prepares the ground for the material in the following sections, where the differential refinement of these twisted structures is considered and the *differential* anomaly-free field configurations are derived from these.

- 5.4.6.1 – The type II superstring and twisted Spin^c-structures;
- 5.4.6.2 – The heterotic/type I superstring and twisted String-structures;
- 5.4.6.3 – The M2-brane and twisted String^{2a}-structures;
- 5.4.6.4 – The NS-5-brane and twisted Fivebrane-structures;
- 5.4.6.5 – The M5-brane and twisted $\mathrm{Fivebrane}^{2a\cup 2a}$-structures

The content of this section is taken from [SSS09c].

The physics of all the cases we consider involves a manifold X – the *target space* – or a submanifold $Q \hookrightarrow X$ thereof– a *D-brane* –, equipped with

- two principal bundles with their canonically associated vector bundles:
  - a Spin-principal bundle underlying the tangent bundle TX (and we will write TX also to denote that Spin-principal bundle),
  - and a complex vector bundle $E \to X$ – the "gauge bundle" – associated to a $SU(n)$-principal bundle or to an E_8-principal bundle with respect to a unitary representation of E_8;
- and an n-gerbe / circle $(n+1)$-bundle with class $H^{n+2}(X, \mathbb{Z})$ – the higher background gauge field – denoted $[H_3]$ or $[G_4]$ or similar in the following.

All these structures are equipped with a suitable notion of *connetions*, locally given by some differential-form data. The connection on the Spin-bundle encodes the field of gravity, that on the gauge bundle a Yang-Mills field and that on the n-gerbe a higher analog of the electromagnetic field.

The σ-model quantum field theory of a super-brane propagating in such a background (for instance the superstring, or the super 5-brane) has an effective action functional on its bosonic worldvolume fields that takes values, in general, in the fibers of the Pfaffian line bundle of a worldvolume Dirac operator, tensored with a line bundle that remembers the electric and magnetic charges of the higher gauge field. Only if this tensor product *anomaly line bundle* is trivializable is the effective bosonic action a well-defined starting point for quantization of the σ-model. Therefore the Chern-class of this line bundle over the bosonic configuration space is called the *global anomaly* of the system. Conditions on the background gauge fields that ensure that this class vanishes are called *global anomaly cancellation conditions*. These turn out to be conditions on cohomology classes that are characteristic of the above background fields. This is what we discuss now.

But moreover, the anomaly line bundle is canonicaly equipped with a *connection*, induced from the connections of the background gauge fields, hence induced from their *differential cohomology* data. The curvature 2-form of this connection over the bosonic configuration space is called the *local anomaly* of the σ-model. Conditions on the differential data of the background gauge field that canonically induce a trivialization of this 2-fom are called *local anomaly cancellation conditions*. These we consider below in section 5.4.7.3.

The phenomenon of anomaly line bundles of σ-models induced from background field differential cohomology is classical in the physics literature, if only in broad terms. A clear exposition is in [Free00]. Only recently the special case of the heterotic string σ-model for trivial background gauge bundle has been made fully precise in [Bunk09], using a certain model [Wal09] for the differential string structures that we discuss in section 5.4.7.3.

5.4.6.1 The type II superstring and twisted Spin^c-structures The open type II string propagating on a Spin-manifold X in the presence of a background B-field with class $[H_3] \in H^3(X, \mathbb{Z})$ and with endpoints fixed on a D-brane given by an oriented submanifold $Q \hookrightarrow X$, has a global worldsheet anomaly that vanishes if [FrWi] and only if [EvSa06] the condition

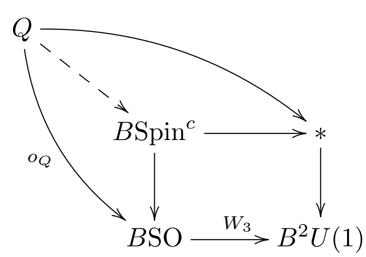
$$[W_3(Q)] + [H_3]|_Q = 0 \quad \in H^3(Q; \mathbb{Z}) \,, \tag{5.4}$$

holds. Here $[W_3(Q)]$ is the third integral Stiefel-Whitney class of the tangent bundle TQ of the brane and $[H_3]_Q$ denotes the restriction of $[H_3]$ to Q.

Notice that $[W_3(Q)]$ is the obstruction to lifting the orientation structure on Q to a Spin^c-structure. More precisely, in terms of homotopy theory this is formulated as follows, 5.2.1. There is a homotopy pullback diagram

$$\begin{array}{ccc} B\mathrm{Spin}^c & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathrm{SO} & \xrightarrow{W_3} & B^2U(1) \end{array} \tag{5.5}$$

of topological spaces, where $B\mathrm{SO}$ is the classifying space of the special orthogonal group, where $B^2U(1) \simeq K(\mathbb{Z}, 3)$ is homotopy equivalent to the Eilenberg-MacLane space that classifies degree-3 integral cohomology, and where the continuous map denoted W_3 is a representative of the universal class $[W_3]$ under this classification. This homotopy pullback exhibits the classifying space of the group Spin^c as the homotopy fiber of W_3. The universal property of the homotopy pullback says that the space of continuous maps $Q \to B\mathrm{Spin}^c$ is the same (is homotopy equivalent to) the space of maps $o_Q : Q \to B\mathrm{SO}$ that are equipped with a homotopy from the composite $Q \xrightarrow{o_Q} B\mathrm{SO} W_3 \longrightarrow B^3U(1)$ to the trivial cocycle $Q \to * \to B^3U(1)$. In other words, for every choice of homotopy filling the outer diagram of



there is a contractible space of choices for the dashed arrow such that everything commutes up to homotopy. Since a choice of map $o_Q : Q \to B\mathrm{SO}$ is an *orienation structure* on Q, and a choice of map $Q \to B\mathrm{Spin}^c$

is a Spin^c-*structure*, this implies that $[W_3(o_Q)]$ is the obstruction to the existence of a Spin^{c}structure on Q (equipped with o_Q.

Moreover, since Q is a manifold, the functor $\mathrm{Maps}(Q, -)$ that forms mapping spaces out of Q preserves homotopy pullbacks. Since $\mathrm{Maps}(Q, B\mathrm{SO})$ is the *space* of orientation structures, we can refine the discussion so far by noticing that the *space of* Spin^c-*structures on* Q, $\mathrm{Maps}(Q, B\mathrm{Spin}^c)$, is itself the homotopy pullback in the diagram

$$\begin{array}{ccc} \mathrm{Maps}(Q, B\mathrm{Spin}^c) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Maps}(Q, B\mathrm{SO}) & \xrightarrow{\mathrm{Maps}(Q,W_3)} & \mathrm{Maps}(Q, B^2U(1)) \end{array} . \tag{5.6}$$

A variant of this characterization will be crucial for the definition of (spaces of) *twisted* such structures below.

These kinds of arguments, even though elementary in homotopy theory, are of importance for the interpretation of anomaly cancellation conditions that we consider here. Variants of these arguments (first for other topological structures, then with twists, then refined to smooth and differential structures) will appear over and over again in our discussion

So in the case that the class of the B-field vanishes on the D-brane, $[H_3]|_Q = 0$, hence that its representative $H_3 : Q \to K(\mathbb{Z}, 3)$ factors through the point, up to homotopy, condition (5.4) states that the oriented D-brane Q must admit a Spin^c-structure, namely a choice of null-homotopy η in

$$\begin{array}{ccc} Q & \xrightarrow{o_Q} & B\mathrm{SO} \\ & \searrow^{H_3|_Q \simeq *} \;\; \swarrow\!\!\!\!\!\swarrow_{\eta} & \downarrow^{W_3} \\ & & K(\mathbb{Z},3) \end{array} . \tag{5.7}$$

(Beware that there are such homotopies filling *all* our diagrams, but only in some cases, such as here, do we want to make them explicit and given them a name.) If, generally, $[H_3]_Q$ does not necessarily vanish, then condition (5.4) still is equivalent to the existence of a homotopy η in a diagram of the above form:

$$\begin{array}{ccc} Q & \xrightarrow{o_Q} & B\mathrm{SO} \\ & \searrow^{H_3|_Q} \;\; \swarrow\!\!\!\!\!\swarrow_{\eta} & \downarrow^{W_3} \\ & & K(\mathbb{Z},3) \end{array} . \tag{5.8}$$

We may think of this as saying that η still "trivializes" $W_3(o_Q)$, but not with respect to the canonical trivial cocycle, but with respect to the given reference background cocycle $H_3|_Q$ of the B-field. Accordingly, following [Wa08], we may say that such an η exhibits not a Spin^c-structure on Q, but an $[H_3]_Q$-*twisted* Spin^c-*structure*.

For this notion to be useful, we need to say what an equivalence or homotopy between two twisted Spin^c-structures is, what a homotopy between such homotopies is, etc., hence what the *space* of twisted Spin^c-structures is. But by generalization of (5.6) we naturally have such a space.

Definition 5.4.47. For X a manifold and $[c] \in H^3(X, \mathbb{Z})$ a degree-3 cohomology class, we say that the space $W_3\mathrm{Struc}(Q)_{[c]}$ defined as the homotopy pullback

$$\begin{array}{ccc} W_3\mathrm{Struc}(Q)_{[H_3]|_Q} \lrcorner & \longrightarrow & * \\ \downarrow & & \downarrow^{c} \\ \mathrm{Maps}(Q, B\mathrm{SO}) & \xrightarrow{\mathrm{Maps}(Q,W_3)} & \mathrm{Maps}(Q, B^2U(1)) \end{array} , \tag{5.9}$$

is the *space of* $[c]$-*twisted* Spin^c-*structures* on X, where the right vertical morphism picks any representative $c : X \to B^2U(1) \simeq K(\mathbb{Z}, 3)$ of $[c]$.

In terms of this notion, the anomaly cancellation condition (5.4) is now read as encoding *existence of structure*:

Observation 5.4.48. On an oriented manifold Q, condition (5.4) precisely guarantees the existence of *$[H_3]|_Q$-twisted W_3-structure*, provided by a lift of the orientation structure o_Q on TQ through the left vertical morphism in def. 5.9.

This makes good sense, because that extra structure is the extra structure of the background field of the σ-model background, subjected to the condition of anomaly freedom. This we will see in more detail in the following examples, and then in section 5.4.7.3.

5.4.6.2 The heterotic/type I superstring and twisted String-structures The heterotic/type I string, propagating on a Spin-manifold X and coupled to a gauge field given by a Hermitean complex vector bundle $E \to X$, has a global anomaly that vanishes if the *Green-Schwarz anomaly cancellation condition* [GrSc]

$$\frac{1}{2}p_1(TX) - \mathrm{ch}_2(E) = 0 \quad \in H^4(X;\mathbb{Z}) \,. \tag{5.10}$$

holds. Here $\frac{1}{2}p_1(TX)$ is the *first fractional Pontryagin class* of the Spin-bundle, and $\mathrm{ch}_2(E)$ is the second Chern-class of E.

As before, this means that at the level of cocycles a certain homotopy exists. Here it is this homotopy which is the representative of the B-field that the string couples to.

In detail, write $\frac{1}{2}p_1 : B\mathrm{Spin} \to B^3U(1)$ for a representative of the universal first fractional Pontryagin class, prop. 5.1.5, and similarly $\mathrm{ch}_2 : B\mathrm{SU} \to B^3U(1)$ for a representative of the universal second Chern class, where now $B^3U(1) \simeq K(\mathbb{Z}, 4)$ is equivalent to the Eilenberg-MacLane space that classifies degree-4 integral cohomology. Then if $TX : X \to B\mathrm{Spin}$ is a classifying map of the Spin-bundle and $E : X \to B\mathrm{SU}$ one of the gauge bundle, the anomaly cancellation condition above says that there is a homotopy, denoted H_3, in the diagram

$$\begin{array}{ccc} X & \xrightarrow{E} & B\mathrm{SU} \\ {\scriptstyle TX}\downarrow & \swarrow_{H_3} & \downarrow{\scriptstyle \mathrm{ch}_2} \\ B\mathrm{Spin} & \xrightarrow[\frac{1}{2}p_1]{} & B^3U(1) \end{array} \,. \tag{5.11}$$

Notice that if both $\frac{1}{2}p_1(TX)$ as well as $\mathrm{ch}_2(E)$ happen to be trivial, such a homotopy is equivalently a map $H_3 : X \to \Omega B^3U(1) \simeq B^2U(1)$. So in this special case the B-field in the background of the heterotic string is a $U(1)$-gerbe, a circle 2-bundle, as in the previous case of the type II string in section 5.4.6.1. Generally, the homotopy H_3 in the above diagram exhibits the B-field as a *twisted* gerbe, whose twist is the difference class $[\frac{1}{2}p_1(TX)] - [\mathrm{ch}_2(E)]$. This is essentially the perspective adopted in [Free00].

For the general discussion of interest here it is useful to slightly shift the perspective on the twist. Recall that a *String structure*, 5.1.4, on the Spin bundle $TX : X \to B\mathrm{Spin}$ is a homotopy filling the outer square of

$$\begin{array}{ccccc} X & & \longrightarrow & & * \\ & \searrow^{\text{(dashed)}} & & \nearrow & \\ {\scriptstyle TX}\downarrow & & B\mathrm{String} & \longrightarrow & * \\ & & \downarrow & & \downarrow \\ & \searrow & B\mathrm{Spin} & \xrightarrow{\frac{1}{2}p_1} & B^3U(1) \end{array} \,,$$

or, which is equivalent by the universal property of homotopy pullbacks, a choice of dashed morphism filling the interior of this square, as indicated.

Therefore, now by analogy with (5.8), we say that a $[\mathrm{ch}_2(E)]$-*twisted string structure* is a choice of homotopy H_3 filling the diagram (5.11).

This notion of twisted string structures was originally suggested in [Wa08]. For it to be useful, we need to say what homotopies of twisted String-structures are, homotopies between these, etc. Hence we need to say what the *space* of twisted String-structures is. This is what the following definition provides, analogous to 5.9.

Definition 5.4.49. For X a manifold, and for $[c] \in H^4(X, \mathbb{Z})$ a degree-4 cohomology class, we say that the space of *c-twisted* String-*structures* on X is the homotopy pullback $\frac{1}{2}p_1\mathrm{Struc}_{[c]}(X)$ in

$$\begin{array}{ccc} \frac{1}{2}p_1\mathrm{Struc}_{[c]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle c} \\ \mathrm{Maps}(X, B\mathrm{Spin}) & \xrightarrow{\mathrm{Maps}(X, \frac{1}{2}p_1)} & \mathrm{Maps}(X, B^3U(1)) \end{array} ,$$

where the right vertical morphism picks a representative c of $[c]$.

In terms of this then, we find

Observation 5.4.50. The anomaly cancellation condition (5.10) is, for a fixed gauge bundle E, precisely the condition that ensures a lift of the given Spin-structure to a $[\mathrm{ch}_2(E)]$-twisted String-structure on X, through the left vertical morphism of def. 5.4.49.

Of course the full background field content involves more than just this topological data, it also consists of local differential form data, such as a 1-form connection on the bundles E and on TX and a connection 2-form on the 2-bundle H_3. Below in section 5.4.7.3 we identify this *differential* anomaly-free field content with a *differential* twisted String-structure.

5.4.6.3 The M2-brane and twisted String^{2a}-structures

The string theory backgrounds discussed above have lifts to 11-dimensional supergravity/M-theory, where the bosonic background field content consists of just the Spin-bundle TX as well as the C-field, which has underlying it a 2-gerbe – or *circle 3-bundle* – with class $[G_4] \in H^4(X, \mathbb{Z})$. The M2-brane that couples to these background fields has an anomaly that vanishes [Wi97a] if

$$2[G_4] = [\frac{1}{2}p_1(TX)] - 2[a(E)] \quad \in H^4(X, \mathbb{Z}) \,, \tag{5.12}$$

where $E \to X$ is an auxiliary E_8-principal bundle, whose class is defined by this condition.

Since E_8 is 15-coskeletal, this condition is equivalent to demanding that $[\frac{1}{2}p_1(TX)] \in H^4(X, \mathbb{Z})$ is further divisible by 2. In the absence of smooth or differential structure, one could therefore replace the E_8-bundle here by a circle 2-gerbe, hence by a $B^2U(1)$-principal bundle, and replace condition (5.12) by

$$2[G_4] = [\frac{1}{2}p_1(TX)] - 2[\mathrm{DD}_2] \,,$$

where $[\mathrm{DD}_2]$ is the canonical 4-class of this 2-gerbe (the "second Dixmier-Douady class"). While topologically this condition is equivalent, over an 11-dimensional X, to (5.12), the spaces of solutions of smooth refinements of these two conditions will differ, because the space of smooth gauge transformations between E_8 bundles is quite different from that of smooth gauge transformations between circle 2-bundles. In the Hořava-Witten reduction [HoWi96] of the 11-dimensional theory down to the heterotic string in 10 dimensions, this difference is supposed to be relevant, since the heterotic string in 10 dimensions sees the smooth E_3-bundle with connection.

In either case, we can understand the situation as a refinement of that described by (twisted) String-structures via a higher analogue of the passage from Spin-structures to Spin^c-structures. To that end recall prop. 5.2.4, which provides an alternative perspective on (5.5).

Due to the universal property of the homotopy pullback, this says, in particular, that a lift from an orientation structure to a Spin^c-structure is a cancelling by a Chern-class of the class obstructing a Spin-structure. In this way lifts from orientation structures to Spin^c-structures are analogous to the divisibility condition (5.12), since in both cases the obstruction to a further lift through the Whitehead tower of the orthogonal group is absorbed by a universal "unitary" class.

In order to formalize this we make the following definition.

Definition 5.4.51. For G some topological group, and $c : BG \to K(\mathbb{Z}, 4)$ a universal 4-class, we say that String^c is the loop group of the homotopy pullback

$$\begin{array}{ccc} B\mathrm{String}^c & \longrightarrow & BG \\ \downarrow & & \downarrow^{c} \\ B\mathrm{Spin} & \xrightarrow{\frac{1}{2}p_1} & B^3U(1) \end{array}$$

of c along the first fractional Pontryagin class.

For instance for $c = \mathrm{DD}_2$ we have that a Spin-structure lifts to a $\mathrm{String}^{2\mathrm{DD}_2}$-structure precisely if $\frac{1}{2}p_1$ is further divisible by 2. Similarly, with $a : BE_8 \to B^3U(1)$ the canonical universal 4-class on E_8-bundles and X a manifold of dimension $\mathrm{dim}X \leq 14$ we have that a Spin-structure on X lifts to a String^{2a}-structure precisely if $\frac{1}{2}p_1$ is further divisible by 2.

$$\begin{array}{ccccc} & & B\mathrm{String}^{2a} & \longrightarrow & BE_8 \\ & \nearrow & \downarrow & \searrow^{\frac{1}{4}p_1} & \downarrow^{2a} \\ X & \longrightarrow & B\mathrm{Spin} & \xrightarrow{\frac{1}{2}p_1} & B^3U(1) \end{array} \quad . \tag{5.13}$$

Using this we can now reformulate the anomaly cancellation condition (5.12) as follows.

Definition 5.4.52. For X a manifold and for $[c] \in H^4(X, \mathbb{Z})$ a cohomology class, the space $(\frac{1}{2}p_1 - 2a)\mathrm{Struc}_{[c]}(X)$ of $[c]$-*twisted* String^{2a}-*structures* on X is the homotopy pullback

$$\begin{array}{ccc} (\frac{1}{2}p_1 - 2a)\mathrm{Struc}_{[c]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow^{c} \\ \mathrm{Maps}(X, B\mathrm{Spin} \times E_8) & \xrightarrow{\frac{1}{2}p_1 - 2a} & \mathrm{Maps}(X, B^3U(1)) \end{array} ,$$

where the right vertical map picks a cocycle c representing the class $[c]$.

In terms of this definition, we have

Observation 5.4.53. Condition (5.12) is precisely the condition guaranteeing a lift of the given Spin- and the given E_8-principal bundle to a $[G_4]$-*twisted* String^{2a}-*structure* along the left vertical map from def. 5.4.52.

There is a further variation of this situation, that is of interest. In the Hořava-Witten reduction of this situation in 11 dimensions down to the sitation of the heterotic string in 10 dimensions, X has a boundary, $Q := \partial X \hookrightarrow X$, and there is a boundary condition on the C-field, saying that the restriction of its 4-class to the boundary has to vanish,

$$[G_4]|_Q = 0 \, .$$

This implies that over Q the anomaly-cancellation conditon (5.12) becomes

$$[\frac{1}{2}p_1(TX)]|_Q = 2[a(E)]|_Q \quad \in H^4(Q, \mathbb{Z}) \, .$$

Notice that this is the Green-Schwarz anomaly cancellation condition (5.10) of the heterotic string, but refined by a further cohomological divisibility condition. The following statement says that this may equivalently be reformulated in terms of String^{2a} structures.

Proposition 5.4.54. *For $E \to X$ a fixed E_8-bundle, we have an equivalence*

$$\mathrm{Maps}(X, B\mathrm{String}^{2a})|_E \simeq (\frac{1}{2}p_1)\mathrm{Struc}(X)_{[2a(E)]}$$

between, on the right, the space of $[2a(E)]$-twisted String*-structures from def. 5.4.49, and, on the left, the space of* String^{2a}*-structures with fixed class* $2a$*, hence the homotopy pullback* $\mathrm{Maps}(X, B\mathrm{String}^{2a}) \times_{\mathrm{Maps}(X,BE_8)} \{E\}$.

Proof. Consider the diagram

$$\begin{array}{ccc}
\mathrm{Maps}(X, \mathrm{String}^{2a})|_E & \longrightarrow & * \\
\downarrow & & \downarrow^{E} \\
\mathrm{Maps}(X, \mathrm{String}^{2a}) & \longrightarrow & \mathrm{Maps}(X, BE_8) \\
\downarrow & & \downarrow^{\mathrm{Maps}(X,2a)} \\
\mathrm{Maps}(X, B\mathrm{Spin}) & \xrightarrow{\mathrm{Maps}(X,\frac{1}{2}p_1)} & \mathrm{Maps}(X, B^3U(1))
\end{array}$$

The top square is a homotopy pullback by definition. Since $\mathrm{Maps}(X, -)$ preserves homotopy pullbacks (for X a manifold, hence a CW-complex), the bottom square is a homotopy pullback by definition 5.4.51. Therefore, by the pasting law, also the total rectangle is a homotopy pullback. With def. 5.4.49 this implies the claim. □

Therefore the boundary anomaly cancellation condition for the M2-brane has the following equivalent formulation.

Observation 5.4.55. For X a Spin-manifold equipped with a complex vector bundle $E \to X$, condition (5.4.6.3) precisely guarantees the existence of a lift to a String^{2a}-structure through the left vertical map in the proof of prop. 5.4.54.

5.4.6.4 The NS-5-brane and twisted Fivebrane-structures

The magnetic dual of the (heterotic) string is the NS-5-brane. Where the string is electrically charged under the B_2-field with class $[H_3] \in H^3(X, \mathbb{Z})$, the NS-5-brane is electrically charged under the B_6-field with class $[H_7] \in H^7(X, \mathbb{Z})$ [Ch81]. In the presence of a String-structure, hence when $[\frac{1}{2}p_1(TX)] = 0$, the anomaly of the 5-brane σ-model vanishes [SaSe85] [GaNi85] if the background fields satisfy

$$[\frac{1}{6}p_2(TX)] = 8[\mathrm{ch}_4(E)] \quad \in H^8(X, \mathbb{Z}) \,, \tag{5.14}$$

where $\frac{1}{6}p_2(TX)$ is the second fractional Pontryagin class of the String-bundle TX.

It is clear now that a discussion entirely analogous to that of section 5.4.6.2 applies. For the untwisted case the following terminology was introduced in [SSS09b].

Definition 5.4.56. Write Fivebrane for the loop group of the homotopy fiber BFivebrane of a representative $\frac{1}{6}p_2$ of the universal second fractional Pontryagin class

$$\begin{array}{ccc}
B\mathrm{Fivebrane} & \longrightarrow & * \\
\downarrow & & \downarrow \\
B\mathrm{String} & \xrightarrow{\frac{1}{6}p_2} & B^7U(1)
\end{array} .$$

In direct analogy with def. 5.4.49 we therefore have the following notion.

Definition 5.4.57. For X a manifold and $[c] \in H^8(X, \mathbb{Z})$ a class, we say that the *space of* $[c]$*-twisted* Fivebrane-*structures* on X, denoted $(\frac{1}{6}p_2)\mathrm{Struc}_{[c]}(X)$, is the homotopy pullback

$$\begin{array}{ccc} (\frac{1}{6}p_2)\mathrm{Struc}_{[c]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow^{c} \\ \mathrm{Maps}(X, B\mathrm{String}) & \xrightarrow{\mathrm{Maps}(X, \frac{1}{6}p_2)} & \mathrm{Maps}(X, B^7U(1)) \end{array},$$

In terms of this we have

Observation 5.4.58. For X a manifold with String-structure and with a background gauge bundle $E \to X$ fixed, condition (5.14) is precisely the condition for the existence of $[8\,\mathrm{ch}(E)]$*-twisted* Fivebrane-*structure* on X.

5.4.6.5 The M5-brane and twisted $\mathrm{Fivebrane}^{2a\cup 2a}$-structures

The magnetic dual of the M2-brane is the M5-brane. Where the M2-brane is electrically charged under the C_3-field with class $[G_4] \in H^4(X, \mathbb{Z})$, the M5-brane is electrically charged under the dual C_6-field with class $[G_8] \in H^8(X, \mathbb{Z})$.

If X admits a String-structure, then one finds a relation for the background fields analogous to (5.12) which reads

$$8[G_8] = 4[a(E)] \cup [a(E)] - [\frac{1}{6}p_2(TX)]\,. \qquad (5.15)$$

The Fivebrane-analog of Spin^c is then the following.

Definition 5.4.59. For G a topological group and $[c] \in H^8(BG, \mathbb{Z})$ a universal 8-class, we say that $\mathrm{Fivebrane}^c$ is the loop group of the homotopy pullback

$$\begin{array}{ccc} B\mathrm{Fivebrane}^c & \longrightarrow & BG \\ \downarrow & & \downarrow^{c} \\ B\mathrm{String} & \xrightarrow{\frac{1}{6}p_2} & B^3U(1) \end{array}.$$

In analogy with def. 5.4.52 we have a notion of twisted $\mathrm{Fivebrane}^c$-structures.

Definition 5.4.60. For X a manifold and for $[c] \in H^8(X, \mathbb{Z})$ a cohomology class, the space $(\frac{1}{6}p_2 - 2a \cup 2a)\mathrm{Struc}_{[c]}(X)$ of $[c]$-*twisted* $\mathrm{Fivebrane}^{2a\cup 2a}$-*structures* on X is the homotopy pullback

$$\begin{array}{ccc} (\frac{1}{6}p_2 - 2a \cup 2a)\mathrm{Struc}_{[c]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow^{c} \\ \mathrm{Maps}(X, B\mathrm{String} \times E_8) & \xrightarrow{\frac{1}{6}p_2 - 2a\cup 2a} & \mathrm{Maps}(X, B^7U(1)) \end{array},$$

where the right vertical map picks a cocycle c representing the class $[c]$.

In terms of these notions we thus see that

Observation 5.4.61. Over a manifold X with String-structure and with a fixed gauge bundle E, condition (5.15) is precisely the condition that guarantees existence of a lift to $[8G_8]$-*twisted* $\mathrm{Fivebrane}^{2a\cup 2a}$-*structure* through the left vertical morphism in def. 5.4.60.

5.4.7 Twisted differential structures in quantum anomaly cancellation

We discuss now the differential refinements of the twisted topological structures from 5.4.6.

This section draws from [SSS09c].

5.4.7.1 Twisted differential $\mathbf{c}_1$-structures We discuss the differential refinement $\hat{\mathbf{c}}_1$ of the universal first Chern class, indicated before in 1.2.14.1. The corresponding $\hat{\mathbf{c}}_1$-structures are simply $\mathrm{SU}(n)$-principal connections, but the derivation of this fact may be an instructive warmup for the examples to follow.

For any $n \in \mathbb{N}$, let $\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1)$ in $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ be the canonical representative of the universal smooth first Chern class, described in 1.2.106. In terms of the standard presentations $\mathbf{B}U(n)_{\mathrm{ch}}$, $\mathbf{B}U(1)_{\mathrm{ch}} \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ of its domain and codomain from prop. 4.4.19 this is given by the determinant function, which over any $U \in \mathrm{CartSp}$ sends

$$\det : C^\infty(U, U(n)) \to C^\infty(U, U(1)) .$$

Write $\mathbf{B}U(n)_{\mathrm{conn}}$ for the differential refinement from prop. 1.2.78. Over a test space $U \in \mathrm{CartSp}$ the set of objects is the set of $\mathfrak{u}(n)$-valued differential forms

$$\mathbf{B}U(n)_{\mathrm{conn}}(U)_0 = \Omega^1(U, \mathfrak{u}(n))$$

and the set of morphisms is that of smooth $U(n)$-valued differential forms, acting by gauge transformations on the $\mathfrak{u}(n)$-valued 1-forms

$$\mathbf{B}U(n)_{\mathrm{conn}}(U)_1 = \Omega^1(U, \mathfrak{u}(n)) \times C^\infty(U, U(n)) .$$

Proposition 5.4.62. *The smooth universal first Chern class has a differential refinement*

$$\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$$

given on $\mathfrak{u}(n)$-valued 1-forms by taking the trace

$$\mathrm{tr} : \mathfrak{u}(n) \to \mathfrak{u}(1) .$$

The existence of this refinement allows us to consider differential and twisted differential $\hat{\mathbf{c}}_1$-structures.

Lemma 5.4.63. *There is an ∞-pullback diagram*

$$\begin{array}{ccc} \mathbf{B}\mathrm{SU}(n)_{\mathrm{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}U(n)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$$

in $\mathrm{Smooth}\infty\mathrm{Grpd}$.

Proof. We use the factorization lemma, 2.3.9, to resolve the right vertical morphism by a fibration

$$\mathbf{E}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$$

in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. This gives that an object in $\mathbf{E}U(1)_{\mathrm{conn}}$ over some test space U is a morphism of the form $0 \xrightarrow{g} g^{-1} d_U g$ for $g \in C^\infty(U, U(1))$, and a morphism in $\mathbf{E}U(1)_{\mathrm{conn}}$ is given by a commuting diagram

$$\mathbf{E}U(1)_{\mathrm{conn}} = \left\{ \begin{array}{ccc} & 0 & \\ {}^{g_1}\swarrow & & \searrow^{g_2} \\ g_1^{-1} d_U g_1 & \xrightarrow{h} & g_2^{-1} d_U g_2 \end{array} \right\} ,$$

where on the right we have $h \in C^\infty(U, U(1))$ such that $h g_1 = g_2$. The morphism to $\mathbf{B}U(1)_{\mathrm{conn}}$ is given by the evident projection onto the lower horizontal part of these triangles.

Then the ordinary 1-categorical pullback of $\mathbf{E}U(1)_{\mathrm{conn}}$ along $\hat{\mathbf{c}}_1$ yields the smooth groupoid $\hat{\mathbf{c}_1}^* \mathbf{E}U(1)_{\mathrm{conn}}$ given over any test space U as follows.

- objects are pairs consisting of a $\mathfrak{u}(n)$-valued 1-form $A \in \Omega^1(U, \mathfrak{u}(n))$ and a smooth function $\rho \in C^\infty(U, U(1))$ such that
$$\mathrm{tr} A = \rho^{-1} d\rho\,;$$
- morphisms $g : (A_1, \rho_1) \to (A_2, \rho_2)$ are labeled by a smooth function $g \in C^\infty(U, U(n))$ such that $A_2 = g^{-1}(A_1 + d_U)g$.

Therefore there is a canonical functor

$$\mathbf{B}\mathrm{SU}(n)_{\mathrm{conn}} \to \hat{\mathbf{c}_1}^* \mathbf{E}U(1)_{\mathrm{conn}}$$

induced from the defining inclusion $\mathrm{SU}(n) \to \mathrm{U}(n)$, which hits precisely the objects for which ρ is the constant function on $1 \in \mathrm{U}(1)$ and which is a bijection to the morphisms between these objects, hence is full and faithful. The functor is also essentially surjective, since every 1-form of the form $h^{-1}dh$ is gauge equivalent to the identically vanishing 1-form. Therefore it is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. By prop. 2.3.13 this proves the claim. □

Proposition 5.4.64. *For X a smooth manifold, we have an ∞-pullback of smooth groupoids*

$$\begin{array}{ccc} \mathrm{SU}(n)\mathrm{Bund}_\nabla(X) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{U}(n)\mathrm{Bund}_\nabla(X) & \xrightarrow{\hat{\mathbf{c}}_1} & \mathrm{U}(1)\mathrm{Bund}_\nabla(X) \end{array} .$$

Proof. This follows from lemma 5.4.63 and the facts that for a Lie group G we have $\mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}}) \simeq G\mathrm{Bund}_\nabla(X)$ and that the hom-functor $\mathbf{H}(X, -)$ preserves ∞-pullbacks. □

5.4.7.2 Twisted differential spin^c**-structures** As opposed to the Spin-group, which is a $\mathbb{Z}_2$-extension of the special orthogonal group, the Spin^c-group, def. 5.2.3, is a $U(1)$-extension of SO. This means that twisted Spin^c-structures have interesting smooth refinements. These we discuss here.

Two standard properties of Spin^c are the following (see [LaMi89]).

Observation 5.4.65. There is a short exact sequence

$$U(1) \to \mathrm{Spin}^c \to \mathrm{SO}$$

of Lie groups, where the first morphism is the canonical inclusion.

Proposition 5.4.66. *There is a fiber sequence*

$$B\mathrm{Spin}^c(n) \to B\mathrm{SO}(n) \stackrel{W_3}{\to} K(\mathbb{Z}, 3)$$

of classifying spaces in Top*, where W_3 is a representative of the universal third integral Stiefel-Whitney class.*

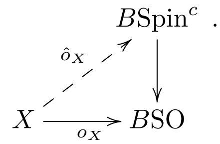
Here W_3 is a classical definition, but, as we will show below, the reader can think of it as being defined as the geometric realization of the smooth characteristic class $\mathbf{W}_3$ from example 1.2.112. Before turning to that, we record the notion of twisted structure induced by this fact:

Definition 5.4.67. For X an oriented manifold of dimension n, a Spin^c-*structure* on X is a trivialization

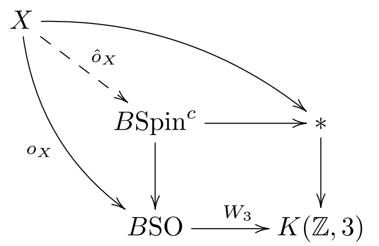
$$\eta : * \stackrel{\simeq}{\to} W_3(o_X)\,,$$

where $o_X : X \to B\mathrm{SO}$ is the given orientation structure.

Observation 5.4.68. This is equivalently a lift $\hat{o}_X$ of o_X:



Proof. By prop. 5.4.66 and the univsersal property of the homotopy pullback:



□

From the general reasoning of twisted cohomology, def. 3.6.225, in the language of twisted **c**-structures, def. 3.9.61, we are therefore led to consider the following.

Definition 5.4.69. The ∞-groupoid of *twisted* spin^c*-structures* on X is $W_3\mathrm{Struc}_{\mathrm{tw}}(X)$.

Remark 5.4.70. It follows from the definition that twisted spin^c-structures over an orientation structure o_X, def. 5.1.2, are naturally identified with equivalences (homotopies)

$$\eta : c \stackrel{\simeq}{\to} W_3(o_X)\,,$$

where $c \in \infty\mathrm{Grpd}(X, B^2U(1))$ is a given twisting cocycle.

In this form twisted spin^c-structures have been considered in [Do06] and in [Wa08]. We now establish a smooth refinement of this situation.

Observation 5.4.71. There is an essentially unique lift in Smooth∞Grpd of W_3 through the geometric realization

$$|-| : \mathrm{Smooth}\infty\mathrm{Grpd} \stackrel{\Pi}{\to} \infty\mathrm{Grpd} \stackrel{\simeq}{\to} \mathrm{Top}$$

(discussed in 4.4.4) of the form

$$\mathbf{W}_3 : \mathbf{B}\mathrm{SO} \to \mathbf{B}^2U(1)\,,$$

where $\mathbf{B}\mathrm{SO}$ is the delooping of the Lie group SO in Smooth∞Grpd and $\mathbf{B}^2U(1)$ that of the smooth circle 2-group, as in 4.4.2.

Proof. This is a special case of theorem 4.4.33. □

Theorem 5.4.72. *In* Smooth∞Grpd *we have a fiber sequence of the form*

$$\mathbf{B}\mathrm{Spin}^c \to \mathbf{B}\mathrm{SO} \stackrel{\mathbf{W}_3}{\to} \mathbf{B}^2U(1)\,,$$

which refines the sequence of prop. 5.4.66.

We consider first a lemma.

Lemma 5.4.73. *A presentation of the essentially unique smooth lift of W_3 from observation 5.4.71, is given by the morphism of simplicial presheaves*

$$\mathbf{W}_3 : \mathbf{B}\mathrm{SO}_{\mathrm{ch}} \stackrel{\mathrm{w}_2}{\to} \mathbf{B}^2\mathbb{Z}_2 \stackrel{\boldsymbol{\beta}_2}{\to} \mathbf{B}^2U(1)_{\mathrm{ch}} ,$$

where the first morphism is that of example 1.2.110 and where the second morphism is the one induced from the canonical subgroup embedding.

Proof. The bare Bockstein homomorphism is presented, by example 1.2.111, by the ∞-anafunctor

$$\begin{array}{ccccc} \mathbf{B}^2(\mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{Z}) & \longrightarrow & \mathbf{B}^2(\mathbb{Z} \to 1) & = & \mathbf{B}^3\mathbb{Z} \\ \downarrow^{\simeq} & & & & \\ \mathbf{B}^2\mathbb{Z}_2 & & & & \end{array} .$$

Accordingly we need to consider the lift of the the morphism

$$\boldsymbol{\beta}_2 : \mathbf{B}^2\mathbb{Z}_2 \to \mathbf{B}^2U(1)$$

induced form subgroup inclusion to to a comparable ∞-anafunctor. This is accomplished by

$$\begin{array}{ccc} \mathbf{B}^2(\mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{Z}) & \stackrel{\hat{\boldsymbol{\beta}}_2}{\longrightarrow} & \mathbf{B}^2(\mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{R}) \\ \downarrow^{\simeq} & & \downarrow^{\simeq} \\ \mathbf{B}^2\mathbb{Z}_2 & \stackrel{\beta_{\mathbf{2}}}{\longrightarrow} & \mathbf{B}^2U(1) \end{array} .$$

Since $\mathbb{R}$ is contractible, we have indeed under geometric realization, 4.3.4, an equivalence

$$\begin{array}{ccc} |\mathbf{B}^2(\mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{Z})| & \stackrel{|\hat{\boldsymbol{\beta}}_2|}{\longrightarrow} & |\mathbf{B}^2(\mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{R})| \\ \downarrow^{\simeq} & & \downarrow^{\simeq} \\ |\mathbf{B}^2(\mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{Z})| & \longrightarrow & |\mathbf{B}^2(\mathbb{Z} \to 1)| \\ \downarrow^{\simeq} & & \downarrow^{\simeq} \\ |\mathbf{B}^2\mathbb{Z}_2| & \stackrel{|\beta_2|}{\longrightarrow} & |\mathbf{B}^3\mathbb{Z}| \end{array} ,$$

where $|\beta_2|$ is the geometric realization of β_2, according to definition 4.3.24.

