Principal ∞-bundles – Theory and applications

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Notes for a talk, May 2012
reporting on joint work [NSS] with
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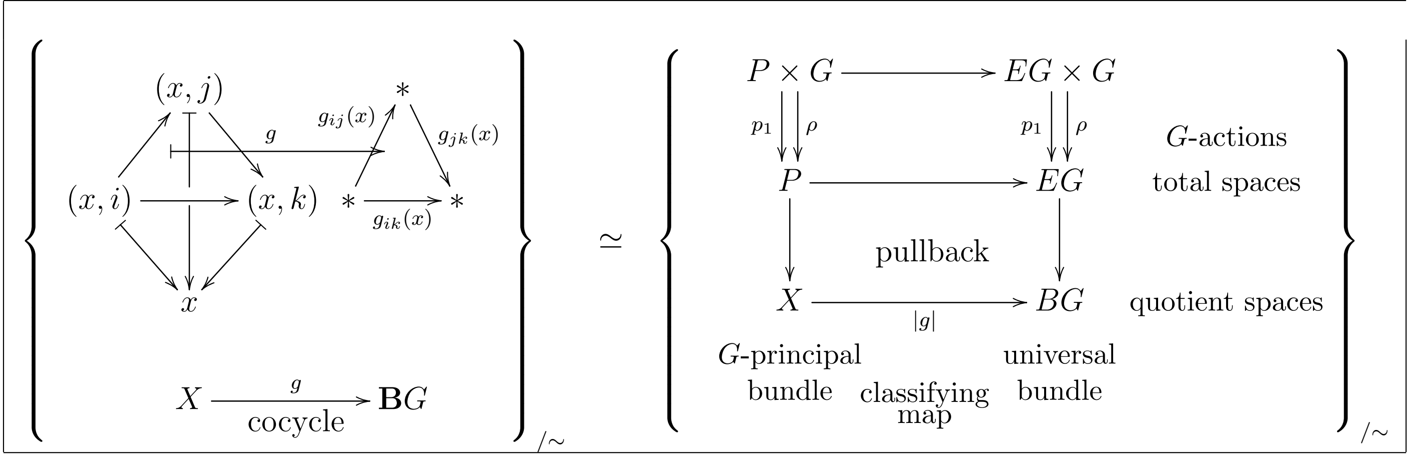
with precursors in [NiWa, RoSt, SSS, S]

Contents

1 Motivation

Classical fact. For X a manifold and G a topological/Lie group, regarded as a sheaf of groups $C(-, G)$ on X, there is an equivalence:

algebraic data on X		**geometric data on X**
$\left\{ \begin{array}{c} \text{degree-1 nonabelian} \\ \text{sheaf cohomology} \end{array} \right\}$ $H^1(X, G)$	$\simeq$	$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{G-principal bundles over } X \end{array} \right\}$ $G\mathrm{Bund}(X)$



Problem. In *higher differential geometry* [S], for instance in *String-geometry* [SSS], Lie groups G are replaced by *grouplike smooth A_∞-spaces*: by *∞-groups* (examples below in 5). Need to generalize the above classical fact to this case.

2 Higher geometry

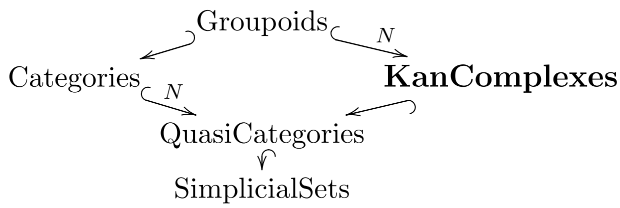
We need

$$\boxed{\text{geometry}} + \boxed{\text{homotopy theory}} = \boxed{\text{higher geometry} \simeq \infty\text{-topos theory}}.$$

Here is a way to think of the above classical fact that will generalize: let

- C := SmthMfd be the category of all smooth manifolds (or some other site, here assumed to have enough points);
- $\mathrm{gSh}(C)$ be the category of groupoid-valued sheaves over C, for instance $X = \{ X \rightrightarrows X \}, \mathbf{B}G = \{ G \rightrightarrows * \} \in \mathrm{gSh}(C)$;
- $\mathrm{Ho}_{\mathrm{gSh}(C)}$ the *homotopy category* obtained by universally turning the *stalkwise groupoid-equivalences* into isomorphisms.

