Chern-Simons terms on higher moduli stacks

Talk at *Hausdorff Institute Bonn, 2011*

Urs Schreiber

November 16, 2011

With

- Domenico Fiorenza
- Chris Rogers
- Hisham Sati
- Jim Stasheff

Details and references at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

Outline

Toy example: 1d U-Chern-Simons theory

General theory

3d Spin-Chern-Simons theory

7d String-Chern-Simons theory

Motivating toy example:

1d $U(n)$-Chern-Simons theory

The first universal Chern class

$$[c_1] \in H^2(BU, \mathbb{Z})$$

is represented by a map between classifying spaces

$$c_1 : BU \to BU(1) \simeq K(\mathbb{Z}, 2)$$

which induces a map on nonabelian cohomology

$$[X, c_1] : \mathrm{VectBund}(X)/_\sim \to H^2(X, \mathbb{Z})$$

that refines to differential cohomology

$$[X, \hat{c}_1] : \mathrm{VectBund}_{\mathrm{conn}}(X)/_\sim \to \mathrm{LineBund}_{\mathrm{conn}}(X)/_\sim$$

(classes of bundles with connection).

For $\Sigma = S^1$ the *line holonomy* $\nabla \mapsto \int_\Sigma \nabla$ induces an *action functional*

$$\exp(iS_{c_1}(-)) : \mathrm{VectBund}_{\mathrm{conn}}(\Sigma)/_\sim \stackrel{[\Sigma,\hat{c}_1]}{\to} \mathrm{LineBund}_{\mathrm{conn}}(\Sigma)/_\sim \stackrel{\int_\Sigma}{\to} U(1)$$

on *gauge equivalence classes* of a $U(n)$-gauge theory on Σ.
It assigns

$$(A \in \Omega^1(\Sigma, \mathfrak{u}(n)) \mapsto \exp(i \int_\Sigma \mathrm{tr}(A))\,.$$

This is "1d $U(n)$-Chern-Simons theory", "spectral action" (Connes), dimensional reduction of large-N gauge theory (e.g.hep-th/0605007).

Problem:
The passage to cohomology destroys many good properties.

For instance for quantization one needs the action functional on the BRST complex of field configurations. This is the Lie algebroid of the *Lie groupoid* of connections and gauge transformations.

Task:
Since Lie groupoid = differentiable stack this means: pass to *smooth moduli stack* of connections.

Let $\mathbf{H} = \mathrm{Sh}_2(\mathrm{SmthMfd})$ be the 2-category of stacks on smooth manifolds.
The Lie groupoid

$$\mathbf{B}U(n) = \{ * \xrightarrow{g \in U(n)} * \}$$

is the moduli stack of smooth $U(n)$-bundles:

$$\mathbf{H}(X, \mathbf{B}U(n)) \simeq \mathrm{VectBund}(X)$$

(no quotient by equivalences). The Lie group homomorphism

$$\mathbf{det} : U(n) \to U(1)$$

induces the evident Lie groupoid homomorphism

$$\mathbf{c}_1 := \mathbf{Bdet} : \mathbf{B}U(n) \to \mathbf{B}U(1)\,.$$

This refines c_1 from classes to cocycles

$$\mathbf{c}_1 : \mathrm{VectBund}(X) \to \mathrm{LineBund}(X)$$

sending bundles to their *determinant line bundle*.

The differentially refined moduli stack

$$\mathbf{B}U(n)_{\mathrm{conn}} = \left\{ A \xrightarrow{g} A^g \,|g \in C^\infty(-, U(n)), A \in \Omega^1(-, \mathfrak{u}(n)) \right\}$$

of unitary connections

$$\mathbf{H}(X, \mathbf{B}U(n)_{\mathrm{conn}}) = \mathrm{VectBund}(X)_{\mathrm{conn}}$$

supports the differential characteristic map

$$\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$$

between moduli stacks of connections, which gives the refined action functional

$$\exp(iS_{\mathbf{c}_1}(-)) : \mathbf{H}(\Sigma, \mathbf{B}U(n)_{\mathrm{conn}}) \to \mathbf{H}(\Sigma, \mathbf{B}U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

on the moduli stack of gauge fields on Σ.