□

Proof of theorem 5.4.72. Consider the pasting diagram in Smooth∞Grpd

$$\begin{array}{ccccc} \mathbf{B}\mathrm{Spin}^c & \longrightarrow & \mathbf{B}U(1) & \longrightarrow & * \\ \downarrow & & \downarrow^{\mathbf{c}_1 \bmod 2} & & \downarrow \\ \mathbf{B}\mathrm{Spin} & \stackrel{\mathbf{w}_2}{\longrightarrow} & \mathbf{B}^2\mathbb{Z}_2 & \stackrel{\boldsymbol{\beta}_2}{\longrightarrow} & \mathbf{B}^2U(1) \end{array} .$$

The square on the right is an ∞-pullback by prop. 4.4.38. The square on the left is an ∞-pullback by proposition 5.2.4. Therefore by the pasting law 2.3.2 the total outer rectanle is an ∞-pullback. By lemma 5.4.73 the composite bottom morphism is indeed the smooth lift $\mathbf{W}_3$ from observation 5.4.71. □
Therefore we are entitled to the following smooth refinement of def. 5.4.69.

Remark 5.4.74. $\mathbf{B}\mathrm{Spin}^c$ is the moduli stack of Spin^c-structures, or, equivalently Spin^c-principal bundles.

Definition 5.4.75. For any $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$, the 1-groupoid of smooth *twisted* spin^c*-structures* $\mathbf{W}_3\mathrm{Struc}_{\mathrm{tw}}(X)$ is the homotopy pullback

$$\begin{array}{ccc} \mathbf{W}_3\mathrm{Struc}_{\mathrm{tw}}(X) & \longrightarrow & H^3(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}\infty\mathrm{Grpd}(X,\mathbf{B}\mathrm{SO}) & \xrightarrow{\mathbf{W}_3} & \mathrm{Smooth}\infty\mathrm{Grpd}(X,\mathbf{B}^2U(1)) \end{array} .$$

We briefly discuss an application of smooth twisted spin^c-structures in physics.

Remark 5.4.76. The action functional of the σ-model of the open type II superstring on a 10-dimensional target X has in general an anomaly, in that it is not a function, but just a section of a possibly non-trivial line bundle over the bosonic configuration space. In [FrWi] it was shown that in the case that the D-branes $Q \hookrightarrow X$ that the open string ends on carry a rank-1 Chan-Paton bundle, this anomaly vanishes precisely if this Chan-Paton bundle is a twisted line bundle exhibiting an equivalence $\mathbf{W}_3(\mathbf{o}_Q) \simeq H|_Q$ between the lifting gerbe of the spin^c-structure and the restriction of the background Kalb-Ramond 2-bundle to Q. By the above discussion we see that this is precisely the datum of a smooth twisted spin^c-structure on Q, where the Kalb-Ramond field serves as the twist. Below in 5.4.7.3.2 we shall see that the quantum anomaly cancellation for the closed *heterotic* superstring is analogously given by twisted string-structures, which follow the same general pattern of twisted $\mathbf{c}$-structures, but in one degree higher.

But in general this quantum anomaly cancellation involves twists mediated by a higher rank twisted bundle. This situation we turn to now.

Definition 5.4.77. For X equipped with orientation structure o_X, def. 5.1.2, and $c \in \mathbf{H}(X, \mathbf{B}^2U(1))$ a twisting circle 2-bundle, we say that the 2-groupoid of *weakly c-twisted* spin^c*-structures* on X is $(W_3(o_X)-c)$-twisted cohomology with respect to the morphism $\mathbf{c} : \mathbf{B}PU \to \mathbf{B}^2U(1)$ discussed in 4.4.8.

Remark 5.4.78. By the discussion in 4.4.8 in weakly twisted spin^c-structure the two cocycles $W_3(o_X)$ and c are not equivalent, but their difference is an *n-torsion* class (for some n) in $H^3(X,\mathbb{Z})$ which twists a unitary rank-n vector bundle on X

Remark 5.4.79. By a refinement of the discussion of [FrWi] in [Ka99] this structure is precisely what removes the quantum anomaly from the action functional of the type II superstring on oriented D-branes that carry a rank n Chan-Paton bundle. A review is in [La09].

Notice that for $i : Q \to X$ a Spin^c-D-brane inclusion into spacetimes X, the 2-groupoid of B-field and brane gauge field bundles is the *relative* $(\mathbf{B}\mathrm{PU} \to \mathbf{B}^2U(1))$-cohomology on i, according to def. 3.6.277.

5.4.7.3 Twisted differential string structures We consider now the obstruction theory for lifts through the smooth and differential refinement, from 5.1, of the Whitehead tower of O.

Definition 5.4.80. For X a Riemannian manifold, equipping it with

1. orientation
2. topological spin structure
3. topological string structure
4. topological fivebrane structure

means equipping it with choices of (homotopy classes of) lifts of the classifying map $TX : X \to BO$ of its tangent bundle through the respective steps of the Whitehead tower of BO

$$
\begin{array}{ccccl}
 & & B\mathrm{Fivebrane} & & \text{fivebrane structure} \quad . \\
 & \nearrow & \downarrow & & \\
 & & B\mathrm{String} & & \text{string structure} \\
 & \nearrow & \downarrow & & \\
 & & B\mathrm{Spin} & & \text{spin structure} \\
 & \nearrow & \downarrow & & \\
 & & B\mathrm{SO} & & \text{orientation} \\
 & \nearrow & \downarrow & & \\
X & \xrightarrow{TX} & B\mathrm{O} & & \text{Riemannian structure}
\end{array}
$$

More in detail:

1. The set (homotopy 0-type) of orientations of a Riemannian manifold is the homotopy fiber of the first Stiefel-Whitney class
$$(w_1)_* : \mathrm{Top}(X, B\mathrm{O}) \to \mathrm{Top}(X, B\mathbb{Z}_2)\,.$$

2. The groupoid (homotopy 1-type) of topological spin structures of an oriented manifold is the homotopy fiber of the second Stiefel-Whitney class
$$(w_2)_* : \mathrm{Top}(X, B\mathrm{SO}) \to \mathrm{Top}(X, B^2\mathbb{Z}_2)\,.$$

3. The 3-groupoid (homotopy 3-type) of topological string structures of a spin manifold is the homotopy fiber of the first fractional Pontryagin class
$$(\frac{1}{2}p_1)_* : \mathrm{Top}(X, B\mathrm{Spin}) \to \mathrm{Top}(X, B^4\mathbb{Z})\,,$$

4. The 7-groupoid (homotopy 7-type) of topological fivebrane structures of a string manifold is the homotopy fiber of the second fractional Pontryagin class
$$(\frac{1}{6}p_2)_* : \mathrm{Top}(X, B\mathrm{String}) \to \mathrm{Top}(X, B^8\mathbb{Z})\,,$$

See [SSS09b] for background and the notion of fivebrane structure. Using the results of 5.1 we may lift this setup from discrete ∞-groupoids to smooth ∞-groupoids and discuss the twisted cohomology, 3.6.12, relative to the smooth fractional Pontryagin classes $\frac{1}{2}\mathbf{p}_1$ and $\frac{1}{6}\mathbf{p}_2$ and their differential refinements $\frac{1}{2}\hat{\mathbf{p}}_1$ and $\frac{1}{6}\hat{\mathbf{p}}_2$

Definition 5.4.81. Let $X \in \mathrm{Smooth}\infty\mathrm{Grpd}$ be any object.

1. The 2-groupoid of *smooth string structures* on X is the homotopy fiber of the lift of the first fractional Pontryagin class $\frac{1}{2}\mathbf{p}_1$ to $\mathrm{Smooth}\infty\mathrm{Grpd}$, prop. 5.1.9:
$$\mathbf{String}(X) \to \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\mathrm{Spin}) \stackrel{(\frac{1}{2}\mathbf{p}_1)}{\to} \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^3U(1))\,.$$

2. The 6-groupoid of *smooth fivebrane stuctures* on X is the homotopy fiber of the lift of the second fractional Pontryagin class $\frac{1}{6}\mathbf{p}_2$ to $\mathrm{Smooth}\infty\mathrm{Grpd}$, prop. 5.1.32:
$$\mathbf{Fivebrane}(X) \to \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\mathrm{String}) \stackrel{(\frac{1}{6}\mathbf{p}_2)}{\to} \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^7U(1))\,.$$

More generally,

1. The 2-groupoid of *smooth twisted string sructures* on X is the ∞-pullback

$$\begin{array}{ccc} \mathbf{String}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & H^3_{\mathrm{smooth}}(X, U(1)) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\mathrm{Spin})[r] & \xrightarrow{(\frac{1}{2}\mathbf{p}_1)} & \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^3 U(1)) \end{array}$$

   in ∞Grpd.

2. The 6-groupoid of *smooth twisted fivebrane stuctures* on X is the ∞-pullback

$$\begin{array}{ccc} \mathbf{Fivebrane}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & H^7_{\mathrm{smooth}}(X, U(1)) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\mathrm{String})[r] & \xrightarrow{(\frac{1}{6}\hat{\mathbf{p}}_2)} & \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^7 U(1)) \end{array}$$

   in ∞Grpd.

Finally, with $\frac{1}{2}\hat{\mathbf{p}_1}$ and $\frac{1}{4}\hat{\mathbf{p}_2}$ the differential characteristic classes, 3.9.7, we set

1. The 2-groupoid of *smooth twisted differential string sructures* on X is the ∞-pullback

$$\begin{array}{ccc} \mathbf{String}_{\mathrm{tw,diff}}(X) & \xrightarrow{\mathrm{tw}} & H^4_{\mathrm{diff}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\mathrm{Spin}_{conn})[r] & \xrightarrow{(\frac{1}{2}\hat{\mathbf{p}}_1)} & \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^3 U(1)_{conn}) \end{array}$$

   in ∞Grpd.

2. The 6-groupoid of *smooth twisted differential fivebrane stuctures* on X is the ∞-pullback

$$\begin{array}{ccc} \mathbf{Fivebrane}_{\mathrm{tw,diff}}(X) & \xrightarrow{\mathrm{tw}} & H^8_{\mathrm{diff}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) & \xrightarrow{(\frac{1}{6}\hat{\mathbf{p}}_2)} & \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}^7 U(1)_{\mathrm{conn}}) \end{array}$$

   in ∞Grpd.

The image of a twisted (differential) String/Fivebrane structure under tw is its *twist*. The restriction to twists whose underlying class vanishes we also call *geometric string structures* and *geometric fivebrane structures.*

Observation 5.4.82. 1. These ∞-pullbacks are, up to equivalence, independent of the choise of the right vertical morphism, as long as this hits precisely one cocycle in each cohomology class.

2. The restriction of the n-groupoids of twisted structures to vanishing twist reproduces the untwisted structures.

The local L_∞-algebra valued form data of differential twisted string- and fivebrane structures has been considered in [SSS09c], as we explain in 5.4.7.3.1. Differential string structures for twists with underlying trivial class (*geometric string structures*) have been considered in [Wal09] modeled on bundle 2-gerbes.

We have the following immediate consequences of the definition:

Observation 5.4.83. The spaces of choices of string structures extending a given spin structure S are as follows

- if $[\frac{1}{2}\mathbf{p}_1(S)] \neq 0$ it is empty: $\mathrm{String}_S(X) \simeq \emptyset$;
- if $[\frac{1}{2}\mathbf{p}_1(S)] = 0$ it is $\mathrm{String}_S(X) \simeq \mathbf{H}(X, \mathbf{B}^2 U(1))$.

In particular the set of equivalence classes of string structures lifting S is the cohomology set

$$\pi_0 \mathrm{String}_S(X) \simeq H^2_{\mathrm{Smooth}}(X, \mathbf{B}^2 U(1)) .$$

If X is a smooth manifold, then this is $\simeq H^3(X, \mathbb{Z})$.

Proof. Apply the pasting law for ∞-pullbacks, prop. 2.3.2 on the diagram

$$\begin{array}{ccccc} \mathrm{String}_S(X) & \longrightarrow & \mathrm{String}(X) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{S} & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}(n)) & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{H}(X, \mathbf{B}^3 U(1)) \end{array} .$$

The outer diagram defines the loop space object of $\mathbf{H}(X, \mathbf{B}^3 U(1))$. Since $\mathbf{H}(X, -)$ commutes with forming loop space objects we have

$$\mathrm{String}_S(X) \simeq \Omega \mathbf{H}(X, \mathbf{B}^3 U(1)) \simeq \mathbf{H}(X, \mathbf{B}^2 U(1)) .$$

□

Sometimes it is useful to express string structures on X in terms of circle 2-bundles/bundle gerbes on the total space of the given spin bundle $P \to X$ [Redd06]:

Proposition 5.4.84. *A smooth string structure on X over a smooth* Spin*-principal bundle $P \to X$ induces a circle 2-bundle $\hat{P}$ on P which restricted to any fiber $P_x \simeq \mathrm{Spin}$ is equivalent to the String 2-group extension* $\mathrm{String} \to \mathrm{Spin}$.

Proof. By prop. 3.6.254. □

5.4.7.3.1 L_∞-Čech cocycles for differential string structures

We use the presentation of the ∞-topos Smooth∞Grpd by the local model structure on simplicial presheaves $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ to give an explicit construction of twisted differential string structures in terms of Čech-cocycles with coefficients in L_∞-algebra valued differential forms. We will find a twisted version of the $\mathfrak{string}$-2-connections discussed above in 1.2.13.7.2.

We need the following fact from [FSS10].

Proposition 5.4.85. *The differential fractional Pontryagin class $\frac{1}{2}\hat{\mathbf{p}}_1$ is presented in* $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ *by the top morphism of simplicial presheaves in*

$$\begin{array}{ccc} \mathbf{cosk}_3 \exp(\mathfrak{so})_{\mathrm{ChW,smp}} & \xrightarrow{\exp(\mu, \mathrm{cs})} & \mathbf{B}^3 \mathbb{R}/\mathbb{Z}_{\mathrm{ChW,smp}} \\ \downarrow & & \downarrow \\ \mathbf{cosk}_3 \exp(\mathfrak{so})_{\mathrm{diff,smp}} & \xrightarrow{\exp(\mu, \mathrm{cs})} & \mathbf{B}^3 \mathbb{R}/\mathbb{Z}_{\mathrm{smp}} \\ \downarrow \simeq & & \\ \mathbf{B}\mathrm{Spin}_c & & \end{array} .$$

Here the middle morphism is the direct Lie integration of the L_∞-algebra cocycle, 4.4.14, while the top morphisms is its restriction to coefficients for ∞-connections, 4.4.17.

In order to compute the homotopy fibers of $\frac{1}{2}\hat{\mathbf{p}}_1$ we now find a resolution of this morphism $\exp(\mu, \mathrm{cs})$ by a fibration in $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. By the fact that this is a simplicial model category then also the hom of any cofibrant object into this morphism, computing the cocycle ∞-groupoids, is a fibration, and therefore, by the general natur of homotopy pullbacks, we obtain the homotopy fibers as the ordinary fibers of this fibration.

We start by considering such a factorization before differential refinement, on the underlying characteristic class $\exp(\mu)$. To that end, we replace the Lie algebra $\mathfrak{g} = \mathfrak{so}$ by an equivalent but bigger Lie 3-algebra (following [SSS09c]). We need the following notation:

- $\mathfrak{g} = \mathfrak{so}$, the special orthogonal Lie algebra (the Lie algebra of the spin group);
- $b^2\mathbb{R}$, the line Lie 3-algebra, def. 4.4.58, the single generator in degee 3 of its Chevalley-Eilenberg algebra we denote $c \in CE(b^2\mathbb{R})$, $dc = 0$.
- $\langle -, - \rangle \in \mathrm{W}(\mathfrak{g})$ is the Killing form invariant polynomial, regarded as an element of the Weil algebra of $\mathfrak{so}$;
- $\mu := \langle -, [-, -] \rangle \in \mathrm{CE}(\mathfrak{g})$, the degree 3 Lie algebra cocycle, identified with a morphism

  $$\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{CE}(b^2\mathbb{R}) : \mu$$

  of Chevalley-Eilenberg algebras; and normalized such that its continuation to a 3-form on Spin is the image in de Rham cohomology of Spin of a generator of $H^3(\mathrm{Spin}, \mathbb{Z}) \simeq \mathbb{Z}$;
- $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ is a Chern-Simons element, def. 4.4.116, interpolating between the two;
- $\mathfrak{g}_\mu$, the string Lie 2-algebra, def. 5.1.15.

Definition 5.4.86. Let $(b\mathbb{R} \to \mathfrak{g}_\mu)$ denote the L_∞-algebra whose Chevalley-Eilenberg algebra is

$$\mathrm{CE}(b\mathbb{R} \to \mathfrak{g}_\mu) = (\wedge^\bullet(\mathfrak{g}^* \oplus \langle b \rangle \oplus \langle c \rangle), d)\,,$$

with b a generator in degree 2, and c a generator in degree 3, and with differential defined on generators by

$$\begin{aligned} d|_{\mathfrak{g}^*} &= [-,-]^* \\ db &= -\mu + c\,. \\ dc &= 0 \end{aligned}$$

Observation 5.4.87. The 3-cocycle $\mathrm{CE}(\mathfrak{g}) \stackrel{\mu}{\leftarrow} \mathrm{CE}(b^2\mathbb{R})$ factors as

$$\mathrm{CE}(\mathfrak{g}) \xleftarrow{(c \mapsto \mu, b \mapsto 0)} \mathrm{CE}(b\mathbb{R} \to \mathfrak{g}) \leftarrow \xleftarrow{(c \mapsto c)} \mathrm{CE}(CE(b^2\mathbb{R}) \;: \mu\,,$$

where the morphism on the left (which is the identity when restricted to $\mathfrak{g}^*$ and acts on the new generators as indicated) is a quasi-isomorphism.

Proof. To see that we have a quasi-isomorphism, notice that the dg-algebra is somorphic to the one with generators $\{t^a, b, c'\}$ and differentials

$$\begin{aligned} d|_{\mathfrak{g}^*} &= [-,-]^* \\ db &= c' \\ dc' &= 0 \end{aligned} \quad ,$$

where the isomorphism is given by the identity on the t^as and on b and by

$$c \mapsto c' + \mu\,.$$

The primed dg-algebra is the tensor product $\mathrm{CE}(\mathfrak{g}) \otimes \mathrm{CE}(inn(b\mathbb{R}))$, where the second factor is manifestly cohomologically trivial. $\square$

The point of introducing the resolution $(b\mathbb{R} \to \mathfrak{g}_\mu)$ in the above way is that it naturally supports the obstruction theory of lifts from $\mathfrak{g}$-connections to string Lie 2-algebra 2-connections

Observation 5.4.88. The defining projection $\mathfrak{g}_\mu \to \mathfrak{g}$ factors through the above quasi-isomorphism $(b\mathbb{R} \to \mathfrak{g}_\mu) \to \mathfrak{g}$ by the canonical inclusion

$$\mathfrak{g}_\mu \to (b\mathbb{R} \to \mathfrak{g}_\mu)\,,$$

which dually on CE-algebras is given by

$$t^a \mapsto t^a$$
$$b \mapsto -b$$
$$c \mapsto 0\,.$$

In total we are looking at a convenient presentation of the long fiber sequence of the string Lie 2-algebra extension:

$$\begin{array}{ccccc} & & (b\mathbb{R} \to \mathfrak{g}_\mu) & \longrightarrow & b^2\mathbb{R}\ . \\ & \nearrow & \downarrow{\scriptstyle \simeq} & & \\ b\mathbb{R} \longrightarrow \mathfrak{g}_\mu & \longrightarrow & \mathfrak{g} & & \end{array}$$

(The signs appearing here are just unimportant convention made in order for some of the formulas below to come out nice.)

Proposition 5.4.89. *The image under Lie integration of the above factorization is*

$$\exp(\mu) : \mathbf{cosk}_3 \exp(\mathfrak{g}) \to \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{g}_\mu) \to \mathbf{B}^3\mathbb{R}/\mathbb{Z}_c$$

where the first morphism is a weak equivalence followed by a fibration in the model structure on simplicial presheaves $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. To see that the left morphism is objectwise a weak homotopy equivalence, notice that a $[k]$-cell of $\exp(b\mathbb{R} \to \mathfrak{g}_\mu)$ is identified with a pair consisting of a based smooth function $f : \Delta^k \to \mathrm{Spin}$ and a vertical 2-form $B \in \Omega^2_{\mathrm{si,vert}}(U \times \Delta^k)$, (both suitably with sitting instants perpendicular to the boundary of the simplex). Since there is no further condition on the 2-form, it can always be extended from the boundary of the k-simplex to the interior (for instance simply by radially rescaling it smoothly to 0). Accordingly the simplicial homotopy groups of $\exp(b\mathbb{R} \to \mathfrak{g}_\mu)(U)$ are the same as those of $\exp(\mathfrak{g})(U)$. The morphism between them is the identity in f and picks $B = 0$ and is hence clearly an isomorphism on homotopy groups.

We turn now to discussing that the second morphism is a fibration. The nontrivial degrees of the lifting problem

$$\begin{array}{ccc} \Lambda[k]_i & \longrightarrow & \exp(b\mathbb{R} \to \mathfrak{g}_\mu)(U) \\ \downarrow & & \downarrow \\ \Delta[k] & \longrightarrow & \mathbf{B}^3\mathbb{R}/\mathbb{Z}_c(U) \end{array}$$

are $k = 3$ and $k = 4$.

Notice that a 3-cell of $\mathbf{B}^3\mathbb{R}/\mathbb{Z}_c(U)$ is a smooth function $c : U \to \mathbb{R}/\mathbb{Z}$ and that the morphism $\exp(b\mathbb{R} \to \mathfrak{g}_\mu) \to \mathbf{B}^3\mathbb{R}/\mathbb{Z}_c$ sends the pair (f, B) to the fiber integration $\int_{\Delta^3}(f^*\langle\theta \wedge [\theta \wedge \theta]\rangle + dB)$.

Given our lifting problem in degree 3, we have given a function $c : U \to \mathbb{R}/\mathbb{Z}$ and a smooth function (with sitting instants at the subfaces) $U \times \Lambda^3_i \to \mathrm{Spin}$ together with a 2-form B on the horn $U \times \Lambda^3_i$.

By pullback along the standard continuous retract $\Delta^3 \to \Lambda^3_i$ which is non-smooth only where f has sitting instants, we can always extend f to a smooth function $f' : U \times \Delta^3 \to \mathrm{Spin}$ with the property that $\int_{\Delta^3}(f')^*\langle\theta \wedge [\theta \wedge \theta]\rangle = 0$. (Following the general discussion at Lie integration.)

In order to find a horn filler for the 2-form component, consider any smooth 2-form with sitting instants and non-vanishing integeral on Δ^2, regarded as the missing face of the horn. By multiplying it with a suitable smooth function on U we can obtain an extension $\tilde{B} \in \Omega^3_{\mathrm{si,vert}}(U \times \partial\Delta^3)$ of B to all of $U \times \partial\Delta^3$ with the property that its integral over $\partial\Delta^3$ is the given c. By Stokes' theorem it remains to extend $\tilde{B}$ to the interior of Δ^3 in any way, as long as it is smooth and has sitting instants.

To that end, we can find in a similar fashion a smooth U-parameterized family of closed 3-forms C with sitting instants on Δ^3, whose integral over Δ^3 equals c. Since by sitting instants this 3-form vanishes in a neighbourhood of the boundary, the standard formula for the Poincare lemma applied to it produces a 2-form $B' \in \Omega^2_{\mathrm{si,vert}}(U \times \Delta^3)$ with $dB' = C$ that itself is radially constant at the boundary. By construction the difference $\tilde{B} - B'|_{\partial\Delta^3}$ has vanishing surface integral. By the argument in the proof of prop. 4.4.61 it follows that the difference extends smoothly and with sitting instants to a closed 2-form $\hat{B} \in \Omega^2_{\mathrm{si,vert}}(U \times \Delta^3)$. Therefore the sum $B' + \hat{B} \in \Omega^2_{\mathrm{si,vert}}(U \times \Delta^3)$ equals B when restricted to Λ^k_i and has the property that its integral over Δ^3 equals c. Together with our extension f', this constitutes a pair that solves the lifting problem.

The extension problem in degree 4 amounts to a similar construction: by coskeletalness the condition is that for a given $c : U \to \mathbb{R}/\mathbb{Z}$ and a given vertical 2-form on $U \times \partial\Delta^3$ such that its integral equals c, as well as a function $f : U \times \partial\Delta^3 \to Spin$, we can extend the 2-form and the functionalong $U \times \partial\Delta^3 \to U \times \Delta^3$. The latter follows from the fact that $\pi_2\mathrm{Spin} = 0$ which guarantees a continuous filler (with sitting instants), and using the Steenrod-Wockel approximation theorem [Wock09] to make this smooth. We are left with the problem of extending the 2-form, which is the same problem we discussed above after the choice of $\tilde{B}$. □

We now proceed to extend this factorization to the exponentiated differential coefficients, 4.4.17. The direct idea would be to use the evident factorization of differential L_∞-cocycles of the form

$$\begin{array}{ccccc} \mathrm{CE}(\mathfrak{so}) & \longleftarrow & \mathrm{CE}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{CE}(b^2\mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{W}(\mathfrak{so}) & \longleftarrow & \mathrm{W}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{W}(b^2\mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{inv}(\mathfrak{so}) & \longleftarrow & \mathrm{inv}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{inv}(b^2\mathbb{R}) \end{array} .$$

For computations we shall find it convenient to consider this after a change of basis.

Observation 5.4.90. The Weil algebra $\mathrm{W}(b\mathbb{R} \to \mathfrak{g}_\mu)$ of $(b^2\mathbb{R} \to \mathfrak{g})$ is given on the extra shifted generators $\{r^a = \sigma t^a, h = \sigma b, g = \sigma c\}$ by

$$\begin{aligned} dt^a &= C^a{}_{bc} t^b \wedge t^c + r^a \\ dr^a &= -C^a{}_{bc} t^b \wedge r^a \\ db &= -\mu + c + h \\ dh &= \sigma\mu - g \\ dc &= g \end{aligned}$$

(where σ is the shift operator extended as a graded derivation).

Definition 5.4.91. Define $\tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{g}_\mu)$ to be the dg-algebra with the same underlying graded algebra as $\mathrm{W}(b\mathbb{R} \to \mathfrak{g}_\mu)$ but with the differential modified as follows

$$\begin{aligned} dt^a &= C^a{}_{bc} t^b \wedge t^c + r^a \\ dr^a &= -C^a{}_{bc} t^b \wedge r^a \\ db &= -\mathrm{cs} + c + h \\ dh &= \langle -, - \rangle - g \\ dc &= g \end{aligned} \quad .$$

Moreover, define $\tilde{\mathrm{inv}}(b\mathbb{R} \to \mathfrak{string})$ to be the dg-algebra

$$\tilde{\mathrm{inv}}(b\mathbb{R} \to \mathfrak{string}) := (\mathrm{inv}(\mathfrak{so}) \otimes \langle g, h\rangle)/(dh = \langle -,-\rangle - g)\,.$$

Observation 5.4.92. We have a commutative diagram of dg-algebras

$$\begin{array}{ccccc} \mathrm{CE}(\mathfrak{so}) & \xleftarrow{\simeq} & \mathrm{CE}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{CE}(b^2\mathbb{R}) \\ \uparrow && \uparrow && \uparrow \\ \mathrm{W}(\mathfrak{so}) & \xleftarrow{\simeq} & \tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{W}(b^2\mathbb{R}) \\ \uparrow && \uparrow && \uparrow \\ \mathrm{inv}(\mathfrak{so}) & \xleftarrow{\simeq} & \tilde{\mathrm{inv}}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{inv}(b^2\mathbb{R}) \end{array}$$

where $\tilde{W}(b\mathbb{R} \to \mathfrak{string}) \to W(\mathfrak{so})$ acts as

$$\begin{aligned} t^a &\mapsto t^a \\ r^a &\mapsto r^a \\ b &\mapsto 0 \\ c &\mapsto \mathrm{cs} \\ h &\mapsto 0 \\ g &\mapsto \langle -,-\rangle \end{aligned}$$

and we identify $W(b^2\mathbb{R}) = (\wedge^\bullet\langle c, g\rangle, dc = g)$. The left horizontal morphisms are quasi-isomorphisms, as indicated.

Definition 5.4.93. We write $\exp(b\mathbb{R} \to \mathfrak{string})_{\tilde{\mathrm{ChW}}}$ for the simplicial presheaf defined as $\exp(b\mathbb{R} \to \mathfrak{string})_{\mathrm{ChW}}$, but using $\mathrm{CE}(b\mathbb{R} \to \mathfrak{string}) \leftarrow \tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{string}) \leftarrow \tilde{\mathrm{inv}}(b\mathbb{R} \to \mathfrak{string})$ instead of the untwiddled version of these algebras.

Proposition 5.4.94. *Under differential Lie integration the above factorization, observation 5.4.92, maps to a factorization*

$$\exp(\mu, \mathrm{cs}) : \mathbf{cosk}_3 \exp(\mathfrak{g})_{\mathrm{ChW}} \xrightarrow{\simeq} \mathbf{cosk}_3 \exp((b\mathbb{R} \to \mathfrak{g}_\mu))_{\tilde{\mathrm{ChW}}} \to \mathbf{B}^3 U(1)_{\mathrm{ChW,ch}}$$

of $\exp(\mu, \mathrm{cs})$ *in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$*, where the first morphism is a weak equivalence and the second a fibration.*

Proof. We discuss that the first morphism is an equivalence. Clearly it is injective on homotopy groups: if a sphere of A-data cannot be filled, then also adding the (B, C)-data does not yield a filler. So we need to check that it is also surjective on homotopy groups: any two choices of (B, C)-data on a sphere are homotopic: we may interpolate B in any smooth way and then solve the equation $dB = -\mathrm{cs}(A) + C + H$ for the interpolation of C.

We now check that the second morphism is a fibration. It is itself the composite

$$\mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{g}_\mu)_{\mathrm{ChW}} \to \exp(b^2\mathbb{R})_{\mathrm{ChW}}/\mathbb{Z} \xrightarrow{\int_{\Delta^\bullet}} \mathbf{B}^3\mathbb{R}/\mathbb{Z}_{\mathrm{ChW,ch}}\,.$$

Here the second morphism is a degreewise surjection of simplicial abelian groups, hence a degreewise surjection under the normalized chain complex functor, hence is itself already a projective fibration. Therefore it is sufficient to show that the first morphism here is a fibration.

In degree $k = 0$ to $k = 3$ the lifting problems

$$\begin{array}{ccc} \Lambda[k]_i & \longrightarrow & \exp(b\mathbb{R} \to \mathfrak{g}_\mu)_{\tilde{\mathrm{ChW}}}(U) \\ \downarrow && \downarrow \\ \Delta[k] & \longrightarrow & \exp(b^2\mathbb{R})_{\mathrm{ChW}}/\mathbb{Z}(U) \end{array}$$

may all be equivalently reformulated as lifting against a cylinder $D^k \hookrightarrow D^k \times [0,1]$ by using the sitting instants of all forms.

We have then a 3-form $H \in \Omega^3_{\mathrm{si}}(U \times D^{k-1} \times [0,1])$ and differential form data (A, B, C) on $U \times D^{k-1}$ given. We may always extend A along the cylinder direction $[0,1]$ (its vertical part is equivalently a based smooth function to Spin which we may extend constantly). H has to be horizontal so is already constantly extended along the cylinder.

We can then use the kind of formula that proves the Poincaré lemma to extend B. Let $\Psi : (D^k \times [0,1]) \times [0,1] \to (D^k \times [0,1])$ be a smooth contraction. Then while $d(H - \mathrm{CS}(A) - C)$ may be non-vanishing, by horizonatlity of their curvature characteristic forms we still have that $\iota_{\partial_t} \Psi_t^* d(H - CS(A) - C)$ vanishes (since the contraction vanishes).

Therefore the 2-form

$$\tilde{B} := \int_{[0,1]} \iota_{\partial_t} \Psi_t^* (H - \mathrm{CS}(A) - C)$$

satisfies $d\tilde{B} = (H - CS(A) - C)$. It may however not coincide with our given B at $t = 0$. But the difference $B - \tilde{B}_{t=0}$ is a closed form on the left boundary of the cylinder. We may find some closed 2-form on the other boundary such that the integral around the boundary vanishes. Then the argument from the proof of the Lie integration of the line Lie n-algebra applies and we find an extension λ to a closed 2-form on the interior. The sum

$$\hat{B} := \tilde{B} + \lambda$$

then still satisfies $d\hat{B} = H - CS(A) - C$ and it coincides with B on the left boundary.

Notice that here $\tilde{B}$ indeed has sitting instants: since H, $\mathrm{CS}(A)$ and C have sitting instants they are constant on their value at the boundary in a neighbourhood perpendicular to the boundary, which means for these 3-forms in the degrees ≤ 3 that they *vanish* in a neighbourhood of the boundary, hence that the above integral is towards the boundary over a vanishing integrand.

In degree 4 the nature of the lifting problem

$$\begin{array}{ccc} \Lambda[4]_i & \longrightarrow & \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{g}_\mu)(U) \\ \downarrow & & \downarrow \\ \Delta[4] & \longrightarrow & \mathbf{B}^3\mathbb{R}/\mathbb{Z}_{\mathrm{ChW,ch}} \end{array}$$

starts out differently, due to the presence of $\mathbf{cosk}_3$, but it then ends up amounting to the same kind of argument:

We have four functions $U \to \mathbb{R}/\mathbb{Z}$ which we may realize as the fiber integration of a 3-form H on $U \times (\partial\Delta[4] \setminus \delta_i\Delta[3])$, and we have a lift to (A, B, C, H)-data on $U \times (\partial\Delta[4] \setminus \delta_i(\Delta[3]))$ (the boundary of the 4-simplex minus one of its 3-simplex faces).

We observe that we can

- always extend C smoothly to the remaining 3-face such that its fiber integration there reproduces the signed difference of the four given functions corresponding to the other faces (choose any smooth 3-form with sitting instants and with non-vanishing integral and rescale smoothly);
- fill the A-data horizonatlly due to the fact that $\pi_2(\mathrm{Spin}) = 0$.
- the C-form is already horizontal, hence already filled.

Moreover, by the fact that the 2-form B already is defined on all of $\partial\Delta[4] \setminus \delta_i(\Delta[3])$ its fiber integral over the boundary $\partial\Delta[3]$ coincides with the fiber integral of $H - \mathrm{cs}(A) - C$ over $\partial\Delta[4] \setminus \delta_i(\Delta[3])$). But by the fact that we have lifted C and the fact that $\mu(A_{\mathrm{vert}}) = \mathrm{cs}(A)|_{\Delta^3}$ is an integral cocycle, it follows that this equals the fiber integral of $C - \mathrm{cs}(A)$ over the remaining face.

Use then as above the vertical Poincaré lemma-formula to find $\tilde{B}$ on $U \times \Delta^3$ with sitting instants that satisfies the equation $dB = H - \mathrm{cs}(A) - C$ there. Then extend the closed difference $B - \tilde{B}|_0$ to a closed smooth 2-form on Δ^3. As before, the difference

$$\hat{B} := \tilde{B} + \lambda$$

is an extension of B that constitutes a lift. □

Corollary 5.4.95. *For any $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$, for any choice of differentiaby good open cover with corresponding cofibrant presentation $\hat{X} = C(\{C_i\}) \in [\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ we have that the 2-groupoids of twisted differential string structures are presented by the ordinary fibers of the morphism of Kan complexes*

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\hat{X}, \exp(\mu, \mathrm{cs}))$$

$$[\mathrm{CartSp}^{op}, \mathrm{sSet}](\hat{X}, \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{g}_\mu)_{\mathrm{ChW}}) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\hat{X}, \mathbf{B}^3 U(1)_{\mathrm{ChW}})\,.$$

over any basepoints in the connected components of the Kan complex on the right, which correspond to the elements $[\hat{\mathbf{C}}_3] \in H^4_{\mathrm{diff}}(X)$ in the ordinary differential cohomology of X.

Proof. Since $[\mathrm{CartSp}_{\mathrm{smooth}}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is a simplicial model category the morphism $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\hat{X}, \exp(\mu, \mathrm{cs}))$ is a fibration because $\exp(\mu, \mathrm{cs})$ is and $\hat{X}$ is cofibrant.

It follows from the general theory of homotopy pullbacks that the ordinary pullback of simplicial presheaves

$$\begin{array}{ccc} \mathbf{String}_{\mathrm{diff,tw}}(X) & \longrightarrow & H^4_{\mathrm{diff}}(X) \\ \downarrow & & \downarrow \\ [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\hat{X}, \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{g}_\mu)_{\mathrm{ChW}}) & \longrightarrow & [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\hat{X}, \mathbf{B}^3 U(1)_{\mathrm{ChW}}) \end{array}$$

is a presentation for the defining ∞-pullback for $\mathbf{String}_{\mathrm{diff,tw}}(X)$. □

We unwind the explicit expression for a twisted differential string structure under this equivalence. Any twisting cocycle is in the above presentation given by a Čech-Deligne-cocycle, as discussed at 4.4.16.

$$\hat{\mathbf{H}}_3 = ((H_3)_i, \cdots)$$

with local connection 3-form $(H_3)_i \in \Omega^3(U_i)$ and globally defined curvature 4-form $\mathcal{G}_4 \in \Omega^4(X)$.

Observation 5.4.96. A twisted differential string structure on X, twisted by this cocycle, is on patches U_i a morphism

$$\Omega^\bullet(U_i) \leftarrow \tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{g}_\mu)$$

in dgAlg, subject to some horizontality constraints. The components of this are over each U_i a collection of differential forms of the following structure

$$\left(\begin{array}{ll} F_\omega = & d\omega + \frac{1}{2}[\omega \wedge \omega] \\ H_3 = & \nabla B := dB + CS(\omega) - C_3 \\ \mathcal{G}_4 = & dC_3 \\ dF_\omega = & -[\omega \wedge F_\omega] \\ dH_3 = & \mathcal{G}_4 - \langle F_\omega \wedge F_\omega \rangle \\ d\mathcal{G}_4 = & 0 \end{array}\right)_i \xleftarrow{\begin{array}{ll} t^a & \mapsto \omega^a \\ r^a & \mapsto F^a_\omega \\ b & \mapsto B \\ c & \mapsto C_3 \\ h & \mapsto H_3 \\ g & \mapsto \mathcal{G}_4 \end{array}} \left(\begin{array}{ll} r^a = & dt^a + \frac{1}{2}C^a{}_{bc} t^b \wedge t^c \\ h = & db + \mathrm{cs} - c \\ g = & dc \\ dr^a = & -C^a{}_{bc} t^b \wedge r^a \\ dh = & \langle -,- \rangle - g \\ dg = & 0 \end{array}\right).$$

Here we are indicating on the right the generators and their relation in $\tilde{W}(b\mathbb{R} \to \mathfrak{g}_\mu)$ and on the left their images and the images of the relations in $\Omega^\bullet(U_i)$. This are first the definitions of the curvatures themselves and then the Bianchi identities satisfied by these.

