LECTURES ON n-CATEGORIES AND COHOMOLOGY

TALKS BY JOHN BAEZ, NOTES BY MICHAEL SHULMAN

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Preface

The goal of these talks was to explain how cohomology and other tools of algebraic topology are seen through the lens of n-category theory. The talks were extremely informal, glossing over the difficulties involved in making certain things precise, just trying to sketch the big picture in an elementary way. A lot of the material is hard to find spelled out anywhere, but nothing new is due to me: anything not already known by experts was invented by James Dolan, Toby Bartels or Mike Shulman (who took notes, fixed lots of mistakes, and wrote the Appendix).

The first talk was one of the 2006 Namboodiri Lectures in Topology at the University of Chicago. It's a quick introduction to the relation between Galois theory, covering spaces, cohomology, and higher categories. The remaining talks, given in the category theory seminar at Chicago, were more advanced. Topics include nonabelian cohomology, Postnikov towers, the theory of 'n-stuff', and n-categories for $n = -1$ and -2. Some questions from the audience have been included.

Mike Shulman's extensive Appendix (§5) clarifies many puzzles raised in the talks. It also ventures into deeper waters, such as the role of posets and fibrations in higher category theory, alternate versions of the periodic table of n-categories, and the theory of higher topoi. For readers who want more details, we have added an annotated bibliography. — JB

1. The Basic Principle of Galois Theory

1.1. **Galois theory.** Around 1832, Galois discovered a basic principle:

> **We can study the ways a little thing k can sit in a bigger thing K:**
>
> $$k \hookrightarrow K$$
>
> **by keeping track of the symmetries of K that fix k. These form a subgroup of the symmetries of K:**
>
> $$\mathrm{Gal}(K|k) \subseteq \mathrm{Aut}(K).$$

For example, a point k of a set K is completely determined by the subgroup of permutations of K that fix k. More generally, we can recover any subset k of a set K from the subgroup of permutations of K that fix k.

However, Galois applied his principle in a trickier example, namely commutative algebra. He took K to be a field and k to be a subfield. He studied this situation by looking at the subgroup $\mathrm{Gal}(K|k)$ of automorphisms of K that fix k. Here this subgroup does not determine k unless we make a further technical assumption, namely that K is a 'Galois extension' of k. In general, we just have a map sending each subfield of K to the subgroup of $\mathrm{Aut}(K)$ that fixes it, and a map sending each subgroup of $\mathrm{Aut}(K)$ to the subfield it fixes. These maps are not inverses; instead, they satisfy some properties making them into what what is called a 'Galois connection'.

When we seem forced to choose between extra technical assumptions or less than optimal results, it often means we haven't fully understood the general principle we're trying to formalize. But, it can be very hard to take a big idea like the basic principle of Galois theory and express it precisely without losing some of its power. That is not my goal here. Instead, I'll start by considering a weak but precise version of Galois' principle as applied to a specific subject: not commutative algebra, but *topology*.

Topology isn't really separate from commutative algebra. Indeed, in the mid-1800s, Dedekind, Kummer and Riemann realized that commutative algebra is a lot like topology, only backwards. Any space X has a commutative algebra $\mathscr{O}(X)$ consisting of functions on it. Any map

$$f\colon X \to Y$$

gives a map

$$f^*\colon \mathscr{O}(Y) \to \mathscr{O}(X).$$

If we're clever we can think of any commutative ring as functions on some space — or on some 'affine scheme':

$$[\text{Affine Schemes}] = [\text{Commutative Rings}]^{\mathrm{op}}.$$

Note how it's backwards: the *inclusion* of commutative rings

$$p^*\colon \mathbb{C}[z] \hookrightarrow \mathbb{C}[\sqrt{z}]$$

corresponds to the *branched cover* of the complex plane by the Riemann surface for $\sqrt{z}$:

$$\begin{array}{rcl} p\colon \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & z^2 \end{array}$$

So: classifying how a little commutative algebra can *sit inside* a big one amounts to classifying how a big space can *cover* a little one. Now the Galois group gets renamed the group of **deck transformations**: in the above example it's $\mathbb{Z}/2$:

$$\sqrt{z} \mapsto -\sqrt{z}.$$

The theme of 'branched covers' became very important in later work on number theory, where number fields are studied by analogy to function fields (fields of functions on Riemann surfaces). However, it's the simpler case of unbranched covers where the basic principle of Galois theory takes a specially simple and pretty form, thanks in part to Poincaré. This is the version we'll talk about now. Later we'll generalize it from unbranched covers to 'fibrations' of various sorts — fibrations of spaces, but also fibrations of n-categories. Classifying fibrations using the basic principle of Galois theory will eventually lead us to cohomology.

1.2. **The fundamental group.** Around 1883, Poincaré discovered that any nice connected space B has a connected covering space that covers all others: its **universal cover**. This has the biggest deck transformation group of all: the **fundamental group** $\pi_1(B)$.

The idea behind Galois theory — turned backwards! — then says that:

Connected covering spaces of B are classified by subgroups

$$H \subseteq \pi_1(B).$$

This is the version we all learn in grad school. To remove the 'connectedness' assumption, we can start by rephrasing it like this:

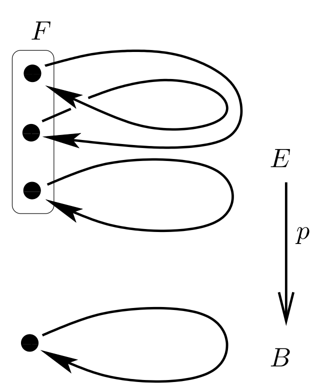
Connected covering spaces of B with fiber F are classified by transitive actions of $\pi_1(B)$ on F.

This amounts to the same thing, since transitive group actions are basically the same as subgroups: given a subgroup $H \subseteq \pi_1(B)$ we can define F to be $\pi_1(B)/H$, and given a transitive action of $\pi_1(B)$ on F we can define H to be the stabilizer group of a point. The advantage of this formulation is that we can generalize it to handle covering spaces where the total space isn't connected:

Covering spaces of B with fiber F are classified by actions of $\pi_1(B)$ on F:

$$\pi_1(B) \to \mathrm{Aut}(F).$$

Here F is any set and $\mathrm{Aut}(F)$ is the group of permutations of this set:



You can see how a loop in the base space gives a permutation of the fiber. The basic principle of Galois theory has become 'visible'!

1.3. **The fundamental groupoid.** So far the base space B has been connected. What if B is not connected? For this, we should replace $\pi_1(B)$ by $\Pi_1(B)$: the **fundamental groupoid** of B. This is the category where:

- objects are points of B: $\bullet x$
- morphisms are homotopy classes of paths in B:

$$x \bullet \xrightarrow{\;f\;} \bullet\, y$$

The basic principle of Galois theory then says this:

> **Covering spaces $F \hookrightarrow E \to B$ are classified by actions of $\Pi_1(B)$ on F: that is, functors**
>
> $$\Pi_1(B) \to \mathrm{Aut}(F).$$

Even better, we can let the fiber F be different over different components of the base B:

> **Covering spaces $E \to B$ are classified by functors**
>
> $$\Pi_1(B) \to \mathbf{Set}.$$

What does this mean? It says a lot in a very terse way. Given a covering space $p\colon E \to B$, we can uniquely lift any path in the base space to a path in E, given a lift of the path's starting point. Moreover, this lift depends only on the homotopy class of the path. So, our covering space assigns a set $p^{-1}(b)$ to each point $b \in B$, and a map between these sets for any homotopy class of paths in B. Since composition of paths gets sent to composition of maps, this gives a *functor* from $\Pi_1(B)$ to **Set**.

Conversely, given any functor $F\colon \Pi_1(B) \to \mathbf{Set}$, we can use it to cook up a covering space of B, by letting the fiber over b be $F(b)$, and so on. So, with some work, we get a one-to-one correspondence between isomorphism classes of covering spaces $E \to B$ and natural isomorphism classes of functors $\Pi_1(B) \to \mathbf{Set}$.

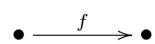
But we actually get more: we get an *equivalence of categories.* The category of covering spaces of B is equivalent to the category where the objects are functors $\Pi_1(B) \to \mathbf{Set}$ and the morphisms are natural transformations between these guys. This is what I've really meant all along by saying "X's are classified by Y's." I mean there's a category of X's, a category of Y's, and these categories are equivalent.

1.4. **Eilenberg–Mac Lane Spaces.** In 1945, Eilenberg and Mac Lane published their famous paper about categories. They *also* published a paper showing that any group G has a 'best' space with G as its fundamental group: the **Eilenberg-Mac Lane space** $K(G,1)$.

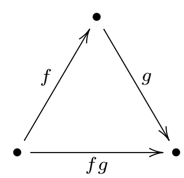
In fact their idea is easiest to understand if we describe it a bit more generally, not just for groups but for groupoids. For any groupoid G we can build a space $K(G,1)$ by taking a vertex for each object of G:

$$\bullet\; x$$

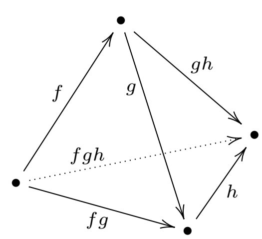
an edge for each morphism of G:



a triangle for each composable pair of morphisms:



a tetrahedron for each composable triple:



and so on. This space has G as its fundamental groupoid, and it's a **homotopy 1-type**: all its homotopy groups above the 1st vanish. These facts characterize it.

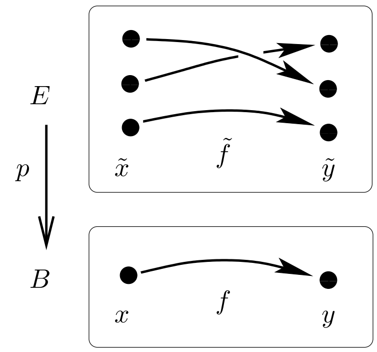
Using this idea, one can show a portion of topology is just groupoid theory: homotopy 1-types are the same as groupoids! To make this precise requires a bit of work. It's not true that the category of homotopy 1-types and maps between them is equivalent to the category of groupoids and functors between them. But, they form Quillen equivalent model categories. Or, if you prefer, they form 2-equivalent 2-categories.

1.5. **Grothendieck's dream.** Since the classification of covering spaces

$$E \to B$$

only involves the fundamental groupoid of B, we might as well assume B is a homotopy 1-type. Then E will be one too.

So, we might as well say E and B are *groupoids!* The analogue of a covering space for groupoids is a **discrete fibration**: a functor $p\colon E \to B$ such that for any morphism $f\colon x \to y$ in B and object $\tilde{x} \in E$ lifting x, there's a unique morphism $\tilde{f}\colon \tilde{x} \to \tilde{y}$ lifting f:



The basic principle of Galois theory then becomes:

Discrete fibrations $E \to B$ are classified by functors

$$B \to \mathbf{Set}.$$

This is true even when E and B are categories, though then people use the term 'opfibrations'. This — and much more — goes back to Grothendieck's 1971 book *Étale Coverings and the Fundamental Group*, usually known as SGA1.

Grothendieck dreamt of a much bigger generalization of Galois theory in his 593-page letter to Quillen, *Pursuing Stacks*. Say a space is a **homotopy n-type** if its homotopy groups above the nth all vanish. Since homotopy 1-types are 'the same' as groupoids, maybe homotopy n-types are 'the same' as n-groupoids! It's certainly true if we use Kan's simplicial approach to n-groupoids — but we want it to emerge from a general theory of n-categories.

For n-groupoids, the basic principle of Galois theory should say something like this:

Fibrations $E \to B$ where E and B are n-groupoids are classified by weak $(n+1)$-functors

$$B \to n\mathbf{Gpd}.$$

Now when we say 'classified by' we mean there's an equivalence of $(n+1)$-categories. 'Weak' n-functors are those where everything is preserved *up to equivalence*. I include the adjective 'weak' only for emphasis: we need all n-categories and n-functors to be weak for Grothendieck's dream to have any chance of coming true, so for us, everything is weak by default.

Grothendieck made the above statement precise and proved it for $n = 1$; later Hermida did it for $n = 2$. Let's see what it amounts to when $n = 1$. To keep things really simple, suppose E, B are just *groups*, and fix the fiber F, also a group. With a fixed fiber our classifying 2-functor will land not in all of **Gpd**, but in $\mathrm{AUT}(F)$, which is the 'automorphism 2-group' of F — I'll say exactly what that is in a minute. In this simple case fibrations are just *extensions*, so we get a statement like this:

Extensions of the group B by the group F, that is, short exact sequences

$$1 \to F \to E \to B \to 1,$$

are classified by weak 2-functors

$$B \to \mathrm{AUT}(F).$$

This is called **Schreier theory**, since a version of this result was first shown by Schreier around 1926. The more familiar classifications of abelian or central group extensions using Ext or H^2 are just watered-down versions of this.

$\mathrm{AUT}(F)$ is the **automorphism 2-group** of F, a 2-category with:

- F as its only object: $\bullet F$
- automorphisms of F as its morphisms:

$$F\bullet \xrightarrow{\ \alpha\ } \bullet F$$

- elements $g \in F$ with $g\alpha(f)g^{-1} = \beta(f)$ as its 2-morphisms:

$$F\bullet \underset{\beta}{\overset{\alpha}{\Downarrow g}} \bullet F$$

In other words, we get $\mathrm{AUT}(F)$ by taking **Gpd** and forming the sub-2-category with F as its only object, all morphisms from this to itself, and all 2-morphisms between these.

Given a short exact sequence of groups, we classify it by choosing a **set-theoretic section**:

$$1 \longrightarrow F \xrightarrow[i]{} E \underset{p}{\overset{s}{\rightleftarrows}} B \longrightarrow 1\ ,$$

meaning a function $s\colon B \to E$ with $p(s(b)) = b$ for all $b \in B$. This gives for any $b \in B$ an automorphism $\alpha(b)$ of F:

$$\alpha(b)(f) = s(b)fs(b)^{-1}.$$

Since s need not be a homomorphism, we may not have

$$\alpha(b)\,\alpha(b') = \alpha(bb')$$

but this holds *up to conjugation* by an element $\alpha(b, b') \in F$. That is,

$$\alpha(b, b')\,[\alpha(b)\,\alpha(b')f]\,\alpha(b, b')^{-1} = \alpha(bb')f$$

where

$$\alpha(b, b') = s(bb')\,(s(b)\,s(b'))^{-1}.$$

This turns out to yield a *weak* 2-functor

$$\alpha\colon B \to \mathrm{AUT}(F).$$

If we consider two weak 2-functors equivalent when there's a 'weak natural isomorphism' between them, different choices of s will give equivalent 2-functors. Isomorphic extensions of B by F also give equivalent 2-functors.

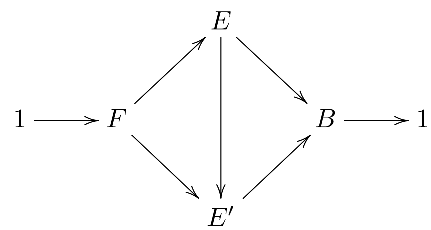
The set of equivalence classes of weak 2-functors $B \to \mathrm{AUT}(F)$ is often called the **nonabelian cohomology** $H(B, \mathrm{AUT}(F))$. So, we've described a map sending isomorphism classes of short short exact sequences

$$1 \to F \to E \to B \to 1$$

to elements of $H(B, \mathrm{AUT}(F))$. And, this map is one-to-one and onto.

This is *part* of what we mean by saying extensions of B by F are classified by weak 2-functors $B \to \mathrm{AUT}(F)$. But as usual, we really have something much better: an equivalence of 2-categories.

There's a well-known category of extensions of B by F, where the morphisms are commutative diagrams like this:



But actually, we get a 2-category of extensions using the fact that groups are special groupoids and **Gpd** is a 2-category. Similarly, $\hom(B, \mathrm{AUT}(F))$ is a 2-category, since $\mathrm{AUT}(F)$ is a 2-category and B is a group, hence a special sort of category, hence a special sort of 2-category. And, the main result of Schreier theory says the 2-category of extensions of B by F is equivalent to $\hom(B, \mathrm{AUT}(F))$. This implies the earlier result we stated, but it's much stronger.

In short, generalizing the fundamental principle of Galois theory to fibrations where everything is a *group* gives a beautiful classification of group extensions in terms of nonabelian cohomology. In the rest of these lectures, we'll explore how this generalizes as we go from groups to n-groupoids and n-categories.

2. The Power of Negative Thinking

2.1. **Extending the periodic table.** Now I want to dig a lot deeper into the relation between fibrations, cohomology and n-categories. At this point I'll suddenly assume that you have some idea of what n-categories are, or at least can fake it. The periodic table of n-categories shows what various degenerate versions of n-category look like. We can think of an $(n+k)$-category with just one j-morphism for $j < k$ as a special sort of n-category. They look like this:

THE PERIODIC TABLE

	$n = \mathbf{0}$	$n = \mathbf{1}$	$n = \mathbf{2}$
$k = \mathbf{0}$	**sets**	**categories**	**2-categories**
$k = \mathbf{1}$	**monoids**	**monoidal categories**	**monoidal 2-categories**
$k = \mathbf{2}$	**commutative monoids**	**braided monoidal categories**	**braided monoidal 2-categories**
$k = \mathbf{3}$	‘’	**symmetric monoidal categories**	**sylleptic monoidal 2-categories**
$k = \mathbf{4}$	‘’	‘’	**symmetric monoidal 2-categories**
$k = \mathbf{5}$	‘’	‘’	‘’
$k = \mathbf{6}$	‘’	‘’	‘’

For example, in the $n = 0$, $k = 1$ spot we have 1-categories with just one 0-morphism, or in normal language, categories with just one object — i.e., monoids. Indeed a ‘monoid’ is the perfect name for a one-object category, because ‘monos’ means ‘one’ — but that's not where the name comes from, of course. It's a good thing, too, or else Eilenberg and Mac Lane might have called categories ‘polyoids’.

