STRUCTURED COSPANS AND OPEN PETRI NETS

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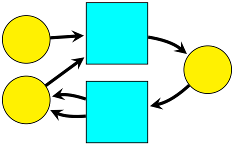
9 April 2020



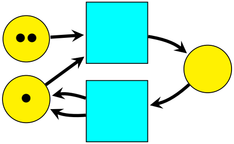
A **Petri net** has:

a finite set of **places** ,

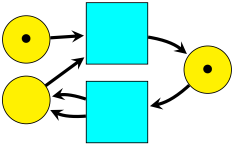
a finite set of **transitions** ,

a natural number of edges from each place to each transition,

a natural number of edges from each transition to each place.



When we run a Petri net, we start by placing a finite number of **tokens** in each place. This is called a **marking**. Then we can repeatedly move the tokens using the transitions.



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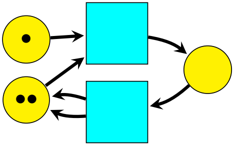

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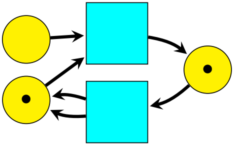


When we run a Petri net, we start by placing a finite number of **tokens** in each place. This is called a **marking**. Then we can repeatedly move the tokens using the transitions.

Mathematically, a **Petri net** is a diagram like this:

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

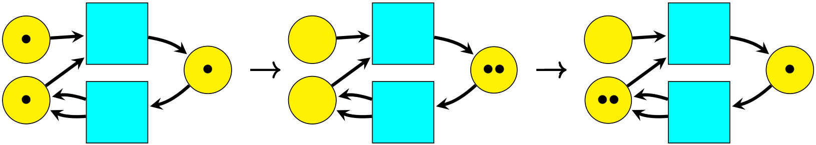
where S and T are sets, and $\mathbb{N}[S]$ is the set of **markings**: formal finite sums of elements of S.

More mathematically, $\mathbb{N}[S]$ is the underlying set of the free commutative monoid on S:

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left=2, "J"] \arrow[r, phantom, "\perp"] & \mathrm{CommMon} \arrow[l, shift left=2, "K"] \end{tikzcd}$$

$$\mathbb{N} = K \circ J$$

Any Petri net gives a symmetric monoidal category where the objects are markings and the morphisms are generated by transitions.



What kind of symmetric monoidal category do we get? A *commutative* monoidal category!

A **commutative monoidal category** is a category in CommMon.

Equivalently, it's a symmetric monoidal category where the

- braidings $\beta_{x,y} \colon x \otimes y \to y \otimes x$
- associators $\alpha_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, and
- left and right unitors $\lambda_x \colon I \otimes x \to x$, $\rho_x \colon x \otimes I \to x$

are all identity morphisms.

Any Petri net P gives a commutative monoidal category FP for which:

- objects are markings of P;
- morphisms are generated from transitions of T by composition and tensor product, subject to the laws of a commutative monoidal category.

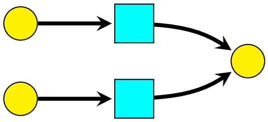
FP is the free commutative monoidal category on the Petri net P, in the following sense.

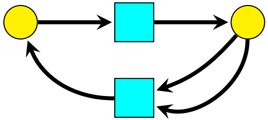
There's a category **Petri**, where:

- an object is a Petri net;
- a morphism from $s, t \colon T \to \mathbb{N}[S]$ to $s', t' \colon T \to \mathbb{N}[S']$ is a pair of functions $f \colon T \to T', g \colon S \to S'$ such that these diagrams commute:

$$\begin{tikzcd} T \arrow[r, "s"] \arrow[d, "f"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[g]"] \\ T' \arrow[r, "s'"'] & \mathbb{N}[S'] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "t"] \arrow[d, "f"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[g]"] \\ T' \arrow[r, "t'"'] & \mathbb{N}[S'] \end{tikzcd}$$

A morphism of Petri nets:





There's also a category **CMC** of commutative monoidal categories, where:

- objects are commutative monoidal categories;
- morphisms are strict monoidal functors (automatically symmetric).

Theorem (Jade Master)

There are adjoint functors

$$
\begin{tikzcd}
\mathrm{Petri} \arrow[r, shift left=1.5ex, "F"] \arrow[r, phantom, "\perp"] & \mathrm{CMC} \arrow[l, shift left=1.5ex, "U"]
\end{tikzcd}
$$

with F sending the Petri net P to the free commutative monoidal category FP described earlier.