By prop. 4.4.124 we have that for $\mathfrak{g}$ an L_∞-algebra and

$$\mathbf{B}G := \mathbf{cosk}_{n+1} \exp(\mathfrak{g})$$

the delooping of the smooth Lie n-group obtained from it by Lie integration, def. 4.4.53 the coefficient for ∞-connections on G-principal ∞-bundles is

$$\mathbf{B}G_{\mathrm{conn}} := \mathbf{cosk}_{n+1} \exp(\mathfrak{g})_{\mathrm{conn}} .$$

Proposition 5.4.97. *The 2-groupoid of entirely untwisted differential string structures, def. 5.4.81, on X (the twist being $0 \in H^4_{\mathrm{diff}}(X)$) is equivalent to that of principal 2-bundles with 2-connection over the string 2-group, def. 5.1.10, as discussed in 1.2.13.7.2:*

$$\mathrm{String}_{\mathrm{diff},\mathrm{tw}=0}(X) \simeq \mathrm{String2Bund}_\nabla(X) .$$

Proof. By 5.4.7.3.1 we compute $\mathrm{String}_{\mathrm{diff},\mathrm{tw}=0}(X)$ as the ordinary fiber of the morphism of simplicial presheaves

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{g}_\mu)) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}^3 U(1)_{\mathrm{diff}})$$

over the identically vanishing cocycle.

In terms of the component formulas of observation 5.4.96, this amounts to restricting to those cocyles for which over each $U \times \Delta^k$ the equations

$$C = 0$$
$$G = 0$$

hold. Comparing this to the explicit formulas for $\exp(b\mathbb{R} \to \mathfrak{g}_\mu)$ and $\exp(b\mathbb{R} \to \mathfrak{g}_\mu)_{\mathrm{conn}}$ in 5.4.7.3.1 we see that these cocycles are exactly those that factor through the canonical inclusion

$$\mathfrak{g}_\mu \to (b\mathbb{R} \to \mathfrak{g}_\mu)$$

from observation 5.4.88. □

5.4.7.3.2 The Green-Schwarz mechanism in heterotic supergravity Local differential form data as in observation 5.4.96 is known in theoretical physics in the context of the Green-Schwarz mechanism for 10-dimensional supergravity. We conclude with some comments on the meaning and application of this result (for background and references on the physics story see for instance [SSS09b]).

The standard action functionals of higher dimensional supergravity theories are generically *anomalous* in that instead of being functions on the space of field configurations, they are just sections of a line bundle over these spaces. In order to get a well defined action principle as input for a path-integral quantization to obtain the corresponding quantum field theories, one needs to prescribe in addition the data of a *quantum integrand*. This is a choice of trivialization of these line bundles, together with a choice of flat connection. For this to be possible the line bundle has to be trivializable and flat in the first place. Its failure to be tivializable – its Chern class – is called the *global anomaly*, and its failure to be flat – its curvature 2-form – is called its local anomaly.

But moreover, the line bundle in question is the tensor product of two different line bundles with connection. One is a Pfaffian line bundle induced from the fermionic degrees of freedom of the theory, the other is a line bundle induced from the higher form fields of the theory in the presence of higher *electric and magnetic charge*. The Pfaffian line bundle is fixed by the requirement of supersymmetry, but there is freedom in choosing the background higher electric and magnetic charge. Choosing these appropriately such as to ensure that the tensor product of the two anomaly line bundles produces a flat trivializable line bundle is called an *anomaly cancellation* by a *Green-Schwarz mechanism*.

Concretely, the higher gauge background field of 10-dimensional heterotic supergravity is the Kalb-Ramond field, which in the absence of *fivebrane magnetic charge* is modeled by a circle 2-bundle (bundle gerbe) with connection and curvature 3-form $H_3 \in \Omega^3_{\mathrm{cl}}(X)$, satisfying the higher *Maxwell equation*

$$dH_3 = 0\,.$$

Notice that we may think of a circle 2-bundle as a homotopy from the trivial circle 3-bundle to itself.

In order to cancel the relevant quantum anomaly it turns out that a magnetic background charge density is to be added to the system whose differential form representative is the difference $j_{\mathrm{mag}} := \langle F_{\nabla_{\mathrm{SU}}} \wedge F_{\nabla_{\mathrm{SU}}} \rangle - \langle F_{\nabla_{\mathrm{Spin}}} \wedge F_{\nabla_{\mathrm{Spin}}} \rangle$ between the Pontryagin forms of the Spin-tangent bundle and a given SU-gauge bundle. This modifies the above Maxwell equation locally, on a patch $U_i \subset X$ to

$$dH_i = \langle F_{A_i} \wedge F_{A_i} \rangle - \langle F_{\omega_i} \wedge F_{\omega_i} \rangle\,.$$

Comparing with prop. 5.4.96 and identifying the curvature of the twist with $\mathcal{G}_4 = \langle F_{A_i} \wedge F_{A_i} \rangle$ we see that, while such H_i can no longer be the curvature 3-form of a circle 2-bundle, it can be the local 3-form component of a *twisted* circle 3-bundle that is part of the data of a twisted differential string-structure. The above differential form equation exhibits a de Rham homotopy between the two Pontryagin forms. This is the local differential aspect of the very defnition of a twisted differential string-structure: a homotopy from the Chern-Simons circle 3-bundle of the Spin-tangent bundle to a given twisting circle 3-bundle.

For many years the anomaly cancellation for the heterotic superstring was known at the level of precision used in the physics community, based on a seminal article by Killingback. Recently [Bunk09] has given a rigorous proof in the special case that underlying topological class of the twisting gauge bundle is trivial. This proof used the model of twisted differential string structures with topologically tivial twist given in [Wal09]. This model is explicitly constructed in terms of bundle 2-gerbes and doees not exhibit the homotopy pullback property of def. 3.9.8 explicitly. However, the author shows that his model satisfies the abstract properties following from the universal property of the homotopy pullback.

When we take into account also gauge transformations of the gauge bundle, we should replace the homotopy pullback defining twisted differential string structurs this by the full homotopy pullback

$$\begin{array}{ccc} \mathrm{GSBackground}(X) & \longrightarrow & \mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}U) \\ \downarrow & & \downarrow{\scriptstyle \hat{\mathbf{c}}_2} \\ \mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}\mathrm{Spin}) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^3 U(1)) \end{array} \,.$$

The look of this diagram makes manifest how in this situation we are looking at the structures that homotopically cancel the differential classes $\frac{1}{2}\hat{\mathbf{p}}$ and $\hat{\mathbf{c}}_2$ against each other.

Since $\mathbf{H}_{dR}(X, \mathbf{B}^3 U(1))$ is abelian, we may also consider the corresponding Mayer-Vietoris sequence by realizing $\mathrm{GSBackground}(X)$ equivalently as the homotopy fiber of the difference of differential cocycles $\frac{1}{2}\hat{\mathbf{p}}_1 - \hat{\mathbf{c}}_2$.

$$\begin{array}{ccc} \mathrm{GSBackground}(X) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{H}_{\mathrm{conn}}(X, \mathbf{B}\mathrm{Spin} \times \mathbf{B}U) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1 - \hat{\mathbf{c}}_2} & \mathbf{H}_{\mathrm{dR}}(X, \mathbf{B}^4 U(1)) \end{array} \,.$$

5.4.8 The supergravity C-field

We consider a slight variant of twisted differential **c**-structures, where instead of having the twist directly in differential cohomology, it is instead first considered just in de Rham cohomology but then supplemented by a lift of the structure ∞-group.

We observe that when such a twist is by the sum of the first fractional Pontryagin class with the second Chern class, and when the second of these two steps is considered over the boundary of the base manifold, then the differental structures obtained this way exhibit some properties that a differential cohomological description of the *C_3-field* in *11-dimensional supergravity*, 5.3.3.2, is expected to have.

This section draws from [FiSaScII] and [FiSaScIII].

The supergravity C-field is subject to a certain $\mathbb{Z}_2$-twist [Wi96] [Wi97a], due to a quadratic refinement of its action functional, which we review below in 5.4.8.1. A formalization of this twist in abelian differential cohomology for fixed background spin structure has been given in [HoSi05], in terms of *differential integral Wu structures*. These we review in 5.4.8.2 and refine them from $\mathbb{Z}_2$-coefficients to circle n-bundles. Then we present a natural moduli 3-stack of C-field configurations that refines this model to nonabelian differential cohomology, generalizing it to dynamical gravitational background fields, in 5.4.8.4. We discuss a natural boundary coupling of these fields to E_8-gauge fields in 5.4.8.6.

5.4.8.1 Higher abelian Chern-Simons theories with background charge The supergravity C-field is an example of a general phenomenon of higher abelian Chern-Simons QFTs in the presence of *background charge*. This phenomenon was originally noticed in [Wi96] and then made precise in [HoSi05]. The holographic dual of this phenomenon is that of self-dual higher gauge theories, which for the supergravity C-field is the nonabelian 2-form theory on the M5-brane [FiSaScIII]. We review the idea in a way that will smoothly lead over to our refinements to nonabelian higher gauge theory in section 5.4.8.

Fix some natural number $k \in \mathbb{N}$ and an oriented manifold (compact with boundary) X of dimension $4k+3$. The gauge equivalence class of a $(2k+1)$-form gauge field $\hat{G}$ on X is an element in the differential cohomology group $\hat{H}^{2k+2}(X)$. The cup product $\hat{G} \cup \hat{G} \in \hat{H}^{4k+4}(X)$ of this with itself has a natural higher holonomy over X, denoted

$$\begin{aligned} \exp(iS(-)) : \hat{H}^{2k+2}(X) &\to U(1) \\ \hat{G} &\mapsto \exp(i \int_X \hat{G} \cup \hat{G}) \,. \end{aligned}$$

This is the exponentiated action functional for bare $(4k+3)$-dimensional abelian Chern-Simons theory. For $k=0$ this reduces to ordinary 3-dimensional abelian Chern-Simons theory. Notice that, even in this case, this is a bit more subtle that Chern-Simons theory for a simply-connected gauge group G. In the latter case all fields can be assumed to be globally defined forms. But in the non-simply-connected case of $U(1)$, instead the fields are in general cocycles in differential cohomology. If, however, we restrict attention to fields C in the inclusion $H^{2k+1}_{\mathrm{dR}}(X) \hookrightarrow \hat{H}^{2k+2}(X)$, then on these the above action reduces to the familiar expression

$$\exp(iS(C)) = \exp(i \int_X C \wedge d_{\mathrm{dR}} C) \,.$$

Observe now that the above action functional may be regarded as a *quadratic form* on the group $\hat{H}^{2k+2}(X)$. The corresponding bilinear form is the ("secondary", since X is of dimension $4k+3$ instead of $4k+4$) *intersection pairing*

$$\langle -,- \rangle : \hat{H}^{2k+2}(X) \times \hat{H}^{2k+2}(X) \to U(1)$$

$$(\hat{a}_1, \hat{a}_2) \mapsto \exp(i \int_X \hat{a}_1 \cup \hat{a}_2) \,.$$

But note that from $\exp(iS(-))$ we do *not* obtain a *quadratic refinement* of the pairing. A quadratic refinement is, by definition, a function

$$q : \hat{H}^{2k+2}(X) \to U(1)$$

(not necessarily homogenous of degree 2 as $\exp(iS(-))$ is), for which the intersection pairing is obtained via the polarization formula

$$\langle \hat{a}_1, \hat{a}_2 \rangle = q(\hat{a}_1 + \hat{a}_2) q(\hat{a}_1)^{-1} q(\hat{a}_2)^{-1} q(0) \,.$$

If we took $q := \exp(iS(-))$, then the above formula would yield not $\langle -, - \rangle$, but the square $\langle -, - \rangle^2$, given by the exponentiation of *twice* the integral.

The observation in [Wi96] was that for the correct holographic physics, we need instead an action functional which is indeed a genuine quadratic refinement of the intersection pairing. But since the differential classes in $\hat{H}^{2k+2}(X)$ refine *integral* cohomology, we cannot in general simply divide by 2 and pass from $\exp(i \int_X \hat{G} \cup \hat{G})$ to $\exp(i \int_X \frac{1}{2} \hat{G} \cup \hat{G})$. The integrand in the latter expression does not make sense in general in differential cohomology. If one tried to write it out in the "obvious" local formulas one would find that it is a functional on fields which is not gauge invariant. The analog of this fact is familiar from nonabelian G-Chern-Simons theory with simply-connected G, where also the theory is consistent only at interger *levels*. The "level" here is nothing but the underlying integral class $G \cup G$. Therefore the only way to obtain a square root of the quadratic form $\exp(iS(-))$ is to *shift it*. Here we think of the analogy with a quadratic form

$$q : x \mapsto x^2$$

on the real numbers (a parabola in the plane). Replacing this by

$$q^{\lambda} : x \mapsto x^2 - \lambda x$$

for some real number λ means keeping the shape of the form, but shifting its minimum from 0 to $\frac{1}{2}\lambda$. If we think of this as the potential term for a scalar field x then its ground state is now at $x = \frac{1}{2}\lambda$. We may say that there is a *background field* or *background charge* that pushes the field out of its free equilibrium.

To lift this reasoning to our action quadratic form $\exp(iS(-))$ on differential cocycles, we need a differential class $\hat{\lambda} \in H^{2k+2}(X)$ such that for every $\hat{a} \in H^{2k+2}(X)$ the composite class

$$\hat{a} \cup \hat{a} - \hat{a} \cup \hat{\lambda} \in H^{4k+4}(X)$$

is even, hence is divisible by 2. Because then we could define a shifted action functional

$$\exp(iS^{\lambda}(-)) : \hat{a} \mapsto \exp\left(i \int_X \frac{1}{2} (\hat{a} \cup \hat{a} - \hat{a} \cup \hat{\lambda}) \right) \,,$$

where now the fraction $\frac{1}{2}$ in the integrand does make sense. One directly sees that if this exists, then this shifted action is indeed a quadratic refinement of the intersection pairing:

$$\exp(iS^{\lambda}(\hat{a} + \hat{b})) \exp(iS^{\lambda}(\hat{a}))^{-1} \exp(iS^{\lambda}(\hat{b}))^{-1} \exp(iS^{\lambda}(0)) = \exp(i \int_X \hat{a} \cup \hat{b}) \,.$$

The condition on the existence of $\hat{\lambda}$ here means, equivalently, that the image of the underlying integral class vanishes under the map

$$(-)_{\mathbb{Z}_2} : H^{2k+2}(X, \mathbb{Z}) \to H^{2k+2}(X, \mathbb{Z}_2)$$

to $\mathbb{Z}_2$-cohomology:

$$(a)_{\mathbb{Z}_2} \cup (a)_{\mathbb{Z}_2} - (a)_{\mathbb{Z}_2} \cup (\lambda)_{\mathbb{Z}_2} = 0 \in H^{4k+4}(X, \mathbb{Z}_2) \,.$$

Precisely such a class $(\lambda)_{\mathbb{Z}_2}$ does uniquely exist on every oriented manifold. It is called the *Wu class* $\nu_{2k+2} \in H^{2k+2}(X, \mathbb{Z}_2)$, and may be *defined* by this condition. Moreover, if X is a Spin-manifold, then every second Wu class, ν_{4k}, has a pre-image in integral cohomology, hence λ does exist as required above

$$(\lambda)_{\mathbb{Z}_2} = \nu_{2k+2} \,.$$

It is given by polynomials in the Pontrjagin classes of X (discussed in section E.1 of [HoSi05]). For instance the degree-4 Wu class (for $k = 1$) is refined by the first fractional Pontrjagin class $\frac{1}{2}p_1$

$$(\tfrac{1}{2}p_1)_{\mathbb{Z}_2} = \nu_4 \,.$$

In the present context, this was observed in [Wi96] (see around eq. (3.3) there).

Notice that the equations of motion of the shifted action $\exp(iS^\lambda(\hat{a}))$ are no longer $\mathrm{curv}(\hat{a}) = 0$, but are now

$$\mathrm{curv}(\hat{a}) = \tfrac{1}{2}\mathrm{curv}(\hat{\lambda}) \,.$$

We therefore think of $\exp(iS^\lambda(-))$ as the exponentiated action functional for *higher dimensional abelian Chern-Simons theory with background charge* $\frac{1}{2}\lambda$.

With respect to the shifted action functional it makes sense to introduce the shifted field

$$\hat{G} := \hat{a} - \tfrac{1}{2}\hat{\lambda} \,.$$

This is simply a re-parameterization such that the Chern-Simons equations of motion again look homogenous, namely $G = 0$. In terms of this shifted field the action $\exp(iS^\lambda(\hat{a}))$ from above equivalently reads

$$\exp(iS^\lambda(\hat{G})) = \exp(i\int_X \tfrac{1}{2}(\hat{G}\cup\hat{G} - (\tfrac{1}{2}\hat{\lambda})^2)) \,.$$

For the case $k = 1$, this is the form of the action functional for the 7d Chern-Simons dual of the 2-form gauge field on the 5-brane first given as (3.6) in [Wi96]

In the language of twisted cohomological structures, def. 3.9.61, we may summarize this situation as follows: *In order for the action functional of higher abelian Chern-Simons theory to be correctly divisible, the images of the fields in $\mathbb{Z}_2$-cohomology need to form a* twisted Wu-structure*, [Sa11c].Therefore the fields themselves need to constitute a* twisted *λ-structure. For $k = 1$ this is a* twisted String-structure *[SSS09c] and explains the quantization condition on the C-field in 11-dimensional supergravity.*

In [HoSi05] a formalization of the above situation has been given in terms of a notion there called *differential integral Wu structures*. In the following section we explain how this follows from the notion of twisted Wu structures with the twist taken in $\mathbb{Z}_2$-coefficients. Then we refine this to a formalization to *twisted differential Wu structures* with the twist taken in smooth circle n-bundles.

5.4.8.2 Differential integral Wu structures We discuss some general aspects of smooth and differential refinements of $\mathbb{Z}_2$-valued universal characteristic classes. For the special case of *Wu classes* we show how these notions reduce to the definition of *differential integral Wu structures* given in [HoSi05]. We then construct a refinement of these structures that lifts the twist from $\mathbb{Z}_2$-valued cocycles to smooth circle n-bundles. This further refinement of integral Wu structures is what underlies the model for the supergravity C-field in section 5.4.8.

Recall from prop. 5.2.4 the characterization of Spin^c as the loop space object of the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin}^c & \longrightarrow & \mathbf{B}U(1) \\ \downarrow & & \downarrow^{\mathbf{c}_1 \,\mathrm{mod}\, 2} \\ \mathbf{B}\mathrm{SO} & \xrightarrow{\mathbf{w}_2} & \mathbf{B}^2\mathbb{Z}_2 \end{array} \,.$$

For general $n \in \mathbb{N}$ the analog of the first Chern class mod 2 appearing here is the higher Dixmier-Douady class mod 2

$$\mathbf{DD}_{\mathrm{mod}\,2} : \mathbf{B}^n U(1) \xrightarrow{\mathrm{DD}} \mathbf{B}^{n+1}\mathbb{Z} \xrightarrow{\mathrm{mod}\,2} \mathbf{B}^{n+1}\mathbb{Z}_2 \,.$$

Let now

$$\nu_{n+1} : \mathbf{B}\mathrm{SO} \to \mathbf{B}^{n+1}\mathbb{Z}_2$$

be a representative of the universal smooth *Wu class* in degree $n+1$, the ($\Pi \dashv$ Disc)-adjunct of the topological universal Wu class using that $\mathbf{B}^{n+1}\mathbb{Z}$ is discrete as a smooth ∞-groupoid, and using that $\Pi(\mathbf{B}\mathrm{SO}) \simeq B\mathrm{SO}$ is the ordinary classifying space, by prop. 4.3.30.

Definition 5.4.98. Let $\mathrm{Spin}^{\nu_{n+1}}$ be the loop space object of the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin}^{\nu_{n+1}} & \longrightarrow & \mathbf{B}\mathrm{SO} \\ \downarrow^{\boldsymbol{\nu}^{\mathrm{int}}_{n+1}} & & \downarrow^{\nu_{n+1}} \\ \mathbf{B}^n U(1) & \xrightarrow{\mathrm{mod}\,2} & \mathbf{B}^{n+1}\mathbb{Z}_2 \end{array} .$$

We call the left vertical morphism $\boldsymbol{\nu}_{n+1}$ appearing here the *universal smooth integral Wu structure* in degree $n+1$.

A morphism of stacks

$$\boldsymbol{\nu}_{n+1} : X \to \mathbf{B}\mathrm{Spin}^{\nu_{n+1}}$$

is a choice of orientation structure on X together with a choice of smooth integral Wu structure lifting the corresponding Wu class ν_{n+1}.

Example 5.4.99. The smooth first fractional Pontrjagin class $\frac{1}{2}\mathbf{p}_2$, prop. 5.1.5, fits into a diagram

$$\begin{array}{ccccc} \mathbf{B}\mathrm{Spin} & & \longrightarrow & & \\ & \searrow^{u} & & \searrow & \\ & & \mathbf{B}\mathrm{Spin}^{\nu_4} & \longrightarrow & \mathbf{B}\mathrm{SO} \\ & {\scriptstyle \frac{1}{2}\mathbf{p}_1} \searrow & \downarrow^{\boldsymbol{\nu}^{\mathrm{int}}_4} & & \downarrow^{\nu_4} \\ & & \mathbf{B}^3 U(1) & \xrightarrow{\mathrm{mod}\,2} & \mathbf{B}^4\mathbb{Z}_2 \end{array} .$$

In this sense we may think of $\frac{1}{2}\mathbf{p}_1$ as being the integral and, moreover, smooth refinement of the universal degree-4 Wu class on $\mathbf{B}\mathrm{Spin}$.

Proof. Using the defining property of $\frac{1}{2}\mathbf{p}_1$, this follows with the results discussed in appendix E.1 of [HoSi05]. ☐

Proposition 5.4.100. *Let X be a smooth manifold equipped with orientation*

$$o_X : X \to \mathbf{B}\mathrm{SO}$$

and consider its Wu-class $[\nu_{n+1}(o_X)] \in H^{n+1}(X, \mathbb{Z}_2)$

$$\nu_{n+1}(o_X) : \; X \xrightarrow{o_X} \mathbf{B}\mathrm{SO} \xrightarrow{\nu_{n+1}} \mathbf{B}^{n+1}\mathbb{Z}_2 \; .$$

The n-groupoid $\hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{2k}]}(X)$ of $[\nu_{n+1}]$-twisted differential $\mathbf{DD}_{\mathrm{mod}2}$-structures, according to def. 3.9.61, hence the homotopy pullback

$$\begin{array}{ccc} \hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{n+1}]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow^{\nu_{n+1}(o_X)} \\ \mathbf{H}(X, \mathbf{B}^3 U(1)_{\mathrm{conn}}) & \xrightarrow{\hat{\mathbf{DD}}_{\mathrm{mod}\,2}} & \mathbf{H}(X, \mathbf{B}^{n+1}\mathbb{Z}_2) \end{array} ,$$

categorifies the groupoid $\hat{\mathcal{H}}^{n+1}_{\nu_{n+1}}(X)$ of differential integral Wu structures *as in def. 2.12 of [HoSi05]: its 1-truncation is equivalent to the groupoid defined there*

$$\tau_1 \hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struct}_{[\nu_{n+1}]}(X) \simeq \hat{\mathcal{H}}^{n+1}_{\nu_{n+1}}(X) \, .$$

Proof. By prop. 4.4.88, the canonical presentation of $\mathbf{DD}_{\mathrm{mod2}}$ via the Dold-Kan correspondence is given by an epimorphism of chain complexes of sheaves, hence by a fibration in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Precisely, the composite

$$\hat{\mathbf{DD}}_{\mathrm{mod}\,2} : \; \mathbf{B}^n U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^n U(1) \xrightarrow{\mathrm{DD}} \mathbf{B}^{n+1}\mathbb{Z} \xrightarrow{\mathrm{mod}\,2} \mathbf{B}^{n+1}\mathbb{Z}_2$$

is presented by the vertical sequence of morphisms of chain complexes

$$\begin{array}{ccccccccc}
\mathbb{Z} & \hookrightarrow & C^\infty(-,\mathbb{R}) & \xrightarrow{d_{\mathrm{dR}}\log} & \Omega^1(-) & \xrightarrow{d_{\mathrm{dR}}} & \cdots & \xrightarrow{d_{\mathrm{dR}}} & \Omega^n(-) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{Z} & \hookrightarrow & C^\infty(-,\mathbb{R}) & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0
\end{array} .$$

By remark 2.3.14 we may therefore compute the defining homotopy pullback for $\hat{\mathbf{DD}}_{\mathrm{mod2}}\mathrm{Struct}_{[\nu_{n+1}]}(X)$ as an ordinary fiber product of the corresponding simplicial sets of cocycles. The claim then follows by inspection. □

Remark 5.4.101. Explicitly, a cocycle in $\tau_1 \hat{\mathbf{DD}}_{\mathrm{mod2}}\mathrm{Struct}_{[\nu_{n+1}]}(X)$ is identified with a Čech cocycle with coefficients in the Deligne complex

$$\left(\mathbb{Z} \hookrightarrow C^\infty(-,\mathbb{R}) \xrightarrow{d_{\mathrm{dR}}\log} \Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-) \right)$$

such that the underlying $\mathbb{Z}[n+1]$-valued cocycle modulo 2 equals the given cocycle for ν_{n+1}. A coboundary between two such cocycles is a gauge equivalence class of ordinary Čech-Deligne cocycles such that their underlying $\mathbb{Z}$-cocycle vanishes modulo 2. Cocycles of this form are precisely those that arise by multiplication with 2 or arbitrary Čech-Deligne cocycles.

This is the groupoid structure discussed on p. 14 of [HoSi05], there in terms of singular instead of Čech cohomology.

We now consider another twisted differential structure, which refines these twisting integral Wu structures to *smooth* integral Wu structures, def. 5.4.98.

Definition 5.4.102. For $n \in \mathbb{N}$, write $\mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}}$ for the homotopy pullback of smooth moduli n-stacks

$$\begin{array}{ccc}
\mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^n U(1)_{\mathrm{conn}} \\
\downarrow & & \downarrow \\
\mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \times \mathbf{B}^n U(1) & \xrightarrow{\boldsymbol{\nu}^{\mathrm{int}}_{n+1} - 2\mathbf{DD}} & \mathbf{B}^n U(1)
\end{array} ,$$

where $\boldsymbol{\nu}^{\mathrm{int}}_{n+1}$ is the universal smooth integral Wu class from def. 5.4.98, and where $2\mathbf{DD} : \mathbf{B}^n U(1) \to \mathbf{B}^n U(1)$ is the canonical smooth refinement of the operation of multiplication by 2 on integral cohomology.

We call this the moduli n-stack of *smooth differential Wu-structures.*

By construction, a morphism $X \to \mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}}$ classifies also all possible orientation structures and smooth integral lifts of their Wu structures. In applications one typically wants to fix an integral Wu structure lifting a given Wu class. This is naturally formalized by the following construction.

Definition 5.4.103. For X an oriented manifold, and

$$\boldsymbol{\nu}_{n+1} : X \to \mathbf{B}\mathrm{Spin}^{\nu_{n+1}}$$

a given smooth integral Wu structure, def. 5.4.98, write $\mathbf{H}_{\boldsymbol{\nu}_{n+1}}(X, \mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}})$ for the n-groupoid of cocycles whose underlying smooth integral Wu structure is $\boldsymbol{\nu}_{n+1}$, hence for the homotopy pullback

$$\begin{array}{ccc} \mathbf{H}_{\boldsymbol{\nu}_{n+1}}(X, \mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}}) & \longrightarrow & \mathbf{H}(X, \mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}}) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}^n U(1)) & \xrightarrow{(\boldsymbol{\nu}_{n+1}, \mathrm{id})} & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \times \mathbf{B}^n U(1)) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\boldsymbol{\nu}_{n+1}} & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}^{\nu_{n+1}}) \end{array} .$$

Proposition 5.4.104. *Cohomology with coefficients in $\mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}}$ over a given smooth integral Wu structure coincides with the corresponding differential integral Wu structures:*

$$\hat{H}^{n+1}_{\nu_{n+1}}(X) \simeq H_{\boldsymbol{\nu}_{n+1}}(X, \mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}}) .$$

Proof. Let $C(\{U_i\})$ be the Čech-nerve of a good open cover of X. By prop. 4.4.88 the canonical presentation of $\mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ is a projective fibration. Since $C(\{U_i\})$ is projectively cofibrant and $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is a simplicial model category, the morphism of Čech cocycle simplicial sets

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}^n U(1)_{\mathrm{conn}}) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](C(\{U_i\}), \mathbf{B}^n U(1))$$

is a Kan fibration. Hence, by remark 2.3.14, its homotopy pullback may be computed as the ordinary pullback of simplicial sets of this map. The claim then follows by inspection.

Explicitly, in this presentation a cocycle in the pullback is a pair $(a, \hat{G})$ of a cocycle a for a circle n-bundle and a Deligne cocycle $\hat{G}$ with underlying bare cocycle G, such that there is an equality of degree-n Čech $U(1)$-cocycles

$$G = \boldsymbol{\nu}_{n+1} - 2a .$$

A gauge transformation between two such cocycles is a pair of Čech cochains $\hat{\gamma}$, α such that $\gamma = 2\alpha$ (the cocycle $\boldsymbol{\nu}_{n+1}$ being held fixed). This means that the gauge transformations acting on a given $\hat{G}$ solving the above constraint are precisely the all Deligne cocychains, but multiplied by 2. This is again the explicit description of $\hat{H}_{\nu_{n+1}}(X)$ from remark 5.4.101. □

5.4.8.3 Twisted differential String(E_8)-structures

We discuss smooth and differential refinements of the canonical degree-4 universal characteristic class

$$a : BE_8 \to K(\mathbb{Z}, 4)$$

for E_8 the largest of the exceptional semimple Lie algebras.

Proposition 5.4.105. *There exists a differential refinement of the canonical integral 4-class on BE_8 to the smooth moduli stack of E_8-connections with values in the smooth moduli 3-stack of circle 3-bundles with 3-connection*

$$\hat{\mathbf{a}} : (\mathbf{B}E_8)_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} .$$

Using the L_∞-algebraic data provided in [SSS09a], this was constructed in [FSS10].

Proposition 5.4.106. *Under geometric realization, prop. 3.8.2, the smooth class* $\mathbf{a}$ *becomes an equivalence*

$$|\mathbf{a}| : BE_8 \simeq_{16} B^3U(1) \simeq K(\mathbb{Z}, 4)$$

on 16-coskeleta.

Proof. By [BoSa58] the 15-coskeleton of the topological space E_8 is a $K(\mathbb{Z}, 4)$. By [FSS10], $\mathbf{a}$ is a smooth refinement of the generator $[a] \in H^4(BE_8, \mathbb{Z})$. By the Hurewicz theorem this is identified with $\pi_4(BE_8) \simeq \mathbb{Z}$. Hence in cohomology $\mathbf{a}$ induces an isomorphism

$$\pi_4(BE_8) \simeq [S^4, BE_8] \simeq H^1(S^4, E_8) \xrightarrow[\simeq]{|\mathbf{a}|} H^4(S^4, \mathbb{Z}) \simeq [S^4, K(\mathbb{Z}, 4)] \simeq \pi_4(S^4) \ .$$

Therefore $|\mathbf{a}|$ is a weak homotopy equivalence on 16 coskeleta. □

5.4.8.4 The moduli 3-stack of the C-field As we have reviewed above in section 5.4.8.1, the flux quantization condition for the C-field derived in [Wi97a] is the equation

$$[G_4] = \tfrac{1}{2}p_1 \mod 2 \quad \text{in} \quad H^4(X, \mathbb{Z}) \tag{5.16}$$

in integral cohomology, where $[G_4]$ is the cohomology class of the C-field itself, and $\frac{1}{2}p_1$ is the first fractional Pontrjagin class of the Spin manifold X. One can equivalently rewrite (5.16) as

$$[G_4] = \tfrac{1}{2}p_1 + 2a \quad \text{in} \quad H^4(X, \mathbb{Z}), \tag{5.17}$$

where a is some degree 4 integral cohomology class on X. By the discussion in section 5.4.8.2, the correct formalization of this for *fixed* spin structure is to regard the gauge equivalence class of the C-field as a differential integral Wu class relative to the integral Wu class $\nu_4^{\mathrm{int}} = \frac{1}{2}p_1$, example 5.4.99, of that spin structure. By prop. 5.4.104 and prop. 5.1.9, the natural refinement of this to a smooth moduli 3-stack of C-field configurations and arbitrary spin connections is the homotopy pullback of smooth 3-stacks

$$\begin{array}{ccc} \mathbf{B}^n U(1)^{\nu_{n+1}}_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^3 U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}^3 U(1) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1 + 2\mathbf{DD}} & \mathbf{B}^3 U(1) \end{array} .$$

Here the moduli stack in the bottom left is that of the field of gravity (spin connections) together with an auxiliary circle 3-bundle / 2-gerbe. Following the arguments in [FiSaScIII] (the traditional ones as well as the new ones presented there), we take this auxiliary circle 3-bundle to be the Chern-Simons circle 3-bundle of an E_8-principal bundle. According to prop. 5.4.105 this is formalized on smooth higher moduli stacks by further pulling back along the smooth refinement

$$\mathbf{a} : \mathbf{B}E_8 \to \mathbf{B}^3 U(1)$$

of the canonical universal 4-class $[a] \in H^4(BE_8, \mathbb{Z})$. Therefore we are led to formalize the E_8*-model for the* C*-field* as follows.

Definition 5.4.107. The *smooth moduli 3-stack of spin connections and* C*-field configurations* in the E_8-model is the homotopy pullback **CField** of the moduli n-stack of smooth differential Wu structures $\mathbf{B}^n U(1)^{\nu_4}_{\mathrm{conn}}$, def. 5.4.102, to spin connections and E_8-instanton configurations, hence the homotopy pullback

$$\begin{array}{ccc} \mathbf{CField} & \longrightarrow & \mathbf{B}^3 U(1)^{\nu_4}_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \xrightarrow{(u, \mathbf{a})} & \mathbf{B}\mathrm{Spin}^{\nu_4} \times \mathbf{B}^3 U(1) \end{array} , \tag{5.18}$$

where u is the canonical morphism from example 5.4.99.

Remark 5.4.108. By the pasting law, prop. 2.3.2, **CField** is equivalently given as the homotopy pullback

$$\begin{array}{ccc} \mathbf{CField} & \xrightarrow{\hat{\mathbf{G}}_4} & \mathbf{B}^3 U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \xrightarrow{\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}} & \mathbf{B}^3 U(1) \end{array} . \tag{5.19}$$

Spelling out this definition, a C-field configuration

$$(\nabla_{\mathfrak{so}}, \nabla_{b^2\mathbb{R}}, P_{E_8}) : X \to \mathbf{CField}$$

on a smooth manifold X is the datum of

1. *a principal* Spin*-bundle with* $\mathfrak{so}$*-connection* $(P_{\mathrm{Spin}}, \nabla_{\mathfrak{so}})$ *on* X*;*
2. *a principal* E_8*-bundle* P_{E_8} *on* X*;*
3. *a* $U(1)$*-2-gerbe with connection* $(P_{\mathbf{B}^2 U(1)}, \nabla_{\mathbf{B}^2 U(1)})$ *on* X*;*
4. *a choice of equivalence of* $U(1)$*-2-gerbes between between* $P_{\mathbf{B}^2 U(1)}$ *and the image of* $P_{\mathrm{Spin}} \times_X P_{E_8}$ *via* $\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}$*.*

It is useful to observe that there is the following further equivalent reformulation of this definition.

Proposition 5.4.109. *The moduli 3-stack* **CField** *from def. 5.4.107 is equivalently the homotopy pullback*

$$\begin{array}{ccc} \mathbf{CField} & \longrightarrow & \Omega^4_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \xrightarrow{(\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})_{\mathrm{dR}}} & \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R} \end{array} , \tag{5.20}$$

where the bottom morphism of higher stacks is presented by the correspondence of simplicial presheaves

$$\begin{array}{ccccccc} \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times (\mathbf{B}E_8)_{\mathrm{diff}} & \longrightarrow & \mathbf{B}\mathrm{Spin}_{\mathrm{diff}} \times (\mathbf{B}E_8)_{\mathrm{diff}} & \xrightarrow{(\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})_{\mathrm{diff}}} & \mathbf{B}^3 U(1)_{\mathrm{diff}} & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \longrightarrow & \mathbf{B}\mathrm{Spin} \times \mathbf{B}E_8 & \xrightarrow{\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}} & \mathbf{B}^3 U(1) & & \end{array} . \tag{5.21}$$

Moreover, it is equivalently the homtopy pullback

$$\begin{array}{ccc} \mathbf{CField} & \longrightarrow & \Omega^4_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \xrightarrow{(\frac{1}{4}\mathbf{p}_1 + \mathbf{a})_{\mathrm{dR}}} & \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R} \end{array} , \tag{5.22}$$

where now the bottom morphism is the composite of the bottom morphism before, postcomposed with the morphism

$$\tfrac{1}{2} : \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R} \to \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R}$$

that is given, via Dold-Kan, by division of differential forms by 2.

Proof. By the pasting law for homotopy pullbacks, prop. 2.3.2, the first homotopy pullback above may be computed as two consecutive homotopy pullbacks

$$
\begin{array}{ccccc}
\mathbf{CField} & \longrightarrow & \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^4_{\mathrm{cl}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \xrightarrow{\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}} & \mathbf{B}^3 U(1) & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R}
\end{array} ,
$$

which exhibits on the right the defining pullback of def. 4.4.88, and thus on the left the one from def. 5.4.107. The statement about the second homotopy pullback above follows analogously after noticing that

$$
\begin{array}{ccc}
\Omega^4_{\mathrm{cl}} & \xrightarrow{1/2} & \Omega^4_{\mathrm{cl}} \\
\downarrow & & \downarrow \\
\flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R} & \xrightarrow{1/2} & \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R}
\end{array} . \tag{5.23}
$$

is a homotopy pullback. □

It is therefore useful to introduce labels as follows.

Definition 5.4.110. We label the structure morphism of the above composite homotopy pullback as

$$
\begin{array}{ccccc}
\mathbf{CField} & \xrightarrow{\hat{G}_4} & \mathbf{B}^3 U(1)_{\mathrm{conn}} & \xrightarrow{\mathcal{G}_4} & \Omega^4_{\mathrm{cl}} \\
\downarrow & \swArrow_{\simeq}^{H_3} & \downarrow{\scriptstyle G_4} & \swArrow_{\simeq} & \downarrow \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 & \xrightarrow[\frac{1}{2}\mathbf{p}_2 + 2\mathbf{a}]{} & \mathbf{B}^3 U(1) & \xrightarrow[\mathrm{curv}]{} & \flat_{\mathrm{dR}} \mathbf{B}^4 U(1)
\end{array} .
$$

Here $\hat{G}_4$ sends a C-field configuration to an underlying circle 3-bundle with connection, whose curvature 4-form is $\mathcal{G}_4$.

Remark 5.4.111. These equivalent reformulations show two things.

1. The C-field model may be thought of as containing E_8-*pseudo-connections*. That is, there is a higher gauge in which a field configuration consists of an E_8-connection on an E_8-bundle – even though there is no dynamical E_8-gauge field in 11d supergravity – but where gauge transformations are allowed to freely shift these connections.
2. There is a precise sense in which imposing the quantization condition (5.17) on integral cohomology is equivalent to imposing the condition $[G_4]/2 = \frac{1}{4}p_1 + a$ in de Rham cohomology / real singular cohomology.