Fact: $\qquad H^1(X, G) \simeq \mathrm{Ho}_{\mathrm{gSh}(C)}(X, \mathbf{B}G)$.
To generalize, let

- 

be the Kan complexes, aka ∞-*groupoids* inside all quasi-categories aka ∞-*categories*
- $\mathrm{sSh}(C)_{\mathrm{lfib}} \hookrightarrow \mathrm{Sh}(C, \mathrm{sSet})$ be the (stalkwise Kan) simplicial sheaves;
- $L_W \mathrm{sSh}(C)_{\mathrm{lfib}}$ the *simplicial localization* obtained by universally turning *stalkwise homotopy equivalences* into *homotopy equivalences.*

Definition/Theorem. This is the ∞-category theory analog of the sheaf topos over C, the ∞-*stack* ∞-*topos*: $\qquad \mathbf{H} := \mathrm{Sh}_\infty(C) \simeq L_W \mathrm{sSh}(C)_{\mathrm{lfib}}$.
Example. Smooth∞Grpd := Sh_∞(SmthMfd) is the ∞-topos of *smooth* ∞-*groupoids* / *smooth* ∞-*stacks.*
Example. For A a sheaf of abelian groups, $\mathbf{B}^{n+1}A := \mathrm{DoldKan}(A[n+1]) \in \mathrm{sSh}(C)$ is the moduli n-stack of $\mathbf{B}^n A$-principal bundles (details in a moment).
Proposition. Every object in Smooth∞Grpd is presented by a simplicial manifold, but not necessarily by a *locally Kan* simplicial manifold (see below).

Definition A *group* in the ∞-topos is a $G \in \mathbf{H}$ equipped with a groupal A_∞-algebra structure: coherently homotopy associative product with coherent homotopy inverses.
Example. In Smooth∞Grpd this is a *smooth* ∞-*group*: for instance a Lie group, or a Lie 2-group, or a differentiable group stack, or a sheaf of simplicial groups on SmthMfd.
Fact. (Milnor-Lurie) There is an equivalence

$$\{ \text{ groups in } \mathbf{H} \} \underset{\text{delooping } \mathbf{B}}{\overset{\text{looping } \Omega}{\underset{\simeq}{\rightleftarrows}}} \left\{ \begin{array}{c} \text{pointed connected} \\ \text{objects in } \mathbf{H} \end{array} \right\}$$

Proposition. Let C have a terminal object. For every ∞-group $G \in \mathrm{Grp}(\mathrm{Sh}_\infty(C))$ there is a sheaf of simplicial groups presenting it under $\mathrm{Sh}_\infty(C) \simeq L_W \mathrm{sSh}(C)$; and every ∞-action $\rho : P \times G \to P$ is presented by a corresponding simplicial action.

3 G-principal ∞-bundles

Definition. A G**-principal bundle** over $X \in \mathbf{H}$ is

- a morphism $P \to X$; with an ∞-action $\rho : P \times G \to P$;
- such that $P \to X$ is ∞-quotient $P \to P//G \overset{(*)}{\Leftrightarrow}$ principality : $P \times G^n \xrightarrow[\simeq]{(p_1,\rho)} P \times_X \cdots \times_X P$

Theorem. There is equivalence of ∞-groupoids $G\mathrm{Bund}(X) \underset{\lim\limits_{\to}}{\overset{\mathrm{hofib}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{H}(X, \mathbf{B}G)$, where

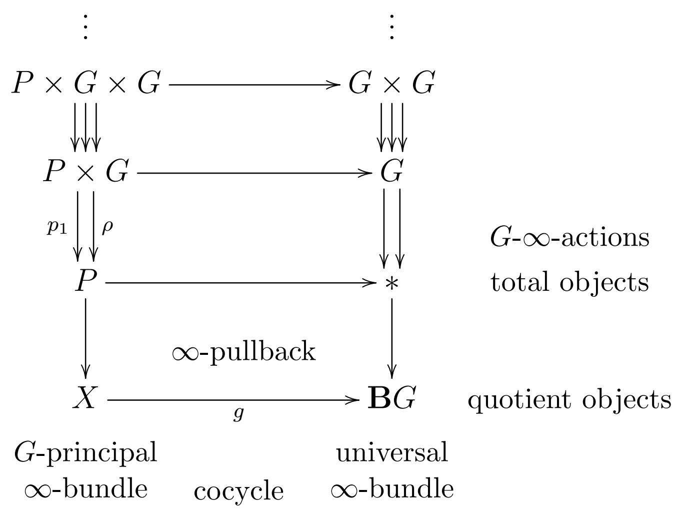
1. hofib sends a cocycle $X \to \mathbf{B}G$ to its homotopy fiber;
2. $\lim\limits_{\longrightarrow}$ sends an ∞-bundle to the map on ∞-quotients $X \simeq P//G \to *//G \simeq \mathbf{B}G$.

In particular, G-principal ∞-bundles are classified by the intrinsic cohomology of $\mathbf{H}$

$$G\mathrm{Bund}(X)/_\sim \simeq H^1(X,G) := \pi_0 \mathbf{H}(X, \mathbf{B}G)\,.$$

Proof. Repeatedly apply two of the
(∗) *Giraud-Rezk-Lurie axioms*
that characterize ∞-toposes:

1. every ∞-quotient is effective;
2. ∞-colimits are preserved by ∞-pullbacks. □



This gives a general abstract theory of principal ∞-bundles in every ∞-topos. We also have the following explicit presentation.