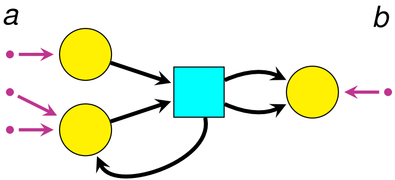
Figuring out the right adjoint U is not as easy as you might think:

- Jade Master, Generalized Petri nets, arXiv:1904.09091

The functor $F\colon \mathrm{Petri} \to \mathrm{CMC}$ provides an 'operational semantics' for Petri nets: it describes a Petri net's *behavior*.

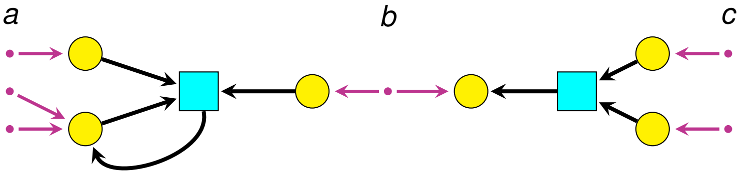
Can we make this operational semantics compositional?

We can build Petri nets from pieces called 'open' Petri nets:

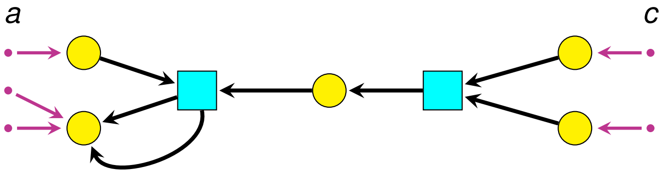


Can we compute a Petri net's behavior from the behavior of its pieces?

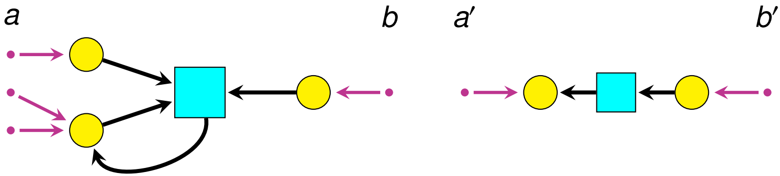
We can compose two open Petri nets:



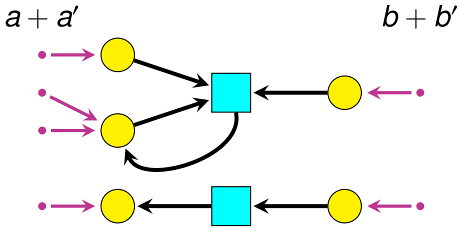
by identifying the outputs of the first with the inputs of the second:



We can also tensor open Petri nets:

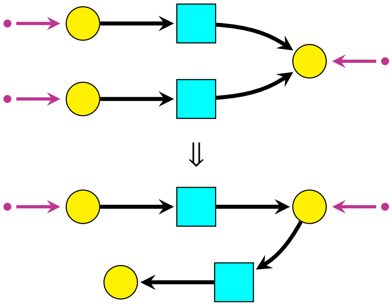


by setting them side by side:



This suggests there's a symmetric monoidal category with open Petri nets as morphisms. And that's almost true!

There are also 2-morphisms *between* open Petri nets:



and composition of open Petri nets is only associative *up to 2-isomorphism*.

So, there's really a symmetric monoidal *double* category with open Petri nets as 'horizontal 1-cells'.

A double category has figures like this:

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"'] & D \end{tikzcd}$$

So, it has:

- **objects** such as A, B, C, D,
- **vertical 1-morphisms** such as f and g,
- **horizontal 1-cells** such as M and N,
- **2-morphisms** such as α.

2-morphisms can be composed vertically and horizontally, and the interchange law holds:

$$
\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
D \arrow[r, "N"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "M'"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\
E \arrow[r, "N'"'] & F
\end{tikzcd}
$$

$$
\begin{tikzcd}
D \arrow[r, "N"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Downarrow\alpha'"] & E \arrow[d, "g'"] \\
G \arrow[r, "O"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "N'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Downarrow\beta'"] & F \arrow[d, "h'"] \\
H \arrow[r, "P"'] & I
\end{tikzcd}
$$

Vertical composition is *strictly* associative and unital, but horizontal composition is *weakly* so.