Observation 5.4.112. When restricted to a fixed Spin-connection, gauge equivalence classes of configurations classified by **CField** naturally form a torsor over the ordinary degree-4 differential cohomology $H^4_{\mathrm{diff}}(X)$.

Proof. By the general discussion of differential integral Wu-structures in section 5.4.8.2. □

5.4.8.5 The homotopy type of the moduli stack

We discuss now the homotopy type of the the 3-groupoid

$$
\mathbf{CField}(X) := \mathbf{H}(X, \mathbf{CField})
$$

of C-field configurations over a given spacetime manifold X. In terms of gauge theory, its 0-th homotopy group is the set of *gauge equivalence classes* of field configurations, its first homotopy group is the set of *gauge-of-gauge equivalence classes* of auto-gauge transformations of a given configuration, and so on.

Definition 5.4.113. For X a smooth manifold, let

$$\begin{array}{ccc} & & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \\ & \nearrow^{\nabla_{\mathfrak{so}}} & \downarrow \\ X & \xrightarrow{P_{\mathrm{Spin}}} & \mathbf{B}\mathrm{Spin} \end{array}$$

be a fixed spin structure with fixed spin connection. The restriction of $\mathbf{CField}(X)$ to this fixed spin connection is the homotopy pullback

$$\begin{array}{ccc} \mathbf{CField}(X)_{P_{\mathrm{Spin}}} & \longrightarrow & \mathbf{CField}(X) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}E_8) & \xrightarrow{((P_{\mathrm{Spin}}, \nabla_{\mathfrak{so}}), \mathrm{id})} & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) \end{array} .$$

Proposition 5.4.114. *The gauge equivalence classes of* $\mathbf{CField}(X)_{P_{\mathrm{Spin}}}$ *naturally surjects onto the differential integral Wu structures on* X*, relative to* $\frac{1}{2}p_1(P_{\mathrm{Spin}}) \bmod 2$*, (example 5.4.99):*

$$\pi_0 \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \twoheadrightarrow \hat{H}^{n+1}_{\frac{1}{2}p_1(P_{\mathrm{Spin}})}(X) \ .$$

The gauge-of-gauge equivalence classes of the auto-gauge transformation of the trivial C-field configuration naturally surject onto $H^2(X, U(1))$*:*

$$\pi_1 \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \twoheadrightarrow H^2(X, U(1)) \ .$$

Proof. By def. 5.4.107 and the pasting law, prop. 2.3.2, we have a pasting diagram of homotopy pullbacks of the form

$$\begin{array}{ccccccc} \mathbf{CField}(X)_{P_{\mathrm{Spin}}} & \twoheadrightarrow & \mathbf{H}_{\frac{1}{2}\mathbf{p}_1(P_{\mathrm{Spin}})}(X, \mathbf{B}^n U(1)^{\nu_4}_{\mathrm{conn}}) & \longrightarrow & & \longrightarrow & \mathbf{H}(X, \mathbf{B}^n U(1)^{\nu_4}_{\mathrm{conn}}) \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathbf{H}(X, \mathbf{B}E_8) & \xrightarrow{\mathbf{H}(X, \mathbf{a})} & \mathbf{H}(X, \mathbf{B}^3 U(1)) & \xrightarrow{(\nabla_{\mathfrak{so}}, \mathrm{id})} & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}^3 U(1)) & \xrightarrow{(u, \mathrm{id})} & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}^{\nu_4} \times \mathbf{B}^3 U(1)) \end{array} ,$$

where in the middle of the top row we identified, by def. 5.4.103, the n-groupoid of smooth differential Wu structures lifting the smooth Wu structure $\frac{1}{2}\mathbf{p}_1(P_{\mathrm{Spin}})$.

Due to prop. 5.4.104 we are therefore reduced to showing that the top left morphism is surjective on π_0.

But the bottom left morphism is surjective on π_0, by prop. 5.4.106. Now, the morphisms surjective on π_0 are precisely the *effective epimorphisms* in ∞Grpd, and these are stable under pullback. Hence the first claim follows.

For the second, we use that

$$\pi_1 \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \simeq \pi_0 \Omega \mathbf{CField}(X)_{P_{\mathrm{Spin}}}$$

and that forming loop space objects (being itself a homotopy pullback) commutes with homotopy pullbacks and with taking cocycles with coefficients in higher stacks, $\mathbf{H}(X, -)$.

Therefore the image of the left square in the above under Ω is the homotopy pullback

$$\begin{array}{ccc} \Omega \mathbf{CField}(X)_{P_{\mathrm{Spin}}} & \twoheadrightarrow & \mathbf{H}_{\frac{1}{2}\mathbf{p}_1(P_{\mathrm{Spin}})}(X, \mathbf{B}^n U(1)^{\nu_4}_{\mathrm{conn}}) \\ \downarrow & & \downarrow \\ C^\infty(X, E_8) & \xrightarrow{\mathbf{H}(X, \Omega \mathbf{a})} & \mathbf{H}(X, \mathbf{B}^2 U()1) \end{array} ,$$

where in the bottom left corner we used

$$\begin{aligned} \Omega \mathbf{H}(X, \mathbf{B}E_8) &\simeq \mathbf{H}(X, \Omega\mathbf{B}E_8) \\ &\simeq \mathbf{H}(X, E_8) \\ &\simeq C^\infty(X, E_8) \end{aligned} ,$$

and similarly for the bottom right corner. This identifies the bottom morphism on connected components as the morphism that sends a smooth function $X \to E_8$ to its homotopy class under the homotopy equivalence $E_8 \simeq_{15} B^2U(1) \simeq K(\mathbb{Z}, 3)$, which holds over the 11-dimensional X.

Therefore the bottom morphism is again surjective on π_0, and so is the top morphism. The claim then follows with prop. 5.4.100. □

5.4.8.6 Boundary moduli of the C-field We consider now ∂X (a neighbourhood of) the boundary of spacetime X, and discuss a variant of the moduli stack CField that encodes the boundary configurations of the supergravity C field.

Two different kinds of boundary conditions for the C-field appear in the literature.

- On an M5-brane boundary, the integral class underlying the C-field vanishes. (For instance page 24 of [Wi96]).
- On the fixed points of a 3-bundle-*orientifold*, def. 5.4.5, for the case that X has an $S^1//\mathbb{Z}_2$-orbifold factor, the C-field vanishes entirely. (This is considered in [HoWi95]. See section 3.1 of [Fal] for details.)

We construct higher moduli stacks for both of these conditions in the following. In addition to being restricted, the supergravity fields on a boundary also pick up additional degrees of freedom

- The E_8-principal bundle over the boundary is equipped with a connection.

We present now a sequence of natural morphisms of 3-stacks

$$C\mathrm{Field}^{\mathrm{bdr}'} \longrightarrow C\mathrm{Field}^{\mathrm{bdr}} \xrightarrow{\iota} C\mathrm{Field}, \qquad \iota' : C\mathrm{Field}^{\mathrm{bdr}'} \to C\mathrm{Field}$$

into the moduli stack of bulk C-fields, such that C-field configurations on X with the above behaviour over ∂X correspond to the *relative cohomology*, def. 3.6.277, with coefficients in ι or ι',respectively, hence to commuting diagrams of the form

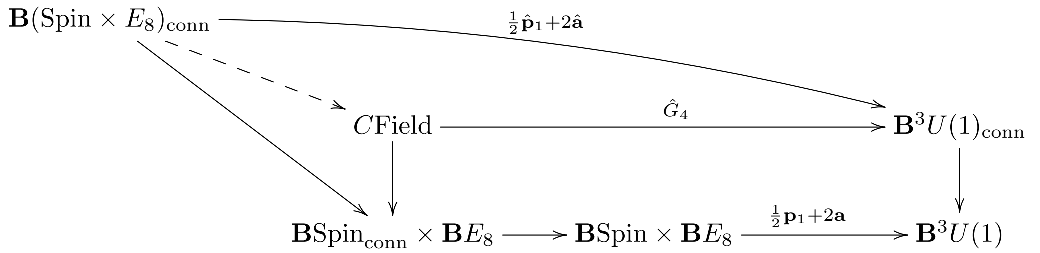
$$\begin{array}{ccc} \partial X & \xrightarrow{\phi_{\mathrm{bdr}}} & C\mathrm{Field}^{\mathrm{bdr}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ X & \xrightarrow{\phi} & C\mathrm{Field} \end{array} ,$$

and analogously for the primed case. (This is directly analogous to the characterization of type II supergravity field configurations in the presence of D-branes as discussed in 5.4.7.2.)

To this end, recall the general diagram of moduli stacks from def. 3.9.59 that relates the characteristic map $\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}$ with its differential refinement $\frac{1}{2}\hat{\mathbf{p}}_1 + 2\hat{\mathbf{a}}$:

$$\begin{array}{ccc} \mathbf{B}(\mathrm{Spin} \times E_8) & \xrightarrow{\flat \frac{1}{2}\mathbf{p}_1 + 2\flat\mathbf{a}} & \flat\mathbf{B}^3U(1) \\ \big\downarrow & & \big\downarrow \\ \mathbf{B}(\mathrm{Spin} \times E_8)_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1 + 2\hat{\mathbf{a}}} & \mathbf{B}^3U(1)_{\mathrm{conn}} \\ \big\downarrow & & \big\downarrow \\ \mathbf{B}(\mathrm{Spin} \times E_8) & \xrightarrow{\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}} & \mathbf{B}^3U(1) \end{array} .$$

The defining ∞-pullback diagram for CField factors the lower square of this diagram as follows

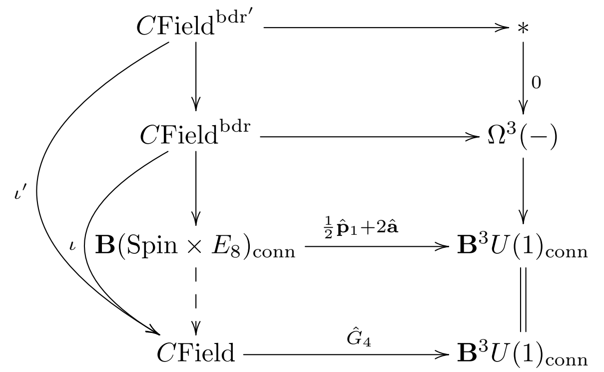


Here the dashed morphism is the universal morphism induced from the commutativity of the previous diagram together with the pullback property of the 3-stack CField. This morphism is the natural map of moduli which induces the relative cohomology that makes the E_8-bundle pick up a connection on the boundary.

It therefore remains to model the condition that G_4 or even $\hat{G}_4$ vanishes on the boundary. This condition is realized by further pulling back along the sequence

$$* \xrightarrow{0} \Omega^3(-) \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} \; .$$

Definition 5.4.115. Write $C\mathrm{Field}^{\mathrm{bdr}}$ and $C\mathrm{Field}^{\mathrm{bdr}'}$, respectively, for the moduli 3-stacks which arise as homotopy pullbacks in the top rectangles of



For X a smooth manifold with boundary, we say that the 3-groupoid of *C-field configurations with boundary data* on X is the hom ∞-groupoid

$$\mathbf{H}^I(\partial X \to X \;,\; C\mathrm{Field}^{\mathrm{bdr}} \xrightarrow{\iota} C\mathrm{Field})\,,$$

in the arrow category of the ambient ∞-topos $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grp}$, where on the right we have the composite morphism indicated by the curved arrow above, and analogously for the primed case.

Observation 5.4.116. The moduli 3-stack $C\mathrm{Field}^{\mathrm{brd}}$ is equivalent to is the moduli 3-stack of twisted $\mathrm{String}^{\mathbf{2a}}$-2-connections whose underlying twist has trivial class. The moduli 3-stack $C\mathrm{Field}^{\mathrm{bdr}'}$ is equivalent to the moduli 3-stack of untwisted $\mathrm{String}^{\mathbf{2a}}$-2-connections

$$C\mathrm{Field}^{\mathrm{bdr}'} \simeq \mathrm{String}^{\mathbf{2a}}_{\mathrm{conn}}\,.$$

This is presented via Lie integration of L_∞-algebras as

$$C\mathrm{Field}^{\mathrm{bdr}'} \simeq \mathbf{cosk}_3 \exp((\mathfrak{so} \oplus \mathfrak{e}_8)_{\mu_3^{\mathfrak{so}} + \mu_3^{\mathfrak{e}_8}})_{\mathrm{conn}}\,.$$

The presentation of $C\mathrm{Field}^{\mathrm{bdr}}$ by Lie integration is locally given by

$$\left(\begin{array}{ll} F_A = & dA + \frac{1}{2}[A \wedge A] \\ H_3 = & \nabla B := dB + \mathrm{CS}(A) - C_3 \\ \mathcal{G}_4 = & dC_3 \\ dF_A = & -[A \wedge F_A] \\ dH_3 = & \langle F_A \wedge F_A \rangle - \mathcal{G}_4 \\ d\mathcal{G}_4 = & 0 \end{array}\right)_i \xleftarrow{\begin{array}{ll} t^a & \mapsto A^a \\ r^a & \mapsto F_A^a \\ b & \mapsto B \\ c & \mapsto C_3 \\ h & \mapsto H_3 \\ g & \mapsto \mathcal{G}_4 \end{array}} \left(\begin{array}{ll} r^a = & dt^a + \frac{1}{2}C^a{}_{bc}t^b \wedge t^c + \\ h = & db + cs - c \\ g = & dc \\ dr^a = & -C^a{}_{bc}t^b \wedge r^a \\ dh = & \langle -, - \rangle - g \\ dg = & 0 \end{array}\right),$$

where

$$\mathfrak{g} = \mathfrak{so} \oplus \mathfrak{e}_8$$

and hence

$$A = \omega + A_{\mathfrak{e}_8} \,.$$

Proof. By definition 3.9.62 and prop. 5.2.13. □

Remark 5.4.117. Notice that with respect to String-connections, there are two levels of twists here:

1. The C-field 3-form twists the $\mathrm{String}^{\mathbf{2a}}$-2-connections.
2. For vanishing C-field 3-form, a $\mathrm{String}^{\mathbf{2a}}$-2-connection is still a twisted String-2-connection, where the twist is now by the Chern-Simons 3-bundle with connection of the underlying E_8-bundle with connection.

5.4.8.7 Hořava-Witten boundaries are membrane orientifolds We now discuss a natural formulation of the origin of the Hořava-Witten boundary conditions [HoWi95] in terms of higher stacks and nonabelian differential cohomology, specifically, in terms of what we call *membrane orientifolds*. From this we obtain a corresponding refinement of the moduli 3-stack of C-field configurations which now explicitly contains the twisted $\mathbb{Z}_2$-equivariance of the Hořava-Witten background.

Recall the notion of higher orientifolds and their identification with twisted differential $\mathbf{J}_n$-structures from 5.4.5.

Observation 5.4.118. Let $U//\mathbb{Z}_2 \hookrightarrow Y//\mathbb{Z}_2$ be a patch on which a given $\hat{\mathbf{J}}_n$-structure has a trivial underlying integral class, such that it is equivalent to a globally defined $(n+1)$-form C_U on U. Then the components of this this 3-form orthogonal to the $\mathbb{Z}_2$-action are *odd* under the action. In particular, if $U \hookrightarrow Y$ sits in the fixed point set of the action, then these components vanish. This is the Hořava-Witten boundary condition on the C-field on an 11-dimensional spacetime $Y = X \times S^1$ equipped with $\mathbb{Z}_2$-action on the circle. See for instance section 3 of [Fal] for an explicit discussion of the $\mathbb{Z}_2$ action on the C-field in this context.

We therefore have a natural construction of the moduli 3-stack of Hořava-Witten C-field configurations as follows

Definition 5.4.119. Let $\mathbf{CField}_J(Y)$ be the homotopy pullback in

$$\begin{array}{ccc} \mathbf{CField}_J(Y) & \longrightarrow & \hat{\mathbf{J}}\mathrm{Struc}_\rho(Y//\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ & & \mathbf{H}(Y, \mathbf{B}^3U(1)_{\mathrm{conn}}) \\ & & \downarrow \\ \mathbf{H}(Y, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) & \xrightarrow{\mathbf{H}(Y, \frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})} & \mathbf{H}(Y, \mathbf{B}^3U(1)) \end{array},$$

where the top right morphism is the map $\hat{G}_\rho \mapsto \hat{G}$ from remark 5.4.46.

The objects of $\mathbf{CField}_J(Y)$ are C-field configurations on Y that not only satisfy the flux quantization condition, but also the Hořava-Witten twisted equivariance condition (in fact the proper globalization of that condition from 3-forms to full differential cocycles). This is formalized by the following.

Observation 5.4.120. There is a canonical morphism $\mathbf{CField}_J(Y) \to \mathbf{CField}(Y)$, being the dashed morphism in

$$\begin{array}{ccc} \mathbf{CField}_J(Y) & \longrightarrow & \hat{\mathbf{J}}\mathrm{Struc}_\rho(Y//\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ \mathbf{CField}(Y) & \longrightarrow & \mathbf{H}(Y, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \\ \downarrow & & \downarrow \\ \mathbf{H}(Y, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) & \xrightarrow{\mathbf{H}(Y, \frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})} & \mathbf{H}(Y, \mathbf{B}^3 U(1)) \end{array} ,$$

which is given by the universal property of the defining homotopy pullback of **CField**, remark 5.4.108.

A supergravity field configuration presented by a morphism $Y \to \mathbf{CField}$ into the moduli 3-stack of configurations that satisfy the flux quantization condition in addition satisfies the Hořava-Witten boundary condition if, as an element of $\mathbf{CField}(Y) := \mathbf{H}(Y, \mathbf{CField})$ it is in the image of $\mathbf{CField}_J(Y) \to \mathbf{CField}(Y)$. In fact, there may be several such pre-images. A choice of one is a choice of membrane orientifold structure.

5.4.9 Differential T-duality

In [KaVa10] (see also the review in section 7.4 of [BuSc10]) a formalization of the differential refinement of topological T-duality is given. We discuss here how this is naturally an example of the twisted differential **c**-structures, 3.9.8.

(...)

5.5 Symplectic higher geometry

The notion of *symplectic manifold* formalizes in physics the concept of a *classical mechanical system.* The notion of *geometric quantization*, 3.9.13, of a symplectic manifold is one formalization of the general concept in physics of *quantization* of such a system to a *quantum mechanical system.*

Or rather, the notion of symplectic manifold does not quite capture the most general systems of classical mechanics. One generalization requires passage to *Poisson manifolds.* The original methods of geometric quantization become meaningless on a Poisson manifold that is not symplectic. However, a Poisson structure on a manifold X is equivalent to the structure of a Poisson Lie algebroid $\mathfrak{P}$ over X. This is noteworthy, because the latter *is* again symplectic, as a Lie algebroid, even if the underlying Poisson manifold is not symplectic: it is a *symplectic Lie 1-algebroid*, prop. 5.5.16.

Based on related observations it was suggested, [Wei89] that a notion of *symplectic groupoid* should naturally replace that of *symplectic manifold* for the purposes of geometric quantization to yield a notion of *geometric quantization of symplectic groupoids.* Since a symplectic manifold can be regarded as a symplectic Lie 0-algebroid, prop. 5.5.16, and also as a symplectic smooth 0-groupoid this step amounts to a kind of categorification of symplectic geometry.

More or less implicitly, there has been evidence that this shift in perspective is substantial: the *deformation quantization* of a Poisson manifold famously turns out [Kon03] to be constructible in terms of correlators of the 2-dimensional TQFT called the *Poisson σ-model*, 5.7.11.4, associated with the corresponding Poisson Lie algebroid. The fact that this is 2-dimensional and not 1-dimensional, as the quantum mechanical system that it thus encodes, is a direct reflection of this categorification shift of degree.

On general abstract grounds this already suggests that it makes sense to pass via higher categorification further to symplectic Lie n-algebroids, def. 5.5.14, as well as to symplectic 2-groupoids, symplectic 3-groupoids, etc. up to symplectic ∞-groupoids, def. 5.5.21.

Formal hints for such a generalization had been noted in [Sev01] (in particular in its concluding table). More indirect – but all the more noteworthy – hints came from quantum field theory, where it was observed that a generalization of symplectic geometry to *multisymplectic geometry* [Hél11] of degree n more naturally captures the description of n-dimensional QFT (notice that quantum mechanics may be understood as $(0+1)$-dimensional QFT). For, observe that the symplectic form on a symplectic Lie n-algebroid is, while always "binary", nevertheless a representative of de Rham cohomology in degree $n+2$.

There is a natural formalization of these higher symplectic structures in the context of any cohesive ∞-topos. Moreover, by 5.5.2 symplectic forms on L_∞-algebroids have a natural interpretation in ∞-Lie theory: they are L_∞-invariant polynomials. This means that the ∞-Chern-Weil homomorphism applies to them.

Observation 5.5.1. From the perspective of ∞-Lie theory, a smooth manifold Σ equipped with a symplectic form ω is equivalently a Lie 0-algebroid equipped with a quadratic and non-degenerate *L_∞-invariant polynomial* (def. 4.4.112).

This observation implies

1. a direct ∞-Lie theoretic analog of symplectic manifolds: *symplectic Lie n-algebroids* and their Lie integration to *symplectic smooth ∞-groupoids*

2. the existence of a canonical ∞-Chern-Weil homomorphism for every symplectic Lie n-algebroid.

This is spelled out below in 5.5.1, 5.5.2, 5.5.3, which is taken from [FRS11a]. The ∞-group extensions, def. 3.6.245, that are induced by the unrefined ∞-Chern-Weil homomorphism, 3.9.7, on a symplectic ∞-groupoid are their *prequantum circle $(n+1)$-bundles*, the higher analogs of prequantum line bundles in the geometric quantization of symplectic manifolds. This we discuss in 4.4.20. Further below in 5.7.11 we show that the *refined* ∞-Chern-Weil homomorphism, 3.9.11, on a symplectic ∞-groupoid constitutes the action functional of the corresponding *AKSZ σ-model* (discussed below in 5.7.11).

- 5.5.1 – Symplectic dg-geometry;

- 5.5.2– Symplectic L_∞-algebroids;
- 5.5.3 – Symplectic smooth ∞-groupoids;

The parts 5.5.1 and 5.5.2 are taken from [FRS11a].

5.5.1 Symplectic dg-geometry

In 4.5 we considered a general abstract notion of infinitesimal thickenings in higher differential geometry and showed how from the point of view of ∞-Lie theory this leads to the notion of L_∞-algebroids, def. 4.5.12. As is evident from that definition, these can also be regarded as objects in *dg-geometry* [ToVe05]. We make explicit now some basic aspects of this identification.

The following definitions formulate a simple notion of *affine smooth graded manifolds* and *affine smooth dg-manifolds*. Despite their simplicity these definitions capture in a precise sense all the relevant structure: namely the *local* smooth structure. Globalizations of these definitions can be obtained, if desired, by general abstract constructions.

Definition 5.5.2. The category of *affine smooth* $\mathbb{N}$*-graded manifolds* – here called *smooth graded manifolds* for short – is the full subcategory

$$\mathrm{SmoothGrMfd} \subset \mathrm{GrAlg}_{\mathbb{R}}^{\mathrm{op}}$$

of the opposite category of $\mathbb{N}$-graded-commutative $\mathbb{R}$-algebras on those isomorphic to Grassmann algebras of the form

$$\wedge^\bullet_{C^\infty(X_0)}\Gamma(V^*)\,,$$

where X_0 is an ordinary smooth manifold, $V \to X_0$ is an $\mathbb{N}$-graded smooth vector bundle over X_0 degreewise of finite rank, and $\Gamma(V^*)$ is the graded $C^\infty(X)$-module of smooth sections of the dual bundle.

For a smooth graded manifold $X \in \mathrm{SmoothGrMfd}$, we write $C^\infty(X) \in \mathrm{cdgAlg}_{\mathbb{R}}$ for its corresponding dg-algebra *of functions*.

Remarks.

- The full subcategory of these objects is equivalent to that of all objects isomorphic to one of this form. We may therefore use both points of view interchangeably.
- Much of the theory works just as well when V is allowed to be $\mathbb{Z}$-graded. This is the case that genuinely corresponds to *derived* (instead of just higher) differential geometry. An important class of examples for this case are BV-BRST complexes which motivate much of the literature. For the purpose of this short note, we shall be content with the $\mathbb{N}$-graded case.
- For an $\mathbb{N}$-graded $C^\infty(X_0)$-module $\Gamma(V^*)$ we have

  $$\wedge^\bullet_{C^\infty}\Gamma(V^*) = C^\infty(X_0) \oplus \Gamma(V_0^*) \oplus \left(\Gamma(V_0^*) \wedge_{C^\infty(X_0)} \Gamma(V_0^*) \oplus \Gamma(V_1^*)\right) \oplus \cdots\,,$$

  with the leftmost summand in degree 0, the next one in degree 1, and so on.
- There is a canonical functor

  $$\mathrm{SmoothMfd} \hookrightarrow \mathrm{SmthGrMfd}$$

  which identifies an ordinary smooth manifold X with the smooth graded manifold whose function algebra is the ordinary algebra of smooth functions $C^\infty(X_0) := C^\infty(X)$ regarded as a graded algebra concentrated in degree 0. This functor is full and faithful and hence exhibits a full subcategory.

All the standard notions of differental geometry apply to differential graded geometry. For instance for $X \in \mathrm{SmoothGrMfd}$, there is the graded vector space $\Gamma(TX)$ of vector fields on X, where a vector field is identified with a graded *derivation* $v : C^\infty(X) \to C^\infty(X)$. This is naturally a graded (super) Lie algebra with super Lie bracket the graded commutator of derivations. Notice that for $v \in \Gamma(TX)$ of odd degree we have $[v, v] = v \circ v + v \circ v = 2v^2 : C^\infty(X) \to C^\infty(X)$.

Definition 5.5.3. The category of (affine, $\mathbb{N}$-graded) *smooth differential-graded manifolds* is the full subcategory

$$\mathrm{SmoothDgMfd} \subset \mathrm{cdgAlg}_{\mathbb{R}}^{\mathrm{op}}$$

of the opposite of differential graded-commutative $\mathbb{R}$-algebras on those objects whose underlying graded algebra comes from SmoothGrMfd.

This is equivalently the category whose objects are pairs (X, v) consisting of a smooth graded manifold $X \in \mathrm{SmoothGrMfd}$ and a grade 1 vector field $v \in \Gamma(TX)$, such that $[v, v] = 0$, and whose morphisms $(X_1, v_1) \to (X_2, v_2)$ are morphisms $f : X_1 \to X_2$ such that $v_1 \circ f^* = f^* \circ v_2$.

Remark 5.5.4. The dg-algebras appearing here are special in that their degree-0 algebra is naturally not just an $\mathbb{R}$-algebra, but a *smooth algebra* (a "C^∞-ring", see [Stel10] for review and discussion).

Definition 5.5.5. The *de Rham complex functor*

$$\Omega^\bullet(-) : \mathrm{SmoothGrMfd} \to \mathrm{cdgAlg}_{\mathbb{R}}^{\mathrm{op}}$$

sends a dg-manifold X with $C^\infty(X) \simeq \wedge^\bullet_{C^\infty(X_0)} \Gamma(V^*)$ to the Grassmann algebra over $C^\infty(X_0)$ on the graded $C^\infty(X_0)$-module

$$\Gamma(T^*X) \oplus \Gamma(V^*) \oplus \Gamma(V^*[-1]) \,,$$

where $\Gamma(T^*X)$ denotes the ordinary smooth 1-form fields on X_0 and where $V^*[-1]$ is V^* with the grades *increased* by one. This is equipped with the differential $\mathbf{d}$ defined on generators as follows:

- $\mathbf{d}|_{C^\infty(X_0)} = d_{\mathrm{dR}}$ is the ordinary de Rham differential with values in $\Gamma(T^*X)$;
- $\mathbf{d}|_{\Gamma(V^*)} \to \Gamma(V^*[-1])$ is the degree-shift isomorphism
- and $\mathbf{d}$ vanishes on all remaining generators.

Definition 5.5.6. Observe that $\Omega^\bullet(-)$ evidently factors through the defining inclusion $\mathrm{SmoothDgMfd} \hookrightarrow \mathrm{cdgAlg}_{\mathbb{R}}$. Write

$$\mathfrak{T}(-) : \mathrm{SmoothGrMfd} \to \mathrm{SmoothDgMfd}$$

for this factorization.

The dg-space $\mathfrak{T}X$ is often called the *shifted tangent bundle* of X and denoted $T[1]X$.

Observation 5.5.7. For Σ an ordinary smooth manifold and for X a graded manifold corresponding to a vector bundle $V \to X_0$, there is a natural bijection

$$\mathrm{SmoothGrMfd}(\mathfrak{T}\Sigma, X) \simeq \Omega^\bullet(\Sigma, V)$$

where on the right we have the set of V-valued smooth differential forms on Σ: tuples consisting of a smooth function $\phi_0 : \Sigma \to X_0$, and for each $n > 1$ an ordinary differential n-form $\phi_n \in \Omega^n(\Sigma, \phi_0^* V_{n-1})$ with values in the pullback bundle of V_{n-1} along ϕ_0.

The standard Cartan calculus of differential geometry generalizes directly to graded smooth manifolds. For instance, given a vector field $v \in \Gamma(TX)$ on $X \in \mathrm{SmoothGrMfd}$, there is the *contraction derivation*

$$\iota_v : \Omega^\bullet(X) \to \Omega^\bullet(X)$$

on the de Rham complex of X, and hence the *Lie derivative*

$$\mathcal{L}_v := [\iota_v, \mathbf{d}] : \Omega^\bullet(X) \to \Omega^\bullet(X) \,.$$

Definition 5.5.8. For $X \in$ SmoothGrMfd the *Euler vector field* $\epsilon \in \Gamma(TX)$ is defined over any coordinate patch $U \to X$ to be given by the formula

$$\epsilon|_U := \sum_a \deg(x^a) x^a \frac{\partial}{\partial x^a} \,,$$

where $\{x^a\}$ is a basis of generators and $\deg(x^a)$ the degree of a generator. The *grade* of a homogeneous element α in $\Omega^\bullet(X)$ is the unique natural number $n \in \mathbb{N}$ with

$$\mathcal{L}_\epsilon \alpha = n\alpha \,.$$

Remarks.

- This implies that for x^i an element of grade n on U, the 1-form $\mathbf{d}x^i$ is also of grade n. This is why we speak of *grade* (as in "graded manifold") instead of *degree* here.
- Since coordinate transformations on a graded manifold are grading-preserving, the Euler vector field is indeed well-defined. Note that the degree-0 coordinates do not appear in the Euler vector field.

The existence of ϵ implies the following useful statement (amplified in [Royt02]), which is a trivial variant of what in grade 0 would be the standard Poincaré lemma.

Observation 5.5.9. On a graded manifold, every closed differential form ω of positive grade n is exact: the form

$$\lambda := \frac{1}{n} \iota_\epsilon \omega$$

satisfies

$$\mathbf{d}\lambda = \omega \,.$$

Definition 5.5.10. A *symplectic dg-manifold* of grade $n \in \mathbb{N}$ is a dg-manifold (X, v) equipped with 2-form $\omega \in \Omega^2(X)$ which is

- non-degenerate;
- closed;

as usual for symplectic forms, and in addition

- of grade n;
- v-invariant: $\mathcal{L}_v \omega = 0$.

In a local chart U with coordinates $\{x^a\}$ we may find functions $\{\omega_{ab} \in C^\infty(U)\}$ such that

$$\omega|_U = \frac{1}{2} \mathbf{d}x^a \, \omega_{ab} \wedge \mathbf{d}x^b \,,$$

where summation of repeated indices is implied. We say that U is a *Darboux chart* for (X, ω) if the ω_{ab} are constant.

Observation 5.5.11. The function algebra of a symplectic dg-manifold (X, ω) of grade n is naturally equipped with a Poisson bracket

$$\{-,-\} : C^\infty(X) \otimes C^\infty(X) \to C^\infty(X)$$

which decreases grade by n. On a local coordinate patch $\{x^a\}$ this is given by

$$\{f, g\} = \frac{f6}{x^a6} \omega^{ab} \frac{\partial g}{\partial x^b} \,,$$

where $\{\omega^{ab}\}$ is the inverse matrix to $\{\omega_{ab}\}$, and where the graded differentiation in the left factor is to be taken from the right, as indicated.

Definition 5.5.12. For $\pi \in C^\infty(X)$ and $v \in \Gamma(TX)$, we say that π *is a Hamiltonian for* v, or equivalently, that v *is the of* π if

$$\mathbf{d}\pi = \iota_v \omega \,.$$

Note that the convention $(-1)^{n+1}\mathbf{d}\pi = \iota_v\omega$ is also frequently used for defining Hamiltonians in the context of graded geometry.

Remark 5.5.13. In a local coordinate chart $\{x^a\}$ the defining equation $\mathbf{d}\pi = \iota_v\omega$ becomes

$$\mathbf{d}x^a \frac{\partial \pi}{\partial x^a} = \omega_{ab} v^a \wedge \mathbf{d}x^b = \omega_{ab}\mathbf{d}x^a \wedge v^b \,,$$

implying that

$$\omega_{ab} v^b = \frac{\partial \pi}{\partial x^a} \,.$$

5.5.2 Symplectic L_∞-algebroids

Here we discuss L_∞-algebroids, def. 4.5.12, equipped with *symplectic structure*, which we conceive of as: equipped with de Rham cocycles that are *invariant polynomials*, def. 4.4.112.

Definition 5.5.14. A *symplectic Lie* n*-algebroid* $(\mathfrak{P}, \omega)$ is a Lie n-algebroid $\mathfrak{P}$ equipped with a quadratic non-degenerate invariant polynomial $\omega \in W(\mathfrak{P})$ of degree $n+2$.

This means that

- on each chart $U \to X$ of the base manifold X of $\mathfrak{P}$, there is a basis $\{x^a\}$ for $\mathrm{CE}(\mathfrak{a}|_U)$ such that

$$\omega = \frac{1}{2}\mathbf{d}x^a\, \omega_{ab} \wedge \mathbf{d}x^b$$

  with $\{\omega_{ab} \in \mathbb{R} \hookrightarrow C^\infty(X)\}$ and $\deg(x^a) + \deg(x^b) = n$;

- the coefficient matrix $\{\omega_{ab}\}$ has an inverse;

- we have

$$d_{\mathrm{W}(\mathfrak{P})}\omega = d_{\mathrm{CE}(\mathfrak{P})}\omega + \mathbf{d}\omega = 0 \,.$$

The following observation essentially goes back to [Sev01] and [Royt02].

Proposition 5.5.15. *There is a full and faithful embedding of symplectic dg-manifolds of grade* n *into symplectic Lie* n*-algebroids.*

Proof. The dg-manifold itself is identified with an L_∞-algebroid by def. 4.5.12. For $\omega \in \Omega^2(X)$ a symplectic form, the conditions $\mathbf{d}\omega = 0$ and $\mathcal{L}_v\omega = 0$ imply $(\mathbf{d} + \mathcal{L}v)\omega = 0$ and hence that under the identification $\Omega^\bullet(X) \simeq \mathrm{W}(\mathfrak{a})$ this is an invariant polynomial on $\mathfrak{a}$.

It remains to observe that the L_∞-algebroid $\mathfrak{a}$ is in fact a Lie n-algebroid. This is implied by the fact that ω is of grade n and non-degenerate: the former condition implies that it has no components in elements of grade $> n$ and the latter then implies that all such elements vanish. □

The following characterization may be taken as a definition of Poisson Lie algebroids and Courant Lie 2-algebroids.

Proposition 5.5.16. *Symplectic Lie* n*-algebroids are equivalently:*

- *for* $n = 0$*: ordinary symplectic manifolds;*
- *for* $n = 1$*: Poisson Lie algebroids;*
- *for* $n = 2$*: Courant Lie 2-algebroids.*

See [Royt02, Sev01] for more discussion.

Proposition 5.5.17. *Let $(\mathfrak{P}, \omega)$ be a symplectic Lie n-algebroid for positive n in the image of the embedding of proposition 5.5.15. Then it carries the canonical L_∞-algebroid cocycle*

$$\pi := \frac{1}{n+1} \iota_\epsilon \iota_v \omega \in \mathrm{CE}(\mathfrak{P})$$

which moreover is the Hamiltonian, according to definition 5.5.12, of $d_{\mathrm{CE}(\mathfrak{P})}$.

Proof. Since $\mathbf{d}\omega = \mathcal{L}_v \omega = 0$, we have

$$\begin{aligned} \mathbf{d}\iota_\epsilon \iota_v \omega &= \mathbf{d}\iota_v \iota_\epsilon \omega \\ &= (\iota_v \mathbf{d} - \mathcal{L}_v)\iota_\epsilon \omega \\ &= \iota_v \mathcal{L}_\epsilon \omega - [\mathcal{L}_v, \iota_\epsilon]\omega \\ &= n \iota_v \omega - \iota_{[v,\epsilon]}\omega \\ &= (n+1)\iota_v \omega, \end{aligned}$$

where Cartan's formula $[\mathcal{L}_v, \iota_\epsilon] = \iota_{[v,\epsilon]}$ and the identity $[v, \epsilon] = -[\epsilon, v] = -v$ have been used. Therefore $\pi := \frac{1}{n+1}\iota_\epsilon \iota_v \omega$ satisfies the defining equation $\mathbf{d}\pi = \iota_v \omega$ from definition 5.5.12. □

Remark 5.5.18. On a local chart with coordinates $\{x^a\}$ we have

$$\pi\big|_U = \frac{1}{n+1}\omega_{ab} \ \deg(x^a) x^a \ \wedge v^b \, .$$

Our central observation now is the following.

Proposition 5.5.19. *The cocycle $\frac{1}{n}\pi$ from prop. 5.5.17 is in transgression with the invariant polynomial ω. A Chern-Simons element witnessing the transgression according to def. 4.4.116 is*

$$\mathrm{cs} = \frac{1}{n}\left(\iota_\epsilon \omega + \pi\right) \, .$$

Proof. It is clear that $i^* \mathrm{cs} = \frac{1}{n}\pi$. So it remains to check that $d_{\mathrm{W}(\mathfrak{P})}\mathrm{cs} = \omega$. As in the proof of proposition 5.5.17, we use $\mathbf{d}\omega = \mathcal{L}_v \omega = 0$ and Cartan's identity $[\mathcal{L}_v, \iota_\epsilon] = \iota_{[v,\epsilon]} = -\iota_v$. By these, the first summand in $d_{\mathrm{W}(\mathfrak{P})}(\iota_\epsilon \omega + \pi)$ is

$$\begin{aligned} d_{\mathrm{W}(\mathfrak{P})}\iota_\epsilon \omega &= (\mathbf{d} + \mathcal{L}_v)\iota_\epsilon \omega \\ &= [\mathbf{d} + \mathcal{L}_v, \iota_\epsilon]\omega \\ &= n\omega - \iota_v \omega \\ &= n\omega - \mathbf{d}\pi \end{aligned} .$$

The second summand is simply

$$d_{\mathrm{W}(\mathfrak{P})}\pi = \mathbf{d}\pi$$

since π is a cocycle. □

Remark 5.5.20. In a coordinate patch $\{x^a\}$ the Chern-Simons element is

$$\mathrm{cs}\big|_U = \frac{1}{n}\left(\omega_{ab} \deg(x^a) x^a \ \wedge \mathbf{d}x^b + \pi\right) \, .$$

In this formula one can substitute $\mathbf{d} = d_{\mathrm{W}} - d_{\mathrm{CE}}$, and this kind of substitution will be crucial for the proof our main statement in proposition 5.7.52 below. Since $d_{\mathrm{CE}} x^i = v^i$ and using remark 5.5.18 we find

$$\sum_a \omega_{ab} \deg(x^a) x^a \wedge d_{\mathrm{CE}} x^b = (n+1)\pi \, ,$$

and hence

$$\mathrm{cs}\big|_U = \frac{1}{n}\left(\deg(x^a)\,\omega_{ab}x^a \wedge d_{\mathrm{W}(\mathfrak{P})}x^b - n\pi\right) \,.$$

In the section 5.7.11 we show that this transgression element cs *is* the AKSZ-Lagrangian.