We'll be thinking about all these things in the most weak manner possible, so ‘2-category’ means ‘weak 2-category’, aka ‘bicategory’, and so on. Everything I'm going to tell you *should* be true, once we really understand what is going on. Right now it's more in the nature of dreams and speculations, but I don't think we'll be able to prove the theorems until we dream enough.

Eckmann and Hilton algebraicized a topological argument going back to Hurewicz, which proves that strict monoidal categories with one object are commutative monoids. Eugenia Cheng and Nick Gurski have studied this carefully, and they've shown that things are a little more complicated when we consider *weak* monoidal categories, but I'm going to proceed in a robust spirit and leave such issues to smart young people like them.

Things get interesting in the second column, when we get *braided* monoidal categories. These are not the most obvious sort of 'commutative' monoidal categories that Mac Lane first wrote down, namely the symmetric ones. James Dolan and I were quite confused about why braided and symmetric both exist, until we started getting the hang of the periodic table.

Noticing that in the first column we stabilize after 2 steps at commutative monoids, and in the second after 3 steps at symmetric monoidal categories, we enunciated the **stabilization hypothesis**, which says that the nth column should stabilize at the $(n+2)$nd row. We believed this because of the Freudenthal suspension theorem in homotopy theory, which says that if you keep suspending and looping a space, it gets nicer and nicer, and if it only has n nonvanishing homotopy groups, eventually it's as nice as it can get, and it stabilizes after $n+2$ steps. This is related, because any space gives you an n-groupoid of points, paths, paths-of-paths, and so on.

Such a beautiful pattern takes the nebulous, scary subject of n-categories and imposes some structure on it. There are all sorts of operations that take you hopping between different squares of this chart.

We call this chart the 'Periodic Table' — not because it's periodic, but because we can use it to predict new phenomena, like Mendeleev used the periodic table to predict new elements.

After we came up with the periodic table I showed it to Chris Isham, who does quantum gravity at Imperial College. I was incredibly happy with it, but he said: "That's obviously not right — you didn't start the chart at the right place. First there should be a column with just one interesting row, then a column with two, and *then* one with three."

I thought he was crazy, but it kept nagging me. It's sort of weird to start counting at three, after all. But there are no (-1)-categories or (-2)-categories! Are there?

It turns out there are! Eventually Toby Bartels and James Dolan figured out what they are. And they realized that Isham was right. The periodic table really looks like this:

THE EXTENDED PERIODIC TABLE

	$n = -2$	$n = -1$	$n = 0$	$n = 1$	$n = 2$
$k = 0$	**?**	**?**	**sets**	**categories**	**2-categories**
$k = 1$	''	**?**	**monoids**	**monoidal categories**	**monoidal 2-categories**
$k = 2$	''	''	**commutative monoids**	**braided monoidal categories**	**braided monoidal 2-categories**
$k = 3$	''	''	''	**symmetric monoidal categories**	**sylleptic monoidal 2-categories**
$k = 4$	''	''	''	''	**symmetric monoidal 2-categories**

You should be dying to know what fills in those question marks. Just for fun, I'll tell you what two of them are now. You probably won't think these answers are obvious — but you will soon:

- (-1)-categories are just truth values: there are only two of them, True and False.
- (-2)-categores are just 'necessarily true' truth values: there is only one of them, which is True.

I know this sounds crazy, but it sheds lots of light on many things. Let's see why (-1)- and (-2)-categories really work this way.

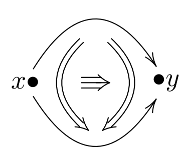
2.2. **The categorical approach.** Before describing (-1)-categories and (-2)-categories, we need to understand a couple of facts about the n-categorical world.

The first is that in the n-categorical universe, every n-category is secretly an $(n+1)$-category with only identity $(n+1)$-morphisms. It's common for people to talk about sets as discrete categories, for example. A way to think about it is that these identity $(n+1)$-morphisms are really *equations*.

When you play the n-category game, there's a rule that you should never say things are equal, only isomorphic. This makes sense up until the top level, the level of n-morphisms, when you break down and allow yourself to say that n-morphisms are equal. But actually you aren't breaking the rule here, if you think of your n-category as an $(n+1)$-category with only identity morphisms. Those equations are really isomorphisms: it's just that the only isomorphisms existing at the $(n+1)$st level are identities. Thus when we assert equations, we're refusing to think about things still more categorically, and saying "all I can take today is an n-category" rather than an $(n+1)$-category.

We can iterate this and go on forever, so every n-category is really an ∞-category with only identity j-morphisms for $j > n$.

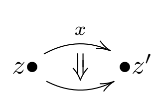
The second thing is that big n-categories have lots of little n-categories inside them. For example, between two objects x, y in a 3-category, there's a little 2-category $\hom(x, y)$. James and I jokingly call this sort of thing a 'microcosm', since it's like a little world within a world:



A microcosm

In general, given objects x, y in an n-category, there is an $(n-1)$-category called $\hom(x, y)$ (because it's the 'thing' of morphisms from x to y) with the morphisms $f : x \to y$ as its objects, etc..

We can iterate this and look at microcosms of microcosms. A couple of objects in $\hom(x, y)$ give an $(n-2)$-category, and so on. It's handy to say that two j-morphisms x and y are **parallel** if they look like this:



This makes sense if $j > 0$; if $j = 0$ we decree that all j-morphisms are parallel. The point is that it only makes sense to talk about something like $f\colon x \to y$ when x and y are parallel. Given parallel j-morphisms x and y, we get an $(n-j-1)$-category $\hom(x,y)$ with $(j+1)$-morphisms $f\colon x \to y$ as objects, and so on. This is a little microcosm.

In short: *given parallel j-morphisms x and y in an n-category,* $\hom(x,y)$ *is an $(n-j-1)$-category.* Now take $j = n$. Then we get a (-1)-category! If x and y are parallel n-morphisms in an n-category, then $\hom(x,y)$ is a (-1)-category. What is it?

You might say "that's cheating: you're not allowed to go that high." But it isn't really cheating, since, as we said, every n-category is secretly an ∞-category. We just need to work out the answer, and that will tell us what a (-1)-category is.

The objects of $\hom(x,y)$ are $(n+1)$-morphisms, which here are just identities. So, if $x = y$ there is one object in $\hom(x,y)$, otherwise there's none. So there are really just two possible (-1)-categories. There aren't any (-1)-categories that don't arise in this way, since in general any n-category can be stuck in between two objects to make an $(n+1)$-category.

Thus, there are just two (-1)-categories. You could think of them as the 1-element set and the empty set, although they're not exactly sets. We can also call them '$=$' and '$\neq$', or 'True' and 'False.' The main thing is, there are just two.

I hope I've convinced you this is right. You may think it's silly, but you should think it's right.

Now we should take $j = n+1$ to get the (-2)-categories. If we have two parallel $(n+1)$-morphisms in an n-category, they are both identities, so being parallel they must both be $1_z\colon z \to z$ for some z, so they're equal. At this level, they *have* to be equal, so there *is* an identity from one from to the other, necessarily. It's like the previous case, except we only have one choice.

So there's just one (-2)-category. When there's just one of something, you can call it anything you want, since you don't have to distinguish it from anything. But it might be good to call this guy the 1-element set, or '$=$', or 'True'. Or maybe we should call it 'Necessarily True', since there's no other choice.

So, we've worked out (-1)-categories and (-2)-categories. We could keep on going, but it stabilizes past this point: for $n > 2$, $(-n)$-categories are all just 'True.' I'll leave it as a puzzle for you to figure out what a monoidal (-1)-category is.

2.3. **Homotopy n-types.** I really want to talk about what this has to do with topology. We're going to study very-low-dimensional algebra and apply it to very-low-dimensional topology—in fact, so low-dimensional that they never told you about it in school. Remember Grothendieck's dream:

Hypothesis 1 (Grothendieck's Dream, aka the Homotopy Hypothesis)**.** *n-Groupoids are the same as homotopy n-types.*

Here 'n-groupoids' means *weak* n-groupoids, in which everything is invertible up to higher-level morphisms, in the weakest possible way. Similarly 'the same' is meant in the weakest possible way, which we might make precise using something called 'Quillen equivalence'. A **homotopy n-type** is a nice space (e.g. a CW-complex) with vanishing homotopy groups π_j for $j > n$. (If it's not connected, we have to take π_j at every basepoint.) We could have said 'all spaces' instead of 'nice spaces,' but then we'd need to talk about *weak* homotopy equivalence instead of homotopy equivalence.

People have made this precise and shown that it's true for various low values of n, and we're currently struggling with it for higher values. It's known (really well) for $n = 1$, (pretty darn well) for $n = 2$, (partly) for $n = 3$, and somewhat fuzzier for higher n. Today we're going to do it for *lower* values of n, like $n = -1$ and $n = -2$.

Now, they never told you about the negative second homotopy group of a space—or 'homotopy thingy', since after all we know π_0 is only a set, not a group. In fact we won't define these negative homotopy thingies as thingies yet, but only what it means for them to vanish:

Definition 2. We say $\pi_j(X)$ **vanishes for all basepoints** if given any $f\colon S^j \to X$ there exists $g\colon D^{j+1} \to X$ extending f. We say X is a **homotopy n-type** if $\pi_j(X)$ vanishes for all basepoints whenever $j > n$.

We'll use this to figure out what a homotopy 0-type is, then use it to figure out what a homotopy (-1)-type and a homotopy (-2)-type are.

X is a *homotopy 0-type* when all circles and higher-dimensional spheres mapped into X can be contracted. So, X is just a disjoint union of connected components, all of which are contractible. (The fact that being able to contract all spheres implies the space is contractible, for nice spaces, is Whitehead's theorem.) From the point of view of a homotopy theorist, such a space might as well just be a set of points, i.e. a discrete space, but the points could be 'fat'. This is what Grothendieck said should happen; all 0-categories are 0-groupoids, which are just sets.

Now let's figure out what a homotopy (-1)-type is. If you pay careful attention, you'll see the following argument is sort of the same as what we did before to figure out what (-1)-categories are.

By definition, a *homotopy (-1)-type* is a disjoint union of contractible spaces (i.e. a homotopy 0-type) with the extra property that maps from S^0 can be contracted. What can X be now? It can have just one contractible component (the easy case)—or it can have none (the sneaky case). So X is a disjoint union of 0 or 1 contractible components. From the point of view of homotopy theory, such a space might as well be an empty set or a 1-point set. This is the same answer that we got before: the absence or presence of an equation, 'False' or "True'.

Finally, a *homotopy (-2)-type* is thus a space like this such that any map $S^{-1} \to X$ extends to D^0. Now we have to remember what S^{-1} is. The n-sphere is the unit sphere in $\mathbb{R}^{n+1}$, so S^{-1} is the unit sphere in $\mathbb{R}^0$. It consists of all unit vectors in this zero-dimensional vector space, i.e. it is the empty set, $S^{-1} = \emptyset$. Well, a map from the empty set into X is a really easy thing to be given: there's always just one. And D^0 is the unit disc in $\mathbb{R}^0$, so it's just the origin, $D^0 = \{0\}$. So this extension condition says that X has to have at least one point in it. Thus a homotopy (-2)-type is a disjoint union of precisely one contractible component. Up to homotopy, it is thus a one-point set, or 'True'. So Grothendieck's idea works here too.

Now I think you all agree; I gave you *two* proofs!

What about (-3)-types? Even I get a little scared about S^{-2} and D^{-1}.

[Peter May: *They're both empty, so it stabilizes here.*]

2.4. **Stuff, structure, and properties.** What's all this nonsense about? In math we're often interested in equipping things with extra structure, stuff, or properties, and people are often a little vague about what these mean. For example, a group is a set (*stuff*) with operations (*structure*) such that a bunch of equations hold (*properties*).

You can make these concepts very precise by thinking about forgetful functors. It always bugged me when reading books that no one ever defined 'forgetful functor'. Some functors are more forgetful than others. Consider a functor $p\colon E \to B$ (the notation reflects that later on, we're going to turn it into a fibration when we use Grothendieck's idea). There are various amounts of forgetfulness that p can have:

- p **forgets nothing** if it is an equivalence of categories, i.e. faithful, full, and essentially surjective. For example the identity functor $\mathbf{AbGp} \to \mathbf{AbGp}$ forgets nothing.
- p **forgets at most properties** if it is faithful and full. E.g. $\mathbf{AbGp} \to \mathbf{Gp}$, which forgets the property of being abelian, but a homomorphism of abelian groups is just a homomorphism between groups that happen to be abelian.
- p **forgets at most structure** if it is faithful. E.g. the forgetful functor from groups to sets, $\mathbf{AbGp} \to \mathbf{Sets}$, forgets the structure of being an abelian group, but it's still faithful.
- p **forgets at most stuff** if it is arbitrary. E.g. $\mathbf{Sets}^2 \to \mathbf{Sets}$, where we just throw out the second set, is not even faithful.

There are different ways of slicing this pie. For now, we are thinking of each level of forgetfulness as subsuming the previous ones, so 'forgetting at most structure' means forgetting structure and/or properties and/or nothing, but we can also try to make them completely disjoint concepts. Later I'll define a concept of 'forgetting *purely* structure' and so on.

What's going on here is that in every case, what you can do is take objects downstairs and look at their *fiber* or really *homotopy fiber* upstairs. An object in the homotopy fiber upstairs is an object together with a morphism from its image to the object downstairs, as follows.

Given $p\colon E \to B$ and $x \in B$, its **homotopy fiber** or **essential preimage**, which we write $p^{-1}(x)$, has:

- objects $e \in E$ equipped with isomorphism $p(e) \cong x$
- morphisms $f\colon e \to e'$ in E compatible with the given isomorphisms:

$$\begin{array}{ccc} p(e) & \xrightarrow{\;p(f)\;} & p(e') \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & x & \end{array}$$

It turns out that the more forgetful the functor is, the bigger and badder the homotopy fibers can be. In other words, switching to the language of topology, they can have bigger **homotopical dimension**: they can have nonvanishing homotopy groups up to dimension d for bigger d.

Fact 3. *If E and B are groupoids (we'll consider the case of categories in §5.2), then*

- *p forgets stuff if all $p^{-1}(x)$ are arbitrary (1-)groupoids. For example, there's a whole* groupoid *of ways to add an extra set to some set.*

- *p forgets structure if all $p^{-1}(x)$ are 0-groupoids, i.e. groupoids which are (equivalent to) sets. For example, there's just a* set *of ways of making a set into a group.*
- *p forgets properties if all $p^{-1}(x)$ are (-1)-groupoids. For example, there's just a* truth value *of ways of making a group into an abelian group: either you can or you can't (i.e. it is or it isn't).*
- *p forgets nothing if all $p^{-1}(x)$ are (-2)-groupoids. For example, there's just a* 'necessarily true' truth value *of ways of making an abelian group into an abelian group: you always can, in one way.*

(In the examples above, we are considering the groupoids *of sets, pairs of sets, groups and abelian groups. We'll consider the case of categories in §5.2.)*

We can thus study how forgetful a functor is by looking at what homotopy dimension its fibers have.

Note that to make this chart work, we really needed the negative dimensions. You should want to also go in the other direction, say if we had 2-groupoids or 3-groupoids; then we'll have something 'even more substantial than stuff.' James Dolan dubbed that *eka-stuff*, by analogy with how Mendeleev called elements which were missing in the periodic table 'eka-?', e.g. 'eka-silicon' for the missing element below silicon, which now we call germanium. He guessed that eka-silicon would be a lot like silicon, but heavier, and so on. Like Mendeleev, we can use the periodic table to guess things, and then go out and check them.

2.5. **Questions and comments.**

2.5.1. *What should forgetful mean?*

Peter May: *Only functors with left adjoints should really be called 'forgetful.' Should the free group functor be forgetful?*

MS: *A set is the same as a group with the property of being free and the structure of specified generators.*

JB: Right, you can look at it this way. Then the free functor $F\colon \mathbf{Set} \to \mathbf{Grp}$ 'forgets at most structure': it's faithful, but neither full nor essentially surjective. Whether you want to call it 'forgetful' is up to you, but this is how I'm using the terminology now.

2.5.2. *Monoidal* (-1)*-categories.*

Puzzle: What's a monoidal (-1)-category?

Answer: A (-1)-category is a truth value, and the only monoidal (-1)-category is True.

To figure this out, note that a monoidal (-1)-category is what we get when we take a 0-category with just one object, say x, and look at $\hom(x, x)$. A 0-category with one object is just a one-element set $\{x\}$, and $\hom(x, x)$ is just the equality $x = x$, which is 'True.'