Summary and outlook.

Refine characteristic classes

- from classifying spaces to higher moduli stacks;
- from cohomology to differential cohomology
- from Lie groups to higher smooth groups;

Obtain

- higher Chern-Simons action functionals;
- higher Wess-Zumino-Witten action functionals;
- higher twisted differential Spin-structures;
- higher twisted differential Spin^c-structures;

General theory

The collection of *smooth ∞-groupoids* or *smooth ∞-stacks* is

$$\mathbf{H} := L_W \mathrm{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{sSet})\,,$$

the (simplicial) localization of the category of simplicial presheaves at the class W of morphisms that are stalkwise weak homotopy equivalences.

This has an adjoint quadruple of derived functors to Top ("cohesion")

$$\mathbf{H} \underoverset{\underset{\mathrm{coDisc}}{\hookleftarrow}}{\overset{\Pi}{\longrightarrow}}{\overset{\xleftarrow{\mathrm{Disc}}}{\underset{\Gamma}{\longrightarrow}}} \mathrm{Top}\,,$$

where Π preserves finite products.
We say that a lift through Π is a **smooth refinement**.

Example.

The Eilenberg-MacLane space $K(\mathbb{Z}, n+1)$ has a smooth refinement by the moduli n-stack

$$\mathbf{B}^n U(1) \simeq \mathrm{DoldKan}(U(1)[n])$$

of circle n-bundles (bundle $(n-1)$-gerbes).

This has moreover a differential refinement to the moduli n-stack

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \simeq \mathrm{DoldKan}(\text{Deligne-Beilinson complex})$$

of circle n-bundles with connection.

3d Spin-Chern-Simons theory

The first Pontryagin class

$$\frac{1}{2}p_1 : B\mathrm{Spin} \to B^3U(1) \simeq \mathbb{K}(\mathbb{Z}, 4)$$

has a unique (up to equivalence) smooth refinement

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3U(1)$$

whose homotopy fiber is the smooth *String 2-group* (any of the existing models)

$$\mathbf{B}\mathrm{String} \to \mathbf{B}\mathrm{Spin} \stackrel{\frac{1}{2}\mathbf{p}_1}{\to} \mathbf{B}^3U(1)\,.$$

$$\mathbf{H}(X, \mathbf{B}String) \longrightarrow \mathbf{H}(X, \mathbf{B}\mathrm{Spin}) \longrightarrow \mathbf{H}(X, \mathbf{B}^3U(1)) \quad .$$

String 2-bundle lifts $\longrightarrow$ Spin bundles $\longrightarrow$ obstructing Chern-Simons 3-bundles

This has a differential refinement

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

$$\mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} \quad .$$

String 2-connection lifts	$\longrightarrow$	Spin connections	$\longrightarrow$	obstructing Chern-Simons 3-connections

The corresponding action functional is that of 3d Chern-Simons theory

$$\exp(iS_{\frac{1}{2}\mathbf{p}_1}) : \ \mathbf{H}(\Sigma, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) \longrightarrow \mathbf{H}(\Sigma, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

$$A \mapsto \exp(i \int_\Sigma \langle A \wedge dA \rangle + \frac{1}{3}\langle A \wedge A \wedge A \rangle)$$

Advantage.

Refinement to higher stacks provides *homotopy fibers*:

for $\hat{G}_4$ a fixed circle 3-connection, the 2-groupoid of $[\hat{G}_4]$-**twisted** String-**2-connections** is the homotopy fiber of $\frac{1}{2}\hat{\mathbf{p}}_1$ over $\hat{G}_4$.

Application.