5.5.3 Symplectic smooth ∞-groupoids

We define *symplectic smooth ∞-groupoids* in terms of their underlying symplectic L_∞-algebroids.

Recall that for any $n \in \mathbb{N}$, a *symplectic Lie n-algebroid* $(\mathfrak{P}, \omega)$ is (def. 5.5.14) an L_∞-algebroid $\mathfrak{P}$ that is equipped with a quadratic and non-degenerate L_∞-invariant polynomial. Under Lie integration, def. 4.4.53, $\mathfrak{P}$ integrates to a smooth n-groupoid $\tau_n \exp(\mathfrak{P}) \in \mathrm{Smooth}\infty\mathrm{Grpd}$. Under the ∞-Chern-Weil homomorphism, 4.4.17, the invariant polynomial induces a differential form on the smooth ∞-groupoid, 3.9.3:

$$\omega : \tau_n \exp(\mathfrak{P}) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+2}\mathbb{R}$$

representing a class $[\omega] \in H^{n+2}_{dR}(\tau_n \exp(\mathfrak{P}))$.

Definition 5.5.21. Write

$$\mathrm{SymplSmooth}\infty\mathrm{Grpd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}/(\coprod_n \flat_{\mathrm{dR}}\mathbf{B}^{n+2}\mathbb{R})$$

for the full sub-∞-category of the over-∞-topos of Smooth∞Grpd over the de Rham coefficient objects on those objects in the image of this construction.

We say an object on SymplSmooth∞Grpd is a *symplectic smooth ∞-groupoid*.

Remark 5.5.22. There are evident variations of this for the ambient Smooth∞Grpd replaced by some variant, such as SynthDiffInfGrpd∞Grpd, or SmoothSuper∞Grpd, 4.6).

We now spell this out for $n = 1$. The following notion was introduced in [Wei89] in the study of geometric quantization.

Definition 5.5.23. A *symplectic groupoid* is a Lie groupoid $\mathcal{G}$ equipped with a differential 2-form $\omega_1 \in \Omega^2(\mathcal{G}_1)$ which is

1. a symplectic 2-form on $\mathcal{G}_1$;
2. closed as a simplicial form:

$$\delta\omega_1 = \partial_0^*\omega_1 - \partial_1^*\omega_1 + \partial_2^*\omega_1 = 0\,,$$

   where $\partial_i : \mathcal{G}_2 \to \mathcal{G}_1$ are the face maps in the nerve of $\mathcal{G}$.

Example 5.5.24. Let (X, ω) be an ordinary symplectic manifold. Then its fundamental groupoid $\Pi_1(X)$ canonically is a symplectic groupoid with $\omega_1 := \partial_1^*\omega - \partial_0^*\omega$.

Proposition 5.5.25. *Let $\mathfrak{P}$ be the symplectic Lie 1-algebroid (Poisson Lie algebroid), def. 5.5.14, induced by the Poisson manifold structure corresponding to (X, ω). Write*

$$\omega : \mathfrak{T}\mathfrak{P} \to \mathfrak{T}b^3\mathbb{R}$$

for the canonical invariant polynomial.

Then the corresponding ∞-Chern-Weil homomorphism according to 4.4.17

$$\exp(\omega) : \exp(\mathfrak{P})_{\mathrm{diff}} \to \mathbf{B}^3_{\mathrm{dR}}\mathbb{R}$$

exhibits the symplectic groupoid from example 5.5.24.

Proof. We start with the simple situation where (X, ω) has a global Darboux coordinate chart $\{x^i\}$. Write $\{\omega_{ij}\}$ for the components of the symplectic form in these coordinates, and $\{\omega^{ij}\}$ for the components of the inverse.

Then the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{P})$ is generated from $\{x^i\}$ in degree 0 and $\{\partial_i\}$ in degree 1, with differential given by

$$d_{\mathrm{CE}} x^i = -\omega^{ij}\partial_j$$

$$d_{\mathrm{CE}} \partial_i = \frac{\partial \pi^{jk}}{\partial x^i}\partial_j \wedge \partial_k = 0\,.$$

The differential in the corresponding Weil algebra is hence

$$d_{\mathrm{W}} x^i = -\omega^{ij}\partial_j + \mathbf{d}x^i$$

$$d_{\mathrm{W}} \partial_i = \mathbf{d}\partial_i\,.$$

By prop. 5.5.16, the symplectic invariant polynomial is

$$\omega = \mathbf{d}x^i \wedge \mathbf{d}\partial_i \in W(\mathfrak{P})\,.$$

Clearly it is useful to introduce a new basis of generators with

$$\partial^i := -\omega^{ij}\partial_j\,.$$

In this new basis we have a manifest isomorphism

$$\mathrm{CE}(\mathfrak{P}) = \mathrm{CE}(\mathfrak{T}X)$$

with the Chevalley-Eilenberg algebra of the tangent Lie algebroid of X.

Therefore the Lie integration of $\mathfrak{P}$ is the fundamental groupoid of X, which, since we have assumed global Darboux oordinates and hence contractible X, is just the pair groupoid:

$$\tau_1 \exp(\mathfrak{P}) = \Pi_1(X) = (\; X \times X \rightrightarrows X \;)\,.$$

It remains to show that the symplectic form on $\mathfrak{P}$ makes this a symplectic groupoid.

Notice that in the new basis the invariant polynomial reads

$$\begin{aligned} \omega &= -\omega_{ij}\mathbf{d}x^i \wedge \mathbf{d}\partial^j \\ &= \mathbf{d}(\omega_{ij}\partial^i \wedge \mathbf{d}x^j) \end{aligned}.$$

The corresponding ∞-Chern-Weil homomorphism, 4.4.17, that we need to compute is given by the ∞-anafunctor

$$\begin{array}{ccccc} \exp(\mathfrak{P})_{\mathrm{diff}} & \xrightarrow{\exp(\omega)} & \exp(b^3\mathbb{R})_{\mathrm{dR}} & \xrightarrow{\int_{\Delta^\bullet}} & \flat_{dR}\mathbf{B}^3\mathbb{R} \\ \downarrow{\simeq} & & & & \\ \exp(\mathfrak{P}) & & & & \end{array}\;.$$

Over a test space $U \in \mathrm{CartSp}$ and in degree 1 an element in $\exp(\mathfrak{P})_{\mathrm{diff}}$ is a pair (X^i, η^i)

$$X^i \in C^\infty(U \times \Delta^1)$$

$$\eta^i \in \Omega^1_{\mathrm{vert}}(U \times \Delta^1)$$

subject to the constraint that along Δ^1 we have

$$d_{\Delta^1} X^i + \eta^i_{\Delta^1} = 0\,.$$

The vertical morphism $\exp(\mathfrak{P})_{\mathrm{diff}} \to \exp(\mathfrak{P})$ has in fact a section whose image is given by those pairs for which η^i has no leg along U. We therefore find the desired form on $\exp(\mathfrak{P})$ by evaluating the top morphism on pairs of this form.

Such a pair is taken by the top morphism to

$$\begin{aligned}(X^i, \eta^j) \mapsto & \int_{\Delta^1} \omega_{ij} F_{X^i} \wedge F_{\partial^j} \\ &= \int_{\Delta^1} \omega_{ij} (d_{dR} X^i + \eta^i) \wedge d_{\mathrm{dR}} \eta^j \in \Omega^3(U)\end{aligned} .$$

Using the above constraint and the condition that η^i has no leg along U, this becomes

$$\cdots = \int_{\Delta^1} \omega_{ij} d_U X^i \wedge d_U d_{\Delta^1} X^j \,.$$

By the Stokes theorem the integration over Δ^1 yields

$$\begin{aligned}\cdots &= \omega_{ij} d_{\mathrm{dR}} X^i \wedge d_{\mathrm{dR}} X^j|_0 - \omega_{ij} d_{\mathrm{dR}} X^i \wedge d_{\mathrm{dR}} X^j|_1 \\ &= \partial_1^* \omega - \partial_0^* \omega\end{aligned} .$$

□

5.6 Higher geometric prequantization

We discuss here the application of cohesive higher geometric prequantization, 3.9.13, to the natural action functionals that we found above in 5.7 and 5.8.

- 5.6.1 – Prequantum mechanics;
- 5.6.2 – Prequantum 2d field theory;
- 5.6.3 – Prequantum Chern-Simons theory;
- 5.6.4 – Prequantization of symplectic Lie n-algebroids

5.6.1 $n = 1$ – prequantum mechanics

Let $V = \mathbb{C}$ be the 0-groupoid of complex numbers and $V//U(1)$ the action groupoid with respect to the standard action.

Proposition 5.6.1. *For $P \to X$ a principal $U(1)$-bundle, we have that $\Gamma(X, P \times_{U(1)} \mathbb{C})$ is the ordinary space of smooth sections of the associated line bundle.*

Corollary 5.6.2. *For $n = 1$ the definition of prequantum operators in def. 3.9.102 is the traditional one.*

5.6.2 $n = 2$ – prequantum 2d field theory

Let $V = \mathrm{Core}(\mathrm{Vect}(-)) \in \mathrm{Smooth}\infty\mathrm{Grpd}$ be the maximal groupoid-valued stack inside the stack of smooth vector bundles of finite rank. Let $V//\mathbf{B}U(1) \to \mathbf{B}^2U(1)$ be the canonical action.

Proposition 5.6.3. *For given circle 2-bundle $P \to X$, the groupoid $\Gamma(X, P \times_{\mathbf{B}U(1)} V)$ is the groupoid of P-twisted vector bundles on X, discussed in 4.4.8.*

5.6.3 $n = 3$ – prequantum Chern-Simons theory

Let G be a simply connected semisimple Lie group. The Lagrangian for G-Chern-Simons theory is refined to the moduli stack of G-connections

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}} .$$

Proposition 5.6.4. *Let Σ_3 be a compact smooth manifold of dimension 3. Then the composite*

$$\exp(iS(-)) : \ [\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\hat{\mathbf{c}}} [\Sigma_3, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

is the action functional of Chern-Simons theory.

Proof. By theorem 5.1.9. □

Proposition 5.6.5. *Let Σ_2 be a smooth manifold of dimension 2. Then the curvature 4-form of the circle 3-bundle with connection given by the the composite*

$$\Sigma_2 \times [\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}G_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}} \mathbf{B}^3U(1)_{\mathrm{conn}}$$

is the canonical symplectic current plus terms whose fiber integral over Σ_2 vanishes.

It follows that the transgression of the Chern-Simons circle 3-bundle $\hat{\mathbf{c}}$ to the phase space $[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}]$ is the prequantum circle bundle with connection for ordinary Chern-Simons theory.

5.6.4 Prequantization of symplectic Lie n-algebroids

By the discussion in 5.5, the symplectic form on a symplectic n-groupoid, def. 5.5.14, may be regarded as the image of an invariant polynomial under the unrefined ∞-Chern-Weil homomorphism, 4.4.17,

$$\omega : X \to \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R} .$$

Therefore the passage to the prequantum n-bundle with connection on X corresponds to passing to the *refined* ∞-Chern-Weil homomorphism

$$\hat{\omega} : X \to \mathbf{B}^n U(1)_{\mathrm{conn}} .$$

Definition 5.6.6. Let (X, ω) be a symplectic ∞-groupoid, def. 5.5.3. Then ω represents a class

$$[\omega] \in H^{n+1}_{\mathrm{dR}}(X) .$$

We say this form is *integral* if it is in the image of the curvature-projection,

$$\mathrm{curv} : H_{\mathrm{diff}}(X, \mathbf{B}^n U(1)) \to H^{n+1}_{\mathrm{dR}}(X)$$

from the ordinary differential cohomology, 4.4.16, of X

In this case we say a *prequantum circle n-bundle with connection* for (X, ω) is a lift of ω to $\mathbf{H}_{\mathrm{diff}}(X, \mathbf{B}^{n+1}U(1))$.

Write $\hat{X} \to X$ for the underlying circle $(n+1)$-group-principal ∞-bundle.

Proposition 5.6.7. *If (X, ω) indeed comes from the Lie integration of a symplectic Lie n-algebroid $(\mathfrak{P}, \omega)$ such that the periods of the L_∞-cocycle π that ω transgresses to are integral, then $\hat{X}$ is the Lie integration of the L_∞-extension, def. 4.4.102,*

$$b^n\mathbb{R} \to \hat{\mathfrak{P}} \to \mathfrak{P}$$

classified by π:

$$\hat{X} \simeq \tau_{n+1} \exp(\hat{\mathfrak{P}}) .$$

Example 5.6.8. For $n = 1$ this reduces to the discussion in [WeXu91].

Example 5.6.9. For $\mathfrak{g}$ a semisimple Lie algebra with quadratic invariant polynomial ω, the pair $(b\mathfrak{g}, \omega)$ is a symplectic Lie 2-algebroid (Courant Lie 2-algebroid) over the point.

In this case the infinitesimal prequantum line 2-bundle is the delooping of the string Lie 2-algebra, def. 5.1.15

$$\widehat{b\mathfrak{g}} \simeq b\mathfrak{string}$$

and the prequantum circle 2-group-principal 2-bundle is the delooping of the smooth string 2-group, def. 5.1.10

$$(\hat{X} \to X) = (\mathbf{B}\mathrm{String} \to \mathbf{B}G) .$$

5.7 Higher extended Chern-Simons theory

We consider the realization of the general abstract ∞-*Chern-Simons functionals* from 3.9.11 in the context of smooth, synthetic-differential and super-cohesion. We discuss general aspects of the class of quantum field theories defined this way and then identify a list of special cases of interest. This section builds on [FRS11a] and [FRS11b].

- 5.7.1 – Higher extended ∞-Chern-Simons theory
  - 5.7.1.1 – Fiber integration and extended Chern-Simons functionals
  - 5.7.1.2 – Construction from L_∞-cocycles
- 5.7.2 – Higher cup-product Chern-Simons theories
- Examples
  - 5.7.3 – Volume holonomy
  - 5.7.4 – 1d Chern-Simons functionals
  - 5.7.5 – 3d Chern-Simons functionals
    - 5.7.5.1 – Ordinary Chern-Simons theory
    - 5.7.5.3 – Ordinary Dijkgraaf-Witten theory
  - 5.7.6 – 4d Chern-Simons functionals
    - 5.7.6.1 – 4d BF theory and topological Yang-Mills theory
    - 5.7.6.2 – 4d Yetter model
  - 5.7.7 – Abelian gauge coupling of branes
  - 5.7.8 – Higher abelian Chern-Simons functionals
    - 5.7.8.1 – $(4k+3)$d $U(1)$-Chern-Simons functionals;
    - 5.7.8.2 – Higher electric coupling and higher gauge anomalies.
  - 5.7.9.2 – 7d Chern-Simons functionals
    - 5.7.9.1 – The cup product of a 3d CS theory with itself;
    - 5.7.9.2 – 7d CS theory on string 2-connection fields;
    - 5.7.9.3 – 7d CS theory in 11d supergravity on AdS_7.
  - 5.7.8.2 – Higher electric coupling and higher gauge anomalies
  - 5.7.10 – Action of closed string field theory type
  - 5.7.11 – AKSZ σ-models
    - 5.7.11.3 – Ordinary Chern-Simons as AKSZ theory
    - 5.7.11.4 – Poisson σ-model
    - 5.7.11.5 – Courant σ-model
    - 5.7.11.6 – Higher abelian Chern-Simons theory in dimension $4k+3$

5.7.1 ∞-Chern-Simons field theory

By prop. 5.1.9 the action functional of ordinary Chern-Simons theory [Fre] for a simple Lie group G may be understood as being the volume holonomy, 4.4.19, of the Chern-Simons circle 3-bundle with connection that the refined Chern-Weil homomorphism assigns to any connection on a G-principal bundle.

We may observe that all the ingredients of this statement have their general abstract analogs in any cohesive ∞-topos $\mathbf{H}$: for any cohesive ∞-group G and any representatative $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n A$ of a characteristic class for G there is canonically the induced ∞-Chern-Weil homomorphism, 3.9.7

$$L_{\mathbf{c}} : \mathbf{H}_{\mathrm{conn}}(-, \mathbf{B}G) \to \mathbf{H}^n_{\mathrm{diff}}(-)$$

that sends intrinsic G-connections to cocycles in intrinsic differential cohomology with coefficients in A. This may be thought of as the *Lagrangian* of the ∞-Chern-Simons theory induced by $\mathbf{c}$.

In the cohesive ∞-topos Smooth∞Grpd of smooth ∞-groupoids, 4.4, we deduced in 4.4.19 a natural general abstract procedure for integration of $L_{\mathbf{c}}$ over an n-dimensional parameter space $\Sigma \in \mathbf{H}$ by a realization of the general abstract construction described in 3.9.11. The resulting smooth function

$$\exp(S_{\mathbf{c}}) : [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \to U(1)$$

is the exponentiated action functional of ∞-Chern-Simons theory on the smooth ∞-groupoid of field configurations. It may be regarded itself as a degree-0 characteristic class on the space of field configurations. As such, its differential refinement $d\exp(S_{\mathbf{c}}) : [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \to \flat_{\mathrm{dR}}\mathbf{B}U(1)$ is the Euler-Lagrange equation of the theory.

We show that this construction subsumes the action functional of ordinary Chern-Simons theory, of Dijkgraaf-Witten theory, of BF-theory coupled to topological Yang-Mills theory, of all versions of AKSZ theory including the Poisson sigma-model and the Courant sigma model in lowest degree, as well as of higher Chern-Simons supergravity.

5.7.1.1 Fiber integration and extended Chern-Simons functionals We discuss fiber integration in ordinary differential cohomology refined to smooth higher stacks and how this turns every differential characteristic maps into a tower of extended higher Chern-Simons action functionals in all codimensions.

This section draws from [FiSaScIV].

One of the basic properties of ∞-toposes is that they are *cartesian closed*. This means that:

Fact 5.7.1. *For every two objects $X, A \in \mathbf{H}$ – hence for every two smooth higher stacks – there is another object denoted $[X, A] \in \mathbf{H}$ that behaves like the "space of smooth maps from X to A." in that for every further $Y \in \mathbf{H}$ there is a natural equivalence of cocycle ∞-groupoids of the form*

$$\mathbf{H}(X \times Y, A) \simeq \mathbf{H}(Y, [X, A]) \,,$$

saying that cocycles with coefficients in $[X, A]$ on Y are naturally equivalent to A-cocycles on the product $X \times Y$.

Remark 5.7.2. The object $[X, A]$ is in category theory known as the *internal hom* object, but in applications to physics and stacks it is often better known as the "families version" of A-cocycles on Y, in that for each smooth parameter space $U \in$ SmthMfd, the elements of $[X, A](U)$ are "U-parameterized families of A-cocycles on X", namely A-cocycles on $X \times U$. This follows from the above characterizing formula and the Yoneda lemma:

$$[X, A](U) \xrightarrow[\text{Yoneda}]{\simeq} \mathbf{H}(U, [X, A]) \xrightarrow{\simeq} \mathbf{H}(X \times U, A) \,.$$

Notably for G a smooth ∞-group and $A = \mathbf{B}G_{\mathrm{conn}}$ a moduli ∞-stack of smooth G-principal ∞-bundles with connection the object

$$[\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \in \mathbf{H}$$

is the smooth higher moduli stack of G-connection *on* Σ_k. It assigns to a test manifold U the ∞-groupoid of U-parameterized families of G-∞-connections, namely of G-∞-connections on $X \times U$. This is the smooth higher stack incarnation of the configuration space of higher G-gauge theory on Σ_k.

Example 5.7.3. In the discussion of anomaly polynomials in heterotic string theory over a 10-dimensional spacetime X one encounters degree-12 differential forms $I_4 \wedge I_8$, where I_i is a degree i polynomial in characteristic forms. Clearly these cannot live on X, as every 12-form on X, given by an element in the hom-∞-groupoid

$$\mathbf{H}(X, \Omega^{12}(-)) \xrightarrow[\text{Yoneda}]{\simeq} \Omega^{12}(X)$$

is trivial. Instead, these differential forms are elements in the internal hom $[X, \Omega^{12}(-)]$, which means that for every choice of smooth parameter space U there is a smooth 12-form on $X \times U$, such that this system of forms transforms naturally in U.

Below we discuss how such anomaly forms appear from morphisms of higher moduli stacks

$$\mathbf{c}_{\text{conn}} : \mathbf{B}G_{\text{conn}} \to \mathbf{B}^{11}U(1)_{\text{conn}}$$

for $\mathbf{B}G_{\text{conn}}$ the higher moduli stack of supergravity field configurations by sending the families of moduli of field configurations on spacetime X to their anomaly form:

$$[X, \mathbf{B}G_{\text{conn}}] \xrightarrow{[X, \mathbf{c}_{\text{conn}}]} [X, \mathbf{B}^{11}U(1)_{\text{conn}}] \xrightarrow{[X, \text{curv}]} [X, \Omega^{12}(-)] \; .$$

We now discuss how such families of n-cocycles on some X can be integrated over X to yield $(n - \dim(X))$-cocycles.

Proposition 5.7.4. *Let Σ_k be a closed (= compact and without boundary) oriented smooth manifold of dimension k. Then for every $n \geq k$ there is a natural morphism of smooth higher stacks*

$$\exp(2\pi i \int_{\Sigma_k} (-)) : [\Sigma_k, \mathbf{B}^n U(1)_{\text{conn}}] \to \mathbf{B}^{n-k}U(1)_{\text{conn}}$$

from the moduli n-stack of circle n-bundles with connection on Σ_k to the moduli $(n-k)$-stack of smooth circle $(n-k)$-bundles with connection such that

1. *for $k = n$ this yields a $U(1)$-valued gauge invariant smooth function*

$$\exp(2\pi i \int_{\Sigma_k} (-)) : [\Sigma_n, \mathbf{B}^n U(1)_{\text{conn}}] \to U(1) \; ,$$

   which is the n-volume holonomy *of a circle n-connection over the "n-dimensional Wilson volume" Σ_n;*

2. *for $k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 \leq n$ we have*

$$\exp(2\pi i \int_{\Sigma_{k_1}} (-)) \circ \exp(2\pi i \int_{\Sigma_{k_2}} (-)) \simeq \exp(2\pi i \int_{\Sigma_{k_1} \times \Sigma_{k_2}} (-)) \, .$$

Proof. Since $\mathbf{B}^n U(1)_{\text{conn}}$ is fibrant in the projective local model structure $[\text{CartSp}^{\text{op}}, \text{sSet}]_{\text{proj,loc}}$ (since every circle n-bundle with connection on a Cartesian space is trivializable) the mapping stack $[\Sigma_k, \mathbf{B}^n U(1)_{\text{conn}}]$ is presented for any choice of good open cover $\{U_i \to \Sigma_k\}$ by the simplicial presheaf

$$U \mapsto [\text{CartSp}^{\text{op}}, \text{sSet}](\check{C}(\mathcal{U}) \times U, \mathbf{B}^n U(1)_{\text{conn}}) \, ,$$

where $\check{C}(\mathcal{U})$ is the Čech nerve of the open cover $\{U_i \to \Sigma_k\}$. Therefore a morphism as claimed is given by natural fiber integration of Deligne hypercohomology along product bundles $\Sigma_k \times U \to U$ for closed Σ_k. This has been constructed for instance in [GoTe00]. □

Definition 5.7.5. Let $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ be a differential characteristic map. Then for Σ_k a closed smooth manifold of dimension $k \leq n$, we call

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) : \; [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_k, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k} U(1)_{\mathrm{conn}}$$

the *off-shell prequantum* $(n-k)$*-bundle of extended* $\mathbf{c}_{\mathrm{conn}}$*-*$\infty$*-Chern-Simons theory.* For $n = k$ we have a *circle 0-bundle*

$$\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\mathrm{conn}}]) : \; [\Sigma_n, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_n, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_n, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_n}(-))} U(1) \;\;,$$

which we call the *action functional* of the theory.

This construction subsumes several fundamental aspects of Chern-Simons theory:

1. gauge invariance and smoothness of the (extended) action functionals, remark 5.7.6;
2. inclusion of instanton sectors (nontrivial gauge ∞-bundles), remark 5.7.7;
3. level quantization, remark 5.7.8;
4. definition on non-bounding manifolds and relation to (higher) topological Yang-Mills on bounding manifolds, remark 5.7.9.

We discuss these in more detail in the following remarks, as indicated.

Remark 5.7.6 (Gauge invariance and smoothness)**.** Since $U(1) \in \mathbf{H}$ is an ordinary manifold (after forgetting the group structure), a 0-stack with no non-trivial morphisms (no gauge transformation), the action functional $\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\mathrm{conn}}])$ takes every morphism in the moduli stack of field configurations to the identity. But these morphisms are the *gauge transformations*, and so this says that $\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\mathrm{conn}}])$ is *gauge invariant*, as befits a gauge theory action functional. To make this more explicit, notice that

$$\mathbf{H}(\Sigma_n, \mathbf{B}G_{\mathrm{conn}}) \simeq [\Sigma_n, \mathbf{B}G_{\mathrm{conn}}](*)$$

is the evaluation of the moduli stack on the point, hence the ∞-groupoid of smooth families of field configurations which are trivially parameterized. Moreover

$$H^1_{\mathrm{conn}}(\Sigma_n, G) := \pi_0 \mathbf{H}(\Sigma_n, \mathbf{B}G_{\mathrm{conn}})$$

is the set of gauge equivalent such field configurations. Then the statement that the action functional is both gauge invariant and smooth is the statement that it can be extended from $H^1_{\mathrm{conn}}(\Sigma_n, G)$ (supposing that it were given there as a function $\exp(iS(-))$ by other means) via $\mathbf{H}(\Sigma_n, \mathbf{B}G_{\mathrm{conn}})$ to $[\Sigma_n, \mathbf{B}G_{\mathrm{conn}}]$

$$\begin{array}{cccc}
H^1_{\mathrm{conn}}(\Sigma_n, G) & \xrightarrow{\exp(iS(-))} & U(1) & \\
\downarrow & & \nearrow & \\
\mathbf{H}(\Sigma_n, \mathbf{B}G_{\mathrm{conn}}) & \nearrow_{\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\mathrm{conn}}])} & & \text{gauge invariance} \\
\downarrow & & & \\
[\Sigma_n, \mathbf{B}G_{\mathrm{conn}}] & & & \text{smoothness .}
\end{array}$$

Remark 5.7.7 (Definition on instanton sectors)**.** Ordinary 3-dimensional Chern-Simons theory is often discussed for the special case only when the gauge group G is connected and simply connected. This yields a drastic simplification compared to the general case; since for every Lie group the second homotopy group $\pi_2(G)$ is trivial, and since the homotopy groups of the classifying space BG are those of G shifted up in degree by one, this implies that BG is 3-connected and hence that every continuous map $\Sigma_3 \to BG$ out of a 3-manifold is homotopic to the trivial map. This implies that every G-principal bundle over Σ_3 is trivializable. As a result, the moduli stack of G-gauge fields on Σ_3, which a priori is $[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}]$, becomes in this case equivalent to just the moduli stack of trivial G-bundles with (non-trivial) connection on Σ_3, which is identified with the groupoid of just $\mathfrak{g}$-valued 1-forms on Σ_3, and gauge transformations between these, which is indeed the familiar configurations space for 3-dimensional G-Chern-Simons theory.

One should compare this to the case of 4-dimensional G-gauge theory on a 4-dimensional manifold Σ_4, such as G-Yang-Mills theory. By the same argument as before, in this case G-principal bundles may be nontrivial, but are classified enirely by the second Chern class (or first Pontrjagin class) $[c_2] \in H^4(\Sigma_4, \pi(G))$. In Yang-Mills theory with $G = SU(n)$, this class is known as the *instanton number* of the gauge field.

The simplest case where non-trivial classes occur already in dimension 3 is the non-simply connected gauge group $G = U(1)$, discussed in section 5.7.5.2 below. Here the moduli stack of fields $[\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}]$ contains configurations which are not given by globally defined 1-forms, but by connections on non-trivial circle bundles. By analogy with the case of $SU(n)$-Yang-Mills theory, we will loosely refer to such field configurations as instanton field congurations, too. In this case it is the first Chern class $[c_1] \in H^2(X, \mathbb{Z})$ that measures the non-triviality of the bundle. If the first Chern-class of a $U(1)$-gauge field configurations happens to vanish, then the gauge field is again given by just a 1-form $A \in \Omega^1(\Sigma_3)$, the familiar gauge potential of electromagnetism. The value of the 3d Chern-Simons action functional on such a non-instanton configuration is simply the familiar expression

$$\exp(iS(A)) = \exp(2\pi i \int_{\Sigma_3} A \wedge d_{\mathrm{dR}} A)\,,$$

where on the right we have the ordinary integration of the 3-form $A \wedge dA$ over Σ_3.

In the general case, however, when the configuration in $[\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}]$ has non-trivial first Chern class, the expression for the value of the action functional on this configuration is more complicated. If we pick a good open cover $\{U_i \to \Sigma_3\}$, then we can arrange that locally on each patch U_i the gauge field is given by a 1-form A_i and the contribution of the action functional over U_i by $\exp(2\pi i \int_{\Sigma_3} A_i \wedge dA_i)$ as above. But in such a decomposition there are further terms to be included to get the correct action functional. This is what the construction in Prop. 5.7.5 achieves.

Remark 5.7.8 (Level quantization)**.** Traditionally, Chern-Simons theory in 3-dimensions with gauge group a connected and simply connected group G comes in a family parameterized by a *level* $k \in \mathbb{Z}$. This level is secretly the cohomology class of the differential characteristic map

$$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

(constructed in [FSS10]) in

$$H^3_{\mathrm{smooth}}(BG, U(1)) \simeq H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}\,.$$

So the traditional level is a cohomological shadow of the differential characteristic map that we interpret as the off-shell prequantum n-bundle in full codimension n (down on the point). Notice that for a general smooth ∞-group G the cohomology group $H^{n+1}(BG, \mathbb{Z})$ need not be equivalent to $\mathbb{Z}$ and so in general the level need not be an integer. For for every smooth ∞-group G, and given a morphism of moduli stacks $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, also every integral multiple $k\mathbf{c}_{\mathrm{conn}}$ gives an n-dimensional Chern-Simons theory, "at k-fold level". The converse is in general hard to establish: whether a given $\mathbf{c}_{\mathrm{conn}}$ can be divided by an integer. For instance for 3-dimensional Chern-Simons theory division by 2 may be possible for Spin-structure. For 7-dimensional Chern-Simons theory division by 6 may be possible in the presence of String-structure [FiSaScIII].

Remark 5.7.9. Ordinary 3-dimensional Chern-Simons theory is often defined on bounding 3-manifolds Σ_3 by

$$\exp(iS(\nabla)) = \exp(2\pi i k \int_{\Sigma_4} \langle F_{\widehat{\nabla}} \wedge F_{\widehat{\nabla}} \rangle)\,,$$

where Σ_4 is any 4-manifold with $\Sigma_3 = \partial \Sigma_4$ and where $\widehat{\nabla}$ is any extension of the gauge field configuration from Σ_3 to Σ_4. Similar expressions exist for higher dimensional Chern-Simons theories. If one takes these expressions to be the actual definition of Chern-Simons action functional, then one needs extra discussion for which manifolds (with desired structure) are bounding, hence which vanish in the respective cobordism ring, and, more seriously, one needs to include those which are not bounding from the discussion. For example, in type IIB string theory one encounters the cobordism group $\Omega_{11}^{\mathrm{Spin}}(K(\mathbb{Z},6))$ [Wi96], which is proven to vanish in [KS05], meaning that all the desired manifolds happen to be bounding.

We emphasize that our formula in Prop. 5.7.5 applies generally, whether or not a manifold is bounding. Moreover, it is guaranteed that *if* Σ_n happens to be bounding after all, then the action functional is equivalently given by integrating a higher curvature invariant over a bounding $(n+1)$-dimensional manifold. At the level of differential cohomology classes $H^n_{\mathrm{conn}}(-, U(1))$ this is the well-known property (a review and further pointers are given in [HoSi05]) which is an explicit axiom in the equivalent formulation by Cheeger-Simons differential characters: a Cheeger-Simons differential character of degree $(n+1)$ is by definition a group homomorphism from closed n-manifolds to $U(1)$ such that whenever the n-manifold happens to be bounding, the value in $U(1)$ is given by the exponentiated integral of a smooth $(n+1)$-form over any bounding manifold.

With reference to such differential characters Chern-Simons action functions have been formulated for instance in [Wi96, Wi98b]. The sheaf hypercohomology classes of the Deligne complex that we are concerned with here are well known to be equivalent to these differential characters, and Čech-Deligne cohoomology has the advantage that with results such as [GoTe00] invoked in Prop. 5.7.4 above, it yields explict formulas for the action functional on non-bounding manifolds in terms of local differential form data.

5.7.1.2 Construction from L_∞-cocycles We discuss the construction of ∞-Chern-Simons functionals from differential refinements of L_∞-algebra cocycles.

This section draws from [FiSaScI].

Recall for the following the construction of the ∞-Chern-Weil homomorphism by Lie integration of Chern-Simons elements, 4.4.17, for L_∞-algebroids, 4.5.1.

A Chern-Simons element cs witnessing the transgression from an invariant polynomial $\langle - \rangle$ to a cocycle μ is equivalently a commuting diagram of the form

$$\begin{array}{ccccc} \mathrm{CE}(\mathfrak{a}) & \stackrel{\mu}{\longleftarrow} & \mathrm{CE}(b^n\mathbb{R}) & & \text{cocycle} \\ \uparrow & & \uparrow & & \\ \mathrm{W}(\mathfrak{a}) & \stackrel{cs}{\longleftarrow} & \mathrm{W}(b^n\mathbb{R}) & & \text{Chern-Simons element} \\ \uparrow & & \uparrow & & \\ \mathrm{inv}(\mathfrak{a}) & \stackrel{\langle - \rangle}{\longleftarrow} & \mathrm{inv}(b^n\mathbb{R}) & & \text{invariant polynomial} \end{array}$$

in $\mathrm{dgAlg}_{\mathbb{R}}$. On the other hand, an n-connection with values in a Lie n-algebroid $\mathfrak{a}$ is a span of simplicial presheaves

$$\begin{array}{ccc} \hat{\Sigma} & \stackrel{\nabla}{\longrightarrow} & \mathbf{cosk}\exp(\mathfrak{a})_{\mathrm{conn}} \\ \downarrow{\simeq} & & \\ \Sigma & & \end{array}$$

with coefficients in the simplicial presheaf $\mathbf{cosk}_{n+1}\exp(\mathfrak{a})_{conn}$, def. 4.4.123, that sends $U \in \mathrm{CartSp}$ to the $(n+1)$-coskeleton, def. 3.6.28, of the simplicial set, which in degree k is the set of commuting diagrams

$$\begin{array}{ccccc}
\Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{a}) & & \text{transition function} \\
\uparrow & & \uparrow & & \\
\Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{a}) & & \text{connection forms} \\
\uparrow & & \uparrow & & \\
\Omega^\bullet(U) & \xleftarrow{\langle F_A \rangle} & \mathrm{inv}(\mathfrak{a}) & & \text{curvature characteristic forms}
\end{array} ,$$

such that the curvature forms F_A of the ∞-Lie algebroid valued differential forms A on $U \times \Delta^k$ with values in $\mathfrak{a}$ in the middle are horizontal.

If μ is an ∞-Lie algebroid cocycle of degree n, then the ∞-Chern-Weil homomorphism operates by sending an ∞-connection given by a Čech cocycle with values in simplicial sets of such commuting diagrams to the obvious pasting composite

$$\begin{array}{ccccccc}
\Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{a}) & \xleftarrow{\mu} & \mathrm{CE}(b^n\mathbb{R}) & : \mu(A_{\mathrm{vert}}) & \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{a}) & \xleftarrow{cs} & \mathrm{W}(b^n\mathbb{R}) & : \mathrm{cs}(A) & \text{Chern-Simons form} \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^\bullet(U) & \xleftarrow{\langle F_A \rangle} & \mathrm{inv}(\mathfrak{a}) & \xleftarrow{\langle - \rangle} & \mathrm{inv}(b^n\mathbb{R}) & : \langle F_A \rangle & \text{curvature}
\end{array} .$$

Under the map to the coskeleton the group of such cocycles for line n-bundle with connection is quotiented by the discrete group Γ of periods of μ, such that the ∞-Chern-Weil homomorphism is given by sending the ∞-connection ∇ to

$$\begin{array}{ccccc}
\hat{\Sigma} & \xrightarrow{\nabla} & \mathbf{cosk}_n \exp(\mathfrak{a})_{\mathrm{conn}} & \xrightarrow{\exp(\mathrm{cs})} & \mathbf{B}^n(\mathbb{R}/\Gamma)_{\mathrm{conn}} \\
\downarrow{\simeq} & & & & \\
\Sigma & & & &
\end{array} .$$

This presents a circle n-bundle with connection, 4.4.16, whose connection n-form is locally given by the Chern-Simons form $\mathrm{cs}(A)$. This is the Lagrangian of the ∞-Chern-Simons theory defined by $(\mathfrak{a}, \langle - \rangle)$ and evaluated on the given ∞-connection. If Σ is a smooth manifold of dimension n, then the higher holonomy, 4.4.19, of this circle n-bundle over Σ is the value of the Chern-Simons action. After a suitable gauge transformation this is given by the integral

$$\exp(iS(A)) = \exp(i \int_\Sigma \mathrm{cs}(A)) ,$$

the value of the ∞-Chern-Simons action functional on the ∞-connection A.

Proposition 5.7.10. *Let $\mathfrak{g}$ be an L_∞-algebra and $\langle -, \cdots, - \rangle$ an invariant polynomial on $\mathfrak{g}$. Then the ∞-connections A with values in $\mathfrak{g}$ that satisfy the equations of motion of the corresponding ∞-Chern-Simons theory are precisely those for which*

$$\langle -, F_A \wedge F_A \wedge \cdots F_A \rangle = 0 ,$$

as a morphism $\mathfrak{g} \to \Omega^\bullet(\Sigma)$, where F_A denotes the (in general inhomogeneous) curvature form of A.

In particular for binary and non-degenerate invariant polynomials the equations of motion are

$$F_A = 0 .$$

Proof. Let $A \in \Omega(\Sigma \times I, \mathfrak{g})$ be a 1-parameter variation of $A(t=0)$, that vanishes on the boundary $\partial\Sigma$. Here we write $t : [0,1] \to \mathbb{R}$ for the canonical coordinate on the interval.