Note that monoidal (-1)-categories are stable, so we are just adding a property to the previous one, just like when we pass from monoids to commutative monoids, or braiding to symmetry.

In general, we have forgetful functors marching *up* the periodic table, which forget different amounts of things. We forget nothing until we get up to the end of the stable range, then we forget a property (symmetry, or commutativity), then a structure (monoid structure, or braiding), then stuff (a monoidal structure, of which there are a whole category of ways to add to a given category), then eka-stuff (ways to make a 2-category into a monoidal 2-category), and so on.

2.5.3. *Maps of truth values.* Here's a funny thing. Note that for categories, we have a composition *function*

$$\hom(x,y)\times\hom(y,z)\to\hom(x,z).$$

For a set, the substitute is transitivity:

$$(x=y)\ \&\ (y=z)\Rightarrow(x=z).$$

In other words, we can 'compose equations.' But here $\Rightarrow$ is acting as a map between truth values. What sort of morphisms of truth values do we have? We just have a 0-category of (-1)-categories, so there should be only identity morphisms. The implication $F\Rightarrow T$ doesn't show up in this story.

That's a bit sad. Ideally, a bunch of *propositional logic* would show up at the level of -1-categories. Toby Bartels has a strategy to fix this. In his approach, posets play a much bigger role in the periodic table, to include the notion of implication between truth values.

In fact, it seems that the periodic table is just a slice of a larger 3-dimensional table relating higher categories and logic... see §5.1 for more on this.

3. Cohomology: The Layer-Cake Philosophy

We're going to continue heading in the direction of cohomology, but we'll get there by a perhaps unfamiliar route. Last time we led up to the concept of n-*stuff*, although we stopped right after discussing ordinary stuff and only mentioned eka-stuff briefly.

3.1. **Factorizations.** What does it mean to forget *just* stuff, or *just* properties?

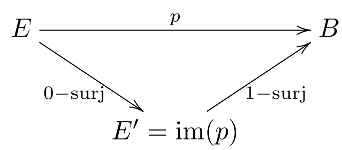
Definition 4. An ∞-functor $p\colon E\to B$ is n**-surjective** (perhaps **essentially** n**-surjective**) if given any parallel $(n-1)$-morphisms e and e' in E, and any n-morphism $f\colon p(e)\to p(e')$, there is an n-morphism $\widetilde{f}\colon e\to e'$ such that $p(\widetilde{f})$ is equivalent to f: $p(\widetilde{f})\simeq f$.

For example, suppose $p\colon E\to B$ is a function between *sets*. It is

- 0-surjective if it is *surjective* in the usual sense, since in this case equivalences are just equalities. (The presence of e and e' here is a bit confusing, unless you believe that all n-categories go arbitrarily far down as well.)
- 1-surjective if it is *injective*; since in this case all 1-morphisms are identities, 1-surjective means that if $p(e)=p(e')$, then $e=e'$.

That's a nice surprise: *injective means 'surjective on equations'!*

Now, another thing that you can do in the case of sets is to take any old function $p\colon E \to B$ and factor it as first a surjection and then an injection:



The interesting thing is that this keeps generalizing as we go up to higher categories.

To see how this works, first suppose E and B are categories. A functor $p\colon E \to B$ is:

- 0-surjective if it is essentially surjective;
- 1-surjective if it is full;
- 2-surjective if it is faithful;
- 3-surjective always, and so on.

Do you see why?

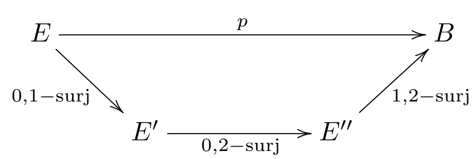
- Crudely speaking, 0-surjective means 'surjective on objects'. But you have to be a bit careful: it's sufficient that every object in B is *isomorphic* to $p(e)$ for some $e \in E$. So, 0-surjective really means *essentially* surjective.
- Crudely speaking, 1-surjective means 'surjective on arrows,' but you just have to be a little careful: it's not fair to ask that an arrow downstairs be the image of something unless we already know that its source and target are the images of something. So, it really means our functor is *full*
- Similarly, 2-surjective means 'surjective on equations between morphisms,' i.e. injective on hom-sets. So, it means our functor is *faithful*.

Note that the conjunction of all three of these means our functor is an equivalence, just as a surjective and injective function is an isomorphism of sets.

The notions of forgetting at most stuff, structure, or properties can also be defined using conjunctions of these conditions, namely:

- Forgets nothing: 0, 1, and 2-surjective
- Forgets at most properties: 1 and 2-surjective
- Forgets at most structure: 2-surjective
- Forgets stuff: arbitrary

As with functions between sets, we get a factorization result for functors between categories. Any functor factors like this:



I think this is a well-known result. You build these other categories as 'hybrids' of E and B: E gradually turns into B from the top down. We start with E; then we throw in new 2-morphisms (equations between morphisms) that we get from B; then we throw in new 1-morphisms (morphisms), and finally new 0-morphisms (objects). It's like a horse transforming into a person from the head down. First it's a horse, then it's a centaur, then it's a faun-like thing that's horse from the legs down, and finally it's a person.

In more detail:

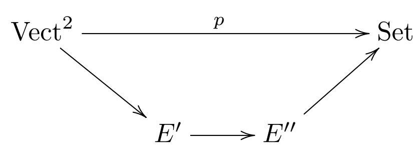
- The objects of E' are the same objects as E, but a morphism from e to e' is a morphism $p(e) \to p(e')$ in B which is in the image of p; and
- The objects of E'' are the objects of B in the (essential) image of p, with all morphisms between them.

This is the same thing as is happening for sets, but there it's happening so fast that you can't see it happening.

The next example will justify this terminology:

- A functor which is 0- and 1-surjective **forgets purely stuff**;
- A functor which is 0- and 2-surjective **forgets purely structure**;
- A functor which is 1- and 2-surjective **forgets purely properties**.

Example 5. Let's take the category of *pairs of vector spaces* and forget down to just the underlying set of the *first* vector space (so that we have an interesting process at every stage).



The objects of E' are again pairs of vector spaces, but its morphisms are just linear maps between the *first* ones. We write this as

$$\big[\text{pairs of vector spaces, linear maps between first ones}\big].$$

In fact, this category is equivalent to Vect; the extra vector space doesn't participate in the morphisms, so it might as well not be there. Our factorization cleverly managed to forget the stuff (the second vector space) but still keep the structure on what remains.

The objects of E'' are sets with the *property* that they can be made into vector spaces, and its morphisms are arbitrary functions between them. Here we forgot just the structure of being a vector space, but we cleverly didn't forget the *property* of being vector-space-izable. That's sort of cool.

Having seen how these factorizations work for sets and categories, we can guess how they go for ∞-categories. Let's say an ∞-functor $p\colon E \to B$ **forgets purely j-stuff** if it's i-surjective for all $i \neq j+1$. Note the funny '+1' in there: we need this to make things work smoothly. For ordinary categories, we have:

- A functor that 'forgets purely 1-stuff' forgets purely stuff;
- A functor that 'forgets purely 0-stuff' forgets purely structure;
- A functor that 'forgets purely (-1)-stuff' forgets purely properties;
- A functor that 'forgets purely (-2)-stuff' forgets nothing.

If we go up to 2-categories we get a new concept, '2-stuff'. This is what we called 'eka-stuff' last time.

Given the pattern we're seeing here, and using what they knew about Postnikov towers, James Dolan and Toby Bartels guessed a factorization result like this:

Hypothesis 6. *Given a functor $p\colon E \to B$ between n-categories, it admits a factorization:*

$$\begin{array}{ccccccc} E_n = E & & & \xrightarrow{\quad p \quad} & & & B = E_{-2} \\ & \searrow^{p_n} & & & & \nearrow_{p_{-1}} & \\ & & E_{n-1} \xrightarrow{p_{n-1}} \cdots & & \cdots \xrightarrow{p_0} E_{-1} & & \end{array}$$

where p_j forgets purely j-stuff.

Someone must have already made this precise and proved this for 2-categories. If not, someone should go home and do it tonight; it shouldn't be hard.

In fact, this result is already known for all n-groupoids, but only if you believe Grothendieck that they are the same as homotopy n-types. In this case, there's a topological result which says that every map between homotopy n-types factors like this. Such a factorization is called a **Moore–Postnikov tower**. When we factor a map from a space to a *point* this way, it's called a **Postnikov tower**. As we'll see, this lets us view a space as being made up out of layers, one for each homotopy group. And this lets us *classify homotopy types using cohomology* — at least in principle.

3.2. **Cohomology and Postnikov towers.** In a minute we'll see that from the viewpoint of homotopy theory, a space is a kind of 'layer cake' with one layer for each dimension. I claim that cohomology is fundamentally the study of classifying 'layer cakes' like this. There are many other kinds of layer cakes, like chain complexes (which are watered-down versions of spaces), L_∞-algebras and A_∞-algebras (which are chain complexes with extra bells and whistles), and so on. But let's start with spaces.

How does it work? If E is a homotopy n-type, we study it as follows. We map it to something incredibly boring, namely a point, and then work out the Postnikov tower of this map:

$$\begin{array}{ccccccc} E_n = E & & & \xrightarrow{\quad p \quad} & & & * = E_{-2} \\ & \searrow^{p_n} & & & & \nearrow_{p_{-1}} & \\ & & E_{n-1} \xrightarrow{p_{n-1}} \cdots & & \cdots \xrightarrow{p_0} E_{-1} & & \end{array}$$

Here we think of E as an n-groupoid and the point as the terminal n-groupoid, which has just one object, one morphism and so on. The Postnikov tower keeps crushing E down, so that E_j is really just a j-groupoid. This process is called **decategorification**. At the end of the day, E_{-2} is the only (-2)-groupoid there is: the point.

Of course, in the world of topology, they don't use our category-theoretic terminology to describe these maps. Morally, p_j forgets purely j-stuff — but topologists call this 'killing the jth homotopy group.' More precisely, the map

$$p_j\colon E_j \to E_{j-1}$$

induces isomorphisms

$$\pi_i(E_j) \to \pi_i(E_{j-1})$$

for all i *except* $i = j$, in which case it induces the zero map.

We won't get into how topologists actually construct Postnikov towers. Once Grothendieck's dream comes true, it will be a consequence of the result for n-categories.

It's fun to see how this Postnikov tower works in the shockingly low-dimensional cases $j = 0$ and $j = -1$, where the jth 'homotopy thingy' isn't a group — just a set, or truth value. When we get down to E_0, our space is just a set, at least up to homotopy equivalence. Killing its 0th 'homotopy set' then collapses all its points to the same point (if it has any to begin with). We're left with either the one-point set or the empty set. Killing the (-1)st 'homotopy truth value' then gives us the one-point set.

But enough of this negative-dimensional madness. Let's see how people use Postnikov towers to classify spaces up to homotopy equivalence. Consider any simplification step $p_j \colon E_j \to E_{j-1}$ in our Postnikov tower. By the wonders of homotopy theory, we can describe this as a fibration. One of the great things about homotopy theory is that even a map that doesn't look anything like a 'bundle' is always equivalent to a fibration, so we can think of it as some kind of bundle-type thing. Thus we can consider the homotopy fiber F_j of this map, which can either be constructed directly, the way we constructed the 'essential preimage' for a functor (here using paths in the base space) or by first converting the map into an actual fibration and then taking its literal fiber.

(We say 'the' fiber as if they were all the same, but if the space isn't connected they won't necessarily all be the same. Let's assume for now that E was connected, for simplicity.)

So, we get a fibration

$$F_j \to E_j \xrightarrow{p_j} E_{j-1}$$

Since p_j doesn't mess with any homotopy groups except the jth, the long exact sequence of homotopy groups for a fibration

$$\ldots \longrightarrow \pi_i(F_j) \longrightarrow \pi_i(E_j) \longrightarrow \pi_i(E_{j-1}) \longrightarrow \pi_{i-1}(F_j) \longrightarrow \ldots$$

tells us that the homotopy fiber must have only one non-vanishing homotopy group: $\pi_i(F_j) = 0$ unless $i = j$. We killed the jth homotopy group, so where did that group go? It went up into the fiber.

Such an F_j, with only one non-vanishing homotopy group, is called an **Eilenberg–Mac Lane space**. The great thing is that a space with only its jth homotopy group nonzero is completely determined by that group — up to homotopy equivalence, that is. For fancier spaces, the homotopy groups aren't enough to determine the space: we also have to say how the homotopy groups talk to each other, which is what this Postnikov business is secretly doing. The Eilenberg–Mac Lane space with G as its jth homotopy group is called $K(G, j)$. (Of course we need G abelian if $j > 1$.)

So, we've got a way of building any homotopy n-type E as a 'layer cake' where the layers are Eilenberg–Mac Lane spaces, one for each dimension. At the jth stage of this procress, we get a space E_j as the total space of this fibration:

$$K(\pi_j(E), j) = F_j \to E_j \xrightarrow{p_j} E_{j-1}.$$

These spaces E_j become better and better approximations to our space as j increases, and $E_n = E$.

If we know the homotopy groups of the space E, the main task is to understand the fibrations p_j. The basic principle of Galois theory says how to classify fibrations:

Fibrations of n-groupoids

$$F \to E \xrightarrow{p} B$$

with a given base B and fiber F are classified by maps

$$k\colon B \to \mathrm{AUT}(F).$$

Here $\mathrm{AUT}(F)$ is the **automorphism $(n+1)$-group** of F, i.e. an $(n+1)$-groupoid with one object. For example, a set has an automorphism group, a category has an automorphism 2-group, and so on. So, the map k is an $(n+1)$-functor, of the weakest possible sort.

How do we get the map k from the fibration? Topologists have a trick that involves turning $\mathrm{AUT}(F)$ into a space called the 'classifying space' for F-bundles, at least when B and F are spaces.

But now I want you to think about it n-categorically. How does it work? Think of B as an ∞-category. Then a path in B (a 1-morphism) lifts to a path in E, which when we move along it, induces some automorphism of the fiber (a 1-morphism in the one-object $(n+1)$-groupoid $\mathrm{AUT}(F)$). Similarly, a path of paths induces a morphism between automorphisms. This continues all the way up, which is how we get a map $k\colon B \to \mathrm{AUT}(F)$.

This is a highbrow way of thinking about cohomology theory. We may call it 'nonabelian cohomology.' You've probably seen cohomology with coefficients in some *abelian* group, which is a special case that's easy to compute; here we are talking about a more general version that's supposed to explain what's really going on.

Specifically, we call the set of $(n+1)$-functors $k\colon B \to \mathrm{AUT}(F)$ modulo equivalence the **nonabelian cohomology** of B with coefficients in $\mathrm{AUT}(F)$. We denote it like this:

$$H(B, \mathrm{AUT}(F)).$$

We purposely leave off the little superscript i's that people usually put on cohomology; our point of view is more global. The element $[k]$ in the cohomology $H(B, \mathrm{AUT}(F))$ corresponding to a given fibration is called its **Postnikov invariant**.

So, to classify a space E, we think of it as an n-groupoid and break it down with its Postnikov tower, getting a whole *list* of guys

$$k_j\colon E_{j-1} \to \mathrm{AUT}(F_j)$$

and thus a list of Postnikov invariants

$$[k_j] \in H(E_{j-1}, \mathrm{AUT}(F_j)).$$

Together with the homotopy groups of E (which determine the fibers F_j), these Postnikov invariants classify the space E up to homotopy equivalence. Doing this in practice, of course, is terribly hard. But the idea is simple.

Next time we'll examine certain low-dimensional cases of this and see what it amounts to. In various watered-down cases we'll get various famous kinds of cohomology. The full-fledged n-categorical version is beyond what anyone knows how to handle except in low dimensions — even for n-groupoids, except by appealing to Grothendieck's dream. Street has a nice paper on cohomology with coefficients in an ∞-category; he probably knew this stuff I'm talking about way back when I was just a kid. I'm just trying to bring it to the masses.

3.3. **Questions and comments.**

3.3.1. *Internalizing n-surjectivity.*

Tom Fiore: *You can define epi and mono categorically and apply them in any category, not just sets. Can you do a similar thing and define analogues of 0-, 1-, and 2-surjectivity using diagrams in any 2-category, etc.?*

JB: I don't know. That's a great question.

Eugenia Cheng: *You can define a concept of 'essentially epic' in any 2-category, by weakening the usual definition of epimorphism. But in* **Cat***, 'essentially epic' turns out to mean essentially surjective and full. I expect that in n-categories, it will give 'essentially surjective on j-morphisms below level n'. You can't isolate the action on objects from the action on morphisms, so you can't characterize a property that just refers to objects.*

(For a more thorough discussion see §5.5: there is a way to characterize essentially surjective functors 2-categorically, as the functors that are left orthogonal to full and faithful functors.)

3.3.2. *How normal people think about this stuff.*

Aaron Lauda: *What do mortals call this* $(n+1)$*-group* $\mathrm{AUT}(F)$ *that you get from an n-groupoid* F*?*

JB: Well, suppose we have any $(n+1)$-group, say G. The first thing to get straight is that there are *two ways* to think of this in terms of topology.