Definition For $G \in \mathrm{Grp}(\mathrm{sSh}(C))$, and $X \in \mathrm{sSh}(C)_{\mathrm{lfib}}$, a *weakly G-principal simplicial bundle* is a G-action ρ over X such that the *principality morphism* $(\rho, p_1) : P \times G \to P \times_X P$ is a stalkwise weak equivalence.

Theorem.

$$\mathrm{Nerve}\left\{ \begin{array}{c} \text{weakly } G\text{-principal} \\ \text{simplicial bundles} \\ \text{over } X \end{array} \right\} \simeq G\mathrm{Bund}(X)\,.$$

Example. For X terminal over C and restricted to cohomology classes, this is [JL].
Remark. We need more than that, notably $X = \mathbf{B}G$ itself, see next page.
Example. For $C = *$ we have $\mathrm{sSh}(C)_{\mathrm{lfib}} = \mathrm{KanComplexes}$. Classical theory considers *strictly* principal simplicial bundles [Ma].
Proposition. Strictly principal simplicial bundles over $C = *$ do present the cohomology $H^1(X,G)$, but not in general the full cocycle space $\mathbf{H}(X, \mathbf{B}G)$. For C nontrivial they do in general not even present $H^1(X,G)$.
Proposition. For G a simplicial Lie group, which is "CartSp-acyclic" (e.g. String), every G-principal ∞-bundle over a smooth manifold is presented by a locally Kan simplicial smooth manifold.

4 Associated and twisted ∞-bundles

Observation. By the above theorem, every G-∞-action $\rho : V \times G \to G$ has a classifying map:

$$\begin{array}{ccc} V & \longrightarrow & V/\!/G \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}G \end{array}$$

Proposition. This is the *universal ρ-associated V-bundle.*

Observation. Sections σ of the associated ∞-bundle are *lifts* of the cocycle through ρ; and these locally factor through V:

$$\left\{ \begin{array}{ccc} P \times_G V & \longrightarrow & V/\!/G \\ {\scriptstyle \sigma}\uparrow\downarrow & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{\;g\;} & \mathbf{B}G \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & & V/\!/G \\ & {\scriptstyle \sigma}\nearrow & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{\;g\;} & \mathbf{B}G \end{array} \right\} \quad \begin{array}{ccccc} & & V & \longrightarrow & V/\!/G \\ & {\scriptstyle \sigma|_U}\nearrow & & & \downarrow{\scriptstyle \rho} \\ U & \twoheadrightarrow & X & \xrightarrow{\;g\;} & \mathbf{B}G \end{array} .$$

Hence sections are cocycles in *g-twisted ΩV-cohomology* relative ρ:

$$\Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g, \rho) .$$

Theorem. Equivalently this classifies *P-twisted ∞-bundles*: twisted G-equivariant ΩV-∞-bundles on P:

$$\begin{array}{ccccc} Q & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \\ P & \longrightarrow & V & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\;\sigma\;} & V/\!/G & \xrightarrow{\;\rho\;} & \mathbf{B}G \end{array} \qquad (X \xrightarrow{\;g\;} \mathbf{B}G)$$

P-twisted ΩV-principal ∞-bundle

G-principal ∞-bundle

section of ρ-associated V-∞-bundle

$$\left\{ \begin{array}{c} \text{sections of} \\ \rho\text{-associated } V\text{-}\infty\text{-bundle} \end{array} \right\} \simeq \left\{ \begin{array}{c} g\text{-twisted } \Omega V\text{-cohomology} \\ \text{relative } \rho \end{array} \right\} \simeq \left\{ \begin{array}{l} \Omega V\text{-}\infty\text{-bundles} \\ \text{twisted by } P \end{array} \right\}$$

First example. Associated *connected-fiber* ∞-bundles are ∞-*gerbes.*

- A (nonabelian/Giraud-)*gerbe* on X is a connected 1-truncated object in $\mathbf{H}_{/X}$ (a *connected stack* on X).
- A (nonabelian/Giraud-Breen) ∞-*gerbe* over X is a connected object in $\mathbf{H}_{/X}$.
- A G-∞-*gerbe* is an $\mathrm{Aut}(\mathbf{B}G)$-associated ∞-bundle. Its *band* is the underlying $\mathrm{Out}(G)$-principal ∞-bundle.

Observation. G-∞-gerbes *bound by a band* are classified by $(\mathbf{B}\mathrm{Aut}(\mathbf{B}G) \to \mathbf{B}\mathrm{Out}(G))$-twisted cohomology.

