∞-Chern-Simons functionals

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Details and references at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

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General theory

Some examples

- 3d Spin-Chern-Simons theory
- 3d Dijkgraaf-Witten theory
- 4d Yetter model
- 4d nonabelian BF theory
- 7d String-Chern-Simons theory
- $(4k+3)$d abelian Chern-Simons theory
- AKSZ σ-models
  - Poisson σ-model
  - Courant σ-model
- Closed string field theory

Motivating toy example:

1d $U(n)$-Chern-Simons theory

Let

$$\mathbf{H} = \mathrm{Sh}_2(\mathrm{SmthMfd})$$

be the 2-category of stacks on smooth manifolds;
the *groupoid-valued presheaves*

$$\mathbf{H} \simeq L_W \mathrm{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{Grpd})\,.$$

after “simplicial localization” at $W = \{\text{stalkwise equivalences}\}$.

Example. The Lie groupoid

$$\mathbf{B}U(n) = \{ * \xrightarrow{g} * \,|\, g \in C^\infty(-, U(n))\}$$

is the moduli stack of smooth unitary bundles:

$$\mathbf{H}(X, \mathbf{B}U(n)) \simeq \mathrm{VectBund}(X)$$

(no quotient by equivalences).

The Lie groupoid homomorphism

$$\mathbf{c}_1 := \mathbf{B}\,\mathbf{det} : \mathbf{B}U(n) \to \mathbf{B}U(1)\,.$$

is a smooth refinement of the first Chern-class:

$$\begin{array}{ccc} \mathrm{VectBund}(X) & \xrightarrow{\mathbf{c}_1} & \mathrm{LineBund}(X) \\ \downarrow & & \downarrow \\ [X, BU(n)] & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) \end{array}$$

This has a *differential refinement*:

$$\mathbf{B}U(n)_{\mathrm{conn}} = \left\{ A \xrightarrow{g} A^g \;\middle|\; \begin{array}{l} g \in C^\infty(-, U(n)) \\ A \in \Omega^1(-, \mathfrak{u}(n)) \end{array} \right.$$

the moduli stack of unitary connections

$$\mathbf{H}(X, \mathbf{B}U(n)_{\mathrm{conn}}) = \mathrm{VectBund}(X)_{\mathrm{conn}}$$

The differential refinement of $\mathbf{c}_1$

$$\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$$

induces by *holonomy* $\int_\Sigma$ over 1d Σ an action functional $S_{\mathbf{c}_1}$

$$\begin{array}{ccc} \mathbf{H}(\Sigma, \mathbf{B}U(n)_{\mathrm{conn}}) & \xrightarrow{\hat{\mathbf{c}}_1} & \mathbf{H}(\Sigma, \mathbf{B}U(1)_{\mathrm{conn}}) \\ & \searrow^{\exp(iS_{\mathbf{c}_1})} & \downarrow{\scriptstyle \int_\Sigma} \\ & & U(1) \end{array}$$

on the moduli stack of gauge fields on Σ.

This is "1d U-Chern-Simons theory":

$$\exp(iS_{\mathbf{c}_1}) : (A \in \Omega^1(\Sigma, \mathfrak{u}(n))) \mapsto \exp(i \int_\Sigma \mathrm{tr} A) \quad .$$

"spectral action" (Connes), dimensional reduction of large-N gauge theory (e.g. hep-th/0605007).

Summary and outlook.

Generalize this story to

- higher characteristic classes

by working with

- higher smooth stacks.

Obtain

- higher gauge theories

of Chern-Simons type.

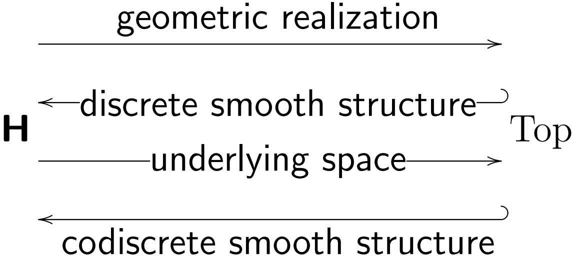
General theory

The collection of *smooth ∞-groupoids* or *smooth ∞-stacks* is

$$\mathbf{H} := L_W \mathrm{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{sSet}) \,,$$

the simplicial localization of simplicial presheaves at stalkwise weak homotopy equivalences.

This is *cohesive*: comes with an adjoint quadruple of derived functors



Lift through geometric realization is a **smooth refinement**.

Example. The Eilenberg-MacLane space $K(\mathbb{Z}, n+1)$ has a smooth refinement by the moduli n-stack

$$\mathbf{B}^n U(1) \simeq \mathrm{DoldKan}(U(1)[n])$$

of circle n-bundles (bundle $(n-1)$-gerbes).

This has moreover a differential refinement to the moduli n-stack

$$\mathbf{B}^n U(1)_{\mathrm{conn}}$$

$$\simeq$$

$$\mathrm{DoldKan}(\text{Deligne-Beilinson complex})$$

of circle n-bundles with connection.