First, by Grothendieck's dream, we can think of G as a topological group, say $|G|$, that just happens to be a homotopy n-type. Why do the numbers go down one like this? It's just like when people see a category with one object: they call it a monoid. There's a level shift here: the *morphisms* of the 1-object category get called *elements* of the monoid.

Second, we can just admit that our $(n+1)$-group G is a special sort of $(n+1)$-groupoid. Following Grothendieck's dream, we can think of this as a homotopy $(n+1)$-type, called $B|G|$. But, we should always think of this as a 'connected pointed' homotopy $(n+1)$-type.

Why? Well, an $(n+1)$-group is just an $(n+1)$-groupoid with one object. More generally, an $(n+1)$-groupoid is *equivalent* to an $(n+1)$-group if it's **connected** — if all its objects are equivalent. But to actually turn a connected $(n+1)$-groupoid into an $(n+1)$-group, we need to pick a distinguished object, or 'basepoint'. So, an $(n+1)$-group is essentially the same as a connected pointed $(n+1)$-groupoid. If we translate this into the language of topology, we see that an $(n+1)$-group amounts to a connected pointed homotopy $(n+1)$-type. This is usually called $B|G|$, the **classifying space** of the topological group $|G|$.

These two viewpoints are closely related. The homotopy groups of $B|G|$ are the same as those of $|G|$, just shifted:

$$\pi_{j+1}(B|G|) = \pi_j(|G|).$$

You may think this is unduly complicated. Why bother thinking about an $(n+1)$-group in *two different ways* using topology? In fact, both are important. Given your n-groupoid F, it's good to use both tricks just described to study the $(n+1)$-group $\mathrm{AUT}(F)$.

The first trick gives a topological group $|\mathrm{AUT}(F)|$ that happens to be a homotopy n-type. This group is often called the group of **homotopy self-equivalences** of $|F|$, the homotopy n-type corresponding to F. The reason is that its elements are homotopy equivalences $f\colon |F| \xrightarrow{\sim} |F|$.

The second trick gives a connected pointed homotopy $(n+1)$-type $B|\mathrm{AUT}(F)|$. We can use this to classify fibrations with F as fiber.

Aaron Lauda: *How does that work?*

JB: Well, we've seen that fibrations whose base B and fiber F are n-groupoids should be classified by $(n+1)$-functors

$$k\colon B \to \mathrm{AUT}(F)$$

where $\mathrm{AUT}(F)$ is a $(n+1)$-groupoid with one object. But normal people think about this using topology. So, they turn B into a homotopy n-type, say $|B|$. They turn $\mathrm{AUT}(F)$ into a connected pointed homotopy $(n+1)$-type: the classifying space $B|\mathrm{AUT}(F)|$. And, they turn k into a map, say

$$|k|\colon |B| \to B|\mathrm{AUT}(F)|.$$

So, instead of thinking of the Postnikov invariant as an equivalence class of $(n+1)$-functors

$$[k] \in H(B, \mathrm{AUT}(F))$$

they think of it as a homotopy class of maps:

$$[|k|] \in [\,|B|,\, B|\mathrm{AUT}(F)|\,]$$

where now the square brackets mean 'homotopy classes of maps' — that's what equivalence classes of j-functors become in the world of topology. And, they show fibrations with base $|B|$ and fiber $|F|$ are classified by this sort of Postnikov invariant.

Since I'm encouraging you to freely hop back and forth between the language of n-groupoids and the language of topology, from now on I won't write $|\cdot|$ to describe the passage from n-groupoids to spaces, or n-functors to maps. I just wanted to sketch how it worked, here.

Aaron Lauda: *So, all this is part of some highbrow approach to cohomology... but how does this relate to plain old cohomology, like the kind you first learn about in school?*

JB: Right. Suppose we're playing the Postnikov tower game. We have a homotopy n-type E, and somehow we know its homotopy groups π_j. So, we get this tower of fibrations

$$K(\pi_j, j) = F_j \to E_j \xrightarrow{p_j} E_{j-1}$$

where E_n is the space with started with and E_{-1} is just a point. To classify our space E just need to classify all these fibrations. That's what the Postnikov invariants do:

$$[k_j] \in H(E_{j-1}, \mathrm{AUT}(K(\pi_j, j))).$$

Now I'm using the language of topology, where AUT stands for the group of homotopy self-equivalences. But in the language of topology, the Postnikov invariants are homotopy classes of maps

$$k_j \colon E_{j-1} \to B\,\mathrm{AUT}(K(\pi_j, j)).$$

So, in general, our cohomology involves the space $B\,\mathrm{AUT}(K(\pi_j, j))$, which sounds pretty complicated. But we happen to know some very nice automorphisms of $K(\pi_j, j)$. It's an abelian topological group, at least for $j > 1$, so it can act on itself by left translations. Thus, sitting inside $B\,\mathrm{AUT}(K(\pi_j, j))$ we actually have $BK(\pi_j, j)$, which is actually the same as $K(\pi_j, j+1)$, since applying B shifts things up one level.

If the map k_j happens to land in this smaller space, at least up to homotopy, we call our space **simple**. Then we can write the Postnikov invariant as

$$[k_j] \in [E_{j-1}, K(\pi_j, j+1)]$$

and the thing on the right is just what people call the **ordinary cohomology** of our space E_{j-1} with coefficients in the group π_j, at least if $j > 1$. So, they write

$$[k_j] \in H^{j+1}(E_{j-1}, \pi_j).$$

Note that by now the indices are running all the way from $j-1$ to $j+1$, since we've played so many sneaky level-shifting tricks.

MS: *Actually, Postnikov towers have a nice interpretation in terms of cohomology even for spaces that aren't 'simple'. The trick is to use 'cohomology with local coefficients'. Given a space X and an abelian group A together with an action ρ of $\pi_1(X)$ on A, you can define cohomology groups $H^n_\rho(X, A)$ where the coefficients are 'twisted' by ρ. It then turns out that*

$$\begin{aligned} H(X, \mathrm{AUT}(K(\pi, j))) &= [X, B\,\mathrm{AUT}(K(\pi, j))] \\ &= \coprod_\rho H^{j+1}_\rho(X, \pi). \end{aligned}$$

So for a space that isn't necessarily simple, a topologist would consider its Postnikov invariants to live in some cohomology with local coefficients. In the simple case, the action ρ is trivial, so we don't need local coefficients.

4. A Low-Dimensional Example

4.1. **Review of Postnikov towers.** Last time we discussed a big idea; this time let's look at an example. Let's start with a single fibration:

$$F \to E \to B.$$

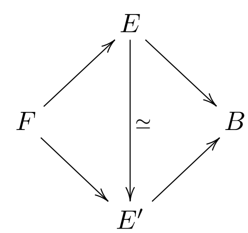
This means that we have some point $* \in B$ and $F = p^{-1}(*)$ is the homotopy fiber, or 'essential preimage' over $*$. This won't depend on the choice of $*$ if B is connected. Let's restrict ourselves to that case: this is no great loss, since any base is a disjoint union of connected components.

We can then classify these fibrations via their 'classifying maps'

$$k\colon B \to \mathrm{AUT}(F)$$

where $\mathrm{AUT}(F)$ is an $(n+1)$-group if F is an n-groupoid. A lowbrow way to state this classification is that there's a notion of equivalence for both these guys, and the equivalence classes of each are in one-to-one correspondence. We could also try to state a highbrow version, which asserts that $\hom(B, \mathrm{AUT}(F))$ is equivalent to some $(n+1)$-category of fibrations with B as base and F as fiber. But let's be lowbrow today.

In both cases, hopefully the notion of equivalence is sort of obvious. 'Equivalence of fibrations' looks a lot like equivalence for extensions of groups — which are, in fact, a special case. In other words, fibrations are equivalent when there exists a vertical map making this diagram commute (weakly):



On the other side, the notion of equivalence for classifying maps

$$k, k'\colon B \to \mathrm{AUT}(F)$$

is equivalence of $(n+1)$-functors, if we think of E and F as n-groupoids, or homotopy of maps if we think of E and F spaces.

This sounds great, but of course we're using all sorts of concepts from n-category theory that haven't been made precise yet. So, today we'll do an example, where we cut things down to a low enough level that we can handle it.

But first — why are we so interested in this? I hope you remember why it's so important. We have a grand goal: we want to classify n-groupoids. This is a Sisyphean task. We'll never actually complete it — but nonetheless, we can learn a lot by trying.

For example, consider the case $n = 1$: the classification of groupoids. Every groupoid is a disjoint union of groups, so we just need to classify groups. Let's say we start by trying to classify finite groups. Well, it's not so easy — after 10,000 pages of work people have only managed to classify the finite *simple* groups. Every finite group can be built up out of those by repeated extensions:

$$1 \to F \to E \to B \to 1.$$

These extensions are just a special case of the fibrations we've been talking about. So we can classify them using cohomology, at least in principle. But it won't be easy, because each time we do an extension we get a new group whose cohomology we need to understand. So, we'll probably never succeed in giving a useful classification of all finite groups. Luckily, even what we've learned so far can help us solve a lot of interesting problems.

Now suppose we want to classify n-groupoids for $n > 1$. We do it via their Postnikov towers, which are iterated fibrations. Given an n-groupoid E, we successively squash it down, dimension by dimension, until we get a single point:

$$\begin{array}{ccccccc} E_n = E & & & \xrightarrow{\qquad\qquad p \qquad\qquad} & & & * = E_{-2} \\ & \searrow^{p_n} & & & & \nearrow^{p_{-1}} & \\ & & E_{n-1} \xrightarrow[p_{n-1}]{} \cdots & & \cdots \xrightarrow[p_0]{} E_{-1} & & \end{array}$$

At each step, we're **decategorifying**: to get the $(j-1)$-groupoid E_{j-1} from the j-groupoid E_j, we promote all the j-isomorphisms to *equations*. That's really what's going on, although I didn't emphasize it last time. Last time I emphasized that the map

$$p_j \colon E_j \to E_{j-1}$$

'forgets purely j-stuff'. What that means here is that decategorification throws out the top level, the j-morphisms, while doing as little damage as possible to the lower levels.

So, we get fibrations

$$F_j \to E_j \to E_{j-1}$$

where each homotopy fiber F_j, which records the stuff that's been thrown out, is a j-groupoid with only nontrivial j-morphisms: it has at most one i-morphism for any $i < j$. If you look at the periodic table, you'll see this means F_j is secretly a *group* for $j = 1$, and an *abelian group* for $j \geq 2$. Another name for this group is $\pi_j(E)$, which is more intuitive if you think of E as a space. If you think of F_j as a space, then it's the Eilenberg–Mac Lane space $K(\pi_j(E), j)$.

What do we learn from this business? That's where the basic principle of Galois theory comes in handy. We take the fibrations

$$F_j \to E_j \xrightarrow{p_j} E_{j-1}$$

and describe them via their classifying maps

$$k_j \colon E_{j-1} \to \mathrm{AUT}(F_j).$$

These give cohomology classes

$$[k_j] \in H(E_{j-1}, \mathrm{AUT}(F_j))$$

called **Postnikov invariants**.

So, we ultimately classify n-groupoids by a list of groups, namely $\pi_1, \ldots, \pi_n$, and all these cohomology classes $[k_j]$. What I want to do is show you how this works in detail, in a very low-dimensional case.

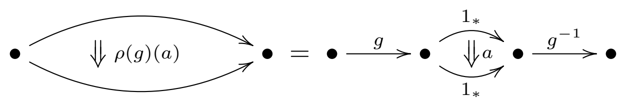
4.2. **Example: the classification of 2-groupoids.** Let's illustrate this for $n = 2$ and classify connected 2-groupoids. Since we're assuming things are connected, we might as well, for the purposes of classification, consider our connected 2-groupoids to be 2-groups. These have one object, a bunch of 1-morphisms from it to itself which are weakly invertible, and a bunch of 2-morphisms from these to themselves which are strictly invertible.

We can classify these using cohomology. Here's how. Given a 2-group, take a skeletal version of it, say E, and form these four things:

1. The group $G = \pi_1(E)$ = the group of '1-loops', i.e. 1-morphisms that start and end at the unique object. Composition of these would, a priori, only be associative up to isomorphism, but we said we picked a *skeletal* version, so these isomorphic objects have to be, in fact, equal.

2. The group $A = \pi_2(E) =$ the group of '2-loops', i.e. 2-morphisms which start and end at the identity 1-morphism 1_*. They form a group more obviously, and the Eckmann–Hilton argument shows this group is abelian.

3. An action ρ of G on A, where $\rho(g)(a)$ is defined by 'conjugation' or 'whiskering':



You can think of the loops as starting and ending at anything, if you like, by doing more whiskering. Then you have to spend a year figuring out whether you want to use left whiskering or right whiskering. This is supposed to be familiar from topology: there π_1 always acts on π_2.

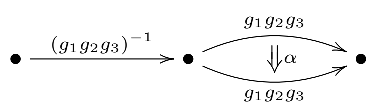
4. The associator

$$\alpha_{g_1 g_2 g_3} \colon (g_1 g_2) g_3 \to g_1 (g_2 g_3)$$

gives a map

$$\alpha \colon G^3 \to A$$

as follows. Take three group elements and get an interesting automorphism of $g_1 g_2 g_3$. Automorphisms of anything can be identified with automorphisms of the identity, by whiskering. Explicitly, we cook up an element of A as follows:



(Don't ask why I put the whisker on the left instead of the right; you can do it either way and it doesn't really matter, though various formulas work out slightly differently.)

That's the stuff and structure, but there's also a property: the associator satisfies the pentagon identity, which means that α satisfies some equation. You all know the pentagon identity. It turns out this equation on α is something that people had been talking about for ages, before Mac Lane invented the pentagon identity. In fact, one of the people who'd been talking about it for ages was Mac Lane himself, because he'd also helped invent cohomology of groups. It's called the **3-cocycle equation** in group cohomology:

$$\rho(g_0)\alpha(g_1, g_2, g_3) - \alpha(g_0 g_1, g_2, g_3) + \alpha(g_0, g_1 g_2, g_3) - \alpha(g_0, g_1, g_2 g_3) + \alpha(g_0, g_1, g_2) = 0$$

Here we write the group operation in A additively, since it's abelian.

This equation is secretly just the pentagon identity satisfied by the associator; that's why it has 5 terms. But, people in group cohomology often write it simply as $d\alpha = 0$, because they know a standard trick for getting function of $(n+1)$ elements of G from a function of n elements, and this trick is called the 'differential' d in group cohomology. If you have trouble remembering this trick, just think of a bunch of kids riding a school bus, but today there's one more kid than seats on the bus. What can we do? Either the first kid can jump out and sit on the hood, or the first two kids can squash into the first seat, and so on... or you can throw the last kid out the back window. That's a good way to remember the formula I just wrote.

Note that our *skeletal* 2-group is not necessarily *strict*! Making isomorphic objects equal doesn't mean making isomorphisms into identities. The associator isomorphism is still nontrivial, but it just happens in this case to be an *automorphism* from one object to the same object.

Theorem 7 (Sinh)**.** *Equivalence classes of 2-groups are in one-to-one correspondence with equivalence classes of 4-tuples*

$$(G, A, \rho, \alpha)$$

consisting of a group, an abelian group, an action, and a 3-cocycle.

The equivalence relation on the 4-tuples is via isomorphisms of G and A which get along with ρ, and get along with α up to a coboundary. Since cohomology is precisely cocycles modulo coboundaries, we really get the traditional notion of group cohomology showing up.

Just as the 3-cocycle equation comes from the pentagon identity for monoidal categories, the coboundary business comes from the notion of a monoidal *natural transformation*: monoidally equivalent monoidal categories won't have the same α, but their α's will differ by a coboundary.

This particular kind of group cohomology is called the 'third' cohomology since α is a function of three variables. We say that

$$[\alpha] \in H^3_\rho(G, A),$$

the third group cohomology of G with coefficients in a G-module A (where the action is defined by ρ). Sometimes this is called 'twisted' cohomology.

So, in short, once we fix G, A, and ρ, the equivalence classes of 2-groups we can build are in one-to-one correspondence with $H^3_\rho(G, A)$.

4.3. **Relation to the general case.** Now I'm going to show why this stuff is a special case of the general notion of cohomology we introduced last time. Why is $H^3_\rho(G, A)$ a special case of what we were calling $H(B, \mathrm{AUT}(F))$?

Consider a little Postnikov tower where we start with a 2-group E and decategorify it getting a group B. We get a fibration $F \to E \to B$. To relate this to what we were just talking about, think of B as the group G. The 2-group F is the 2-category with one object, one morphism and some abelian group A as 2-morphisms. So, seeing how the 2-group E is built out of the base B and the fiber F should be the same as seeing how its built out of G and A. We want to see how the classifying map

$$k\colon B \to \mathrm{AUT}(F)$$

is the same as an element of $H^3_\rho(G, A)$.

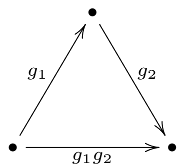
It's good to think about this using a little topology. As a *space*, B is called $K(G, 1)$. It is made by taking one point:

$$\bullet$$

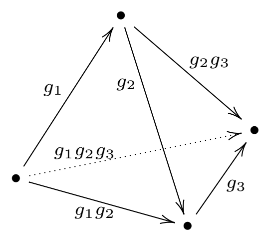
one edge for element $g_1 \in G$:

$$\bullet \xrightarrow{\;g\;} \bullet$$

a triangle for each pair of elements:

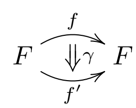


a tetrahedron for each triple:



and so on. The fiber F is A regarded as a 2-group with only an identity 1-cell and 0-cell. So, as a space, F is called $K(A, 2)$, built with one point, one edge, one triangle for every element of A, one tetrahedron whenever $a_1 + a_2 = a_3 + a_4$, and so on.