5 Selected examples

extension / ∞-bundle of coefficients	**twisting ∞-bundle / twisting cohomology**	**twisted ∞-bundle / twisted cohomology**	
$V \longrightarrow V//G \xrightarrow{\rho} \mathbf{B}G$	ρ-associated V-∞-bundle	section	[S]
$\mathbf{B}^2\mathrm{ker}(G) \longrightarrow \mathbf{B}\mathrm{Aut}(\mathbf{B}G) \longrightarrow \mathbf{B}\mathrm{Out}(G)$	band (*lien*)	nonabelian (Giraud-Breen) G-∞-gerbe	[NSS] [S]
$\mathrm{GL}(d)/O(d) \longrightarrow \mathbf{B}O(d) \longrightarrow \mathbf{B}\mathrm{GL}(d)$	tangent bundle	orthogonal structure / Riemannian geometry	[S]
$O(d)\backslash O(d,d)/O(d) \to \mathbf{B}(O(d) \times O(d)) \longrightarrow \mathbf{B}O(d,d)$	generalized tangent bundle	generalized (type II) Riemannian geometry	[S]
$\mathbf{B}U \longrightarrow \mathbf{B}PU \xrightarrow{\mathbf{dd}} \mathbf{B}^2U(1)$	circle 2-bundle / bundle gerbe	twisted vector bundle / twisted K-cocycle / bundle gerbe module	[S]
$\mathbf{B}^nU(1) \longrightarrow \mathbf{B}^nU(1)//\mathbb{Z}_2 \xrightarrow{\mathbf{J}_{n-1}} \mathbf{B}\mathbb{Z}_2$	double cover	higher orientifold / $n = 2$: Jandl bundle gerbe	[FSSb] [SSW]
$V \longrightarrow \mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \xrightarrow{\nu^{\mathrm{int}}_{n+1}} \mathbf{B}^nU(1)$	circle n-bundle	smooth integral Wu structure	[FSSb]
$\mathbf{B}\mathrm{String} \longrightarrow \mathbf{B}\mathrm{Spin} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3U(1)$	circle 3-bundle / bundle 2-gerbe	twisted String 2-bundle	[SSS] [FSSa]
$V \longrightarrow \mathbf{B}(\mathbb{T} \times \mathbb{T}^*) \xrightarrow{\langle \mathbf{c}_1 \cup \mathbf{c}_1 \rangle} \mathbf{B}^3U(1)$	circle 3-bundle / bundle 2-gerbe	twisted T-duality structure	[S]
$\mathbf{B}\mathrm{Fivebrane} \longrightarrow \mathbf{B}\mathrm{String} \xrightarrow{\frac{1}{6}\mathbf{p}_2} \mathbf{B}^7U(1)$	circle 7-bundle	twisted Fivebrane 6-bundle	[SSS] [FSSa]
$\flat\mathbf{B}^nU(1) \longrightarrow \mathbf{B}^nU(1) \xrightarrow{\mathrm{curv}} \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$	curvature $(n+1)$-form	circle n-bundle with connection	[S]

6 Outlook: ∞-Geometric Prequantization

Observation. There is a canonical ∞-action γ of $\mathrm{Aut}_{\mathbf{H}_{/\mathbf{B}G}}(g)$ on the space of ∞-sections $\Gamma_X(P \times_G V)$.
Claim. Since $\mathrm{Sh}_\infty(\mathrm{SmthMfd})$ is "cohesive" [S], there is a notion of *differential refinement* of the above discussion, yielding *connections* on ∞-bundles.
Example. Let $\mathbb{C} \to \mathbb{C}//U(1) \to \mathbf{B}U(1)$ be the canonical complex-linear circle action. Then

- $g_{\mathrm{conn}} : X \to \mathbf{B}U(1)_{\mathrm{conn}}$ classifies a circle bundle with connection, a *prequantum line bundle* of its curvature 2-form;
- $\Gamma_X(P \times_{U(1)} \mathbb{C})$ is the corresponding space of smooth sections;
- γ is the exp(Poisson bracket)-group action of preqantum operators, containing the Heisenberg group action.

Example. Let $\mathbf{B}U \to \mathbf{B}\mathrm{PU} \to \mathbf{B}^2U(1)$ be the canonical 2-circle action. Then

- $g_{\mathrm{conn}} : X \to \mathbf{B}^2U(1)_{\mathrm{conn}}$ classifies a circle 2-bundle with connection, a *prequantum line 2-bundle* of its curvature 3-form;
- $\Gamma_X(P \times_{\mathbf{B}U(1)} \mathbf{B}U)$ is the corresponding groupoid of smooth sections = twisted bundles;
- γ is the exp(2-plectic bracket)-2-group action of 2-plectic geometry, containing the *Heisenberg 2-group* action [RoSc].

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