∞-Chern-Simons functionals.

Given a

- smooth ∞-group G;
- a characteristic class $c \in H^n(BG, \mathbb{Z})$
- a smooth refinement $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$;
- a differential refinement
  $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$;

higher holonomy $\int_\Sigma$ induces an ∞-*Chern-Simons functional* $\exp(iS_{\mathbf{c}})$

$$\begin{array}{ccc} \mathbf{H}(\Sigma, \mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{H}(\Sigma, \mathbf{B}^n U(1)_{\mathrm{conn}}) \\ & \searrow^{\exp(iS_{\mathbf{c}})} & \downarrow^{\int_\Sigma} \\ & & U(1) \end{array}$$

Some Examples.

- 3d Chern-Simons theory
- 3d Dijkgraaf-Witten theory
- 4d Yetter model
- 4d BF-theory
- 7d String-Chern-Simons theory
- $(4k+3)$d abelian Chern-Simons theory
- AKSZ σ-models
  - Poisson σ-model
  - Courant σ-model
- Closed string field theory

3d Spin-Chern-Simons theory

The first Pontryagin class

$$\frac{1}{2}p_1 : B\mathrm{Spin} \to B^3U(1) \simeq \mathbb{K}(\mathbb{Z}, 4)$$

has a unique (up to equivalence) smooth refinement

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3U(1)$$

whose homotopy fiber is the smooth *String 2-group* (any of the existing models)

$$\mathbf{B}\mathrm{String} \to \mathbf{B}\mathrm{Spin} \stackrel{\frac{1}{2}\mathbf{p}_1}{\to} \mathbf{B}^3U(1)\,.$$

$$\mathbf{H}(X, \mathbf{B}String) \longrightarrow \mathbf{H}(X, \mathbf{B}\mathrm{Spin}) \longrightarrow \mathbf{H}(X, \mathbf{B}^3U(1)) \quad .$$

String 2-bundle lifts $\longrightarrow$ Spin bundles $\longrightarrow$ obstructing Chern-Simons 3-bundles

This has a differential refinement

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

$$\mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} \quad .$$

String 2-connection lifts $\longrightarrow$ Spin connections $\longrightarrow$ obstructing Chern-Simons 3-connections

The corresponding action functional is that of 3d Chern-Simons theory

$$\exp(iS_{\frac{1}{2}\mathbf{p}_1}) : \mathbf{H}(\Sigma, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) \longrightarrow \mathbf{H}(\Sigma, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

$$A \mapsto \exp(i \int_\Sigma \langle A \wedge dA \rangle + \frac{2}{3}\langle A \wedge A \wedge A \rangle)$$

3d Dijkgraaf-Witten theory

For G a *discrete* group, morphisms

$$\alpha : \mathbf{B}G \to \mathbf{B}^n U(1)$$

are the same as ordinary group cocycles:

$$\pi_0 \mathbf{H}(BG, \mathbf{B}^n U(1)) \simeq H^n_{\mathrm{grp}}(G, U(1))\,.$$

Field configurations: G-principal bundles.
For $n = 3$ the ∞-Chern-Simons functional

$$\exp(iS_\alpha) : \mathbf{H}(\Sigma, \mathbf{B}G) \to U(1)$$

is the action functional of Dijkgraaf-Witten theory:

$$(P \to \Sigma) \mapsto \langle [\Sigma], \alpha(P) \rangle\,.$$

4d Yetter model

For G any discrete ∞-group and $\alpha : \mathbf{B}G \to \mathbf{B}^n U(1)$ any n-cocycle, we get an n-dimensional analog of DW theory.

Let

$$G = (G_1 \to G_0)$$

be a *discrete* strict 2-group.

Field configurations: *principal 2-bundles*.

For

$$\alpha : \mathbf{B}G \to \mathbf{B}^4 U(1)$$

a 4-cocycle, the ∞-Chern-Simons functional $\exp(iS_\alpha)$ produces the *Yetter model*.

4d nonabelian BF-theory

Let

$$\mathfrak{g} = (\mathfrak{g}_1 \to \mathfrak{g}_0)$$

be a strict Lie 2-algebra (2-term dg Lie algebra) with $\mathfrak{g}_0$ of compact type. Let

$$\mathbf{B}G := \exp(\mathfrak{g})$$

be its universal ∞-Lie integration.
A field configuration

$$\phi : \Sigma \to \mathbf{B}G_{\mathrm{conn}}$$

is a 2-connection

$$\phi = \left(A \in \Omega^1(\Sigma, \mathfrak{g}_0)\,;\, B \in \Omega^2(\Sigma, \mathfrak{g}_1)\right)$$

There is canonical characteristic map

$$\mathbf{c} = \exp(\langle -, - \rangle) : \exp(\mathfrak{g})_{\mathrm{conn}} \to \mathbf{B}^4\mathbb{R}_{\mathrm{conn}}$$

whose ∞-Chern-Simons action functional is $(A, B) \mapsto$

$$\int_\Sigma (\langle F_A \wedge F_A \rangle - 2\langle F_A \wedge \partial B \rangle + 2\langle \partial B \wedge \partial B \rangle) \; .$$

topological YM + BF + cosmological constant.