Now, $\mathrm{AUT}(F)$ is a *3-group* which looks roughly like this. It has one object (which we can think of as 'being' F), one 1-morphism for every automorphism $f\colon F \to F$, one 2-morphism



for each pseudonatural isomorphism, and one 3-morphism for each modification.

That seems sort of scary; what you have to do is figure out what that actually amounts to in the case when F is as above. Let me just tell you. It turns out that in fact, in our case $\mathrm{AUT}(F)$ has

- one object;
- its morphisms are just the group $\mathrm{Aut}(A)$;
- only identity 2-morphisms (as you can check);
- A as the endo-3-morphisms of any 2-morphism.

As a space, this is called '$B(\mathrm{AUT}(F))$', and it is made from one point, an edge for each automorphism f, a triangle for each equation $f_1 f_2 = f_3$, and tetrahedrons whose boundaries commute which are labeled by arbitrary elements of A.

Now, we can think about our classifying map as a weak 3-functor

$$k\colon B \to \mathrm{AUT}(F)$$

but we can also think of it as a map of spaces

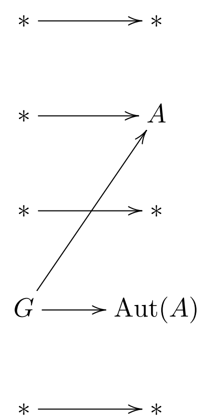
$$k\colon K(G, 1) \to B(\mathrm{AUT}(F))$$

Let's just do it using spaces — or actually, simplicial sets. We have to map each type of simplex to a corresponding type. Here's how it goes:

- This map is boring on 0-cells, since there's only one choice.
- We get a map $\rho\colon G \to \mathrm{Aut}(A)$ for the 1-cells, which is good because that's what we want.

- The map on 2-cells says that this a group homomorphism, since it sends equations $g_1 g_2 = g_3$ to equations $\rho(g_1)\rho(g_2) = \rho(g_3)$.
- The map on 3-cells sends tetrahedra in $K(G,1)$, which are determined by triples of elements of G, to elements of A. This gives the may $\alpha\colon G^3 \to A$.
- The map on 4-cells is what forces α to be a 3-cocycle.

Our map $B \to \mathrm{AUT}(F)$ is *weak*, which is all-important. Even though our 2-morphisms are trivial, which makes the action on 1-morphisms actually a strict homomorphism, and our domain has no interesting 3-morphisms, we also get the higher data which gives the 3-cocycle $G^3 \to A$:



In fact, all sorts of categorified algebraic gadgets should be classified as 'layer cakes' built using Postnikov invariant taking values in the cohomology theory for that sort of gadget. We get group cohomology when we classify n-groupoids. Similarly, to classify categorified Lie algebras, which are called L_∞ algebras, we need Lie algebra cohomology — Alissa Crans has checked this in the simple case of an L_∞ algebra with only two nonzero chain groups. The next case is A_∞ algebras, which are categorified associative algebras. I bet that classifying these involves Hochschild cohomology — but I haven't ever sat down and checked it. And, it should keep on going. There should be a general theorem about this. That's what I mean by the 'layer cake philosophy' of cohomology.

4.4. **Questions and comments.**

4.4.1. *Other values of n.*

Aaron Lauda: *what about other H^n?*

JB: Imagine an alternate history of the world in which people knew about n-categories and had to learn about group cohomology from them.

We can figure out H^3 using 2-groups.

$$\begin{array}{c} \vdots \\ * \\ \pi_2 \\ \pi_1 \\ * \end{array}$$

To get the classical notion of H^4, we would have to think about classifying 3-groups that only have interesting 3-morphisms and 1-morphisms.

$$\begin{array}{c} \vdots \\ * \\ \pi_3 \\ * \\ \pi_1 \\ * \end{array}$$

We still get the whiskering action of 1-morphisms on 3-morphisms, and the pentagonator gives a 4-cocycle. Group cohomology, as customarily taught, is about classifying these 'fairly wimpy' Postnikov towers in which there are just two nontrivial groups: $H^{n+2}(\pi_1, \pi_n)$. This is clearly just a special case of something, and that something is a lot more complicated.

MS: *What about H^2?*

JB: Ah, that's interesting! In general, $H^n(G, A)$ classifies ways of building an $(n-1)$-group with G as its bottom layer (what I was just calling π_1) and A as its top layer (namely π_{n-1}). So, it's all about layer-cakes with two nontrivial layers: the first and $(n-1)$st layers. For simplicity I'm assuming the action of G on A is trivial here.

But the case $n = 2$ is sort of degenerate: now our layer cake has only *one* nontrivial layer, the first layer, built by squashing A right into G. More precisely, $H^2(G, A)$ classifies ways of building a 1-group — an ordinary group — by taking a *central extension* of G by A. We've seen that 3-cocycles come from the associator, so it shouldn't be surprising that 2-cocycles come from something more basic: multiplication, where *two* elements of G give you an element of A.

Ironically, this weird degenerate low-dimensional case is the highest-dimensional case of group cohomology that ordinary textbooks bother to give any clean conceptual interpretation to. They say that H^2 classifies certain ways build a group out of two groups, but they don't say that H^n classifies certain ways to build an $(n-1)$-group out of two groups. They don't say that extensions are just degenerate layer-cakes. And, it gets even more confusing when people start using H^2 to classify 'deformations' of algebraic structures, because they don't always admit that a deformation is just a special kind of extension.

4.4.2. *The unit isomorphisms.*

MS: *What happened to the unit isomorphisms?*

JB: When you make a 2-group skeletal, you can also make its unit isomorphisms equal to the identity. However, you can't make the associator be the identity — that's why it gives some interesting data in our classification of 2-groups, namely the 3-cocycle. This 3-cocycle is the only obstruction to a 2-group being both skeletal and strict.

5. Appendix: Posets, Fibers, and n-Topoi

This appendix is a hodgepodge of (proposed) answers to questions that arose during the lectures, some musings about higher topos theory, and some philosophy about the distinction between pointedness and connectedness.

5.1. **Enrichment and posets.** We observed in §2.5.3 that while (-1)-categories are truth values, having only a 0-category (that is, a set) of them is a bit unsatisfactory, since it doesn't allow us to talk about *implication* between truth values. The notetaker believes the best resolution to this problem is to extend the notion of '0-category' to include not just sets but *posets*. Then we can say that truth values form a poset with two elements, true and false, and a single nonidentity implication false $\Rightarrow$ true. A set can then be regarded as a discrete poset, or equivalently a poset in which every morphism is invertible; that is, a '0-groupoid'.

One way to approach a general theory including posets is to start from very low-dimensional categories and build up higher-dimensional ones using *enrichment*. We said in §2 that one of the general principles of n-category theory is that big n-categories have lots of little n-categories inside them. Another way of expressing this intuition is to say that *An n-category is a category enriched over* $(n-1)$*-categories.*

What does 'enriched' mean? Roughly speaking, a **category enriched over** V consists of

- A collection of objects $x, y, z, \ldots$;
- For each pair of objects x, y, an object in V called $\hom(x, y)$;
- For each triple of objects x, y, z, a morphism in V called
$$\circ\colon \hom(x, y) \times \hom(y, z) \longrightarrow \hom(x, z)$$
- Units, associativity, etc. etc.

Remember than in the world of n-categories, it doesn't make sense to talk about anything being strictly equal except at the top level. So when V is, for instance, the $(n+1)$-category of n-categories, the composition in a V-enriched category should only be associative and unital up to coherent equivalence. For example:

- 1-categories are categories enriched over sets;
- weak 2-categories are categories weakly enriched over categories;
- weak 3-categories are categories weakly enriched over weak 2-categories.

Making this precise for $n > 2$ is tricky, but it's a good intuition.

We may also say that a **V-enriched functor** $p\colon E \to B$ between such categories consists of:

- A function sending objects of E to objects of B;
- Morphisms of V-objects $\hom(x, y) \to \hom(px, py)$;
- various other data (again, as weak as appropriate).

And that's as far up as we need to go. We'll say that a **V-enriched groupoid** is a V-enriched category such that 'every morphism is invertible' in a suitably weak sense.

Let's investigate this notion in our very-low-dimensional world, starting with (-2)-categories, which we take to all be trivial by definition. What is a category enriched over (-2)-categories? Well, it has a collection of objects, together with, for every two objects, the unique (-2)-category as $\hom(x, y)$, and composition maps which are likewise unique. Thus a (-2)-category-enriched category is either:

- empty (has no objects), or
- has some number of objects, each of which is uniquely isomorphic to every other.

Thus it is either empty (false) or contractible (true), agreeing with the notion of (-1)-category that we got from topology. In particular, every (-1)-category is a groupoid.

Our general notion of functor now says that there should be a (-1)-functor from false to true, which we can call 'implication'. This is in line with topology: there is also a continuous map from the empty space to a contractible one.

Continuing on, a category enriched over (-1)-categories has a collection of objects together with, for every pair of objects x, y, a truth value $\hom(x, y)$, and for every triple x, y, z, a morphism

$$\hom(x, y) \times \hom(y, z) \longrightarrow \hom(x, z)$$

Now, the product of two (-1)-categories is empty (that is, false) if and only if one of the factors is. Thus, when we interpret (-1)-categories as truth values, the product $\times$ becomes the logical operation 'and', so if we interpret the truth of $\hom(x, y)$ as meaning $x \leq y$, we see that a category enriched over (-1)-categories is precisely a **poset**. (A non-category theorist would call this a *preordered set* since we don't have antisymmetry, but from a category theorist's perspective that's asking for equality of objects instead of isomorphism, which is perverse.)

Thus, from the enrichment point of view, perhaps '0-category' should mean a poset, rather than a set. As remarked above, we can view a set as a discrete poset (that is, one in which $x \leq y$ only when $x = y$). Categorically, a poset is *equivalent* to a discrete one whenever $x \leq y$ implies $y \leq x$, which is essentially the condition for it to be a *0-groupoid*: since composites are uniquely determined in a poset, a morphism $x \leq y$ is an isomorphism precisely when $y \leq x$ as well.

In a way, it's not surprising that our intuition may have been a little off in this regard, since a lot of it was coming from topology where *everything* is a groupoid.

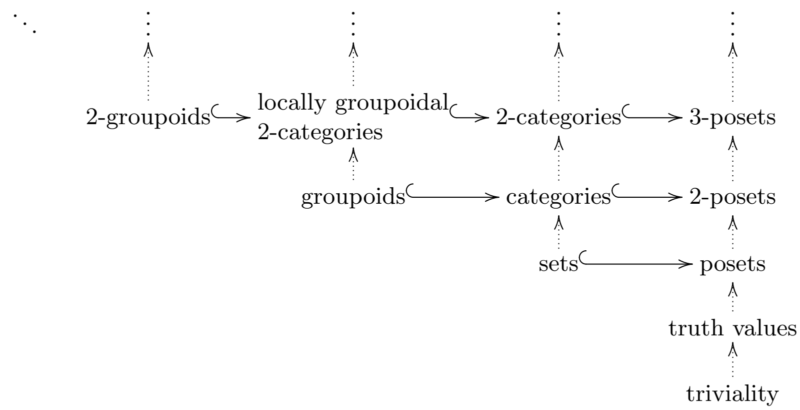
What happens at the next level? Well, if we enrich over sets (0-groupoids), we get what are usually called categories, or 1-categories. If we enrich instead over posets, we get what could variously be called **poset-enriched categories**, **locally posetal 2-categories**, or perhaps **2-posets**.

At the 0-level, we had one extra notion arising: instead of just sets, we got posets as well. At the 1-level, in addition to categories and poset-enriched categories, we also have a third notion: groupoids. These different levels correspond to the different levels of invertibility one can impose. If we start with a 2-poset and make its 2-morphisms all invertible, we get just a category. Then if we go ahead and make its 1-morphisms also invertible, we end up with a groupoid.

We can then go ahead and consider categories enriched over each of these three things, obtaining respectively 3-posets, 2-categories, and locally groupoidal 2-categories. And again there is an extra level that comes in: if we also make the 1-morphisms invertible, we get 2-groupoids.

All these various levels of invertibility can be fit together into the 'enrichment table' below. A dotted arrow $X \dashrightarrow Y$ means that a Y is a category enriched over Xs. The horizontal arrows denote inclusions; as we move to the left along any given line, we make more and more levels of morphisms invertible, coming from the top down. In general, the nth level will have $n+2$ different levels of invertibility stretching off to the left.

THE ENRICHMENT TABLE



What does the enrichment table have to do with the periodic table? Recall that the n in 'n-categories' labels the *columns* of the periodic table, while the rows are labeled with the amount of monoidal structure. Thus we could, if we wanted to, combine the two into a three-dimensional table, replacing the line across the top of the periodic table with the whole table of enrichment.

The enrichment table only includes n-categories for finite n, but we can obtain various different '∞' notions by thinking about passing to some sort of 'limit' in various directions. Of course, these aren't actually limits in any formal sense. For example, it makes intuitive sense to say that the 'vertical limit' along the column

$$\text{sets} \dashrightarrow \text{1-categories} \dashrightarrow \text{2-categories} \dashrightarrow \ldots$$

should be the ∞-categories. Moreover, this should also be the limit along any other column. This is because in (say) the mth column, all the cells of the top m dimensions are invertible, but in the limit all these invertible cells get pushed off to infinity and we end up with noninvertible cells of all dimensions.

We can also consider 'diagonal limits'. It makes intuitive sense to say that the limit along the far-left diagonal, consisting of n-groupoids for increasing n, is the ∞-groupoids, aka homotopy types (à la Grothendieck). The limit along the next diagonal will be the ∞-categories with all morphisms above level 1 invertible. These are often called $(\infty,1)$-categories (but sometimes also $(1,\infty)$-categories); see Bergner's survey article for an introduction to them.

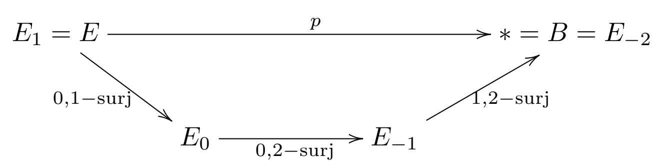
By the way, the term '$(\infty,1)$-categories' may sound strange, but it is just the most frequently used case of a general terminology. An ***(n,m)*-category** is an n-category all of whose j-morphisms for $j>m$ are invertible. Thus a n-category may also be called an (n,n)-category, an n-groupoid may be called an $(n,0)$-category, and a locally groupoidal 2-category may be called a $(2,1)$-category.

To stretch this terminology to its logical limit, we can call a poset-enriched category a $(1,2)$-category, a poset a $(0,1)$-category, and so on for the right-hand column of the enrichment table. If, instead of regarding an n-category as enriched over $(n-1)$-categories, we return to regarding it as an ∞-category in which all cells of dimension $> n$ are identities, we can give the following characterization of (n, m)-categories which includes the case of posets as well.

Definition 8. An **(n,m)-category** is an ∞-category such that

- All j-morphisms for $j > n + 1$ exist and are unique wherever possible. In particular, this implies that all parallel $(n + 1)$-morphisms are 'equal'.
- All j-morphisms for $j > m$ are invertible.

In the next section we'll consider at length one reason that including the n-posets in the periodic table is important. Here's a different, simpler reason. Let E be a category, and consider its Postnikov tower:



As we said in §3.1, E_0 is what we get by making parallel morphisms in E equal if they become equal in B; but here $B = *$, so this just means we identify *all* parallel morphisms. This precisely makes E into a *poset*—not necessarily a set. Thus in order for E_j to be a j-category in the factorization of an n-category which isn't a groupoid, we have to consider posets as a sort of 0-category, poset-enriched categories as a sort of 1-category, and generally j-posets as a sort of j-category.

5.2. **Fibers and fibrations.** Consider the fibers (or, rather, homotopy fibers) of a functor $p\colon E \to B$; we saw in §2.4 that their 'dimension' should reflect how much the functor p forgets. We'd like a generalization of Fact 3 there that applies to categories in addition to groupoids, but it turns out that for this we'll need to include the n-posets again. Consider first the following examples.

Example 9. We know that the functor $p\colon \mathbf{AbGp} \to \mathbf{Gp}$ forgets only properties. What is the (essential) preimage $p^{-1}(G)$ for some group G? It is the category of all abelian groups equipped with isomorphisms to G, and morphisms which preserve the given isomorphisms. This category is contractible if G is abelian, and empty otherwise; in other words, it is essentially a (-1)-category.

Example 10. Even more simply, consider an equivalence of categories $p\colon E \to B$, which forgets nothing. The the preimage $p^{-1}(b)$ is nonempty (since p is essentially surjective), and contractible (since p is full and faithful); thus it is essentially a (-2)-category.