$A(0)$ is critical if

$$\left(\frac{d}{dt}\int_\Sigma \mathrm{cs}(A)\right)_{t=0} = 0$$

for all extensions A of $A(0)$. Using Cartan's magic formula and the Stokes theorem the left hand expression is

$$\begin{aligned}
\left(\frac{d}{dt}\int_\Sigma \mathrm{cs}(A)\right)_{t=0} &= \left(\int_\Sigma \frac{d}{dt}\mathrm{cs}(A)\right)_{t=0} \\
&= \left(\int_\Sigma d\iota_{\partial t}\mathrm{cs}(A) + \int_\Sigma \iota_{\partial_t} d\mathrm{cs}(A)\right)_{t=0} \\
&= \left(\int_\Sigma d_\Sigma(\iota_{\partial t}\mathrm{cs}(A)) + \int_\Sigma \iota_{\partial_t}\langle F_A \wedge \cdots F_A\rangle\right)_{t=0} . \\
&= \left(\int_{\partial\Sigma} \iota_{\partial t}\mathrm{cs}(A) + n\int_\Sigma \langle (\frac{d}{dt}A) \wedge \cdots F_A\rangle\right)_{t=0} \\
&= \left(n\int_\Sigma \langle (\frac{d}{dt}A) \wedge \cdots F_A\rangle\right)_{t=0}
\end{aligned}$$

Here we used that $\iota_{\partial_t} F_A = \frac{d}{dt}A$ and that by assumption this vanishes on $\partial\Sigma$. Since $\frac{d}{dt}A$ can have arbitrary values, the claim follows. □

5.7.2 Higher cup-product Chern-Simons theories

We discuss a class of ∞-Chern-Simons functionals induced from a smooth differential refinement of the *cup-product* on integral cohomology.

This section draws from [FiSaScIV].

5.7.2.1 General construction A crucial property of the Dold-Kan map, as discussed in 2.2.6, is the following.

Proposition 5.7.11. *Let A, B and C be presheaves of chain complexes concentrated in non-negative degrees, and let $\cup : A \otimes B \to C$ be a morphism of presheaves of chain complexes. Then the Dold-Kan map induces a natural morphism of simplicial preseheaves $\cup_{\mathrm{DK}} : \mathrm{DK}(A) \times \mathrm{DK}(B) \to \mathrm{DK}(C)$*

Proof. Both the categories $\mathrm{Ch}_\bullet^+$ and sAb are monoidal categories under the respective standard tensor products (on $\mathrm{Ch}_\bullet^+$ this is given by direct sums of tensor products of abelian groups with fixed total degree and on sAb by the degreewise tensor product of abelian groups), and the functor Γ is lax monoidal with respect to these structures, i.e., for any $V, W \in \mathrm{Ch}_\bullet^+$ we have natural weak equivalences

$$\nabla_{V,W} : \Gamma(V) \otimes \Gamma(W) \to \Gamma(V \otimes W) .$$

These are not isomorphisms, as they would be for a *strong* monoidal functor, but they are weak equivalences. The forgetful functor F is the right adjoint to the functor forming degreewise the free abelian group on a set, therefore it preserves products and hence there are natural isomorphisms

$$F(V \times W) \xrightarrow{\simeq} F(V) \times F(W) ,$$

for all $V, W \in$ sAb. Finally, by the definition of tensor product, there are universal natural quotient maps $V, W \in$ sAb

$$p_{V,W} : V \times W \to V \otimes W .$$

The morphism $\cup_{\mathrm{DK}}$ is then defined as the composition indicated in the following diagram:

$$
\begin{array}{ccccccc}
\mathrm{DK}(A)\times \mathrm{DK}(B) & & & \xrightarrow{\quad\cup_{\mathrm{DK}}\quad} & & & \mathrm{DK}(C) \\
\| & & & & & & \| \\
F(\Gamma(A))\times F(\Gamma(B)) & & & & & & \| \\
\downarrow{\simeq} & & & & & & \| \\
F(\Gamma(A)\times \Gamma(B)) & \xrightarrow{F(p)} & F(\Gamma(A)\otimes \Gamma(B)) & \xrightarrow{F(\nabla)} & F(\Gamma(A\otimes B)) & \xrightarrow{F(\Gamma(\cup))} & F(\Gamma(C))\,.
\end{array}
$$

Given the presentation $\mathbf{H} \simeq L_W[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]$, for every presheaf of chain complexes A on $\mathcal{C}$ we obtain a corresponding ∞-stack, the ∞-stackification of the image of A under the Dold-Kan map, which we will denote by the same symbol: $\mathrm{DK}(A) \in \mathbf{H}$.

Definition 5.7.12. For $A \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}]$ a sheaf of abelian groups, we write $A[n] \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}_\bullet^+]$ for the corresponding presheaf of chain complexes concentrated on A in degree n, and

$$\mathbf{B}^n A \simeq \mathrm{DK}(A[n]) \in \mathbf{H}$$

for the corresponding ∞-stack.

In this case the corresponding cohomology

$$H^n(X, A) = \pi_0 \mathbf{H}(X, \mathbf{B}^n A)$$

is the traditional *sheaf cohomology* of X with coefficients in A. More generally, if $A \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}_\bullet^+]$ is a sheaf of chain complexes not necessarily concentrated in one degree, then

$$H^0(X, A) := \pi_0 \mathbf{H}(X, A)$$

is what traditionally is called the *sheaf hypercohomology* of X with coefficients in A. The central coefficient object in which we are interested here is the sheaf of chain complexes called the *Deligne complex*, to which we now turn.

The *Beilinson-Deligne cup product* is an explicit presentation of the cup product in ordinary differential cohomology for the case that the latter is modeled by the Čech-Deligne cohomology.

Definition 5.7.13. The Beilinson-Deligne cup product is the morphism of sheaves of chain complexes

$$\cup_{\mathrm{BD}} : \mathbb{Z}[p+1]^\infty_D \otimes \mathbb{Z}[q+1]^\infty_D \;\longrightarrow\; \mathbb{Z}[(p+1)+(q+1)]^\infty_D,$$

given on homogeneous elements α, β as follows:

$$\alpha \cup_{\mathrm{BD}} \beta := \begin{cases} \alpha \wedge \beta = \alpha\beta & \text{if } \deg(\alpha) = p+1\,. \\ \alpha \wedge d_{\mathrm{dR}}\beta & \text{if } \deg(\alpha) \leq p \text{ and } \deg(\beta) = 0 \\ 0 & \text{otherwise}\,. \end{cases} .$$

Remark 5.7.14. When restricted to the diagonal in the case that $p = q$, this means that the cup product sends a p-form α to the $(2p+1)$-form $\alpha \wedge d_{\mathrm{dR}}\alpha$. This is of course the local Lagrangian for cup product Chern-Simons theory of p-forms. We discuss this case in detail in section 5.7.8.1.

The Beilinson-Deligne cup product is associative and commutative up to homotopy, so it induces an associative and commutative cup product on ordinary differential cohomology. A survey of this can be found in [Bry00] (around Prop. 1.5.8 there).

Definition 5.7.15. For $p, q \in \mathbb{N}$ the morphism of simplicial presheaves

$$\cup_{\text{conn}} : \mathbf{B}^p U(1)_{\text{conn}} \times \mathbf{B}^q U(1)_{\text{conn}} \to \mathbf{B}^{p+q+1} U(1)_{\text{conn}}$$

is the morphism associated to the Beilinson-Deligne cup product $\cup_{\text{BD}} : \mathbb{Z}[p+1]^\infty_D \otimes \mathbb{Z}[q+1]^\infty_D \longrightarrow \mathbb{Z}[p+q+2]^\infty_D$ by Proposition 5.7.11.

Since the Beilinson-Deligne cup product is associative up to homotopy, this induces a well defined morphism

$$\mathbf{B}^{n_1} U(1)_{\text{conn}} \times \mathbf{B}^{n_2} U(1)_{\text{conn}} \times \cdots \times \mathbf{B}^{n_{k+1}} U(1)_{\text{conn}} \to \mathbf{B}^{n_1 + \cdots + n_{k+1} + k} U(1)_{\text{conn}}.$$

In particular, if $n_1 = \cdots = n_{k+1} = 3$, we find

$$\left(\mathbf{B}^3 U(1)_{\text{conn}}\right)^{k+1} \to \mathbf{B}^{4k+3} U(1)_{\text{conn}}.$$

Furthermore, we see from the explicit expression of the Beilinson-Deligne cup product that, on a local chart U, if the 3-form datum of a connection on a $U(1)$-3-bundle is the 3-form C, then the $4k+3$-form local datum for the corresponding connection on the associated $U(1)$-$(4k+3)$-bundle is

$$C \wedge \underbrace{dC \wedge \cdots \wedge dC}_{k \text{ times}}. \tag{5.24}$$

5.7.3 Higher differential Dixmier-Douady class and higher dimensional $U(1)$-holonomy

The *degenerate* or rather *tautological* case of extended ∞-Chern-Simons theories nevertheless deserves special attention, since it appears universally in all other examples: that where the extended action functional is the *identity* morphism

$$(\mathbf{DD}_n)_{\text{conn}} : \ \mathbf{B}^n U(1)_{\text{conn}} \xrightarrow{\text{id}} \mathbf{B}^n U(1)_{\text{conn}} \ ,$$

for some $n \in \mathbb{N}$. Trivial as this may seem, this is the differential refinement of what is called the (higher) *universal Dixmier-Douady class* the higher universal first Chern class – of circle n-bundles / bundle $(n-1)$-gerbes, which on the topological classifying space $B^n U(1)$ is the weak homotopy equivalence

$$\text{DD}_n : \ B^n U(1) \xrightarrow{\simeq} K(\mathbb{Z}, n+1) \ .$$

Therefore, we are entitled to consider $(\mathbf{DD}_n)_{\text{conn}}$ as the extended action functional of an n-dimensional ∞-Chern-Simons theory. Over an n-dimensional manifold Σ_n the moduli n-stack of field configurations is that of circle n-bundles with connection on Σ_n. In generalization to how a circle 1-bundle with connection has a *holonomy* over closed 1-dimensional manifolds, we note that a circle n-connection has a *n-volume holonomy* over the n-dimensional manifold Σ_n. This is the ordinary (codimension-0) action functional associated to $(\mathbf{DD}_n)_{\text{conn}}$ regarded as an extended action functional:

$$\text{hol} := \exp(2\pi i \int_{\Sigma_n} [\Sigma_n, (\mathbf{DD}_n)_{\text{conn}}]) : [\Sigma_n, \mathbf{B}^n U(1)_{\text{conn}}] \to U(1) \, .$$

This formulation makes it manifest that, for G any smooth ∞-group and $\mathbf{c}_{\text{conn}} : \mathbf{B}G_{\text{conn}} \to \mathbf{B}^n U(1)_{\text{conn}}$ any extended ∞-Chern-Simons action functional in codimension n, the induced action functional is indeed the n-volume holonomy of a family of "Chern-Simons circle n-connections", in that we have

$$\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\text{conn}}]) \simeq \text{hol}_{\mathbf{c}_{\text{conn}}} \, .$$

This is most familiar in the case where the moduli ∞-stack $\mathbf{B}G_{\text{conn}}$ is replaced with an ordinary smooth oriented manifold X (of any dimension and not necessarily compact). In this case $\mathbf{c}_{\text{conn}} : X \to \mathbf{B}^n U(1)_{\text{conn}}$ modulates a circle n-bundle with connection ∇ on this smooth manifold. Now regarding this as an extended Chern-Simons action function in codimension n means to

1. take the moduli stack of fields over a given closed oriented manifold Σ_n to be $[\Sigma_n, X]$, which is simply the space of maps between these manifolds, equipped with its natural ("diffeological") smooth structure (for instance the smooth loop space LX when $n = 1$ and $\Sigma_n = S^1$);

2. take the value of the action functional on a field configuration $\phi : \Sigma_n \to X$ to be the n-volume holonomy of ∇

$$\mathrm{hol}_\nabla(\phi) = \exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\mathrm{conn}}]) : \; [\Sigma_n, X] \xrightarrow{[\Sigma_n, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_n, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi \int_{\Sigma_n}(-))} U(1) \; .$$

Using the proof of Prop. 5.7.4 to unwind this in terms of local differential form data, this reproduces the familiar formulas for (higher) $U(1)$-holonomy.

5.7.4 1d Chern-Simons functionals

We discuss examples of the intrinsic notion of ∞-Chern-Simons action functionals, 4.4.19, over 1-dimensional base spaces.

Example 5.7.16. For some $n \in \mathbb{N}$ let

$$\mathrm{tr} : \mathfrak{u}(n) \to \mathfrak{u}(1) \simeq \mathbb{R}$$

be the trace function, with respect to the canonical identification of $\mathfrak{u}(n)$ with the Lie algebra of skew-Hermitean complex matrices.

This is both a 1-cocycle as well as an invariant polynomial on $\mathfrak{u}(n)$, the former corresponding to a degree-1 element in the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{u}(n))$ and the latter corresponding to an element $d_{\mathrm{W}} c \in \mathrm{W}(\mathfrak{u}(n))$ of degree 2 in the Weil algebra. Hence c is also the corresponding Chern-Simons element, def. 4.4.116. By prop. 5.4.62 this controls the universal differential first Chern class.

The corresponding Chern-Simons action functional is defined on the groupoid of $\mathfrak{u}(n)$-valued differential 1-forms on a line segment Σ and given by

$$A \mapsto \int_\Sigma \mathrm{tr}(A) \, .$$

Any choice of coordinates $\Sigma \hookrightarrow \mathbb{R}$ canonically identifies $A \in \Omega^1(\Sigma, \mathfrak{u}(n))$ with a $\mathfrak{u}(n)$-valued function ϕ. We may think of $\bar{\phi} := \int_\Sigma A = \int_\Sigma \phi dt$ as the average of this function. In terms of this the action functional is simply the trace function itself

$$\bar{\phi} \mapsto \mathrm{tr}(\bar{\phi}) \, .$$

Degenerate as this case is, it is sometimes useful to regard the trace as an example of 1-dimensional Chern-Simons theory, for instance in the context of large-N compactified gauge theory as discussed in [Na06].

Example 5.7.17. Below in 5.7.11 we discuss in detail how (derived) L_∞-algebroids equipped with non-degenerate binary invariant polynomials of *grade* 0 (hence total degree 2) give rise to 1-dimensional Chern-Simons theories.

For derived L_∞-algebroids of the form $T^*\mathbf{B}\mathfrak{g}$ the resulting QFT is discussed in detail in [GrGw11].

5.7.5 3d Chern-Simons functionals

We discuss examples of the intrinsic notion of ∞-Chern-Simons action functionals, 4.4.19, over 3-dimensional base spaces. This includes the archetypical example of ordinary 3-dimensional Chern-Simons theory, but also its discrete analog, Dijkgraaf-Witten theory.

- 5.7.5.1 – Ordinary Chern-Simons theory;
- 5.7.5.3 – Ordinary Dijkgraaf-Witten theory.

5.7.5.1 Ordinary Chern-Simons theory for simply connected simple gauge group

We discuss the action functional of ordinary 3-dimensional Chern-Simons theory (see [Fre] for a survey) from the point of view of intrinsic Chern-Simons action functionals in Smooth∞Grpd.

5.7.5.1.1 Extended Lagrangian and action functional

Theorem 5.7.18. *Let G be a simply connected compact simple Lie group. For*

$$[c] \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$$

a universal characteristic class that generates the degree-4 integral cohomology of the classifying space BG, there is an essentially unique smooth lift $\mathbf{c}$ of the characteristic map c of the form

$$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3 U(1) \qquad \in \mathrm{Smooth}\infty\mathrm{Grpd}$$

on the smooth moduli stack $\mathbf{B}G$ of smooth G-principal bundles with values in the smooth moduli 3-stack of smooth circle 3-bundles. The differential refinement

$$\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} \qquad \in \mathrm{Smooth}\infty\mathrm{Grpd}$$

to the moduli stacks of the corresponding n-bundles with n-connections induces over any any closed oriented 3-dimensional smooth manifold Σ a smooth functional

$$\exp(iS_{\mathrm{CS}}(-)) := \exp(2\pi i \int_\Sigma [\Sigma, \mathbf{L}]) : \ [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\hat{\mathbf{c}}} [\Sigma, \mathbf{B}^3 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_\Sigma (-))} U(1)$$

on the moduli stack of G-principal connections on Σ, which on objects $A \in \Omega^1(\Sigma, \mathfrak{g})$ is the exponentiated Chern-Simons action functional

$$\exp(iS_{\mathrm{CS}}(A)) = \exp(i \int_\Sigma \langle A \wedge d_{\mathrm{dR}} A\rangle + \frac{1}{6}\langle A \wedge [A \wedge A]\rangle) \,.$$

Proof. This is theorem 5.1.9 combined with 4.4.125. □

For more computational details that go into this see also 5.7.11.3 below

5.7.5.1.2 The extended phase spaces

Let G be a connected and simply connected Lie group. We discuss the nature of the moduli of G-principal connections $G\mathbf{Conn}(\Sigma)$ according to 4.4.16.3 for various choices of Σ.

Proposition 5.7.19. *There is an equivalence*

$$\mathrm{hol} : \ G\mathbf{Conn}(S^1) \xrightarrow{\simeq} G//_{\mathrm{Ad}} G$$

in Smooth∞Grpd *between the moduli stack of G-principal connections on the circle, def. 4.4.94, and the quotient groupoid of the adjoint action of G on itself. This is given by sending G-principal connections to their holonomy (for any chosen basepoint on the circle).*

Proof. We show that for each $U \in \mathrm{CartSp}$ the morphism of groupoids hol_U is an equivalence of groupoids.

For $f : U \to G$ a smooth function, since G is connected and U is topologically contractible, we may find a smooth homotopy

$$\eta : [0,1] \times U \to G$$

with $\eta(0)$ constant on the neutral element in a neighbourhood of $\{0\} \times U$ and with $\eta(1) = f$ in a neighbourhood of $\{1\} \times U$. Let then $\eta \mathbf{d}_{[0,1]} \eta^{-1} \in \Omega^1(U \times S^1, \mathfrak{g})$. This is a connection 1-form on $U \times S^1$ whose holonomy is f. Hence hol_U is essentially surjective.

Next, consider $A, A' \in \Omega^1(U \times S^1, \mathfrak{g})$ two connection 1-forms (legs along S^1). Observe that for each point $u \in U$ a gauge transformation $g_u : A_u \to A'_u$ is fixed already by its value at the basepoint of S^1 and moreover it has to satisfy

$$\mathrm{hol}(A_u) = g_u \mathrm{hol}(A'_u) g_u^{-1}\,.$$

This is because for every $t \in [0,1]$ the gauge transformation needs to satisfy the parallel transprt naturality condition

$$\begin{array}{ccc} * & \xrightarrow{g_u(t)} & * \\ {\scriptstyle \mathrm{tra}_{A_u}(0,t)}\uparrow & & \uparrow{\scriptstyle \mathrm{tra}_{A'_u}(0,t)} \\ * & \xrightarrow[g_u(0)]{} & * \end{array} \qquad \in *//G\,,$$

where $\mathrm{tra}_{A_u}(0,t)$ is the parallel transport of the connection A_u along $[0,t]$.

This says that hol_U is also full and faithful. Hence it is an equivalence. □

Remark 5.7.20. We have a dashed lift in

$$\begin{array}{ccc} & & [S^1, \mathbf{B}G_{\mathrm{conn}}] \\ & \nearrow & \downarrow{\scriptstyle \mathrm{conc}} \\ & & G\mathbf{Conn}(S^1) \\ & & {\scriptstyle \simeq}\downarrow{\scriptstyle \mathrm{hol}} \\ G & \longrightarrow & G//_{\mathrm{Ad}}G \end{array}\,,$$

where the top right morphism is the canonical projection of remark 3.9.51, and where the bottom horizontal morphism is the canonical projection map.

Proposition 5.7.21. *There is an equivalence*

$$G\mathbf{Conn}(*) \simeq \mathbf{B}G\,.$$

5.7.5.2 Ordinary 3d $U(1)$-Chern-Simons theory and generalized B_n-geometry Ordinary 3-dimensional $U(1)$-Chern-Simons theory on a closed oriented manifold Σ_3 contains field configurations which are given by globally defined 1-forms $A \in \Omega^1(\Sigma_3)$ and on which the action functional is given by the familiar expression

$$\exp(iS(A)) = \exp(2\pi i k \int_{\Sigma_3} A \wedge d_{\mathrm{dR}} A)\,.$$

More generally, though, a field configuration of the theory is a connection ∇ on a $U(1)$-principal bundle $P \to \Sigma_3$ and this simple formula is modified, from being the exponential of the ordinary integral of the wedge product of two differential forms, to the fiber integration in differential cohomology, Def. 5.7.4, of the differential cup-product, Def. 5.7.15:

$$\exp(iS(\nabla)) = \exp(2\pi i k \int_{\Sigma_3} \nabla \cup_{\mathrm{conn}} \nabla)\,.$$

This defines the action functional on the set $H^1_{\mathrm{conn}}(\Sigma_3, U(1))$ of equivalence classes of $U(1)$-principal bundles with connection

$$\exp(iS(-)) : H^1_{\mathrm{conn}}(\Sigma_3) \to U(1)\,.$$

That the action functional is gauge invariant means that it extends from a function on gauge equivalence classes to a functor on the groupoid $\mathbf{H}^1_{\mathrm{conn}}(\Sigma_3, U(1))$, whose objects are actual $U(1)$-principal connections, and whose morphsims are smooth gauge transformations between these:

$$\exp(iS(-)) : \mathbf{H}^1_{\mathrm{conn}}(\Sigma_3) \to U(1)\,.$$

Finally, that the action functional depends *smoothly* on the connections means that it extends further to the moduli stack of fields to a morphism of stacks

$$\exp(iS(-)) : [\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}] \to U(1)\,.$$

The fully extended prequantum circle 3-bundle of this extended 3d Chern-Simons theory is that of the two-species theory restricted along the diagonal $\Delta : \mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}} \times \mathbf{B}U(1)_{\mathrm{conn}}$. This is the homotopy fiber of the smooth cup square in these degrees.

According to [Hi12] aspects of the differential geometry of the homotopy fiber of a differential refinement of this cup square are captured by the "generalized geometry of B_n-type" that was suggested in section 2.4 of [Ba11]. In view of the relation of the same structure to differential T-duality discussed above one is led to expect that "generalized geometry of B_n-type" captures aspects of the differential cohomology on fiber products of torus bundles that exhibit auto T-duality on differential K-theory. Indeed, such a relation is pointed out in [Bo11][11].

5.7.5.3 Ordinary Dijkgraaf-Witten theory Dijkgraaf-Witten theory (see [FrQu93] for a survey) is commonly understood as the analog of Chern-Simons theory for discrete structure groups. We show that this becomes a precise and systematic statement in Smooth∞Grpd: the Dijkgraaf-Witten action functional is that induced from applying the ∞-Chern-Simons homomorphism to a characteristic class of the form $\mathrm{Disc}BG \to \mathbf{B}^3U(1)$, for $\mathrm{Disc} : \infty\mathrm{Grpd} \to \mathrm{Smooth}\infty\mathrm{Grpd}$ the canonical embeedding of discrete ∞-groupoids, 4.1, into all smooth ∞-groupoids.

Let $G \in \mathrm{Grp} \to \infty\mathrm{Grpd} \stackrel{\mathrm{Disc}}{\to} \mathrm{Smooth}\infty\mathrm{Grpd}$ be a discrete group regarded as an ∞-group object in discrete ∞-groupoids and hence as a smooth ∞-groupoid with discrete smooth cohesion. Write $BG = K(G,1) \in \infty\mathrm{Grpd}$ for its delooping in ∞Grpd and $\mathbf{B}G = \mathrm{Disc}BG$ for its delooping in Smooth∞Grpd.

We also write $\Gamma\mathbf{B}^nU(1) \simeq K(U(1), n)$. Notice that this is different from $B^nU(1) \simeq \Pi\mathbf{B}U(1)$, reflecting the fact that $U(1)$ has non-discrete smooth structure.

Proposition 5.7.22. *For G a discrete group, morphisms $\mathbf{B}G \to \mathbf{B}^nU(1)$ correspond precisely to cocycles in the ordinary group cohomology of G with coefficients in the discrete group underlying the circle group*

$$\pi_0\mathrm{Smooth}\infty\mathrm{Grpd}(\mathbf{B}G, \mathbf{B}^nU(1)) \simeq H^n_{\mathrm{Grp}}(G, U(1))\,.$$

Proof. By the $(\mathrm{Disc} \dashv \Gamma)$-adjunction we have

$$\mathrm{Smooth}\infty\mathrm{Grpd}(\mathbf{B}G, \mathbf{B}^nU(1)) \simeq \infty\mathrm{Grpd}(BG, K(U(1), n))\,.$$

□

Proposition 5.7.23. *For G discrete*

- *the intrinsic de Rham cohomology of $\mathbf{B}G$ is trivial*

$$\mathrm{Smooth}\infty\mathrm{Grpd}(\mathbf{B}G, \flat_{\mathrm{dR}}\mathbf{B}^nU((1)) \simeq *;$$

[11] Thanks, once more, to Alexander Kahle, for discussion of this point, at *String-Math 2012*.

- *all G-principal bundles have a unique flat connection*

$$\mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}G) \simeq \mathrm{Smooth}\infty\mathrm{Grpd}(\Pi(X), \mathbf{B}G)\,.$$

Proof. By the $(\mathrm{Disc} \dashv \Gamma)$-adjunction and using that $\Gamma \circ \flat_{\mathrm{dR}} K \simeq *$ for all K. □

It follows that for G discrete

- any characteristic class $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ is a group cocycle;
- the ∞-Chern-Weil homomorphism coincides with postcomposition with this class

$$\mathbf{H}(\Sigma, \mathbf{B}G) \to \mathbf{H}(\Sigma, \mathbf{B}^n U(1))\,.$$

Proposition 5.7.24. *For G discrete and $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3 U(1)$ any group 3-cocycle, the ∞-Chern-Simons theory action functional on a 3-dimensional manifold Σ*

$$\mathrm{Smooth}\infty\mathrm{Grpd}(\Sigma, \mathbf{B}G) \to U(1)$$

is the action functional of Dijkgraaf-Witten theory.

Proof. By proposition 4.4.125 the morphism is given by evaluation of the pullback of the cocycle $\alpha : BG \to B^3 U(1)$ along a given $\nabla : \Pi(\Sigma) \to BG$, on the fundamental homology class of Σ. This is the definition of the Dijkgraaf-Witten action (for instance equation (1.2) in [FrQu93]).

□

5.7.6 4d Chern-Simons functionals

We discuss some 4-dimensional Chern-Simons functionals

- 5.7.6.1 – 4d BF theory and topological Yang-Mills;
- 5.7.6.2 – 4d Yetter model.

5.7.6.1 BF theory and topological Yang-Mills theory We discuss how the action functional of nonabelian *BF-theory* [Hor89] in 4-dimensions with a "cosmological constant" and coupled to topological Yang-Mills theory is a higher Chern-Simons theory.

Let $\mathfrak{g} = (\mathfrak{g}_2 \stackrel{\partial}{\to} \mathfrak{g}_1)$ be a strict Lie 2-algebra, coming from a differential crossed module, def. 1.2.46, as indicated. Let $\exp(\mathfrak{g})$ be the universal Lie integration, according to def. 4.4.53. Field configurations with values in $\exp(\mathfrak{g})$ are locally Lie 2-algebra valued forms $(A \in \Omega^1(\Sigma, \mathfrak{g}_0))$ and $B \in \Omega^2(\Sigma, \mathfrak{g}_1)$ as in prop. 1.2.86.

The following observation is due to [SSS09a].

Proposition 5.7.25. *We have*

1. *every invariant polynomial $\langle - \rangle_{\mathfrak{g}_1} \in \mathrm{inv}(\mathfrak{g}_1)$ on $\mathfrak{g}_1$ gives rise, under the canonical inclusion $\mathrm{inv}(\mathfrak{g}_1) \hookrightarrow \mathrm{W}(\mathfrak{g})$, not to an invariant polynomial, but to a Chern-Simons element on $\mathfrak{g}$, exhibiting the transgression to a trivial L_∞-algebra cocycle;*
2. *for $\mathfrak{g}_1$ a semisimple Lie algebra and $\langle - \rangle_{\mathfrak{g}_1}$ the Killing form, Σ a 4-dimensional compact manifold, the corresponding Chern-Simons action functional*

$$\exp(iS_{\langle - \rangle_{\mathfrak{g}_1}}) : [\Sigma, \exp(\mathfrak{g})_{\mathrm{conn}}] \to \mathbf{B}^4 \mathbb{R}_{\mathrm{conn}}$$

on Lie 2-algebra valued forms is

$$\Omega^\bullet(X) \xleftarrow{(A,B)} \mathrm{W}(\mathfrak{g}_2 \to \mathfrak{g}_1) \xleftarrow{(\langle - \rangle_{\mathfrak{g}_1}, d_W \langle - \rangle_{\mathfrak{g}_1})} \mathrm{W}(b^{n-1}\mathbb{R})$$

the sum of the action functionals of topological Yang-Mills theory with BF-theory with cosmological constant:

$$\mathrm{cs}_{\langle - \rangle_{\mathfrak{g}_1}}(A,B) = \langle F_A \wedge F_A \rangle_{\mathfrak{g}_1} - 2\langle F_A \wedge \partial B\rangle_{\mathfrak{g}_1} + 2\langle \partial B \wedge \partial B\rangle_{\mathfrak{g}_1}\,,$$

where F_A is the ordinary curvature 2-form of A.

Proof. For $\{t_a\}$ a basis of $\mathfrak{g}_1$ and $\{b_i\}$ a basis of $\mathfrak{g}_2$ we have

$$d_{\mathrm{W}(\mathfrak{g})} : \mathbf{d}t^a \mapsto d_{\mathrm{W}(\mathfrak{g}_1)} + \partial^a{}_i \mathbf{d}b^i\,.$$

Therefore with $\langle - \rangle_{\mathfrak{g}_1} = P_{a_1\cdots a_n}\mathbf{d}r^{a_1} \wedge \cdots \mathbf{d}t^{a_n}$ we have

$$d_{\mathrm{W}(\mathfrak{g})}\langle - \rangle_{\mathfrak{g}_1} = nP_{a_1\cdots a_n}\partial^{a_1}{}_i \mathbf{d}b^i \wedge \cdots \mathbf{d}t^{a_n}\,.$$

The right hand is a polynomial in the shifted generators of $\mathrm{W}(\mathfrak{g})$, and hence an invariant polynomial on $\mathfrak{g}$. Therefore $\langle - \rangle_{\mathfrak{g}_1}$ is a Chern-Simons element for it.

Now for $(A,B) \in \Omega^1(U \times \Delta^k, \mathfrak{g})$ an L_∞-algebra-valued form, we have that the 2-form curvature is

$$F^1_{(A,B)} = F_A - \partial B\,.$$

Therefore

$$\begin{aligned}\mathrm{cs}_{\langle - \rangle_{\mathfrak{g}_1}}(A,B) &= \langle F^1_{(A,B)} \wedge F^1_{(A,B)}\rangle_{\mathfrak{g}_1} \\ &= \langle F_A \wedge F_A \rangle_{\mathfrak{g}_1} - 2\langle F_A \wedge \partial B\rangle_{\mathfrak{g}_1} + 2\langle \partial B \wedge \partial B\rangle_{\mathfrak{g}_1}\end{aligned}.$$

□

5.7.6.2 4d Yetter model The discussion of 3-dimensional Dijkgraaf-Witten theory as in 5.7.5.3 goes through verbatim for discrete groups generalized to discrete ∞-groups G, 4.1.2, and cocycles $\alpha : \mathbf{B}G \to \mathbf{B}^n U(1)$ of any degree n. A field configurations over an n-dimensional manifold Σ is a G-principal ∞-bundle, 4.1.4, necessarily flat, and the induced action functional

$$\exp(iS_\alpha) : \mathbf{H}(\Sigma, \mathbf{B}G) \to U(1)$$

sends a G-principal ∞-bundle classified by a cocycle $g : \Sigma \to \mathbf{B}G$ to the canonical pairing of the singular cocycle corresponging to $\alpha(g) : \Sigma \to \mathbf{B}G \stackrel{\alpha}{\to} \mathbf{B}^n U(1)$ with the fundamental class of Σ.

For $n = 4$ such action functionals sometimes go by the name "Yetter model" [Mack00][MaPo07], in honor of [Yet93], which however did non consider a nontrivial 4-cocycle.

5.7.7 Abelian gauge coupling of branes

The gauge coupling term in the action of an $(n-1)$-brane charged under an abelian n-form background gauge field (electromagnetism, B-field, C-field, etc.) is an example of an ∞-Chern-Simons functional. We spell this out in a moment. Here one typically considers the target space of the $(n-1)$-brane to be a smooth manifold or at most an orbifold. The formal structure, however, allows to consider target spaces that are arbitrary smooth ∞-groupoids / smooth ∞-stacks. When generalized to this class of target spaces, the class of brane gauge coupling functionals in fact coincides with that of *all* ∞-Chern-Simons functionals. Conversely, every ∞-Chern-Simons theory in dimension n may be regarded as the field theory of a "topological $(n-1)$-brane" whose target space is the higher moduli stack of field configurations of the given ∞-Chern-Simons theory.

For X a smooth manifold, let $c \in H^{n+1}(X, \mathbb{Z})$ be a class in integral cohomology, to be called the higher *background magnetic charge*. A smooth refinement of this class to a morphism

$$\mathbf{c} : X \to \mathbf{B}^n U(1)$$

is a circle n-bundle on X, whose topological class is c

$$\hat{\mathbf{c}} : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

A differential refinement of this is a choice of refinement to a circle n-bundle with connection ∇.

Now let Σ the compact n-dimensional worldvolume of an $(n-1)$-brane. Then $[\Sigma, X]$ is the diffeological space (def. 4.4.14) of smooth maps $\phi : \Sigma \to X$. The induced ∞-Chern-Simons functional

$$\exp(iS_{\hat{\mathbf{c}}}) : [\Sigma, X] \xrightarrow{[\hat{\mathbf{c}},\Sigma]} [\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

is the ordinary n-volume holonomy of ∇ over trajectories $\phi : \Sigma \to X$.

5.7.8 Higher abelian Chern-Simons functionals

We discuss higher Chern-Simons functionals on higher abelian gauge fields, notably on circle n-bundles with connection.

- 5.7.8.1 – $(4k+3)$d $U(1)$-Chern-Simons functionals;
- 5.7.8.2 – Higher electric coupling and higher gauge anomalies.

5.7.8.1 $(4k+3)$d $U(1)$-Chern-Simons functionals We discuss higher dimensional abelian Chern-Simons theories in dimension $4k+3$.

The basic ideas can be found in [HoSi05]. We refine the discussion there from differential cohomology classes to higher moduli stacks of differential cocycles. The case in dimension 3 ($k=0$) is discussed for instance in [GuTh08]. The case in dimension 7 ($k=1$) is the higher Chern-Simons theory whose holographic boundary theory encodes the self-dual 2-form gauge theory on the single 5-brane [Wi97b]. Generally, for every k the $(4k+3)$-dimensional abelian Chern-Simons theory induces a self-dual higher gauge theory holographically on its boundary, see [BeMo06].

Proposition 5.7.26. *The cup product in integral cohomology*

$$(-) \cup (-) : H^{k+1}(-,\mathbb{Z}) \times H^{l+1}(-,\mathbb{Z}) \to H^{k+l+2}(-,\mathbb{Z})$$

has a smooth and differential refinement to the moduli ∞-stacks $\mathbf{B}^n U(1)_{\mathrm{conn}}$, prop. 4.4.88, for circle n-bundles with connection

$$(-)\hat{\cup}(-) : \mathbf{B}^k U(1)_{\mathrm{conn}} \times \mathbf{B}^l U(1)_{\mathrm{conn}} \to \mathbf{B}^{k+l+1} U(1)_{\mathrm{conn}} \,.$$

Proof. By the discussion in 4.4.16 we have that $\mathbf{B}^k U(1)_{\mathrm{conn}}$ is presented by the simplicial presheaf

$$\Xi \mathbb{Z}_D^\infty[k+1] \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}].\,,$$

which is the image of the Deligne-Beilinson complex, def. 1.2.102, under the Dold-Kan correspondence, prop. 2.2.6. A lift of the cup product to the Deligne complex is given by the *Deligne-Beilinson cup product* [Del71][Bel85]. Since the Dold-Kan functor $\Xi : [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Ch}_\bullet] \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ is right adjoint, it preserves products and hence this cup product. □

Definition 5.7.27. Let Σ be a compact manifold of dimension $4k+3$ for $k \in \mathbb{N}$. Consider the moduli stack $[\Sigma, \mathbf{B}^k U(1)_{\mathrm{conn}}]$ of circle $(2k+1)$-bundles with connection on Σ.

On this space, the action functional of higher abelian Chern-Simons theory is defined to be the composite

$$\exp(iS(-)) : \; [\Sigma, \mathbf{B}^{2k+1} U(1)_{\mathrm{conn}}] \xrightarrow{(-)\hat{\cup}(-)} [\Sigma, \mathbf{B}^{4k+3} U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1) \;.$$

Observation 5.7.28. When restricted to differential $(2k+1)$-forms, regarded as connections on trivial circle $(2k+1)$-bundles

$$\Omega^{2k+1}(\Sigma) \hookrightarrow [\Sigma, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}]$$

this action functional sends a $(2k+1)$-form C to

$$\exp(iS(C)) = \exp(i \int_\Sigma C \wedge d_{\mathrm{dR}} C)\,.$$

From this expression one sees directly why the corresponding functional is not interesting in the remaining dimensions, because for even degree forms we have $C \wedge dC = \frac{1}{2} d(C \wedge C)$ and hence for these the above functional would be constant.

5.7.8.2 Higher electric coupling and higher gauge anomalies The action functional of ordinary Maxwell electromagnetism in the presence of an electric background current involves a differential cup-product term similar to that in def. 5.7.27. This has a direct generalization to higher electromagnetic fields and the corresponding higher electric currents. If, moreover, a background *magnetic* current is present, then this action functional is, in general, anomalous. The "higher gauge anomalies" in higher dimensional supergravity theories arise this way. This is discussed in [Free00].

Here we refine this discussion from differential cohomology classes to higher moduli stacks of differential cocycles.

Definition 5.7.29. Let Σ be a compact smooth manifold of dimension d.

By prop. 5.7.26 the universal cup product class

$$(-) \cup (-) : B^n U(1) \times B^{d-n-1} U(1) \to B^d U(1)$$

for any $0 \leq n \leq d$ has a smooth and differential refinement $\hat{\cup}$. We write

$$\exp(iS_\cup) : \; [\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}} \times \mathbf{B}^{d-n-1} U(1)_{\mathrm{conn}}] \xrightarrow{(-)\hat{\cup}(-)} [\Sigma, \mathbf{B}^d U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

for the corresponding higher Chern-Simons action functional on the higher moduli stack of *pairs* consisting of an n-connection and an $(d-n-1)$-connection on Σ.

Remark 5.7.30. When restricted to pairs of differential forms

$$(B_1, B_2) \in \Omega^n(\Sigma) \times \Omega^{d-n-1}(\Sigma) \hookrightarrow [\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}} \times \mathbf{B}^{d-n-1} U(1)_{\mathrm{conn}}]$$

this functional sends

$$(B_1, B_2) \mapsto \exp(i \int_\Sigma B_1 \wedge dB_2)\,.$$

The higher Chern-Simons functional of def. 5.7.8.1 is the *diagonal* of this functional, where $B_1 = B_2$. We now consider another variant, where only B_1 is taken to vary, but B_2 is regarded as fixed.