Jade and I constructed a symmetric monoidal double category with open Petri nets as horizontal 1-morphisms:

- JB and Jade Master, Open Petri nets, arXiv:1808.05415

We used the theory of structured cospans:

- JB and Kenny Courser, Structured cospans, arXiv:1911.04630
- Kenny Courser, *Open Systems: A Double Categorical Perspective*, tinyurl.com/courser-thesis

Given a functor

$$L\colon \mathsf{A} \to \mathsf{X}$$

a **structured cospan** is a diagram

$$\begin{tikzcd} & x & \\ L(a) \arrow[ur, "i"] & & L(b) \arrow[ul, "o"'] \end{tikzcd}$$

Theorem (Kenny Courser, JB)

Let A *and* X *be categories with finite colimits, and* $L \colon \mathsf{A} \to \mathsf{X}$ *a left adjoint.*

Then there is a symmetric monoidal double category ${}_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a structured cospan* $L(a) \stackrel{i}{\rightarrow} x \stackrel{o}{\leftarrow} L(b)$
- *a 2-morphism is a commutative diagram*

$$\begin{tikzcd} L(a) \arrow[r, "i"] \arrow[d, "L(f)"'] & x \arrow[d, "h"'] & L(b) \arrow[l, "o"'] \arrow[d, "L(g)"] \\ L(a') \arrow[r, "i'"'] & x' & L(b') \arrow[l, "o'"] \end{tikzcd}$$

Horizontal composition is defined using pushouts in X; composing these:

$$\begin{tikzcd} L(a) \arrow[r] \arrow[d] & x \arrow[d] & L(b) \arrow[l] \arrow[d] \\ L(a') \arrow[r] & x' & L(b') \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} L(b) \arrow[r] \arrow[d] & y \arrow[d] & L(c) \arrow[l] \arrow[d] \\ L(b') \arrow[r] & y' & L(c') \arrow[l] \end{tikzcd}$$

gives this:

$$\begin{tikzcd} L(a) \arrow[r] \arrow[d] & x +_{L(b)} y \arrow[d] & L(c) \arrow[l] \arrow[d] \\ L(a') \arrow[r] & x' +_{L(b')} y' & L(c') \arrow[l] \end{tikzcd}$$

Vertical composition is straightforward.

Tensoring uses binary coproducts in both A and X, and the fact that $L \colon \mathsf{A} \to \mathsf{X}$ preserves these:

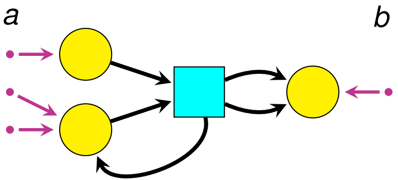
$$\begin{array}{ccccc} L(a_1) & \longrightarrow & x_1 & \longleftarrow & L(b_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2) & \longrightarrow & x_2 & \longleftarrow & L(b_2) \end{array} \quad \otimes \quad \begin{array}{ccccc} L(a'_1) & \longrightarrow & x'_1 & \longleftarrow & L(b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a'_2) & \longrightarrow & x'_2 & \longleftarrow & L(b'_2) \end{array}$$

$$= \quad \begin{array}{ccccc} L(a_1 + a'_1) & \longrightarrow & x_1 + x'_1 & \longleftarrow & L(b_1 + b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2 + a'_2) & \longrightarrow & x_2 + x'_2 & \longleftarrow & L(b_2 + b'_2) \end{array}$$

There's a left adjoint functor

$$L\colon \mathrm{Set} \to \mathrm{Petri}$$

mapping any set to the Petri net with that set of places, and no transitions. With this choice of L, a structured cospan is an **open Petri net**:



Set and Petri have colimits. We thus get a symmetric monoidal double category $\mathbb{O}\mathbf{pen}(\mathrm{Petri})$ with open Petri nets as horizontal 1-morphisms!

We can compose our left adjoints:

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "L"] \arrow[dr, "L'=F\circ L"'] & \mathrm{Petri} \arrow[d, "F"] \\ & \mathrm{CMC} \end{tikzcd}$$

Since CMC also has colimits, the left adjoint

$$L' \colon \mathrm{Set} \to \mathrm{CMC}$$

gives a symmetric monoidal double category $\mathbb{O}\mathbf{pen}(\mathrm{CMC})$ with *open commutative monoidal categories* as horizontal 1-cells.