These examples, along with the groupoid case we considered in Fact 3, lead us to guess that a functor will forget 'at most n-stuff' precisely when its essential preimages are all n-categories. We consider properties to be (-1)-stuff, structure to be 0-stuff, ordinary stuff to be 1-stuff, eka-stuff to be 2-stuff, and so on.

However, this guess is not quite right, as we can see by considering some examples that forget structure.

Example 11. Consider the usual forgetful functor $p\colon \mathbf{Gp} \to \mathbf{Set}$, which we know forgets at most structure. Given a set, such as the 4-element set, its essential preimage $p^{-1}(4)$ is the category of 4-element *labeled* groups (since their underlying sets are equipped with isomorphisms to the given set 4), and homomorphisms that preserve the labeling.

What does this look like? Well, given two labeled 4-element groups, there's exactly one function between them that preserves the labeling and either it's a group homomorphism or it isn't. Since the function preserving labeling is necessarily a bijection, if it is a homomorphism, then it is in fact a group isomorphism; thus this category is (equivalent to) a set.

In this example, we got what we expected, but we had to use a special property of groups: that a bijective homomorphism is an isomorphism. For many other types of structure, this won't be the case.

Example 12. Consider the forgetful functor $p\colon \mathbf{Top} \to \mathbf{Set}$ sending a topological space to its underlying set of points, which also forgets at most structure (in fact, purely structure). In this case, the essential preimage of the 4-element set is the collection of labeled 4-point topological spaces and continuous maps that preserve the labeling. Again, between any two there is exactly one function preserving the labeling, and either it is continuous or it isn't, so this category is a *poset*. In general, however, it won't be a set, since a continuous bijection is not necessarily a homeomorphism.

Thus, in order to get a good characterization of levels of forgetfulness by using essential preimages, we really need to include the n-posets as n-categories.

Let's look at a couple of examples involving higher dimensions.

Example 13. We have a forgetful 2-functor

$$[\text{monoidal categories}] \longrightarrow [\text{categories}]$$

which forgets at most stuff (since it is locally faithful, i.e. 3-surjective). Here the fiber over a category C is the category of ways to add a monoidal structure to C. There are lots of different ways to do this, and in between them we have monoidal functors that are the identity on objects (up to a specified equivalence, if we use the essential preimage), and in between *those* we have monoidal transformations whose components are identities (or specified isomorphisms). Now, there's at most one natural transformation from one functor to another whose components are identities, and either it's monoidal or it isn't. This shows that this collection is in fact a locally posetal 2-category, or a '2-poset', but in fact these monoidal natural transformations are automatically invertible when they exist, so it is in fact it is a 1-category.

Example 14. Let V be a nice category to enrich over, and consider the 'underlying ordinary category' functor

$$(-)_0\colon V\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}.$$

The category C_0 has the same objects as C, and $C_0(X,Y) = V(I, C(X,Y))$. What this functor forgets depends a lot on V:

- In many cases, such as topological spaces, simplicial sets, categories, it is 0-surjective (any ordinary category can be enriched), but in others, such as abelian groups, it is not.
- In general it is not 1-surjective: not every ordinary functor can be enriched.
- In general, it is not 2-surjective: not every natural transformation is V-natural. It is 2-surjective, however, whenever the functor $V(I,-)$ is faithful,

as for topological spaces and abelian groups. But when V is, say, simplicial sets, the functor $V(I,-)$ is not faithful, since a simplicial map is not determined by its action on vertices.
- It is always 3-surjective: a V-natural transformation is determined uniquely by its underlying ordinary natural transformation.

Thus in general, $(-)_0$ forgets at most stuff, but when $V(I,-)$ is faithful, it forgets at most structure.

Now, what is the fiber over an ordinary category C? Its objects are enrichments of C, its morphisms are V-functors whose underlying ordinary functors are the identity, and its 2-cells are V-natural transformations whose components are identities. Such a 2-cell is merely the assertion that two V-functors are equal, so in general this is a 1-category. This is what we expect, since $(-)_0$ forgets at most stuff. However, when $V(I,-)$ is faithful, a V-functor is determined by its underlying functor, so the fiber is in fact a poset, as we expect it to be since in this case the functor forgets at most structure.

It would be nice to have a good example of a 2-functor which forgets at most stuff and whose fibers are 2-posets that are not 1-categories, but I haven't thought of one.

Example 15. In order to do an example that forgets 2-stuff, consider the forgetful 2-functor

$$[\text{pairs of categories}] \longrightarrow [\text{categories}].$$

This functor is not j-surjective for any $j \leq 3$, so it forgets at most 2-stuff. And here the (essential) fiber over a category is a genuine 2-category: we can have arbitrary functors and natural transformations living on that extra category we forgot about.

Can we make this formal and use it as an alternate characterization of how much a functor forgets? The answer is: 'sometimes.' Here's what's true always in dimension one:

- If a functor is an equivalence, then all its essential fibers are contractible ((-2)-categories);
- If it is full and faithful, then all its essential fibers are empty or contractible ((-1)-categories);
- If it is faithful, then all its essential fibers are posets;
- and of course, if it is arbitrary, then its essential fibers can be arbitrary categories.

However, in general none of the implications above can be reversed. This is because a statement about the essential fibers really tells us only about the arrows which live over isomorphisms, while full and faithful tell us something about *all* the arrows.

There are, however, two cases in which the above implications *are* reversible:

(1) When all categories involved are *groupoids*. This is because in this case, all arrows live over isomorphisms, since they all *are* isomorphisms.
(2) If the functor is a *fibration* in the categorical sense.

Being a fibration in the categorical sense is like being a fibration in the topological sense, except that (1) we allow ourselves to lift arrows that have direction, since our categories have such arrows, and (2) we don't allow ourselves to take just any old lift, but require that the lift satisfy a nice universal property. I won't give the formal definition here, since you can find it in many places; instead I want to try to explain what it means.

The notion essentially means that the extra properties, structure, or stuff that lives upstairs in E can be 'transported' along arrows downstairs in B in a *universal* way. When we're transporting along arrows downstairs that are invertible, like paths in topology or arrows in an n-groupoid, this condition is unnecessary since the invertibility guarantees that we aren't making any irreversible changes. My favorite example is the following.

Example 16. Let B be the category of rings and ring homomorphisms. Let E be the category whose objects are pairs (R, M) where R is a ring and M is an R-module, and whose morphisms are pairs $(f, \varphi)\colon (R, M) \to (S, N)$ where $f\colon R \to S$ is a ring homomorphism and $\varphi\colon M \to N$ is an 'f-equivariant map', i.e. $\varphi(rm) = f(r)\varphi(m)$. Then if $f\colon R \to S$ is a ring homomorphism and N is an S-module, there is a canonical associated R-module f^*N—namely, M with R acting through f—and a canonical f-equivariant map $f^*N \to N$—namely the identity map. This map is 'universal' in a suitable sense, and is clearly what we should mean by 'transporting' N backwards along f.

The formal definition of fibration simply makes this notion precise.

The introduction of directionality here also means that we get different things by transporting objects along arrows backwards and forwards. In the above example, the dual construction would be to take an R-module M and construct an S-module $f_!M = S \otimes_R M$ by 'extending scalars' to S. Again this comes with a canonical f-equivariant map $M \to f_!M$. Thus there are actually two notions of categorical fibration; for historical reasons, the 'backwards' one is usually called a **fibration** and the 'forwards' one an **opfibration** (or a 'cofibration', but we eschew that term because it carries the wrong topological intuition). Either one works equally well for the characterization of forgetfulness by fibers.

Another nice thing about the notion of categorical fibration is that while the principle of Galois theory does not apply, in general, to functors between arbitrary categories, it does apply to fibrations. Recall that in the groupoid case, fibrations over a base space (n-groupoid) B with fiber F are equivalent to functors $B \to \mathrm{AUT}(F)$. One can show that for a base category B, fibrations over B are equivalent to (weak) functors $B^{op} \to \mathbf{Cat}$. The way to think of this is that since our arrows are no longer necessarily invertible, the induced morphisms of fibers are no longer necessarily automorphisms, nor are all the fibers necessarily the same. Thus instead of the automorphism n-group of 'the' fiber, we have to use the whole *category* of possible fibers: in this case, **Cat**, since the fibers are categories.

Fibrations also have the nice property that the essential preimage is equivalent to the literal or 'strict' preimage. Since many forgetful functors, like those above, are fibrations, in such cases we can use the strict preimage instead of the essential one. In fact, a much weaker property than being a fibration is enough for this; it suffices that objects upstairs can be transported along 'equivalences' downstairs (which coincides with the notion of fibrations in the n-groupoid case, when all morphisms are equivalences). This is true in many examples which are not full-fledged fibrations.

This advantage also implies, however, that there is a sense in which the notion of categorical fibration is 'not fully weak'. Ross Street has defined a weaker notion of fibration which does not have this property, and which makes sense in any (weak)

2-category. It is easy to check that this weaker notion of fibration also suffices for the characterization of forgetfulness via fibers. Of course, unlike for traditional fibrations, in this case it is essential that we use essential preimages, rather than strict ones, since the two are no longer equivalent.

There ought to be a notion of categorical fibration for n-categories. Some people have studied particular cases of this. Claudio Hermida has studied 2-fibrations between 2-categories. André Joyal, Jacob Lurie, and others have studied various notions of fibration between quasi-categories, which are one model for $(\infty, 1)$-categories; see Joyal's introduction to quasi-categories in this volume for more details.

Let's end this section by formulating a hypothesis about the behavior of fibers for n-categories. Generalizing an idea from the first lecture, let's say that a functor **forgets at most k-stuff** if it is j-surjective for $j > k + 1$.

Hypothesis 17. *If a functor between n-categories forgets at most k-stuff, then its fibers are k-categories (which we take to include poset-enriched k-categories). The converse is true for n-groupoids and for n-categorical fibrations.*

We've checked this hypothesis above for $n = 1$ and for n-groupoids (modulo Grothendieck). Actually, we only checked it for $(1, 1)$-categories, while to be really consistent, we should check it for $(1, 2)$-categories too, but I'll leave that to you. For $n = 0$ it says that an isomorphism of posets has contractible fibers (obvious) and that an inclusion of a sub-poset has fibers which are empty or contractible (also obvious). Surely someone can learn about 2-fibrations and check this hypothesis for $n = 2$ as well.

As one last note, recall that in the topological case, when we studied Postnikov towers in §3.2, we were able, by the magic of homotopy theory, to convert all the maps in our factorization into fibrations. It would be nice if a similiar result were true for categorical fibrations. It isn't true as long as we stick to plain old categories, but there's a sense in which it becomes true once we generalize to things called 'sites' and their corresponding 'topoi'. I won't say any more about this, but it leads us into the next topic.

5.3. n-Topoi.

Knowing about the existence of n-posets and how they fit into the enrichment table also clarifies the notion of topos, and in particular of n-topos.

Topos theory (by which is usually meant what we would call 1-topos theory and 0-topos theory; I'll explain later) is a vastly beautiful and interconnected edifice of mathematics, which can be quite intimidating for the newcomer, not least due to the lack of a unique entry point. In fact, the title of Peter Johnstone's epic compendium of topos theory, *Sketches of an Elephant*, compares the many different approaches to topos theory to the old story of six blind men and an elephant. (The six blind men had never met an elephant before, so when one was brought to them, they each felt part of it to determine what it was like. One felt the legs and said "an elephant is like a tree," one felt the ears and said "an elephant is like a banana leaf," one felt the trunk and said "an elephant is like a snake," and so on. But of course, an elephant is all of these things and none of them.)

So what is a topos anyway? For now, I want you to think of a (1-)topos as *a 1-category that can be viewed as a generalized universe of sets.* What this turns out to mean is the following:

- A topos has limits and colimits;

- A topos is cartesian closed; and
- A topos has a 'subobject classifier'.

It turns out that this much does, in fact, suffice to allow us to more or less replace the category of sets with any topos, and build all of mathematics using objects from that topos instead of our usual notion of sets. (There's one main caveat I will bring up below.)

Now, how can we generalize this to n-topoi for other values of n? I'm going to instead ask the more general question of how we can generalize it to (n, m)-topoi for other values of n and m. I claim that a sensible generalization should allow us to assert that:

> *An (n, m)-topos is an (n, m)-category that can be viewed as a generalized universe of $(n-1, m-1)$-categories.*

One thing this tells us is that we shouldn't expect to have much of a notion of topos for n-groupoids: we don't want to let ourselves drop off the enrichment table. Inspecting the definition of 1-topos confirms this: groupoids generally don't have limits or colimits, let alone anything fancier like exponentials or subobject classifiers. The only groupoid that is a topos is the trivial one. This is a bit unfortunate, since it means we can't test our hypotheses using homotopy theory in any obvious way, but we'll press on anyway.

Let's consider our lower-dimensional world. What should we mean by a '$(0, 1)$-topos'? (I'm going to abuse terminology and call this a '0-topos', since as we saw above, we expect the only $(0, 0)$-topos to be trivial.) Well, our general philosophy tells us that it should be a poset that can be viewed as a generalized universe of truth values. At this point you may think you know what it's going to turn out to be—and you may be right, or you may not be.

What do the characterizing properties of a 1-topos say when interpreted for posets? Limits in a poset are meets (greatest lower bounds), and colimits are joins (least upper bounds), so our 0-topoi will be complete lattices. Being cartesian closed for a poset means that for any elements b, c there exists an object $b \Rightarrow c$ such that

$$a \leq (b \Rightarrow c) \quad \text{if and only if} \quad (a \wedge b) \leq c$$

As we expected, this structure makes our poset look like a generalized collection of truth values: we have a conjunction operation $\wedge$, a disjunction operation $\vee$, and an implication operation $\Rightarrow$. We can define a negation operator by $\neg a = (a \Rightarrow \perp)$, which turns out to behave just as we expect, except that in general $\neg\neg a \neq a$. Thus the logic we get is not *classical* logic, but *constructive* logic, in which the principle of double negation is denied (as are equivalent statements such as the 'law of excluded middle', $a \vee \neg a$). Boolean algebras, which model classical logic, are a special case of these cartesian closed posets, which are called **Heyting algebras**. Thus, a 0-topos is essentially just a complete Heyting algebra.

Now, one of the most exciting things in topos theory is that Heyting algebras turn up in topology! Namely, the lattice of open sets $\mathscr{O}(X)$ of any topological space X is a complete Heyting algebra, and any continuous map $f : X \to Y$ gives rise to a map of posets $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$ which preserves finite meets and arbitrary joins (but not $\Rightarrow$). Thus, we can view complete Heyting algebras as a sort of 'generalized topological space'. When we do this, we call them **locales**. So a more correct thing to say is that 0-topoi are the same as locales.

Now, I didn't mention the subobject classifier. In fact, it doesn't turn out to mean anything interesting for posets. This makes us wonder what its appearance for 1-topoi means. One answer is that *it allows us to apply the principle of Galois theory inside a 1-topos*.

What should that mean? Well, what does the principle of Galois theory (suitably generalized to nonidentical fibers) say for sets? It says, first of all, that functions $p\colon E \to B$, for sets E and B, are equivalent to functors $B \to \mathbf{Set}$. This is straightforward: we take each $b \in B$ to the fiber over it.

But what if we reduce the dimension of the fibers? A function $p\colon E \to B$ whose fibers are (-1)-categories, i.e. truth values, is just a subset of B, and the principle of Galois theory says that these should be equivalent to functors from B to the category of truth values, which is the poset $\{\text{false} \le \text{true}\}$, often written $\mathbf{2}$. This is just the correspondence between subsets and their *characteristic functions*.

Now, a subobject classifier is a categorical way of saying that you have on object Ω which acts like $\mathbf{2}$: it is a target for characteristic functions of subobjects (monomorphisms). Thus, this condition in the definition of 1-topos essentially tells us that we can apply the principle of Galois theory inside the topos. (It turns out that the unrestricted version for arbitrary functions $p\colon E \to B$ is also true in a topos, once you figure out how to interpret it correctly.)

Now, since a 1-topos is a generalized universe of sets and contains an object Ω which acts as a generalization of the poset $\mathbf{2}$ of truth values, we naturally expect Ω to be a generalized universe of truth values, i.e. a 0-topos. This is in fact the case, although there are couple of different ways to make this precise.

One such way is to consider the **subterminal objects** of the topos, which are the objects U such that for any other object E there is *at most* one map $E \to U$. They are called 'subterminal' because they are the subobjects of the terminal object 1, which are by definition the same as the maps $1 \to \Omega$, or the 'points' of Ω. They can also be described as the objects which are 'representably (-1)-categories', since each hom-set $C(E,U)$ has either 0 or 1 element, so it is precisely a truth value. Thus it makes sense that the collection of subterminal objects turns out to be a 0-topos, whose elements are the 'internal truth values' in our given 1-topos. Since in general the logic of a 0-topos is constructive, not classical, the internal logic of a 1-topos is also in general constructive; this is the one caveat I mentioned earlier for our ability to redo all of mathematics in an arbitrary topos.

Thus every 1-topos, or universe of sets, contains inside it a 0-topos, or universe of truth values. We can also go in the other direction: given a locale X (a 0-topos), we can construct its category $\mathrm{Sh}(X)$ of **sheaves**, by an obvious generalization of the notion of sheaves on a topological space, and this turns out to be a 1-topos, which we regard as 'the category of sets in the universe parametrized by X'. As we expect, the subobject classifier in $\mathrm{Sh}(X)$ turns out to be $\mathscr{O}(X)$. In fact, this embeds the category of locales in the (2-)category of topoi, which leads us to consider any 1-topos as a vastly generalized kind of topological space.