For $\nabla_{\mathfrak{e}_8} \in \mathbf{H}(X, \mathbf{B}(E_8)_{\mathrm{conn}})$ an E_8-connection on a Riemannian Spin-manifold X, anomaly-free background gauge field data for the heterotic string on X is a lift σ to the homotopy fiber in

$$\begin{array}{ccc}
\frac{1}{2}\hat{\mathbf{p}}_1\mathrm{Struc}(X) & \longrightarrow & * \\
{\scriptstyle\sigma}\uparrow\;\downarrow & \Downarrow & \downarrow{\scriptstyle \hat{G}_4 := \hat{\mathbf{c}}_2(\nabla_{\mathfrak{e}_8})} \\
\mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{H}(X, \mathbf{B}^3 U(1)_{\mathrm{conn}})
\end{array}$$

7d String-Chern-Simons theory

The second Pontryagin class

$$\frac{1}{6}p_2 : B\mathrm{String} \to B^7U(1) \simeq \mathbb{K}(\mathbb{Z}, 8)$$

has a smooth refinement to higher moduli stacks

$$\frac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{String} \to \mathbf{B}^7U(1)$$

whose homotopy fiber is the smooth *Fivebrane 6-group*

$$\mathbf{B}\mathrm{Fivebrane} \to \mathbf{B}\mathrm{String} \stackrel{\frac{1}{6}\mathbf{p}_2}{\to} \mathbf{B}^3U(1)\,.$$

$$\mathbf{H}(X, \mathbf{B}\mathrm{Fivebrane}) \longrightarrow \mathbf{H}(X, \mathbf{B}\mathrm{String}) \longrightarrow \mathbf{H}(X, \mathbf{B}^7U(1)) \quad .$$

Fivebrane 6-bundle lifts	$\longrightarrow$	String 2-bundles	$\longrightarrow$	obstructing Chern-Simons 7-bundles

This has a differential refinement

$$\frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \stackrel{\frac{1}{6}\hat{\mathbf{p}}_2}{\to} \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

$$\mathbf{H}(X, \mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}}) \xrightarrow{\frac{1}{6}\hat{\mathbf{p}}_2} \mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \longrightarrow \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

Fivebrane 6-connection lifts	$\longrightarrow$	String 2-connections	$\longrightarrow$	obstructing Chern-Simons 7-connections

The corresponding action functional defines a 7d String-Chern-Simons theory

$$\exp(iS_{\frac{1}{6}\mathbf{p}_2}) : \ \mathbf{H}(\Sigma, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \xrightarrow{\frac{1}{6}\hat{\mathbf{p}_2}} \mathbf{H}(\Sigma, \mathbf{B}^7 U(1)_{\mathrm{conn}}) \xrightarrow{\int_\Sigma} U(1)$$

This appears after anomaly cancellation in $\mathrm{AdS}_7/\mathrm{CFT}_6$.

Advantage.
Refinement to higher moduli stacks allows to form the homotopy fibers:

for $\hat{G}_8$ a fixed circle 7-connection, the 6-groupoid of $[\hat{G}_8]$-**twisted** $\mathrm{Fivebrane}$-**6-connections** is the homotopy fiber of $\frac{1}{6}\hat{\mathbf{p}}_2$ over $\hat{G}_8$.

Application: For $\nabla_{\mathfrak{e}_8} \in \mathbf{H}(X, \mathbf{B}(E_8)_{\mathrm{conn}})$ an E_8-connection on a Riemannian String-manifold X, anomaly-free background gauge field data for the NS 5-brane on X is a lift σ to the homotopy fiber in

$$\begin{array}{ccc} \frac{1}{6}\hat{\mathbf{p}}_2\mathrm{Struc}(X) & \longrightarrow & * \\ {\scriptstyle\sigma}\nearrow \ \downarrow & & \downarrow {\scriptstyle \hat{G}_8 := \hat{\mathbf{ch}}_4(\nabla_{\mathfrak{e}_8})} \\ \mathbf{H}(X, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) & \xrightarrow{\frac{1}{6}\hat{\mathbf{p}}_2} & \mathbf{H}(X, \mathbf{B}^7 U(1)_{\mathrm{conn}}) \end{array} .$$

Plenty of
further examples...

Plenty of further examples.

See

http://ncatlab.org/schreiber/show/infinity-Chern-Simons+theory+--+examples

or section 4.6 in

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

End.