7d String-Chern-Simons theory

(arXiv:1011.4735)

The second Pontryagin class

$$\frac{1}{6}p_2 : B\mathrm{String} \to B^7U(1) \simeq \mathbb{K}(\mathbb{Z}, 8)$$

has a smooth refinement to higher moduli stacks

$$\frac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{String} \to \mathbf{B}^7U(1)$$

whose homotopy fiber is the smooth *Fivebrane 6-group*

$$\mathbf{B}\mathrm{Fivebrane} \to \mathbf{B}\mathrm{String} \stackrel{\frac{1}{6}\mathbf{p}_2}{\to} \mathbf{B}^3U(1)\,.$$

$$\mathbf{H}(X, \mathbf{B}\mathrm{Fivebrane}) \longrightarrow \mathbf{H}(X, \mathbf{B}\mathrm{String}) \longrightarrow \mathbf{H}(X, \mathbf{B}^7U(1)) \quad .$$

Fivebrane 6-bundle lifts ⟶ String 2-bundles ⟶ obstructing Chern-Simons 7-bundles

This has a differential refinement

$$\frac{1}{6}\hat{\mathbf{p}}_2 : \; \mathbf{B}\mathrm{String}_{\mathrm{conn}} \longrightarrow \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

$$\mathbf{H}(X, \mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}}) \xrightarrow{\frac{1}{6}\hat{\mathbf{p}}_2} \mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \longrightarrow \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

Fivebrane 6-connection lifts ⟶ String 2-connections ⟶ obstructing Chern-Simons 7-connections

The corresponding action functional defines a 7d String-Chern-Simons theory

$$\exp(iS_{\frac{1}{6}\mathbf{p}_2}) : \; \mathbf{H}(\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \xrightarrow{\frac{1}{6}\hat{\mathbf{p}_2}} \mathbf{H}(\Sigma, \mathbf{B}^7 U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

This appears after anomaly cancellation in $\mathrm{AdS}_7/\mathrm{CFT}_6$.

$(4k+3)$d abelian Chern-Simons theory

For

$$G = \mathbf{B}^n U(1)$$

and

$$\mathbf{DD}^2 = \mathbf{B}^{2k+1} U(1) \stackrel{(-)^{\cup 2}}{\to} \mathbf{B}^{4k+3} U(1)$$

the cup-square of the higher Dixmier-Douady class, the induced ∞-Chern-Simons functional $\exp(iS_{\mathbf{DD}^2})$ is abelian $(4k+3)$d Chern-Simons theory, locally given by

$$C_{2k+1} \mapsto \exp(i \int_\Sigma C \wedge dC) \,.$$

For $k = 1$ this appears in 11d supergravity on AdS_7.

AKSZ σ-models

A *symplectic Lie n-algebroid* is a Lie n-algebroid $\mathfrak{a}$ equipped with binary non-degenerate invariant polynomial $\omega \in \Omega^2(\mathfrak{a})$.

- $n = 0$ – symplectic manifold;
- $n = 1$ – Poisson Lie algebroid;
- $n = 2$ – Courant Lie 2-algebroid.

arXiv:1108.4378: There is a differential refinement

$$\exp(\omega) : \exp(\mathfrak{a})_{\mathrm{conn}} \to \mathbf{B}^n\mathbb{R}_{\mathrm{conn}}$$

and the induced ∞-Chern-Simons functional is that of the AKSZ σ-model.

- Poisson σ-model;
- Courant σ-model.

Closed string field theory

For $(\mathfrak{g}, \{[-, \cdots, -]_k\}_k)$ an arbitrary L_∞-algebra, and $\langle -, - \rangle$ a binary invariant polynomial, the ∞-Chern-Simons Lagrangian is

$$A \mapsto \langle A \wedge d_{\mathrm{dR}} A \rangle + \sum_{k=1}^{\infty} \frac{2}{(k+1)!} \langle A \wedge [A \wedge \cdots A]_k \rangle \,,$$

For $\mathfrak{g}$ the BRST complex of the closed bosonic string and $[-, \cdots, -]_k$ the string's $k+1$-point function, Zwiebach shows that that the Berezinian integral $\int_\Sigma$ of this over three ghost modes is the action functional of *closed string field theory*.

For more details see

http://ncatlab.org/schreiber/show/
infinity-Chern-Simons+theory+--+examples

or section 4.6 in

http://ncatlab.org/schreiber/show/
differential+cohomology+in+a+cohesive+
topos

End.