As a side note, recall that in §5.1 we observed that a set is a groupoid enriched over truth values. Thus you might expect that the objects of $\mathrm{Sh}(X)$, which intuitively are 'sets in the universe where the truth values are $\mathscr{O}(X)$', could be defined as 'groupoids enriched over $\mathscr{O}(X)$'. This is almost right; the problem is that all the objects of such a groupoid turn out to have 'global extent', while an arbitrary sheaf can have objects which are only 'partially defined'. We can, however, make it work if we consider instead groupoids enriched over a suitable *bi*category constructed from $\mathscr{O}(X)$.

Anyway, these relationships between 0-topoi and 1-topoi lead us to hope that in higher dimensions, each (n, m)-topos will contain within it topoi of lower dimensions, and in turn will embed in topoi of higher dimensions via a suitable categorification of sheaves (usually called 'n-stacks' or simply 'stacks'). Notions of (n, m)-topos have already been studied for a few other values of n and m. For instance, there has also been a good deal of interest lately in something that people call '∞-topoi', although from our point of view a better name would be $(\infty, 1)$-topoi. These are special $(\infty, 1)$-categories that can be considered as a generalized universe of homotopy types (i.e. ∞-groupoids). And for a long time algebraic geometers have been studying 'stacks of groupoids', which are pretty close to what we would call a '$(2, 1)$-topos'.

Where does the interest in higher topoi come from? In topology, the principle of Galois theory already works very nicely, and people were working with fibrations, homotopy groups, Postnikov towers, and cohomology long before Grothendieck came along to tell them they were really working with ∞-groupoids. A fancy way to say this is that the category of spaces is already an $(\infty, 1)$-topos.

But in algebraic geometry, the Galois theory fails, because the category under consideration is 'too rigid'. The n-groups $\mathrm{AUT}(F)$ just don't exist. So what the algebraic geometers do is to take their category and *embed* it in a larger category in which the desired objects *do* exist; we would say that they embed it in a $(2, 1)$-topos (if they're only interested in one level of automorphisms) or an $(\infty, 1)$-topos (if they're interested in the full glory of homotopy theory). The way they do this is with a suitable generalization of the sheaf construction to arbitrary categories.

In the case $m > 1$, it is not clear whether there exists a single notion of (n, m)-topos that shares most of the good properties of 1-topoi. Several people have studied this question, however, with some partial encouraging results; the 'fibrational cosmoi' of Ross Street can be viewed as generalized universes of 1-categories, and more recently Mark Weber has studied certain special cosmoi under the name '2-topos'. One of the defining properties of a cosmos is the existence of 'presheaf objects' which allow the application of the principle of Galois theory to internal fibrations in the 2-category (suitably defined). Some people speak of this as "considering sets to be generalized truth values".

5.4. **Geometric morphisms, classifying topoi, and n-stuff.** In this section we'll see that morphisms between topoi admit similar 'Postnikov' factorizations, which in turn tell us interesting things about the logical theories they 'classify'. This section will probably be most interesting to readers with some prior acquaintance with topos theory, but I've tried to make it as accessible as possible.

Recall that a continuous map $f : X \to Y$ of topological spaces gives rise to a function $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$ which preserves finite meets and arbitrary joins. Let X and Y be locales and $\mathscr{O}(X)$ and $\mathscr{O}(Y)$ the corresponding complete Heyting algebras; we *define* a map of locales $f : X \to Y$ to be a function $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$ preserving finite meets and arbitrary joins. We distinguish notationally between the locale X and its poset of 'open sets' $\mathscr{O}(X)$ because the maps go in the opposite direction, even though the locale X technically consists of nothing but $\mathscr{O}(X)$.

Similarly, let X and Y be topoi, and $\mathscr{S}(X)$ and $\mathscr{S}(Y)$ their corresponding 1-categories. We define a map of 1-topoi $f : X \to Y$ to be a functor $f^* : \mathscr{S}(Y) \to$

$\mathscr{S}(X)$ which preserves finite limits and arbitrary colimits; these maps are called **geometric morphisms** for historical reasons.

Now, it turns out that for any (small) category C, the category $\mathbf{Set}^C$ of functors from C to $\mathbf{Set}$ is a topos, and functors $f\colon C \to D$ give rise to geometric morphisms $\widehat{f}\colon \mathbf{Set}^C \to \mathbf{Set}^D$. We can thus ask how properties of the functor f are reflected in properties of the geometric morphism $\widehat{f}$. It turns out that we have the following dictionary (at least, 'modulo splitting idempotents', which is something I don't want to get into—just remember that this is all morally true, but there are some details.)

$$\begin{array}{ccc} f \text{ is full and faithful} & \sim & \widehat{f} \text{ is an 'inclusion'} \\ f \text{ is essentially surjective} & \sim & \widehat{f} \text{ is a 'surjection'} \\ f \text{ is faithful} & \sim & \widehat{f} \text{ is 'localic'} \\ f \text{ is full and essentially surjective} & \sim & \widehat{f} \text{ is 'hyperconnected'} \end{array}$$

What do all those strange terms on the right mean? I'm certainly not going to define them! But I'll try to give you some idea of how to think about them. The notions of 'inclusion' and 'surjection' are suitable generalizations of the correspondingly named notions for topological spaces. Moreover, just as is true for spaces, any geometric morphism factors uniquely as a surjection followed by an inclusion; this also parallels one of our familiar factorizations for functors. This part of the correspondence should make some intuitive sense.

To explain the term 'localic', consider a geometric morphism $p\colon E \to S$. It turns out that we can think of this either as a map between two topoi in the universe of sets, or we can use it to think of E as an *internal* topos in the generalized universe supplied by the topos S. We say that the morphism p is 'localic' if this internal topos is equivalent to the sheaves on some internal locale in S. It turns out that there is another sort of morphism called 'hyperconnected' such that every geometric morphism factors uniquely as a hyperconnected one followed by a localic one, and this too corresponds to a factorization we know and love for functors.

Moreover, every inclusion is localic, and every hyperconnected morphism is a surjection, and it follows that every geometric morphism factors as a hyperconnected morphism, followed by a surjective localic map, followed by an inclusion. This should also look familiar in the world of functors.

Now I want to explain why these classes of geometric morphisms in fact have an *intrinsic* connection to the notions of properties, structure, and stuff, but to do that I have to talk about 'classifying topoi'.

The basic idea of classifying topoi is that we can apply the principle of Galois theory once again, only this time we apply it in the 2-category *of topoi*, and we apply it to classify *models of logical theories*. Let $\mathbb{T}$ be a **typed logical theory**; thus it has some collection of 'types', some 'function and relation symbols' connecting these types, and some 'axioms' imposed on the behavior of these symbols. An example is the theory of categories, which has two types O ('objects') and A ('arrows'), three function symbols $s, t\colon A \to O$, $i : O \to A$, a relation symbol c of type $A \times A \times A$ (here $c(f, g, h)$ is intended to express the assertion that $h = g \circ f$), and various axioms, such as

$$(t(f) = s(g)) \Rightarrow \exists! h\, c(f, g, h)$$

(which says that any two composable arrows have a unique composite). A model of such a theory assigns a set to each type and a function or relation to each symbol, such that the axioms are satisfied; thus a model of the theory of categories is just a small category.

The fact that a topos is a generalized universe of sets implies that we can consider models of such a theory in *any* topos, not just the usual topos of sets. It turns out that for suitably nice theories $\mathbb{T}$ (called 'geometric' theories), there exists a topos $[\mathbb{T}]$ such that for any other topos B, the category of models of $\mathbb{T}$ in B is equivalent to $\hom(B, [\mathbb{T}])$, the category of geometric morphisms from B to $[\mathbb{T}]$ (remember that 1-topoi form a 2-category). Thus, once again, some structure 'in' or 'over' B can be classified by functors from B to a 'classifying object'.

Now suppose that we have two theories $\mathbb{T}$ and $\mathbb{T}'$ such that $\mathbb{T}'$ is $\mathbb{T}$ with some extra types, symbols, and/or axioms added. Since this means that any model of $\mathbb{T}'$ gives, by neglect of structure, a model of $\mathbb{T}$, by the Yoneda lemma we have a geometric morphism $p\colon [\mathbb{T}'] \to [\mathbb{T}]$. It turns out that

p is an inclusion	when	$\mathbb{T}'$ adds only extra axioms to $\mathbb{T}$
p is localic	when	$\mathbb{T}'$ adds extra functions, relations, and axioms to $\mathbb{T}$, but no new types
p is a surjection	when	$\mathbb{T}'$ adds extra types, symbols, and axioms to $\mathbb{T}$, but no new properties of the existing types and symbols in $\mathbb{T}$ are implied by this new structure.
p is hyperconnected	when	$\mathbb{T}'$ adds extra types to $\mathbb{T}$, along with symbols and axioms relating to these new types, but no new functions, relations, or axioms on the existing types in $\mathbb{T}$ are implied by this new structure.

(There are various ways to make these notions precise, which I'm not going to get into.) Thus, these classes of geometric morphisms actually directly encode the notions of forgetting properties (axioms), structure (function and relation symbols), and/or stuff (types).

Notice that localic morphisms are those that add no new types; this is consistent with the fact that locales are 0-topoi, and 0-categories know only about properties ((-1)-stuff) and structure (0-stuff), not stuff (1-stuff). In particular, a classifying topos $[\mathbb{T}]$ is equivalent to a topos of sheaves on a locale precisely when the theory $\mathbb{T}$ has no types. Such a theory, which consists only of propositions and axioms, is called a **propositional theory**; from our point of view, we might also call it a '0-theory', with the more general typed theories considered above being '1-theories'. As far as I know, there has been very little work on notions of n-theories for higher values of n.

Now, given the correspondence between theories and classifying topoi, any factorization for geometric morphisms leads to a factorization for geometric theories. These factorizations are mostly what we would expect, but can be slightly different due to the requirement that all theories in sight be geometric.

Example 18. Consider the forgetful map from monoids to semigroups. (A semigroup is a set with an associative binary operation.) Considered as a functor $\mathbf{Mon} \to \mathbf{SGp}$, it is faithful, but not essentially surjective (since not every semigroup has an identity) or full (since not every semigroup homomorphism between monoids preserves the identity). If we factor it into a full-and-essentially-surjective functor followed by a full-and-faithful one, the intermediate category we obtain is the category of 'semigroups with identity', i.e. the category whose objects are monoids but whose morphisms do not necessarily preserve the identity.

Now, the theories $\mathbb{M}$ of monoids and $\mathbb{S}$ of semigroups are both geometric, so they have classifying topoi $[\mathbb{M}]$ and $[\mathbb{S}]$, and as we expect there is a geometric morphism $[\mathbb{M}] \to [\mathbb{S}]$ which is localic. If we factor it into a surjection followed by an inclusion, however, the intermediate topos we obtain is not the classifying topos for semigroups with identity, because that theory is not geometric. Instead, the intermediate topos we get is the classifying topos for semigroups such that for any finite set of elements, there is an element which behaves as an identity for them. In general, however, the 'identities' for different finite sets could be different.

This theory is, in a sense, the 'closest geometric approximation' to the theory of semigroups with identity. This notion is in accord with the general principle (which we have not mentioned) that geometric logic is the 'logic of finite observation'. In this case, it is evident that if we can only 'observe' finitely many elements of the semigroup, we can't tell the difference between such a model of our weird intermediate geometric theory and a semigroup that has an actual identity.

These considerations may lead us to speculate that morphisms of higher topoi, once defined, will have similar 'Postnikov factorizations'. However, in the absence of confidence that good notions of (n, m)-topos exist for $m > 1$, this must remain a speculation.

5.5. **Monomorphisms and epimorphisms.** A question was asked at one point (in §3.3.1) about whether notions like essential surjectivity can be defined purely 2-categorically, and thereby interpreted in any 2-category, the way that epimorphisms and monomorphisms make sense in any 1-category. This section is an attempt to partially answer that question.

The definitions of monomorphism and epimorphisms in 1-categories are 'representable' in the following sense:

- $m\colon A \to B$ is a monomorphism if for all X, the function
$$C(X, m)\colon C(X, A) \to C(X, B)$$
is injective.
- $e\colon E \to B$ is an epimorphism if for all X, the function
$$C(e, X)\colon C(B, X) \to C(A, X)$$
is injective.

Note that both notions invoke *injectivity* of functions of sets. Thus, the natural notions to consider first are functors which are 'representably' faithful or full-and-faithful. It is easy to check that this works in the covariant direction:

- A functor $p\colon A \to B$ is faithful if and only if it is representably faithful, i.e. all functors
$$\mathbf{Cat}(X, p)\colon \mathbf{Cat}(X, A) \to \mathbf{Cat}(X, B)$$
are faithful; and
- A functor is full and faithful if and only if it representably full and faithful.
- A functor is an equivalence if and only if it is representably an equivalence.

Thus, it makes sense to define a 1-morphism in a 2-category to be **faithful** or **full and faithful** when it is representably so.

We may generalize this (hypothetically) by saying that a functor between n-categories is j**-monic** if it is k-surjective for all $k > j$ (note that this is equivalent to saying that it 'forgets at most $(j-1)$-stuff'), and that a 1-morphism $p\colon A \to B$

in an $(n+1)$-category C is **j-monic** if all functors $C(X,p)$ are j-monic. By analogy with the above observation, we expect that these definitions will be equivalent for the $(n+1)$-category of n-categories.

Thus, every functor is 2-monic, the 1-monic functors are the faithful ones, the 0-monic functors are the full and faithful ones, and the (-1)-monic functors are the equivalences. More degenerately, in a 1-category, every map is 1-monic, the 0-monic morphisms are the usual monomorphisms, and the (-1)-monic morphisms are the isomorphisms.

We may define, dually, a 1-morphism $p\colon E \to B$ in an $(n+1)$-category C to be **j-epic** if all the functors

$$C(p,X)\colon C(B,X) \to C(E,X)$$

are $(n-1-j)$-monic. For example, in a 1-category, every morphism is (-1)-epic, the 0-epic morphisms are the usual epimorphisms, and the 1-epic morphisms are the isomorphisms.

The 'inversion' of numbering here may look a little strange if we remember that every n-category is secretly an ∞-category; when did it suddenly start to matter which n we are using? But it turns out that the transformation above is actually exactly what is required to make the notion independent of n. For example, if $p\colon E \to B$ is a surjective function in **Set**, then $\mathbf{Set}(p,X)$ is injective, hence 0-monic, for any set X; thus p is 0-epic in the 1-category **Set**. But now consider p as a functor between discrete categories. When X is a nondiscrete category, $\mathbf{Cat}(p,X)$ is faithful, but not full; hence it is only 1-monic, but by our definition this is just what is required so that p is again 0-epic.

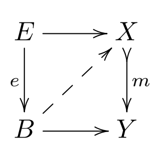
It is easy to check that in a 2-category, every morphism is (-1)-epic, and the 2-epic morphisms are the equivalences. However, even in the 2-category **Cat**, the 0-epic and 1-epic morphisms are not that well-behaved. Here is what is true (proofs are left to the reader):

- If a functor $p\colon E \to B$ is essentially surjective, then it is 0-epic.
- Similarly, if it is full and essentially surjective, then it is 1-epic.

However, neither implication is reversible. For example, the inclusion of the category 2, which has two objects and one nonidentity morphism between them, into the category $\mathscr{I}$, which has two uniquely isomorphic objects, is 1-epic, but not full. And if $p\colon E \to B$ has the property that every object of B is a retract of an object in the image of p, then p is 0-epic, but it need not be essentially surjective.

We thus seek for other characterizations of surjective functions in **Set** which will generalize better to **Cat**. It turns out that the best-behaved notion is the following:

Definition 19. An epimorphism $p\colon E \to B$ in a 1-category is a **strong epimorphism** if it is 'left orthogonal' to monomorphisms, i.e. for any monomorphism $m\colon X \to Y$, every commutative square



has a unique diagonal filler.

In a category with equalizers, the orthogonality property implies that p is already an epimorphism. In **Set**, every epimorphism is strong, but in general this is not true.

Notice that saying $p\colon E \to B$ is left orthogonal to $m\colon X \to Y$ in the category C is equivalent to saying that the following square is a pullback:

$$\begin{array}{ccc} C(B,Y) & \longrightarrow & C(E,Y) \\ \downarrow & & \downarrow \\ C(B,X) & \longrightarrow & C(E,X) \end{array}$$

Therefore, we generalize this to 2-categories as follows.

Definition 20. A 1-morphism $p\colon E \to B$ in a 2-category C is **left orthogonal** to another $m\colon X \to Y$ if the square

$$\begin{array}{ccc} C(B,Y) & \longrightarrow & C(E,Y) \\ \downarrow & & \downarrow \\ C(B,X) & \longrightarrow & C(E,X) \end{array}$$

is a pullback (in a suitable 2-categorical sense).

We can now check that

- A functor $p\colon E \to B$ is essentially surjective if and only if it is left orthogonal to all full and faithful functors, and
- It is full and essentially surjective if and only if it is left orthogonal to all faithful functors.