Let X be an d-dimensional manifold. The configuration space of higher electromagnetic fields of degree n on X is the moduli stack of circle n-bundles with connection $[X, \mathbf{B}^n U(1)_{\mathrm{conn}}]$ on X.

Definition 5.7.31. An *electric background current* on X for degree p electromagnetism is a circle $(d-n-1)$-bundle with connection $\hat{j}_{\mathrm{el}} : X \to \mathbf{B}^{d-n-1} U(1)_{\mathrm{conn}}$.

The *electric coupling action functional* of the higher electromagnetic field in the presence of the background electric current is

$$\exp(iS_{\mathrm{el}}) : \; [X, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{(-)\hat{\cup}\hat{j}_{\mathrm{el}}} [X, \mathbf{B}^d U(1)_{\mathrm{conn}}] \xrightarrow{\int_X} U(1) \;\;,$$

where the first morphism is the differentially refined cup product from prop. 5.7.26.

Remark 5.7.32. For the case of ordinary Maxwell theory, with $n = 1$ and $d = 4$, the electric current is a circle 2-bundle with connection. Its curvature 3-form is traditionally denoted j_{el}. If X is equipped with Lorentzian structure, then its integral over a (compact) spatial slice is the background *electric charge*. Integrality of this value, following from the nature of differential cohomology, is the *Dirac charge quantization* that makes electric charge appear in integral multiples of a fixed unit charge.

For $A \in \Omega^1(X) \to [X, \mathbf{B}U(1)_{\mathrm{conn}}]$ a globally defined connection 1-form, the above action functional is given by

$$A \mapsto \exp(i \int_X A \wedge j_{\mathrm{el}}) \,.$$

In the limiting case that the background electric charge is that carried by a charged point particle, j_{el} is the current which is Poincaré-dual to the trajectory $\gamma : S^1 \to X$ of the particle. In this case the above goes to

$$\cdots \to \exp(i \int_\Sigma A) \,,$$

hence the line holonomy of A along the trajectory of the background charge.

(...)

5.7.9 7d Chern-Simons functionals

We discuss some higher Chern-Simons functionals over 7-dimensional parameter spaces.

- 5.7.9.1 – The cup product of a 3d CS theory with itself;
- 5.7.9.2 – 7d CS theory on string 2-connection fields;
- 5.7.9.3 – 7d CS theory in 11d supergravity on AdS_7.

This section draws from [FiSaScIII].

5.7.9.1 The cup product of a 3d CS theory with itself Let G be a compact and simply connected simple Lie group and consider from 5.7.5.1 the canonical differential characteristic map for the induced 3d Chern-Simons theory

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}} \,.$$

We consider the differentially refined *cup product*, prop. 5.7.26, of this differential characteristic map with itself.

Observation 5.7.33. The topological degree-8 class

$$c \cup c : \; BG \xrightarrow{(c,c)} K(\mathbb{Z},4) \times K(\mathbb{Z},4) \xrightarrow{\cup} K(\mathbb{Z},8)$$

has a smooth and differential refinement of the form

$$\hat{\mathbf{c}} \hat{\cup} \hat{\mathbf{c}} : \; \mathbf{B}G_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}} \mathbf{B}^3U(1)_{\mathrm{conn}} \times \mathbf{B}^3U(1)_{\mathrm{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^7U(1)_{\mathrm{conn}} \,.$$

Proof. By the discussion in 5.7.8.1. □

Definition 5.7.34. Let Σ be a compact smooth manifold of dimension 7. The higher Chern-Simons functional

$$\exp(iS_{\mathrm{CS}}(-)) : \; [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\hat{\mathbf{c}} \hat{\cup} \hat{\mathbf{c}}} [\Sigma, \mathbf{B}^7U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

defines the *cup product Chern-Simons theory* induced by $\mathbf{c}$.

Remark 5.7.35. For ordinary Chern-Simons theory, 5.7.5.1, the assumption that G is simply connected implies that BG is 3-connected, hence that every G-principal bundle on a 3-dimensional Σ is trivializable, so that G-principal connections on Σ can be identified with $\mathfrak{g}$-valued differential forms on Σ. This is no longer in general the case over a 7-dimensional Σ.

Proposition 5.7.36. *If a field configuration $A \in [\Sigma, \mathbf{B}G_{\mathrm{conn}}]$ happens to have trivial underlying bundle, then the value of the cup product CS theory action functional is given by*

$$\exp(iS_{\mathrm{CS}}(A)) = \int_\Sigma \mathrm{CS}(A) \wedge \langle F_A \wedge F_A \rangle \,,$$

where $\mathrm{CS}(-)$ *is the Lagrangian of ordinary Chern-Simons theory, 5.7.5.1.*

5.7.9.2 7d CS theory on string 2-connection fields By theorem 5.1.32 we have a canonical differential characteristic map

$$\frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

from the smooth moduli 2-stack of String-2-connections, 1.2.13.7.2, with values in the smooth moduli 7-stack of circle 7-bundles (bundle 6-gerbes) with connection. This induces a 7-dimensional Chern-Simons theory.

Definition 5.7.37. For Σ a compact 7-dimensional smooth manifold, define $\exp(iS_{\frac{1}{6}p_2}(-))$ to be the Chern-Simons action functional induced by the universal differential second fractional Pontryagin class, theorem 5.1.32,

$$\exp(iS_{\frac{1}{6}p_2}(-)) : \ [\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}] \xrightarrow{\frac{1}{6}\hat{\mathbf{p}}_2} [\Sigma, \mathbf{B}^7 U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1) \ .$$

Recall from 1.2.13.7.2 the different incarnations of the local differential form data for string 2-connections.

Proposition 5.7.38. *Over a 7-dimensional Σ every field configuration $(A, B) \in [\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}]$ is a string 2-connection whose underlying* String*-principal 2-bundle is trivial.*

- *In terms of the strict $\mathfrak{string}$ Lie 2-algebra from def. 1.2.153 this is presented by a pair of nonabelian differential forms $A \in \Omega^1(\Sigma, P_*\mathfrak{so})$, $B \in \Omega^2(\Sigma, \hat{\Omega}_*\mathfrak{so})$. The above action functional takes this to*

$$\begin{aligned}
\exp(iS_{\frac{1}{6}p_2}(A, B)) &= \int_\Sigma \mathrm{CS}_7(A(1)) \\
&= \int_\Sigma (\langle A_e \wedge dA_e \wedge dA_e \wedge dA_e \rangle + k_1 \langle A_e \wedge [A_e \wedge A_e] \wedge dA_e \wedge dA_e \rangle \\
&\quad + k_2 \langle A_e \wedge [A_e \wedge A_e] \wedge [A_e \wedge A_e] \wedge dA_e \rangle + k_3 \langle A_e \wedge [A_e \wedge A_e] \wedge [A_e \wedge A_e] \wedge [A_e \wedge A_e] \rangle)
\end{aligned},$$

  where $A_e \in \Omega^1(\Sigma, \mathfrak{so})$ is the 1-form of endpoint values of A in the path Lie algebra, and where the integrand is the degree-7 Chern-Simons element of the quaternary invariant polynomial on $\mathfrak{so}$.

- *In terms of the skeletal $\mathfrak{string}$ Lie 2-algebra from def. 1.2.152 this is presented by a pair of differential forms $A \in \Omega^1(\Sigma, \mathfrak{so})$, $B \in \Omega^2(\Sigma, \mathbb{R})$. The above action functional takes this to*

$$\exp(iS_{\frac{1}{6}p_2}(A, B)) = \int_\Sigma \mathrm{CS}_7(A) \,.$$

5.7.9.3 7d CS theory in 11d supergravity on AdS_7 The two 7-dimensional Chern-Simons theories from 5.7.9.1 and 5.7.9.2 can be merged to a 7d theory defined on field configurations that are 2-connections with values in the String-2-group from def. 5.2.10. We define and dicuss this higher Chern-Simons theory below in 5.7.9.3.2. In 5.7.9.3.1 we argue that this 7d Chern-Simons theory plays a role in $\mathrm{AdS}_7/\mathrm{CFT}_6$-duality [AGMOO].

5.7.9.3.1 Motivation from $\mathrm{AdS}_7/\mathrm{CFT}_6$-holography We give here an argument that the 7-dimensional nonabelian gauge theory discussed in section 5.7.9.3.2 is the Chern-Simons part of 11-dimensional supergravity on $\mathrm{AdS}_7 \times S^4$ with 4-form flux on the S^4-factor and with quantum anomaly cancellation conditions taken into account. We moreover argue that this implies that the states of this 7-dimensional CS theory over a 7-dimensional manifold encode the conformal blocks of the 6-dimensional worldvolume theory of coincident M5-branes. The argument is based on the available but incomplete knowledge about AdS/CFT-duality, such as reviewed in [AGMOO], and cohomological effects in M-theory as reviewed and discussed in [Sa10a].

There are two, seemingly different, realizations of the *holographic principle* in quantum field theory. On the one hand, Chern-Simons theories in dimension $4k+3$ have spaces of states that can be identified with spaces of correlators of $(4k+2)$-dimensional conformal field theories (spaces of "conformal blocks") on their boundary. For the case $k=0$ this was discussed in [Wi89], for the case $k=1$ in [Wi96]. On the other hand, AdS/CFT duality (see [AGMOO] for a review) identifies correlators of d-dimensional CFTs with states of compatifications of string theory, or M-theory, on asymptotically anti-de Sitter spacetimes of dimension $d+1$ (see [Wi98a]).

In [Wi98b] it was pointed out that these two mechanisms are in fact closely related. A detailed analysis of the $\mathrm{AdS}_5/\mathrm{SYM}_4$-duality shows that the spaces of correlators of the 4-dimensional theory can be identified with the spaces of states obtained by geometric quantization just of the Chern-Simons term in the effective action of type II string theory on AdS_5, which locally reads

$$(B_{\mathrm{NS}}, B_{\mathrm{RR}}) \mapsto N \int_{\mathrm{AdS}_5} B_{\mathrm{NS}} \wedge dB_{\mathrm{RR}} \,,$$

where B_{NS} is the local Neveu-Schwarz 2-form field, B_{RR} is the local RR 2-form field, and where N is the RR 5-form flux picked up from integration over the S^5 factor.

As briefly indicated there, the similar form of the Chern-Simons term of 11-dimensional supergravity (M-theory) on AdS_7 suggests that an analogous argument shows that under $\mathrm{AdS}_7/\mathrm{CFT}_6$-duality the conformal blocks of the $(2,0)$-superconformal theory are identified with the geometric quantization of a 7-dimensional Chern-Simons theory. In [Wi98b] that Chern-Simons action is taken, locally on AdS_7, to be

$$C_3 \mapsto \int_{\mathrm{AdS}_7 \times S^4} C_3 \wedge G_4 \wedge G_4 \quad = N \int_{\mathrm{AdS}_7} C_3 \wedge dC_3 \,,$$

where now C_3 is the local incarnation of the supergravity C-field, 5.3.3.2, where G_4 is its curvature 4-form locally equal to dC_3, and where

$$N := \int_{S^4} G_4$$

is the C-field flux on the 4-spehere factor.

This is the $(4 \cdot 1 + 3 = 7)$-dimensional abelian Chern-Simons theory, 5.7.11.6, shown in [Wi96] to induce on its 6-dimensional boundary the self-dual 2-form – in the *abelian* case.

In order to generalize this to the nonabelian case of interest, we notice that there is a term missing in the above Lagrangian. The quantum anomaly cancellation in 11-dimensional supergravity is known from [DLM95](3.14) to require a corrected Lagrangian whose Chern-Simons term locally reads

$$(\omega, C_3) \mapsto \int_{\mathrm{AdS}_7 \times S^4} C_3 \wedge \left(G_4 \wedge G_4 - I_8^{\mathrm{dR}}(\omega)\right) \,,$$

where ω is the spin connection form, locally, and where $8 I_8^{\mathrm{dR}}(\omega)$ is a de Rham representative of the integral cohomology class

$$8 I_8 = \frac{1}{6} p_2 - 8 (\frac{1}{2} p_1) \cup (\frac{1}{2} p_1) \,, \tag{5.25}$$

with $\frac{1}{2}p_1$ and $\frac{1}{6}p_2$ the first and second fractional Pontrjagin classes, prop. 5.1.5, prop. 5.1.30, respectively, of the given Spin bundle over 11-dimensional spacetime X.

This means that after passing to the effective theory on AdS_7, this corrected Lagrangian picks up another 7-dimensional Chern-Simons term, now one depending on *nonablian* fields (with values in Spin and E_8). Locally this reads

$$S_{7d\mathrm{CS}} : (\omega, C_3) \mapsto N \int_{\mathrm{AdS}_7} C_3 \wedge dC_3 - \frac{N}{8} \int_{\mathrm{Ads}_7} \mathrm{CS}_{8I_8}(\omega) \quad . \tag{5.26}$$

where $\mathrm{CS}_{8I_8}(\omega)$ is a Chern-Simons form for $8I_8^{\mathrm{dR}}(\omega)$, defined locally by

$$d\mathrm{CS}_{8I_8}(\omega) = 8I_8^{\mathrm{dR}}(\omega) \,.$$

But this action functional, which is locally a functional of a 3-form and a Spin-connection, cannot globally be of this form, already because the field that looks locally like a Spin connection cannot globally be a Spin connection. To see this, notice from the discussion of the C-field in 5.4.8, that there is a quantization condition on the supergravity fields on the 11-dimensional X [Wi97a], which in cohomology requires the identity

$$2[G_4] = \frac{1}{2}p_1 + 2a \quad \in H^4(X, \mathbb{Z}) \,,$$

where on the right we have the canonical characteristic 4-class a, prop. 5.2.8, of an 'auxiliary' E_8 bundle on 11-dimensional spacetime. Moreover, we expect that when restricted to the vicinity of the asymptotic boundary of AdS_7,

- the class of G_4 vanishes;
- the E_8-bundle becomes equipped with a connection, too (the E_8-field "becomes dynamical");

in analogy to what happens at the boundary for the Hořava-Witten compactification of the 11-dimensional theory [HoWi95], as discussed in 5.4.8.6. Since, moreover, the states of the topological TFT that we are after are obtained already from geometric quantization, 3.9.13, of the theory in the vicinity $\Sigma \times I$ of a boundary Σ, we find the field configurations of the 7-dimensional theory are to satisfy the constraint in cohomology

$$\frac{1}{2}p_1 + 2a = 0 \,. \tag{5.27}$$

Imposing this condition has two effects.

1. The first is that, according to 3.9.8, what locally looks like a spin-connection is globally instead a *twisted differential String structure*, 5.4.7.3, or equivalently a *2-connection on a twisted* String-*principal 2-bundle*, where the twist is given by the class $2a$. By 1.2.5.3 the total space of such a principal 2-bundle may be identified with a (twisted) *nonabelian bundle gerbe*. Therefore the configuration space of fields of the effective 7-dimensional nonabelian Chern-Simons action above should not involve just Spin connection forms, but String-*2-connection* form data. By 1.2.13.7.2 there is a gauge in which this is locally given by nonabelian 2-form field data with values in the loop group of Spin.

2. The second effect is that on the space of twisted String-2-connections, the differential 4-form $\mathrm{tr}(F_\omega \wedge F_\omega)$, that under the Chern-Weil homomorphism represents the image of $\frac{1}{2}p_1$ in de Rham cohomology, according to 5.4.7.3.1, locally satisfies

   $$dH_3 = \langle F_\omega \wedge F_\omega \rangle - 2\langle F_A \wedge F_A \rangle \,,$$

   where H_3 is the 3-form curvature component of the String-2-connection, and where F_A is the curvature of a connection on the E_8 bundle, locally given by an $\mathfrak{e}_8$-valued 1-form A. Therefore with the quantization condition of the C-field taken into account, the 7-dimensional Chern-Simons action (5.26) becomes

$$S_{7d\mathrm{CS}} = N \int_{\mathrm{AdS}_7} \left(C_3 \wedge dC_3 - \frac{1}{8} H_3 \wedge dH_3 - \frac{1}{4}(H_3 + 2\mathrm{CS}_a(A) \wedge \mathrm{tr}(F_\omega \wedge F_\omega) + \frac{1}{8}\mathrm{CS}_{\frac{1}{6}\hat{\mathbf{p}}_2}(\omega) \right) \,. \tag{5.28}$$

Here the first two terms are 7-dimensional abelian Chern-Simons actions as before, for fields that are both locally abelian three forms (but have very different global nature). The second two terms, however, are action functionals for *nonabelian* Chern-Simons theories. The third term involves the familiar Chern-Simons 3-form of the E_8-connection familiar from 3-dimensional Chern-Simons theory

$$\mathrm{CS}_a(A) = \mathrm{tr}(A \wedge dA) + \frac{2}{3}\mathrm{tr}(A \wedge A \wedge A)\,.$$

Finally the fourth term is the Chern-Simons 7-form that is locally induced, under the Chern-Weil homomorphism, from the quartic invariant polynomial $\langle -,-,-,-\rangle : \mathfrak{so}^{\otimes 4} \to \mathbb{R}$ on the special orthogonal Lie algebra $\mathfrak{so}$, in direct analogy to how standard 3-dimensional Chern-Simons theory is induced under Chern-Weil theory from the quadratic invariant polynomial (the Killing form) $\langle -,-\rangle : \mathfrak{so} \otimes \mathfrak{so} \to \mathbb{R}$:

$$\begin{aligned}\mathrm{CS}_7(\omega) =& \langle \omega \wedge d\omega \wedge d\omega \wedge d\omega\rangle + k_1 \langle \omega \wedge [\omega \wedge \omega] \wedge d\omega \wedge d\omega\rangle \\ &+ k_2 \langle \omega \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \wedge d\omega\rangle + k_3 \langle \omega \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega]\rangle\end{aligned}\,.$$

This line of arguments suggests that the Chern-Simons term that governs 11-dimensional supergravity on $\mathrm{AdS}_7 \times S^4$ is an action functional on fields that are twisted String-2-connections such that the action functional is locally given by (5.28). In 5.7.9.3.2 we show that a Chern-Simons theory satisfying these properties naturally arises from the differential characteristic maps discussed above in 5.7.9.1 and 5.7.9.2.

5.7.9.3.2 Definition and properties We discuss now a twisted combination of the two 7-dimensional Chern-Simons action functionals from 5.7.9.1 and 5.7.9.2 which naturally lives on the moduli 2-stack $C\mathrm{Field}(-)^{\mathrm{bdr}}$ of boundary C-field configurations from 5.4.115. We show that on ∞-connection field configurations whose underlying ∞-bundles are trivial, this functional reduces to that given in equation (5.28).

It is instructive to first consider the simple special case where the E_8 is trivial. In this case the boundary moduli stack $C\mathrm{Field}^{\mathrm{bdr}'}$ from observation 5.4.116 restricts to just that of string 2-connections, $\mathbf{B}\mathrm{String}_{\mathrm{conn}}$.

Definition 5.7.39. Write $8\hat{\mathbf{I}}_8$ for the smooth universal differential characteristic cocycle

$$8\hat{\mathbf{I}}_8 : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \xrightarrow{(\frac{1}{6}\hat{\mathbf{p}}_2) - 8(\frac{1}{2}\hat{\mathbf{p}}_1 \hat{\cup} \frac{1}{2}\hat{\mathbf{p}}_1)} \mathbf{B}^7 U(1)_{\mathrm{conn}} \ ,$$

where $\frac{1}{6}\hat{\mathbf{p}}_2$ is the differential second fractional Pontryagin class from theorem 5.1.32 and where $\frac{1}{2}\hat{\mathbf{p}}_1 \hat{\cup} \frac{1}{2}\hat{\mathbf{p}}_1$ is the differential cup product class from observation 5.7.33.

Definition 5.7.40. For Σ a compact smooth manifold of dimension 7, the canonically induced action functional $\exp(iS_{8I_8}(-))$ from def. 3.9.68, on the moduli 2-stack of String-2-connections is the composite

$$\exp(iS_{8I_8}(-)) : [\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}] \xrightarrow{8\hat{\mathbf{I}}_8} [\Sigma, \mathbf{B}^7 U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1) \ .$$

We give now an explicit description of the field configurations in $[\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}]$ and of the value of $\exp(iS_{8I_8}(-))$ on these in terms of differential form data.

Proposition 5.7.41. *A field configuration in* $[\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}] \in \mathrm{Smooth}\infty\mathrm{Grpd}$ *is presented in the model category* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$*, 4.4, by a correspondence of simplicial presheaves*

$$\begin{array}{ccc} C(\{U_i\}) & \xrightarrow{\phi} & \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{so}_\mu)_{\tilde{\mathrm{conn}}} \ , \\ \downarrow{\simeq} & & \\ \Sigma & & \end{array}$$

where $\mathfrak{so}_\mu$ is the skeletal String Lie 2-algebra, def. 1.2.152, and where on the right we have the adapted differential coefficient object from prop. 5.4.94; such that the projection

$$C(\{U_i\}) \xrightarrow{\phi} \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{so}_\mu)_{\widetilde{\mathrm{conn}}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

has a class.

The underlying nonabelian cohomology class of such a cocycle is that of a String*-principal 2-bundle.*

The local connection and curvature differential form data over a patch U_i is

$$\begin{array}{ll} F_\omega = & d\omega + \frac{1}{2}[\omega \wedge \omega] \\ H_3 = & \nabla B := dB + \mathrm{CS}(\omega) \\ dF_\omega = & -[\omega \wedge F_\omega] \\ dH_3 = & \langle F_\omega \wedge F_\omega \rangle \end{array}$$

Proof. Without the constraint on the C-field this is the description of twisted String-2-connections of observation 5.4.96 where the twist is the C-field. The condition above picks out the untwisted case, where the C-field is trivialized. What remains is an untwisted String-principal 2-bundle.

The local differential form data is found from the modified Weil algebra of $(b\mathbb{R} \to (\mathfrak{so})_{\mu_{\mathfrak{so}}})$ indicated on the right of the following diagram

$$\left(\begin{array}{ll} F_\omega = & d\omega + \frac{1}{2}[\omega \wedge \omega] \\ H_3 = & \nabla B := dB + CS(\omega) - C_3 \\ \mathcal{G}_4 = & dC_3 \\ dF_\omega = & -[\omega \wedge F_\omega] \\ dH_3 = & \langle F_\omega \wedge F_\omega \rangle - \mathcal{G}_4 \\ d\mathcal{G}_4 = & 0 \end{array} \right)_i \xleftarrow{\begin{array}{ll} t^a_{\mathfrak{so}} & \mapsto \omega^a \\ r^a_{\mathfrak{so}} & \mapsto F_\omega \\ b & \mapsto B \\ c & \mapsto C_3 \\ h & \mapsto H_3 \\ g & \mapsto \mathcal{G}_4 \end{array}} \left(\begin{array}{ll} r^a_{\mathfrak{so}} = & dt^a_{\mathfrak{so}} + \frac{1}{2} C^a_{\mathfrak{so}\,bc} t^b_{\mathfrak{so}} \wedge t^c_{\mathfrak{so}} \\ h = & db + \mathrm{cs}_{\mathfrak{so}} - c \\ g = & dc \\ dr^a_{\mathfrak{so}} = & -C^a{}_{bc} t^b_{\mathfrak{so}} \wedge r^a_{\mathfrak{so}} \\ dh = & \langle -, - \rangle - g \\ dg = & 0 \end{array} \right) .$$

□

Remark 5.7.42. While the 2-form B in the presentation used in the above proof is abelian, the total collection of forms is still connection data with coefficients in the nonabelian Lie 2-algebra $\mathfrak{string}$. We explained in remark 1.2.156, that there is a choice of local gauge in which the nonabelianness of the 2-form becomes manifest. For the discussion of the above proposition, however, this gauge is not the most convenient one, and it is more convenient to exhibit the local cocycle data in the above form, which corresponds to the second gauge of remark 1.2.156.

This is an example of a general principle in higher nonabelian gauge theory ("higher gerbe theory"). Due to the higher gauge invariances, the local component presentation of a given structure does not usually manifestly exhibit the gauge-invariant information in an obvious way.

Proposition 5.7.43. *Let $\phi \in [\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}]$ be a field configuration which, in the presentation of prop. 5.7.41, is defined over a single patch $U = \Sigma$.*

Then the action functional of def. 5.7.40 sends this to

$$\exp(iS_{8I_8}(\omega, H_3)) = \exp\left(i \int_\Sigma \left(-8 H_3 \wedge dH_3 + \mathrm{CS}_{\frac{1}{6}\hat{\mathbf{p}}_2}(\omega) \right) \right) .$$

Proof. The first term is that of the cup product theory, 5.7.9.1, after using the identity $\mathrm{tr}(F_\omega \wedge F_\omega) = dH_3$ which holds on the configuration space of String-2-connections by prop. 5.7.41. The second term is that of the $\frac{1}{6}p_2$-Chern-Simons theory from 5.7.9.2. □

Remark 5.7.44. Therefore comparison with equation (5.28) shows that the action functional S_{8I_8} has all the properties that in 5.7.9.3.1 we argued that the effective 7-dimensional Chern-Simons theory inside 11-dimensional supergravity compactified on S^4 should have, in the following special case:

- the C-field flux on S^4 is $N = 8$;

and

- the E_8-field is trivial;
- the C-field on Σ is trivial.

By choosing any multiple of $8\hat{\mathbf{I}}_8$ one can obtain C-field flux of arbitrary multiples of 8. In order to obtain C-field flux that is not a multiple of 8 one needs to discuss further divisibility of $8\hat{\mathbf{I}}_8$.

We discuss now a refinements of S_{8I_8} that generalize away from the last two of these special conditions to obtain the full form of (5.28).

Recall from def. 5.4.115 the higher moduli stack $C\mathrm{Field}^{\mathrm{bdr}}$ of supergravity C-field configurations, which by remark. 5.4.116 is the moduli 3-stack of twisted $\mathrm{String}^{\mathbf{2a}}$-connections. We consider now an action functional on this configuration stack.

Following remark 5.2.14 we write a corresponding field configuration, $\phi \in C\mathrm{Field}^{\mathrm{bdr}}(\Sigma)$, whose underlying topological class is trivial as a tuple of forms

$$(\omega, A, B_2, C_3) \in \Omega^1(\Sigma, \mathfrak{so}) \times \Omega^1(\Sigma, \mathfrak{e}_8) \times \Omega^2(\Sigma) \times \Omega^3(\Sigma)$$

and set

$$H_3 := dB_2 + \mathrm{cs}(\omega) - \mathrm{cs}(A)\,.$$

Recall that by prop. 5.2.13 this object has a presentation by Lie integration as 5.4.7.3.1 as a sub-simplicial set

$$\mathbf{cosk}_3 \exp((\mathbb{R} \to \mathfrak{so} \oplus \mathfrak{e}_8)_{\mu_3^{\mathfrak{so}} - 2\mu_3^{\mathfrak{e}_8}})_{\mathrm{conn}}\,.$$

In terms of this presentation we have an evident differential characteristic class given by the Lie integration of the Chern-Simons element $\mathrm{cs}_{\frac{1}{6}p_2} - 8\mathrm{cs}_{\frac{1}{2}o_1 \cup \frac{1}{2}p_1}$.

Definition 5.7.45. Write $\hat{\mathbf{I}}_8$ for the smooth universal characteristic map given by the composite

$$\mathbf{B}\mathrm{String}^{\mathbf{2a}} \xrightarrow{\exp(\mathrm{cs}_{\frac{1}{6}p_2} - 8\mathrm{cs}_{\frac{1}{2}p_1 \cup \frac{1}{2}p_1})} [\Sigma, \mathbf{B}^7(\mathbb{R}/K)_{\mathrm{conn}}]\ ,$$

where the second morphism is the ∞-Chern-Weil homomorphism of I_8, according to 4.4.17, with $K \subset \mathbb{R}$ the given sublattice of periods.

Write

$$\exp(iS_{I_8}(-)) : \ \mathbf{B}\mathrm{String}^{\mathbf{2a}}_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{I}}_8} [\Sigma, \mathbf{B}^7(\mathbb{R}/K)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} \mathbb{R}/K$$

for the corresponding action functional.

Finally we obtain the refinement of the 7-dimensional Chern-Simons action (5.28) to the full higher moduli stack of boundary C-field configurations.

Proposition 5.7.46. *Let $\phi \in C\mathrm{Field}^{\mathrm{bdr}}(\Sigma))$ be a boundary C-field configuration according to remark. 5.4.116, whose underlying* $\mathrm{String}^{\mathbf{2a}}$*-principal 2-bundle is trivial, which is hence a quadruple of forms*

$$\phi = (\omega, A, B_2, C_3) \in \Omega^1(\Sigma, \mathfrak{so}) \times \Omega^1(\Sigma, \mathfrak{e}_8) \times \Omega^2(\Sigma) \times \Omega^3(\Sigma)\,.$$

The combination of the action functional of def. 5.7.27 and the action functional of def. 5.7.45 sends this to

$$\exp(iS(C_3)) \exp(iS_{8I_8}(\omega, A, B_2)) = \int_\Sigma C_3 \wedge dC_3 + 8\left(H_3 \wedge dH_3 + (H_3 + \mathrm{cs}(A)) \wedge \langle F_\omega \wedge F_\omega \rangle + \frac{1}{8}\mathrm{cs}_{\frac{1}{6}p_2}(\omega)\right) \ \mathrm{mod} K\,,$$

where $H_3 = dB + \mathrm{cs}(\omega) - 2\mathrm{cs}(A)$.

Proof. By the nature of the exp(−)-construction we have

$$\exp(iS_{8I_8}(\omega, A, B)) = \int_\Sigma \left(8\mathrm{cs}(\omega)\wedge d\mathrm{cs}(\omega) + \mathrm{cs}_{\frac{1}{6}p_2}(\omega)\right) \quad .$$

Inserting here the equation for H_3 satisfied by the $\mathrm{String}^{\mathbf{2a}}$-connections yields

$$\begin{aligned} \cdots &= \int_\Sigma \left(8(H_3 + 2\mathrm{cs}(A) - dB)\wedge d(H_3 + 2\mathrm{cs}(A) - dB) + \mathrm{cs}_{\frac{1}{6}p_2}(\omega)\right) \\ &= \int_\Sigma \left(8(H_3 + 2\mathrm{cs}(A))\wedge d(H_3 + 2\mathrm{cs}(A)) + \mathrm{cs}_{\frac{1}{6}p_2}(\omega)\right) \\ &= \int_\Sigma 8\left(H_3\wedge dH_3 + (H_3 + 2\mathrm{cs}(A))\wedge \langle F_\omega \wedge F_\omega\rangle + \frac{1}{8}\mathrm{cs}_{\frac{1}{6}p_2}(\omega)\right) \end{aligned} \quad .$$

□

5.7.10 Action of closed string field theory type

We discuss the form of ∞-Chern-Simons Lagrangians, 5.7.1, on general L_∞-algebras equipped with a quadratic invariant polynomial. The resulting action functionals have the form of that of closed string field theory [Zw93].

Proposition 5.7.47. *Let $\mathfrak{g}$ be any L_∞-algebra equipped with a quadratic invariant polynomial $\langle -,-\rangle$. The ∞-Chern-Simons functional associated with this data is*

$$S : A \mapsto \int_\Sigma \left(\langle A\wedge d_{\mathrm{dR}}A\rangle + \sum_{k=1}^{\infty}\frac{2}{(k+1)!}\langle A\wedge [A\wedge\cdots A]_k\rangle\right) ,$$

where

$$[-,\cdots,-] : \mathfrak{g}^{\otimes k} \to \mathfrak{g}$$

is the k-ary bracket of $\mathfrak{g}$ (prop. 1.2.116).

Proof. There is a canonical contracting homotopy operator

$$\tau : \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$$

such that $[d_{\mathrm{W}}, \tau] = \mathrm{Id}_{\mathrm{W}(\mathfrak{g})}$. Accordingly a Chern-Simons element, def. 4.4.116, for $\langle -,-\rangle$ is given by

$$\mathrm{cs} := \tau\langle -,-\rangle \,.$$

We claim that this is indeed the Lagrangian for the above action functional.

To see this, first choose a basis $\{t_a\}$ and write

$$P_{ab} := \langle t_a, t_b\rangle$$

for the components of the invariant polynomial in that basis and

$$C^a_{a_1,\cdots,a_k} := [t_{a_1},\cdots,t_{a_k}]^a_k$$

as well as

$$C_{a_0,a_1,\cdots,a_k} := P_{a_0 a} C^a_{a_1,\cdots,a_k}$$

for the structure constant of the k-ary brackets.

In terms of this we need to show that

$$\mathrm{cs} = P_{ab} t^a \wedge d_{\mathrm{W}} t^b + \sum_{k=1}^{\infty} \frac{2}{(k+1)!} C_{a_0, \cdots, a_k} t^{a_0} \wedge \cdots \wedge t^{a_k} \,.$$

The computation is best understood via the free dg-algebra $F(\mathfrak{g})$ on the graded vector space $\mathfrak{g}^*$, which in the above basis we may take to be generated by elements $\{t^a, \mathbf{d}t^a\}$. There is a dg-algebra isomorphism

$$F(\mathfrak{g}) \stackrel{\simeq}{\to} \mathrm{W}(\mathfrak{g})$$

given by sending $t^a \mapsto t^a$ and $\mathbf{d}t^a \mapsto d_{\mathrm{CE}(\mathfrak{g})} + r^a$.

On $F(\mathfrak{g})$ the contracting homotopy is evidently given by the map $\frac{1}{L}h$, where L is the word length operator in the above basis and h the graded derivation which sends $t^a \mapsto 0$ and $\mathbf{d}t^a \mapsto t^a$. Therefore τ is given by

$$\begin{array}{ccc} \mathrm{W}(\mathfrak{g}) & \xrightarrow{\tau} & \mathrm{W}(\mathfrak{g}) \\ \downarrow{\simeq} & & \uparrow{\simeq} \\ F(\mathfrak{g}) & \xrightarrow{\frac{1}{L}h} & F(\mathfrak{g}) \end{array} \,.$$

With this we obtain

$$\begin{aligned} \mathrm{cs} &:= \tau\langle -,-\rangle \\ &= \tau P_{ab} \left(d_{\mathrm{W}} t^a + \sum_{k=1}^{\infty} C^a_{a_1, \cdots, a_k} t^{a_1} \wedge \cdots \wedge t^{a_k} \right) \wedge \left(d_{\mathrm{W}} t^b + \sum_{k=1}^{\infty} C^b_{b_1, \cdots, b_k} t^{b_1} \wedge \cdots \wedge t^{b_k} \right) \\ &= P_{ab} t^a \wedge d_{\mathrm{W}} t^b + \sum_{k=1}^{\infty} \frac{2}{k!(k+1)} P_{ab} C^b{}_{b_1, \cdots, b_k} t^a \wedge t^{b_1} \wedge \cdots \wedge t^{b_k} \end{aligned} \,.$$

□

Remark 5.7.48. If here Σ is a completely odd-graded dg-manifold, such as $\Sigma = \mathbb{R}^{0|3}$, then this is the kind of action functional that appears in closed string field theory [Zw93][KaSt08]. In this case the underlying space of the (super-)L_∞-algebra $\mathfrak{g}$ is the BRST complex of the closed (super-)string and $[-, \cdots, -]_k$ is the string's tree-level $(k+1)$-point function.

5.7.11 AKSZ theory

We now consider *symplectic Lie n-algebroids* $\mathfrak{P}$. These carry canonical invariant polynomials ω. We show that the ∞-Chern-Simons action functional associated to such ω is the action functional of the *AKSZ σ-model quantum field theory* with target space $\mathfrak{P}$ (due to [AKSZ97], usefully reviewed in [Royt06]).

This section is based on [FRS11a].

- AKSZ σ-models – 5.7.11.1;
- 5.7.11.2 – The AKSZ action as a Chern-Simons functional ;
- 5.7.11.3 – Ordinary Chern-Simons theory;
- 5.7.11.4 – Poisson σ-model;
- 5.7.11.5 – Courant σ-model;
- 5.7.11.6 – Higher abelian Chern-Simons theory.

5.7.11.1 AKSZ σ-Models The class of topological field theories known as *AKSZ σ-models*[AKSZ97] contains in dimension 3 ordinary Chern-Simons theory (see [Fre] for a comprehensive review) as well as its Lie algebroid generalization (the *Courant σ-model* [Ike03]), and in dimension 2 the Poisson σ-model (see [CaFe00] for a review). It is therefore clear that the AKSZ construction is *some* sort of generalized Chern-Simons theory. Here we demonstrate that this statement is true also in a useful precise sense.

Our discussion proceeds from the observation that the standard Chern-Simons action functional has a systematic origin in Chern-Weil theory (see for instance [GHV] for a classical textbook treatment and [HoSi05] for the refinement to differential cohomology that we need here):

The refined Chern-Weil homomorphism assigns to any invariant polynomial $\langle - \rangle : \mathfrak{g}^{\otimes n} \to \mathbb{R}$ on a Lie algebra $\mathfrak{g}$ of compact type a map that sends $\mathfrak{g}$-connections ∇ on a smooth manifold X to cocycles $[\hat{\mathbf{p}}_{\langle - \rangle}(\nabla)] \in H^{n+1}_{\mathrm{diff}}(X)$ in *ordinary differential cohomology.* These differential cocycles refine the *curvature characteristic class* $[\langle F_\nabla \rangle] \in H^{n+1}_{dR}(X)$ in de Rham cohomology to a fully fledged *line n-bundle with connection*, also known as a *bundle $(n-1)$-gerbe with connection.* And just as an ordinary line bundle (a "line 1-bundle") with connection assigns holonomy to curves, so a line n-bundle with connection assigns holonomy $\mathrm{hol}_{\hat{\mathbf{p}}}(\Sigma)$ to n-dimensional trajectories $\Sigma \to X$. For the special case where $\langle - \rangle$ is the Killing form polynomial and $X = \Sigma$ with $\dim \Sigma = 3$ one finds that this volume holonomy map $\nabla \mapsto \mathrm{hol}_{\hat{\mathrm{p}}_{\langle - \rangle}(\nabla)}(\Sigma)$ is precisely the standard Chern-Simons action functional. Similarly, for $\langle - \rangle$ any higher invariant polynomial this holonomy action functional has as Lagrangian the corresponding higher Chern-Simons form. In summary, this means that Chern-Simons-type action functionals on Lie algebra-valued connections are the images of the refined Chern-Weil homomorphism.

In 3.9.7 a generalization of the Chern-Weil homomorphism to *higher* ("derived") differential geometry has been established. In this context smooth manifolds are generalized first to orbifolds, then to general Lie groupoids, to Lie 2-groupoids and finally to smooth ∞-groupoids (smooth ∞-stacks), while Lie algebras are generalized to Lie 2-algebras etc., up to L_∞-algebras and more generally to Lie n-algebroids and finally to L_∞-algebroids.

In this context one has for $\mathfrak{a}$ any L_∞-algebroid a natural notion of $\mathfrak{a}$-valued ∞-connections on $\exp(\mathfrak{a})$-principal smooth ∞-bundles (where $\exp(\mathfrak{a})$ is a smooth ∞-groupoid obtained by Lie integration from $\mathfrak{a}$). By analyzing the abstractly defined higher Chern-Weil homomorphism in this context one finds a direct higher analog of the above situation: there is a notion of invariant polynomials $\langle - \rangle$ on an L_∞-algebroid $\mathfrak{a}$ and these induce maps from $\mathfrak{a}$-valued ∞-connections to line n-bundles with connections as before .