Theorem (JB, Kenny Courser)

Suppose $\mathsf{A}, \mathsf{X}, \mathsf{X}'$ *have finite colimits and there is a commuting triangle of left adjoints*

$$\begin{tikzcd} \mathsf{A} \arrow[r, "L"] \arrow[dr, "L'"'] & \mathsf{X} \arrow[d, "F"] \\ & \mathsf{X}' \end{tikzcd}$$

Then there is a symmetric monoidal double functor

$$\mathbb{C}\mathbf{sp}(F) \colon {}_L\mathbb{C}\mathbf{sp}(\mathsf{X}) \to {}_{L'}\mathbb{C}\mathbf{sp}(\mathsf{X}')$$

that is the identity on objects and vertical morphisms, and acts as follows on horizontal 1-cells and 2-morphisms:

$$\begin{tikzcd} L(a) \arrow[r, "i"] \arrow[d, "L(f)"'] & x \arrow[d, "\alpha"'] & L(b) \arrow[l, "o"'] \arrow[d, "L(g)"] \\ L(a') \arrow[r, "i'"'] & x' & L(b') \arrow[l, "o'"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} L'(a) \arrow[r, "F(i)"] \arrow[d, "L'f"'] & F(x) \arrow[d, "F(\alpha)"'] & L'(b) \arrow[l, "F(o)"'] \arrow[d, "L'(g)"] \\ L'(a') \arrow[r, "F(i')"'] & F(x') & L'(b') \arrow[l, "F(o')"] \end{tikzcd}$$

So, our commuting triangle of left adjoints

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "L"] \arrow[dr, "L'"'] & \mathrm{Petri} \arrow[d, "F"] \\ & \mathrm{CMC} \end{tikzcd}$$

gives a symmetric monoidal double functor

$$\mathbb{C}\mathbf{sp}(F) \colon \mathbb{O}\mathbf{pen}(\mathrm{Petri}) \to \mathbb{O}\mathbf{pen}(\mathrm{CMC})$$

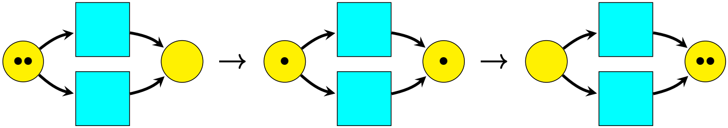
This says the operational semantics of Petri nets is compositional!

We can compose open Petri nets and then turn them into open commutative monoidal categories... or the other way around... and we get isomorphic answers!

Besides the ‘operational’ semantics for open Petri nets, which says how they *behave*, there is also a ‘reachability’ semantics, which says what they *accomplish*.

In fact there are at least two approaches to this! I’ll present one that’s not in my paper with Jade.

Given two markings x, y of a Petri net P, we say y is **reachable** from x if there’s a morphism $f \colon x \to y$ in FP.



If our Petri net P is

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

then its set of markings is $\mathbb{N}[S]$. This becomes a poset where $x \le y$ iff y is reachable from x.

But $\mathbb{N}[S]$ also has the structure of a commutative monoid! It's a **commutative monoidal poset** since

$$x \le y \;\&\; x' \le y' \;\Rightarrow\; x + x' \le y + y'$$

There is a category **CMP** of commutative monoidal posets and order-preserving homomorphisms, and a left adjoint

$$G\colon \mathrm{CMC} \to \mathrm{CMP}$$

that takes any commutative monoidal category and turns its set of objects into a commutative monoidal poset where $x \leq y$ iff there exists a morphism $f\colon x \to y$.

Thus, we get a **reachability semantics** for open Petri nets by composing these symmetric monoidal double functors:

$$\mathbb{O}\mathbf{pen}(\mathrm{Petri}) \xrightarrow{\mathbb{C}\mathbf{sp}(F)} \mathbb{O}\mathbf{pen}(\mathrm{CMC}) \xrightarrow{\mathbb{C}\mathbf{sp}(G)} \mathbb{O}\mathbf{pen}(\mathrm{CMP})$$

In short:

- Each Petri net gives a commutative monoidal category.
- There's a symmetric monoidal *double* category of *open* Petri nets.
- Combining these ideas, we get the operational semantics for open Petri nets.
- Turning commutative monoidal categories into commutative monoidal posets, we get a reachability semantics for open Petri nets.