This construction drastically simplifies when one restricts attention to *trivial* ∞-bundles with (nontrivial) $\mathfrak{a}$-connections. Over a smooth manifold Σ these are simply given by dg-algebra homomorphisms

$$A : \mathrm{W}(\mathfrak{a}) \to \Omega^\bullet(\Sigma) \,,$$

where $\mathrm{W}(\mathfrak{a})$ is the *Weil algebra* of the L_∞-algebroid $\mathfrak{a}$ [SSS09a], and $\Omega^\bullet(\Sigma)$ is the de Rham algebra of Σ (which is indeed the Weil algebra of Σ thought of as an L_∞-algebroid concentrated in degree 0). Then for $\langle - \rangle \in \mathrm{W}(\mathfrak{a})$ an invariant polynomial, the corresponding ∞-Chern-Weil homomorphism is presented by a choice of "Chern-Simons element" $\mathrm{cs} \in \mathrm{W}(\mathfrak{a})$, which exhibits the *transgression* of $\langle - \rangle$ to an L_∞-cocycle (the higher analog of a cocycle in Lie algebra cohomology): the dg-morphism A naturally maps the Chern-Simons element cs of A to a differential form $\mathrm{cs}(A) \in \Omega^\bullet(\Sigma)$ and its integral is the corresponding ∞-Chern-Simons action functional $S_{\langle - \rangle}$

$$S_{\langle - \rangle} : A \mapsto \mathrm{hol}_{\hat{\mathbf{p}_{\langle - \rangle}}}(\Sigma) = \int_\Sigma \mathrm{cs}_{\langle - \rangle}(A) \,.$$

Even though trivial ∞-bundles with $\mathfrak{a}$-connections are a very particular subcase of the general ∞-Chern-Weil theory, they are rich enough to contain AKSZ theory. Namely, here we show that a symplectic dg-manifold of grade n – which is the geometrical datum of the target space defining an AKSZ σ-model – is naturally equivalently an L_∞-algebroid $\mathfrak{P}$ endowed with a quadratic and non-degenerate invariant polynomial ω of grade n. Moreover, under this identification the canonical Hamiltonian π on the symplectic target dg-manifold is identified as an L_∞-cocycle on $\mathfrak{P}$. Finally, the invariant polynomial ω is naturally in transgression with the cocycle π via a Chern-Simons element cs_ω that turns out to be the Lagrangian of the AKSZ σ-model:

$$\int_\Sigma L_{\mathrm{AKSZ}}(-) = \int_\Sigma \mathrm{cs}_\omega(-)\,.$$

(An explicit description of L_{AKSZ} is given below in def. 5.7.50)

In summary this means that we find the following dictionary of concepts:

Chern-Weil theory		**AKSZ theory**
cocycle	π	Hamiltonian
transgression element	cs	Lagrangian
invariant polynomial	ω	symplectic structure

More precisely, we (explain and then) prove here the following theorem:

Theorem 5.7.49. *For $(\mathfrak{P}, \omega)$ an L_∞-algebroid with a quadratic non-degenerate invariant polynomial, the corresponding ∞-Chern-Weil homomorphism*

$$\nabla \mapsto \mathrm{hol}_{\hat{\mathbf{p}_\omega}}(\Sigma)$$

sends $\mathfrak{P}$-valued ∞-connections ∇ to their corresponding exponentiated AKSZ action

$$\cdots = \exp(i \int_\Sigma L_{\mathrm{AKSZ}}(\nabla))\,.$$

.

The local differential form data involved in this statement is at the focus of attention in this section here and contained in prop. 5.7.52 below.

We consider, in definition 5.7.50 below, for any symplectic dg-manifold (X, ω) a functional S_{AKSZ} on spaces of maps $\mathfrak{T}\Sigma \to X$ of smooth graded manifolds. While only this precise definition is referred to in the remainder of the section, we begin by indicating informally the original motivation of S_{AKSZ}. The reader uncomfortable with these somewhat vague considerations can take note of def. 5.7.50 and then skip to the next section.

Generally, a *σ-model field theory* is, roughly, one

1. whose fields over a space Σ are maps $\phi : \Sigma \to X$ to some space X;

2. whose action functional is, apart from a kinetic term, the transgression of some kind of cocycle on X to the mapping space $\mathrm{Map}(\Sigma, X)$.

Here the terms "space", "maps" and "cocycles" are to be made precise in a suitable context. One says that Σ is the *worldvolume*, X is the *target space* and the cocycle is the *background gauge field*.

For instance, an ordinary charged particle (such as an electron) is described by a σ-model where $\Sigma = (0, t) \subset \mathbb{R}$ is the abstract *worldline*, where X is a (pseudo-)Riemannian smooth manifold (for instance our spacetime), and where the background cocycle is a line bundle with connection on X (a degree-2 cocycle in ordinary differential cohomology of X, representing a background *electromagnetic field*). Up to a kinetic term, the action functional is the holonomy of the connection over a given curve $\phi : \Sigma \to X$. A textbook discussion of these standard kinds of σ-models is, for instance, in [DEFJKMMW].

The σ-models which we consider here are *higher* generalizations of this example, where the background gauge field is a cocycle of higher degree (a higher bundle with connection) and where the worldvolume is accordingly higher dimensional. In addition, X is allowed to be not just a manifold, but an approximation to a *higher orbifold* (a smooth ∞-groupoid).

More precisely, here we take the category of spaces to be SmoothDgMfd from def. 5.5.3. We take target space to be a symplectic dg-manifold (X, ω) and the worldvolume to be the shifted tangent bundle $\mathfrak{T}\Sigma$ of a compact smooth manifold Σ. Following [AKSZ97], one may imagine that we can form a smooth $\mathbb{Z}$-graded mapping space $\mathrm{Maps}(\mathfrak{T}\Sigma, X)$ of smooth graded manifolds. On this space the canonical vector fields v_Σ and v_X naturally have commuting actions from the left and from the right, respectively, so that their sum $v_\Sigma + v_X$ equips $\mathrm{Maps}(\mathfrak{T}\Sigma, X)$ itself with the structure of a differential graded smooth manifold.

Next we take the "cocycle" on X (to be made precise in the next section) to be the Hamiltonian π (def. 5.5.12) of v_X with respect to the symplectic structure ω, according to def. 5.5.10. One wants to assume that there is a kind of Riemannian structure on $\mathfrak{T}\Sigma$ that allows to form the transgression

$$\int_{\mathfrak{T}\Sigma} \mathrm{ev}^* \omega := p_! \mathrm{ev}^* \omega$$

by pull-push through the canonical correspondence

$$\mathrm{Maps}(\mathfrak{T}\Sigma, X) \xleftarrow{p} \mathrm{Maps}(\mathfrak{T}\Sigma, X) \times \mathfrak{T}\Sigma \xrightarrow{\mathrm{ev}} X \ .$$

When one succeeds in making this precise, one expects to find that $\int_{\mathfrak{T}\Sigma} \mathrm{ev}^* \omega$ is in turn a symplectic structure on the mapping space.

This implies that the vector field $v_\Sigma + v_X$ on mapping space has a Hamiltonian

$$\mathbf{S} \in C^\infty(\mathrm{Maps}(\mathfrak{T}\Sigma, X)) \,, \ \text{ s.t. } \ \mathbf{dS} = \iota_{v_\Sigma + v_x} \int_{\mathfrak{T}\Sigma} \mathrm{ev}^* \omega \,.$$

The grade-0 component

$$S_{\mathrm{AKSZ}} := \mathbf{S}|_{\mathrm{Maps}(\mathfrak{T}\Sigma, X)_0}$$

constitutes a functional on the space of morphisms of graded manifolds $\phi : \mathfrak{T}\Sigma \to X$. This is the *AKSZ action functional* defining the AKSZ σ-model with target space X and background field/cocycle ω.

In [AKSZ97], this procedure is indicated only somewhat vaguely. The focus of attention there is on a discussion, from this perspective, of the action functionals of the 2-dimensional σ-models called the *A-model* and the *B-model*. In [Royt06] a more detailed discussion of the general construction is given, including an explicit formula for $\mathbf{S}$, and hence for S_{AKSZ}. That formula is the following:

Definition 5.7.50. For (X, ω) a symplectic dg-manifold of grade n with global Darboux coordinates $\{x^a\}$, Σ a smooth compact manifold of dimension $(n+1)$ and $k \in \mathbb{R}$, the *AKSZ action functional*

$$S_{\mathrm{AKSZ}} : \mathrm{SmoothGrMfd}(\mathfrak{T}\Sigma, X) \to \mathbb{R}$$

is

$$S_{\mathrm{AKSZ}} : \phi \mapsto \int_\Sigma \left(\frac{1}{2} \omega_{ab} \phi^a \wedge d_{\mathrm{dR}} \phi^b - \phi^* \pi \right) \,,$$

where π is the Hamiltonian for v_X with respect to ω and where on the right we are interpreting fields as forms on Σ according to prop. 5.5.7.

This formula hence defines an infinite class of σ-models depending on the target space structure (X, ω). (One can also consider arbitrary relative factors between the first and the second term, but below we shall find that the above choice is singled out). In [AKSZ97], it was already noticed that ordinary Chern-Simons theory is a special case of this for ω of grade 2, as is the Poisson σ-model for ω of grade 1 (and hence, as shown there, also the A-model and the B-model). The main example in [Royt06] spells out the general case for ω of grade 2, which is called the *Courant σ-model* there. (We review and re-derive all these examples in detail below.)

One nice aspect of this construction is that it follows immediately that the full Hamiltonian $\mathbf{S}$ on the mapping space satisfies $\{\mathbf{S}, \mathbf{S}\} = 0$. Moreover, using the standard formula for the internal hom of chain complexes, one finds that the cohomology of $(\mathrm{Maps}(\mathfrak{T}\Sigma, X), v_\Sigma + v_X)$ in degree 0 is the space of functions on those fields that satisfy the Euler-Lagrange equations of S_{AKSZ}. Taken together, these facts imply that $\mathbf{S}$ is a solution of the "master equation" of a BV-BRST complex for the quantum field theory defined by S_{AKSZ}. This is a crucial ingredient for the quantization of the model, and this is what the AKSZ construction is mostly used for in the literature (for instance [CaFe00]).

Here we want to focus on another nice aspect of the AKSZ-construction: it hints at a deeper reason for *why* the σ-models of this type are special. It is indeed one of the very few proposals for what a general abstract mechanism might be that picks out among the vast space of all possible local action functionals those that seem to be of relevance "in nature".

We now proceed to show that the class of action functionals S_{AKSZ} are precisely those that higher Chern-Weil theory canonically associates to target data (X, ω). Since higher Chern-Weil theory in turn is canonically given on very general abstract grounds, this in a sense amounts to a derivation of S_{AKSZ} from "first principles", and it shows that a wealth of very general theory applies to these systems.

5.7.11.2 The AKSZ action as an ∞-Chern-Simons functional We now show how an L_∞-algebroid $\mathfrak{a}$ endowed with a triple $(\pi, \mathrm{cs}, \omega)$ consisting of a Chern-Simons element transgressing an invariant polynomial ω to a cocycle π defines an AKSZ-type σ-model action. The starting point is to take as target space the tangent Lie ∞-algebroid $\mathfrak{T}\mathfrak{a}$, i.e., to consider as *space of fields* of the theory the space of maps $\mathrm{Maps}(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{a})$ from the worldsheet Σ to $\mathfrak{T}\mathfrak{a}$. Dually, this is the space of morphisms of dgcas from $\mathrm{W}(\mathfrak{a})$ to $\Omega^\bullet(\Sigma)$, i.e., the space of degree 1 $\mathfrak{a}$-valued differential forms on Σ from definition 1.2.140.

Remark 5.7.51. As we noticed in the introduction, in the context of the AKSZ σ-model a degree 1 $\mathfrak{a}$-valued differential form on Σ should be thought of as the datum of a (notrivial) $\mathfrak{a}$-valued connection on a trivial principal ∞-bundle on Σ.

Now that we have defined the space of fields, we have to define the action. We have seen in definition 1.2.142 that a degree 1 $\mathfrak{a}$-valued differential form A on Σ maps the Chern-Simons element $\mathrm{cs} \in \mathrm{W}(\mathfrak{a})$ to a differential form $\mathrm{cs}(A)$ on Σ. Integrating this differential form on Σ will therefore give an AKSZ-type action which is naturally interpreted as an higher Chern-Simons action functional:

$$\begin{aligned} \mathrm{Maps}(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{a}) &\to \mathbb{R} \\ A &\mapsto \int_\Sigma \mathrm{cs}(A). \end{aligned}$$

Theorem 5.7.49 then reduces to showing that, when $\{\mathfrak{a}, (\pi, \mathrm{cs}, \omega)\}$ is the set of L_∞-algebroid data arising from a symplectic Lie n-algebroid $(\mathfrak{P}, \omega)$, the AKSZ-type action dscribed above is precisely the AKSZ action for $(\mathfrak{P}, \omega)$. More precisely, this is stated as follows.

Proposition 5.7.52. *For $(\mathfrak{P}, \omega)$ a symplectic Lie n-algebroid coming by proposition 5.5.15 from a symplectic dg-manifold of positive grade n with global Darboux chart, the action functional induced by the canonical Chern-Simons element*

$$\mathrm{cs} \in \mathrm{W}(\mathfrak{P})$$

from proposition 5.5.19 is the AKSZ action from definition 5.7.50:

$$\int_\Sigma \mathrm{cs} = \int_\Sigma L_{\mathrm{AKSZ}} \,.$$

In fact the two Lagrangians differ at most by an exact term

$$\mathrm{cs} \sim L_{\mathrm{AKSZ}} \,.$$

Proof. We have seen in remark 5.5.20 that in Darboux coordinates $\{x^a\}$ where

$$\omega = \frac{1}{2}\omega_{ab}\mathbf{d}x^a \wedge \mathbf{d}x^b$$

the Chern-Simons element from proposition 5.5.19 is given by

$$\mathrm{cs} = \frac{1}{n}\left(\deg(x^a)\,\omega_{ab}x^a \wedge d_{\mathrm{W}(\mathfrak{P})}x^b - n\pi\right) .$$

This means that for Σ an $(n+1)$-dimensional manifold and

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(\mathfrak{P}) : \phi$$

a (degree 1) $\mathfrak{P}$-valued differential form on Σ we have

$$\int_\Sigma \mathrm{cs}(\phi) = \frac{1}{n}\int_\Sigma \left(\sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b - n\pi(\phi)\right) \quad ,$$

where we used $\phi(d_{\mathrm{W}(\mathfrak{P})}x^b) = d_{\mathrm{dR}}\phi^b$, as in remark 1.2.141. Here the asymmetry in the coefficients of the first term is only apparent. Using integration by parts on a closed Σ we have

$$\begin{aligned}
\int_\Sigma \sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b &= \int_\Sigma \sum_{a,b}(-1)^{1+\deg(x^a)}\deg(x^a)\,\omega_{ab}(d_{\mathrm{dR}}\phi^a) \wedge \phi^b \\
&= \int_\Sigma \sum_{a,b}(-1)^{(1+\deg(x^a))(1+\deg(x^b))}\deg(x^a)\,\omega_{ab}\phi^b \wedge (d_{\mathrm{dR}}\phi^a)\,, \\
&= \int_\Sigma \sum_{a,b}\deg(x^b)\,\omega_{ab}\phi^a \wedge (d_{\mathrm{dR}}\phi^b)
\end{aligned}$$

where in the last step we switched the indices on ω and used that $\omega_{ab} = (-1)^{(1+\deg(x^a))(1+\deg(x^b))}\omega_{ba}$. Therefore

$$\begin{aligned}
\int_\Sigma \sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b &= \frac{1}{2}\int_\Sigma \sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b + \frac{1}{2}\int_\Sigma \sum_{a,b}\deg(x^b)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b \\
&= \frac{n}{2}\int_\Sigma \omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b\,.
\end{aligned} .$$

Using this in the above expression for the action yields

$$\int_\Sigma \mathrm{cs}(\phi) = \int_\Sigma \left(\frac{1}{2}\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b - \pi(\phi)\right) ,$$

which is the formula for the action functional from definition 5.7.50. □

We now unwind the general statement of proposition 5.7.52 and its ingredients in the central examples of interest, from proposition 5.5.16: the ordinary Chern-Simons action functional, the Poisson σ-model Lagrangian, and the Courant σ-model Lagrangian. (The ordinary Chern-Simons model is the special case of the Courant σ-model for $\mathfrak{P}$ having as base manifold the point. But since it is the archetype of all models considered here, it deserves its own discussion.)

By the very content of proposition 5.7.52 there are no surprises here and the following essentially amounts to a review of the standard formulas for these examples. But it may be helpful to see our general ∞-Lie theoretic derivation of these formulas spelled out in concrete cases, if only to carefully track the various signs and prefactors.

5.7.11.3 Ordinary Chern-Simons theory Let $\mathfrak{P} = b\mathfrak{g}$ be a semisimple Lie algebra regarded as an L_∞-algebroid with base space the point and let $\omega := \langle -, - \rangle \in \mathrm{W}(b\mathfrak{g})$ be its Killing form invariant polynomial. Then $(b\mathfrak{g}, \langle -, - \rangle)$ is a symplectic Lie 2-algebroid.

For $\{t^a\}$ a dual basis for $\mathfrak{g}$, being generators of grade 1 in $\mathrm{W}(\mathfrak{g})$ we have

$$d_{\mathrm{W}} t^a = -\frac{1}{2} C^a{}_{bc} t^a \wedge t^b + \mathbf{d} t^a$$

where $C^a{}_{bc} := t^a([t_b, t_c])$ and

$$\omega = \frac{1}{2} P_{ab} \mathbf{d} t^a \wedge \mathbf{d} t^b \,,$$

where $P_{ab} := \langle t_a, t_b \rangle$. The Hamiltonian cocycle π from prop. 5.5.17 is

$$\begin{aligned}
\pi &= \frac{1}{2+1} \iota_v \iota_\epsilon \omega \\
&= \frac{1}{3} \iota_v P_{ab} t^a \wedge \mathbf{d} t^b \\
&= -\frac{1}{6} P_{ab} C^b_{cd} t^a \wedge t^c \wedge t^d \\
&=: -\frac{1}{6} C_{abc} t^a \wedge t^b \wedge t^c .
\end{aligned}$$

Therefore the Chern-Simons element from prop. 5.5.19 is found to be

$$\begin{aligned}
\mathrm{cs} &= \frac{1}{2} \left(P_{ab} t^a \wedge \mathbf{d} t^b - \frac{1}{6} C_{abc} t^a \wedge t^b \wedge t^c \right) \\
&= \frac{1}{2} \left(P_{ab} t^a \wedge d_{\mathrm{W}} t^b + \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c \right) .
\end{aligned}$$

This is indeed, up to an overall factor 1/2, the familiar standard choice of Chern-Simons element on a Lie algebra. To see this more explicitly, notice that evaluated on a $\mathfrak{g}$-valued connection form

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(b\mathfrak{g}) : A$$

this is

$$2\mathrm{cs}(A) = \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A, A] \rangle = \langle A \wedge d_{dR} A \rangle + \frac{1}{3} \langle A \wedge [A, A] \rangle \,.$$

If $\mathfrak{g}$ is a matrix Lie algebra then the Killing form is proportional to the trace of the matrix product: $\langle t_a, t_b \rangle = \mathrm{tr}(t_a t_b)$. In this case we have

$$\begin{aligned}
\langle A \wedge [A, A] \rangle &= A^a \wedge A^b \wedge A^c \, \mathrm{tr}(t_a (t_b t_c - t_c t_b)) \\
&= 2 A^a \wedge A^b \wedge A^c \, \mathrm{tr}(t_a t_b t_c) \\
&= 2 \, \mathrm{tr}(A \wedge A \wedge A)
\end{aligned}$$

and hence

$$2\mathrm{cs}(A) = \mathrm{tr} \left(A \wedge F_A - \frac{1}{3} A \wedge A \wedge A \right) = \mathrm{tr} \left(A \wedge d_{dR} A + \frac{2}{3} A \wedge A \wedge A \right) \,.$$

5.7.11.4 Poisson σ-model Let $(M, \{-, -\})$ be a Poisson manifold and let $\mathfrak{P}$ be the corresponding Poisson Lie algebroid. This is a symplectic Lie 1-algebroid. Over a chart for the shifted cotangent bundle $T^*[-1]X$ with coordinates $\{x^i\}$ of degree 0 and $\{\partial_i\}$ of degree 1, respectively, we have

$$d_{\mathrm{W}} x^i = -\pi^{ij} \partial_j + \mathbf{d} x^i ;$$

where $\pi^{ij} := \{x^i, x^j\}$ and

$$\omega = \mathbf{d}x^i \wedge \mathbf{d}\partial_i .$$

The Hamiltonian cocycle from prop. 5.5.17 is

$$\pi = \frac{1}{2}\iota_v \iota_\epsilon \omega = -\frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j$$

and the Chern-Simons element from prop. 5.5.19 is

$$\begin{aligned} \mathrm{cs} &= \iota_\epsilon \omega + \pi \\ &= \partial_i \wedge \mathbf{d}x^i - \frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j \end{aligned} .$$

In terms of d_{W} instead of $\mathbf{d}$ this is

$$\begin{aligned} \mathrm{cs} &= \partial_i \wedge d_{\mathrm{W}} x^i - \pi \\ &= \partial_i \wedge d_{\mathrm{W}} x^i + \frac{1}{2}\pi^{ij}\partial_i \partial_j . \end{aligned}$$

So for Σ a 2-manifold and

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(\mathfrak{P}) : (X, \eta)$$

a Poisson-Lie algebroid valued differential form on Σ – which in components is a function $X : \Sigma \to M$ and a 1-form $\eta \in \Omega^1(\Sigma, X^* T^* M)$ – the corresponding AKSZ action is

$$\int_\Sigma \mathrm{cs}(X, \eta) = \int_\Sigma \eta \wedge d_{\mathrm{dR}} X + \frac{1}{2}\pi^{ij}(X)\eta_i \wedge \eta_j .$$

This is the Lagrangian of the Poisson σ-model [CaFe00].

5.7.11.5 Courant σ-model A Courant algebroid is a symplectic Lie 2-algebroid. By the previous example this is a higher analog of a Poisson manifold. Expressed in components in the language of ordinary differential geometry, a Courant algebroid is a vector bundle E over a manifold M_0, equipped with: a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on the fibers, a bilinear bracket $[\cdot, \cdot]$ on sections $\Gamma(E)$, and a bundle map (called the anchor) $\rho \colon E \to TM$, satisfying several compatibility conditions. The bracket $[\cdot, \cdot]$ may be required to be skew-symmetric (Def. 2.3.2 in [Royt02]), in which case it gives rise to a Lie 2-algebra structure, or, alternatively, it may be required to satisfy a Jacobi-like identity (Def. 2.6.1 in [Royt02]), in which case it gives a Leibniz algebra structure.

It was shown in [Royt02] that Courant algebroids $E \to M_0$ in this component form are in 1-1 correspondance with (non-negatively graded) grade 2 symplectic dg-manifolds (M, v). Via this correspondance, M is obtained as a particular symplectic submanifold of $T^*[2]E[1]$ equipped with its canonical symplectic structure.

Let (M, v) be a Courant algebroid as above. In Darboux coordinates, the symplectic structure is

$$\omega = \mathbf{d}p_i \wedge \mathbf{d}q^i + \frac{1}{2}g_{ab}\mathbf{d}\xi^a \wedge \mathbf{d}\xi^b,$$

with

$$\deg q^i = 0, \;\; \deg \xi^a = 1, \;\; \deg p_i = 2,$$

and g_{ab} are constants. The Chevalley-Eilenberg differential corresponds to the vector field:

$$v = P^i_a \xi^a \frac{\partial}{\partial q^i} + g^{ab}\big(P^i_b p_i - \frac{1}{2}T_{bcd}\xi^c \xi^d\big)\frac{\partial}{\partial \xi^a} + \left(-\frac{\partial P^j_a}{\partial q^i}\xi^a p_j + \frac{1}{6}\frac{\partial T_{abc}}{\partial q^i}\xi^a \xi^b \xi^c\right)\frac{\partial}{\partial p_i}.$$

Here $P^i_a = P^i_a(q)$ and $T_{abc} = T_{abc}(q)$ are particular degree zero functions encoding the Courant algebroid structure. Hence, the differential on the Weil algebra is:

$$\begin{aligned} d_W q^i &= P^i_a \xi^a + \mathbf{d} q^i \\ d_W \xi^a &= g^{ab}\big(P^i_b p_i - \frac{1}{2} T_{bcd}\xi^c\xi^d\big) + \mathbf{d}\xi^a \\ d_W p_i &= -\frac{\partial P^j_a}{\partial q^i}\xi^a p_j + \frac{1}{6}\frac{\partial T_{abc}}{\partial q^i}\xi^a\xi^b\xi^c + \mathbf{d} p_i . \end{aligned}$$

Following remark. 5.5.18, we construct the corresponding Hamiltonian cocycle from prop. 5.5.17:

$$\begin{aligned} \pi &= \frac{1}{n+1}\omega_{ab}\deg(x^a) x^a \wedge v^b \\ &= \frac{1}{3}\big(2p_i \wedge v(q^i) + g_{ab}\xi^a \wedge v(\xi^b)\big) \\ &= \frac{1}{3}\big(2p_i P^i_a \xi^a + \xi^a P^i_a p_i - \frac{1}{2} T_{abc}\xi^a\xi^b\xi^c\big) \\ &= P^i_a \xi^a p_i - \frac{1}{6} T_{abc}\xi^a\xi^b\xi^c . \end{aligned}$$

The Chern-Simons element from prop. 5.5.19 is:

$$\begin{aligned} \mathrm{cs} &= \frac{1}{2}\left(\sum_{ab}\deg(x^a)\,\omega_{ab} x^a \wedge d_W x^b - 2\pi\right) \\ &= p_i d_W q^i + \frac{1}{2} g_{ab}\xi^a d_W \xi^b - \pi \\ &= p_i d_W q^i + \frac{1}{2} g_{ab}\xi^a d_W \xi^b - P^i_a \xi^a p_i + \frac{1}{6} T_{abc}\xi^a\xi^b\xi^c . \end{aligned}$$

So for a map

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(\mathfrak{P}) : (X, A, F)$$

where Σ is a closed 3-manifold, we have

$$\int_\Sigma \mathrm{cs}(X,A,F) = \int_\Sigma F_i \wedge d_{\mathrm{dR}} X^i + \frac{1}{2} g_{ab} A^a \wedge d_{\mathrm{dR}} A^b - P^i_a A^a \wedge F_i + \frac{1}{6} T_{abc} A^a \wedge A^b \wedge A^c .$$

This is the AKSZ action for the Courant algebroid σ-model from [Ike03] [Royt02][Royt06].

5.7.11.6 Higher abelian Chern-Simons theory in $d = 4k+3$ We discuss higher abelian Chern-Simons theory, 5.7.8.1, from the point of view of AKSZ theory.

For $k \in \mathbb{N}$, let $\mathfrak{a}$ be the delooping of the line Lie $2k$-algebra, def. 4.4.58: $\mathfrak{a} = b^{2k+1}\mathbb{R}$. By observation 4.4.114 there is, up to scale, a unique binary invariant polynomial on $b^{2k+1}\mathbb{R}$, and this is the wedge product of the unique generating unary invariant polynomial γ in degree $2k+2$ with itself:

$$\omega := \gamma \wedge \gamma \in \mathrm{W}(b^{4k+4}\mathbb{R}) .$$

This invariant polynomial is clearly non-degenerate: for c the canonical generator of $\mathrm{CE}(b^{2k+1}\mathbb{R})$ we have

$$\omega = \mathbf{d}c \wedge \mathbf{d}c .$$

Therefore $(b^{2k+1}\mathbb{R}, \omega)$ induces an ∞-Chern-Simons theory of AKSZ σ-model type in dimension $n+1 = 4k+3$. (On the other hand, on $b^{2k}\mathbb{R}$ there is only the 0 binary invariant polynomial, so that no AKSZ-σ-models are induced from $b^{2k}\mathbb{R}$.)

The Hamiltonian cocycle from prop. 5.5.17 vanishes

$$\pi = 0$$

because the differential $d_{\mathrm{CE}(b^{2k+1}\mathbb{R})}$ is trivial. The Chern-Simons element from prop. 5.5.19 is

$$\mathrm{cs} = c \wedge \mathbf{d}c\,.$$

A field configuration, def. 1.2.140, of this σ-model over a $(2k+3)$-dimensional manifold

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(b^{2k+1}) : C$$

is simply a $(2k+1)$-form. The AKSZ action functional in this case is

$$S_{AKSZ} : C \mapsto \int_\Sigma C \wedge d_{dR} C\,.$$

The simplicity of this discussion is deceptive. It results from the fact that here we are looking at ∞-Chern-Simons theory for universal Lie integrations and for topologically trivial ∞-bundles. More generally the ∞-Chern-Simons theory for $\mathfrak{a} = b^{2k+1}\mathbb{R}$ is nontrivial and rich, as discussed in 5.7.8.1. Its configuration space is that of *circle* $(2k+1)$*-bundles with connection* (4.4.16) on Σ, classified by ordinary differential cohomology in degree $2k+2$, and the action functional is given by the fiber integration in differential cohomology to the point over the Beilinson-Deligne cup product, which is locally given by the above formula, but contains global twists.

5.8 Higher extended WZW theory

We discuss examples of higher WZW functionals, def. 3.9.12.

- 1d WZW functionals
  - 5.8.1 – Massive non-relativistic particle
  - 5.8.2 – Green-Schwarz superparticle
- 2d WZW functionals
  - 5.8.3 – Bosonic string on a Lie group;
  - 5.8.4 – Green-Schwarz superstring;
- 6d WZW functionals
  - 5.8.5 – Bosonic 5-brane on the String-2-group

5.8.1 Massive non-relativistic particle

The action functional of the free massive non-relativistic particle is a special low dimensional case of higher WZW action functionals. A discussion is in section 8.3 of [AzIz95].

5.8.2 Green-Schwarz superparticle

The action functional of the Green-Schwarz superparticle is a special low dimensional case of higher WZW action functionals. A discussion is in section 8.7 of [AzIz95].

5.8.3 Bosonic string on a Lie group

The ordinary 2d WZW model describing a string propagating on a Lie group G (see for instance [Ga00] for a review) is controled by the surface holonomy of a canonical circle 2-bundle on G. We discuss this classical theory from the point of view of the ∞-topos Smooth∞Grpd. We recover the treatment of the differential geometry and differential cohomology of Chern-Simons and WZW-theories as discussed in [CJMSW05] and [Wal08] and generalizes it from cohomology and classifying spaces to cocycles and moduli stacks.

Let now G be a compact and simply connected Lie group and let

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

be the Chern-Simons functional, from 5.7.5.1.

Recall that

- by prop. 4.4.44 the object $\flat_{\mathrm{dR}}\mathbf{B}G$ is presented by the simplicial presheaf given by the sheaf of flat $\mathfrak{g}$-valued forms;
- by theorem 4.4.84 the object $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is presented by the simplicial presheaf which is the image of the Beilinson-Deligne complex under the Dold-Kan map.

Proposition 5.8.1. *Let X be a smooth manifold. In terms of these presentations the composite morphism*

$$\mathbf{H}(X, \flat_{\mathrm{dR}}\mathbf{B}G \to \flat\mathbf{B}G \to \mathbf{B}G_{\mathrm{conn}} \stackrel{\hat{\mathbf{c}}}{\to} \mathbf{B}^{n+1}U(1)_{\mathrm{conn}})$$

from def. 3.9.72 sends a flat $\mathfrak{g}$-valued form $A \in \Omega^1_{\mathrm{flat}}(X, \mathfrak{g})$ to the Deligne cocycle which is trivial except for a globally defined connection 3-form $C = \frac{1}{2}\langle A \wedge [A \wedge A] \rangle$.

Proof. By theorem 5.1.9 the morphism $\hat{\mathbf{c}}$ is presented by $\exp(\mathrm{cs}) : \mathbf{cosk}_3 \exp(\mathfrak{g})_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$. To compute the composite we therefore need to construct a compatible composite of anafunctors. By lemma 4.4.64, prop. 4.4.44 and lemma 4.4.64 this can be given by

$$\begin{array}{ccccccc} \mathbf{cosk}_3 \exp(\mathfrak{g})_{\mathrm{dR}} & \longrightarrow & \mathbf{cosk}_3 \exp(\mathfrak{g})_{\mathrm{flat}} & \longrightarrow & \mathbf{cosk}_3 \exp(\mathfrak{g})_{\mathrm{conn}} & \stackrel{\exp(\mathrm{cs})}{\longrightarrow} & \mathbf{B}^3 U(1)_{\mathrm{conn}} \\ \downarrow \simeq & & \downarrow \simeq & & & & \\ \flat_{\mathrm{dR}} \mathbf{B} G & \longrightarrow & \flat \mathbf{B} G_{\mathrm{ch}} & & & & \end{array} .$$

Chasing an element $A : X \to \flat_{\mathrm{dR}} \mathbf{B} G$ through this diagram shows the claimed statement. □

It follows that a cocycle with coefficients in the differential WZW coefficient object $\mathbf{B}^2 U(1)_{\mathrm{conn}}|_{F=\theta(g)}$, def. 3.9.72, is given by a Deligne cocycle with curvature 3-form $\frac{1}{2}\langle A \wedge [A \wedge A]\rangle$. For a more detailed statement, consider the following.

Lemma 5.8.2. *For $n \in \mathbb{N}$, there is a pasting diagram of ∞-pullbacks*

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}} & \stackrel{\mathrm{F}}{\longrightarrow} & \Omega^{n+1}_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^{n+1}(-) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \end{array} ,$$

where the top morphism is the curvature projection of def. 4.4.82.

Proof. We use the presentation of $\mathbf{B}^{n+1} U(1)_{\mathrm{conn}}$ by the image of the Beilinson-Deligne complex under the Dold-Kan map from theorem 4.4.84.

To compute the ∞-pullback, we produce a fibration replacement in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ of the lower right moprhism. By prop. 2.3.13 it is then sufficient to check that the ordinary fiber of that morphism is weakly equivalent to $\mathbf{B}^n U(1)_{\mathrm{conn}}$.

Consider therefore the simplicial presheaf

$$\tilde{\Omega}^{n+1}(-) := \Xi \left[\begin{array}{ccccccc} C^\infty(-, U(1)) & \stackrel{d_{\mathrm{dR}}}{\longrightarrow} & \Omega^1(-) & \stackrel{d_{\mathrm{dR}}}{\longrightarrow} & \cdots & \stackrel{d_{\mathrm{dR}}}{\longrightarrow} & \Omega^{n+1}(-) \\ & \searrow^{\mathrm{id}} & \oplus & \searrow^{\mathrm{id}} & \cdots & \searrow^{\mathrm{id}} & \oplus \\ & & C^\infty(-, U(1)) & \stackrel{d_{\mathrm{dR}}}{\longrightarrow} & \cdots & \stackrel{d_{\mathrm{dR}}}{\longrightarrow} & \Omega^n(-) \end{array} \right]$$

with Ξ from prop. 2.2.31. One checks that there is a morphism of simplicial presheaves

$$\tilde{\Omega}^{n+1}(-) \to \Omega^{n+1}(-)$$

which in degree 0 is given by

$$\begin{array}{ccc} \Omega^{n+1}(-) & & \\ \oplus & \searrow^{\mathrm{id}} & \\ \Omega^n(-) & \stackrel{d_{\mathrm{dR}}}{\longrightarrow} & \Omega^{n+1}(-) \end{array}$$

and which is a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. This induces a weak equivalence of pullback diagrams

$$\begin{array}{ccccc} * & \longrightarrow & \mathbf{B}^3 U(1)_{\mathrm{conn}} & \twoheadleftarrow & \tilde{\Omega}^3(-) \\ \downarrow = & & \downarrow = & & \downarrow \simeq \\ * & \longrightarrow & \mathbf{B}^3 U(1)_{\mathrm{conn}} & \longleftarrow & \Omega^3(-) \end{array} .$$

Since we manifestly have an ordinary pullback of simplicial presheaves

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \tilde{\Omega}^{n+1}(-) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \end{array}$$

(using that the right adjoint Ξ preserves pullback) this proves the claim. □

Proposition 5.8.3. *There is a canonical morphism*

$$\mathbf{H}(X, \mathbf{B}^2 U(1)_{\mathrm{conn}}|_{F=\theta(g)}) \to \mathbf{H}(X, \mathbf{B}^2 U(1)_{\mathrm{conn}})$$

whose image exhibits the full sub-2-groupoid of circle 2-bundles whose curvature 3-form is of the form $\frac{1}{2}\langle A \wedge [A \wedge A]\rangle$ *for some* $A \in \Omega^1_{\mathrm{flat}}(X, \mathfrak{g})$*. Moreover, the WZW cocycle*

$$\mathrm{WZW}_{\hat{\mathbf{c}}} : G \to \mathbf{B}^2 U(1)_{\mathrm{conn}} \hookrightarrow \mathbf{B}^2 U(1)_{\mathrm{conn}}$$

exhibits, up to equivalence, the traditional WZW gerbe with connection on G.

Proof. By lemma 5.8.2 and prop. 5.8.1 we have a pasting diagram of ∞-bullbacks

$$\begin{array}{ccc} \mathbf{B}^2 U(1)_{\mathrm{conn}}|_{F=\theta(g)} & \longrightarrow & \Omega^1_{\mathrm{flat}}(-, \mathfrak{g}) \\ \downarrow & & \downarrow \\ & & \{C \in \Omega^3(-) | \exists g : C = \frac{1}{2}\langle g^*\theta \wedge [g^*\theta \wedge g^*\theta]\rangle\} \\ \downarrow & & \downarrow \\ \mathbf{B}^2 U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^3(-) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}^3 U(1)_{\mathrm{conn}} \end{array} .$$

□

5.8.4 Green-Schwarz superstring

The *Green-Schwarz superstring* on flat spacetime (see for instance D'Hoker's lecture 10 in [DEFJKMMW] for a standard review) is a WZW coset σ-model whose target space is the supermanifold obtained as the quotient of the super-Poincaré-group by the Lorentz group, and whose background gauge field is a super circle 2-bundle with connection whose super-curvature 3-form is the 3-cocycle from prop. 5.3.13.

In this Lie-theoretic perspective this statement is made explicit for instance in chapter 8 of [AzIz95].

5.8.5 Bosonic 5-brane on String

We consider the 6-dimensional WZW action corresponding to the 7-dimensional Chern-Simons functional of 5.7.9.2.

$$
\begin{array}{ccccc}
\mathrm{String} & \xrightarrow{\mathrm{WZW}_{\frac{1}{6}\hat{\mathbf{p}}_6}} & \mathbf{B}^6 U(1)_{\mathrm{conn}}|_{F=\theta(g)} & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}\mathrm{String} \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}} & \longrightarrow & \mathbf{B}\mathrm{String}_{\mathrm{conn}} \\
 & & \downarrow & & \downarrow{\scriptstyle \frac{1}{6}\hat{\mathbf{p}}_2} \\
 & & * & \longrightarrow & \mathbf{B}^7 U(1)_{\mathrm{conn}}
\end{array}
$$

(...)

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