The forward directions are exercises in category theory. The idea is that we must progressively 'lift' objects, morphisms, and equations (to show functoriality and naturality) from 'downstairs' to 'upstairs'. In both cases, for each j, one of the two functors is j-surjective, so we can use that functor to lift the j-morphisms. The reverse directions are easy using the Postnikov factorization.

We are thus motivated to define, hypothetically, a j-epimorphism in an $(n+1)$-category to be a **strong j-epimorphism** if it is left orthogonal (in a suitably weak sense) to all j-monic morphisms. We have just shown that in **Cat**, the strong 1-epics are precisely the full and essentially surjective functors, while the strong 0-epics are the essentially surjective functors. Clearly all functors are strong (-1)-epic, while only equivalences are strong 2-epic.

We can also prove that in 2-categories with finite limits, any morphism which is left orthogonal to j-monomorphims is automatically a j-epimorphism; we use 2-categorical limits such as 'inserters' and 'equifiers' to take the place of equalizers in the 1-dimensional version. As we have seen, even in **Cat**, not every j-epimorphism is strong.

This leads us to formulate the following hypothesis.

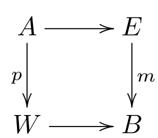
Hypothesis 21. *The strong j-epics in n**Cat** (that is, functors which are left orthogonal to all j-monic functors) are precisely the functors which are k-surjective for $k \leq j$. Not every j-epic is strong, even in n**Cat**.*

Since j-monic functors are those that forget 'at most $(j-1)$-stuff', we might say that the strong j-epics are the functors which 'forget no less than j-stuff'. For example, the strong 0-epics in **Cat** are the essentially surjective functors, which do not forget properties ((-1)-stuff), although they may forget structure (0-stuff) and 1-stuff. Similarly, the strong 1-epics, being essentially surjective and full, do not forget properties or structure, although they may forget 1-stuff.

What does this look like for n-groupoids? For a functor between n-groupoids, being k-surjective is equivalent to inducing a surjection on π_k and an *in*jection on π_{k-1}. Why? Well, remember that π_k of an n-groupoid consists of the automorphisms of the identity $(k-1)$-morphism, modulo the $(k+1)$-morphisms. Thus being surjective on k-morphisms implies being surjective on π_k (although there might be new $(k+1)$-morphisms appearing preventing it from being an isomorphism), but also being injective on π_{k-1}, since everything we quotient by downstairs has to already be quotiented by upstairs (although here there might be entirely new $(k-1)$-morphisms appearing downstairs). This is also equivalent to saying that π_k of the homotopy fiber is trivial.

Thus, a functor between n-groupoids is j-monic if it induces isomorphisms on π_k for $k > j$ and an injection on π_j. Our above conjecture then translates to say that the strong j-epics should be the maps $A \to W$ inducing isomorphisms on π_k for $k < j$ and a surjection on π_j. This is precisely what topologists call a **j-equivalence** or a **j-connected map**, since it corresponds to the vanishing of the 'relative homotopy groups' $\pi_k(W, A)$ for $k \leq j$.

Using this identification, we can then prove our conjecture for n-groupoids (modulo Grothendieck). Suppose we have a square

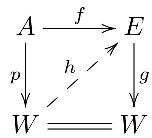


of maps between n-groupoids (i.e. topological spaces), in which p is a j-equivalence and m is j-monic.

By magic of homotopy theory, similar to the way we can transform any map into a fibration, we can transform the map p into a 'relative cell complex'. This means that W is obtained from A by attaching 'cells' D^k along their boundaries S^{k-1}. Since p is a j-equivalence, we can assume that we are only attaching cells of dimension $k > j$. Thus our problem is reduced to defining a lift on each individual D^k. But our assumption on m guarantees that since (inductively) we have a lift of the boundary S^{k-1}, the whole cell D^k must also lift, up to homotopy.

This only shows that single maps lift, but we can also show that the appropriate square is a homotopy pullback by considering various modified squares. (The fanciest way to make this precise is to construct a 'Quillen model structure'.) Thus j-connected maps are in fact left 'homotopy' orthogonal to j-monic maps.

Just as before, we can prove the converse using our knowledge of factorizations. Suppose we have a map $p\colon A \to W$ which is left orthogonal to all j-monic maps. By picking out a particular part of the Postnikov factorization of p, we get a factorization $A \xrightarrow{f} E \xrightarrow{g} W$ in which g is j-monic and f is j-connected. Then in the square



there exists a diagonal lift h. Consider any $k \leq j$; then $\pi_k(f)$ is either an isomorphism (if $k < j$) or surjective (if $k = j$), by assumption. This implies that $\pi_k(h)$ must also be surjective. But $\pi_k(g)\pi_k(h)$ is the identity, so in fact $\pi_k(h)$ must be an isomorphism. Therefore, since $hp \simeq f$, $\pi_k(p)$ must be an isomorphism if $k < j$ and surjective if $k = j$, since $\pi_k(f)$ is so. Thus p was already j-connected.

So, modulo Grothendieck's dream, we have proved Hypothesis 21 in the case of n-groupoids. Joyal's paper on quasi-categories, in this volume, includes a theory of factorization systems in $(\infty, 1)$-categories, generalizing the above arguments for ∞-groupoids to objects of any $(\infty, 1)$-category.

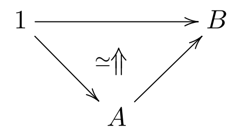
5.6. **Pointedness versus connectedness.** This final section will be even more philosophical, and perhaps controversial, than the others. The central point I wish to make is that the operations of looping and delooping should only be applied to *pointed* n-categories, just as they are only applied in homotopy theory to pointed topological spaces. When we do this, various problems with the periodic table resolve themselves.

What sort of problems? It's well-known that the hypothesis "a k-monoidal n-category is a k-degenerate $(n+k)$-category" is false, even in low dimensions, if you interpret 'is' as referring to a fully categorical sort of equivalence. The simplest example is that while a monoid 'is' a one-object category in a certain sense, the category of monoids is not equivalent to the (2-)category of categories-with-one-object. Similarly, the 2-category of monoidal categories is not equivalent to the (3-)category of one-object bicategories, and so on. In general, the objects and morphisms turn out mostly correct, but the higher-level transformations and so on are wrong. Eugenia Cheng and Nick Gurski have investigated in detail what happens and how you can often carefully chop things off at a particular level in the middle to get an equivalence, but here I want to consider a different point of view.

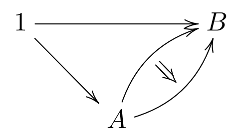
Let's consider the case of groupoids. The topological version of a one-object groupoid is a $K(G, 1)$, so the periodic table leads us to expect that the homotopy theory of $K(G, 1)$s should be equivalent to the category of groups. This is true, but only if we interpret the $K(G, 1)$s as *pointed* spaces and the corresponding homotopy theory likewise. Otherwise, we get the theory of groups and group homomorphisms modulo conjugation.

A related issue is that the homotopy groups π_n, and in particular π_1, are really only defined on *pointed* spaces. While it's true that different choices of basepoint give rise to isomorphic groups (at least for a connected space), the isomorphism is not canonical. In particular, this means that π_n is not *functorial* on the category of unpointed spaces.

Thus, by analogy with topology, we are motivated to consider 'pointed categories'. A **pointed n-category** is an n-category A equipped with a functor $1 \to A$ from the terminal n-category (which has exactly one j-morphism for every j). Note that this is essentially the same as choosing an object in A. A **pointed functor** between two pointed n-categories is a functor $A \to B$ such that

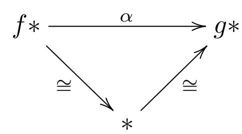


commutes up to a specified natural equivalence. A **pointed transformation** is a transformation α such that



commutes with the specified equivalences up to an invertible modification. And so on for higher data.

What does this look like in low dimensions? A pointed set is just a set with a chosen element, and a pointed function between such sets is a function preserving the basepoints. More interestingly, a pointed category is a category A with a chosen base object $* \in A$, a pointed functor is a functor $f\colon A \to B$ equipped with an isomorphism $f* \cong *$, and a pointed natural transformation is a natural transformation $\alpha\colon f \to g$ such that



commutes.

We have only required our basepoints to be preserved up to coherent equivalence, in line with general n-categorical philosophy, but we now observe that we can always 'strictify' a pointed functor to preserve the basepoints on the nose. Define f' to be f on all objects except $f'* = *$, with the action on arrows defined by conjugating with the given isomorphism $f* \cong *$. Then $f'* = *$, and f' is isomorphic to f via a pointed natural isomorphism. Thus the 2-category of pointed categories, pointed functors, and pointed transformations is biequivalent to the 2-category of pointed categories, *strictly* pointed functors, and pointed transformations. We expect this to be true in higher dimensions as well. Observe that if f and g are strictly pointed functors, then a pointed natural transformation $\alpha\colon f \to g$ is just a natural transformation $f \to g$ such that the component $\alpha_* = 1_*$.

Now we can define a functor Ω from the 2-category of pointed categories to the category of monoids. We take our pointed functors to be strict for convenience, since as we just saw there is no loss in doing so; otherwise we would just have to conjugate by the isomorphisms $f* \cong *$. We define $\Omega A = A(*,*)$ on objects, and for a strictly pointed functor $f\colon A \to B$, we get a monoid homomorphism $A(*,*) \to B(f*, f*) = B(*,*)$. Finally, since as we observed above a pointed transformation between strictly pointed functors is the identity on $*$, these transformations induce simply identities, which is good since those are the only 2-cells we've got in our codomain!

In the other direction, we construct a functor B from the category of monoids to the 2-category of pointed categories, sending a monoid M to the category BM

with one object $*$ and $BM(*, *) = M$, and a monoid homomorphism to the obvious (strictly) pointed functor. We now observe that B is left adjoint to Ω (in a suitable sense), and that moreover the adjunction restricts to an adjoint biequivalence between the category of monoids (regarded as a locally discrete 2-category) and the 2-category of pointed categories with exactly one isomorphism class of objects (which we may call 'pointed and connected').

Similarly, one can construct 'adjoint' functors B and Ω between monoidal categories and pointed bicategories, and show that they restrict to inverse 'triequivalences' between monoidal categories and pointed bicategories with one equivalence class of objects ('pointed connected bicategories'). We can also do this for commutative monoids and 'pointed monoidal categories', but in fact here the word 'pointed' becomes redundant: every monoidal category has an essentially unique basepoint, namely the unit object. (Similarly, any monoid has a unique basepoint, namely its identity. This is because the terminal monoid and the terminal monoidal category are also 'initial' in a suitable sense.) We then obtain an adjoint biequivalence between commutative monoids and connected monoidal categories.

Composing these two adjunctions, we obtain an adjoint pair $B^2 \dashv \Omega^2$ between commutative monoids and pointed bicategories. This restricts to a biequivalence between commutative monoids and pointed bicategories with one equivalence class of objects and one isomorphism class of 1-morphisms ('pointed 1-connected bicategories').

All of these equivalences carry over in an obvious way to the groupoid cases, so that groups are equivalent to pointed groupoids with one isomorphism class of objects, groupal groupoids (2-groups) are equivalent to pointed 2-groupoids with one equivalence class of objects, and abelian groups are equivalent to 2-groups with one isomorphism class of objects, and also to 2-groupoids with one equivalence class of objects and one isomorphism class of morphisms. These are well-known topological results.

This suggests the following pointed version of the correspondence described in the periodic table. Say that an n-category is **i-connected** if it has exactly one equivalence class of j-morphisms for $0 \le j \le i$.

Hypothesis 22 (Delooping Hypothesis)**.** *There is an adjoint pair $B^i \dashv \Omega^i$ between k-monoidal n-categories and (pointed) $(k-i)$-monoidal $(n+i)$-categories, which restricts to an equivalence between k-monoidal n-categories and (pointed) $(i-1)$-connected $(k-i)$-monoidal $(n+i)$-categories.*

We have placed "pointed" in parentheses because it is expected to be redundant for $k > i$.

We have called these functors Ω and B by analogy with the corresponding topological constructions of loop space and delooping (or 'classifying space'). Note that topologists usually say that the left adjoint of Ω is the 'suspension' functor Σ, rather than the 'delooping' functor B. This is because they often consider the functor Ω to take its values just in spaces, rather than monoidal spaces (say, A_∞-spaces). We would get a corresponding adjoint pair in our situation by composing the two adjunctions

$$n\text{-categories} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \begin{matrix}\text{monoidal}\\ n\text{-categories}\end{matrix} \underset{\Omega}{\overset{B}{\underset{\perp}{\rightleftarrows}}} \begin{matrix}\text{pointed}\\ (n+1)\text{-categories}\end{matrix}$$

where $F \dashv U$ is the free-forgetful adjunction. Then $\Sigma A = BF(A)$, the delooping of the free monoidal n-category on A, is what deserves to be called the 'suspension' of A.

Note that there is also a forgetful functor from pointed $(n+1)$-categories to unpointed $(n+1)$-categories, which has a *left* adjoint $(-)_+$ called 'adding a disjoint basepoint'.

Let us investigate further the question of 'connectedness'. Recall from §2.3 that for a space X we say that $\pi_j(X)$ **vanishes for all basepoints** if given any $f\colon S^j \to X$, there exists $g\colon D^{j+1} \to X$ extending f. When X is nonempty, this is equivalent to requiring that the actual groups $\pi_j(X)$ vanish for all base points. Topologists define a nonempty space X to be **k-connected** if $\pi_i(X)$ is trivial for $j \leq k$ and all basepoints. (We'll deal with the empty set later.)

We can generalize this to n-categories in a straightforward way, but we use a different terminology because unlike n-groupoids, n-categories are not characterized by a list of homotopy groups. We say that an n-category **has no j-homotopy** when any two parallel j-morphisms are equivalent. Another way to say this, which is closer to the topology, is to define S^j to be the n-category consisting of two parallel j-morphisms, and D^{j+1} to consist of a $(j+1)$-equivalence between two parallel j-morphisms; then X has no j-homotopy just when all maps $S^j \to X$ extend to D^{j+1}. We can then define an n-category to be **k-connected** if it has no j-homotopy for $j \leq k$.

Note that since we don't know what a (-1)-morphism is, the category S^{-1} can only be empty. And since in general D^j is generated by a single j-equivalence, D^0 should just consist of a single object. Thus an n-category has no (-1)-homotopy just when it is nonempty. Similarly, S^{-2} and D^{-1} should both be empty, so every n-category has no (-2)-homotopy. Therefore, just as for groupoids, an n-category is always (-2)-connected, is (-1)-connected when it is nonempty, and for $k \geq 0$ it is k-connected when it has precisely one isomorphism class of j-cells for $0 \leq j \leq k$. Thus, this definition of connectedness agrees with the one we gave just before Hypothesis 22. Moreover, the new definition allows that hypothesis to make sense even for $i = 0$, in which case it says that all k-monoidal n-categories are nonempty and come equipped with an essentially unique basepoint (the unit object).

Now, what about that pesky empty set? By classical topological definitions, the empty set is unquestionably both connected (it is not the disjoint union of two nonempty open sets) and path-connected (any two points in it are connected by a path). But by our definitions, although it has no 0-homotopy, it is not 0-connected, because it does not have no (-1)-homotopy. (Of course, the empty set is the only space with no 0-homotopy which is not 0-connected.)

This disagreement is perhaps a slight wart on our definitions. However, it is worth pointing out that π_0 of the empty set must also be empty. In particular, it is not equal to $0 = \{0\}$. So the only way the empty set can be 0-connected, if we use the topological definition that X is k-connected $\pi_j(X) = *$ for all $0 \leq j \leq k$, is if we maintain that $\pi_0(X)$, just like the other π_j, requires a base point to be defined (in which case it is a pointed set). In this case, since the empty set has no basepoints, it is still vacuously true that $\pi_0(\emptyset) = 0$ for all basepoints.

We would like to emphasize the crucial distinction between *connected* (having precisely one equivalence class of objects) and being *pointed* (being *equipped* with a chosen object). Clearly, every connected n-category can be pointed in a way which is unique up to equivalence, *but not up to unique equivalence*. Similarly, functors between connected n-categories can be made pointed, but not in a unique way, while transformations and higher data can *not* in general be made pointed at all. Thus the $(n+1)$-categories of connected n-categories and of pointed connected n-categories are not equivalent; the latter is equivalent to the n-category of monoidal $(n-1)$-categories, but the former is not.

This distinction explains an observation due to David Corfield that the periodic table seems to be missing a row. If in the periodic table we replace 'k-monoidal n-categories' by '$(k-1)$-connected $(n+k)$-categories', then the first row is seen to be the (-2)-connected things (that is, no connectivity imposed) while the second row is the 0-connected things. Thus there appears to be a row missing, consisting of the (-1)-connected, or nonempty, things, and morover the top row should be shifted over one to keep the diagonals moving correctly. So we should be looking at a table like this:

THE CONNECTIVITY PERIODIC TABLE

	$n = \mathbf{-1}$	$n = \mathbf{0}$	$n = \mathbf{1}$
$k = \mathbf{-1}$	truth values	sets	categories
$k = \mathbf{0}$	nonempty sets	nonempty categories	nonempty 2-categories
$k = \mathbf{1}$	connected categories	connected 2-categories	connected 3-categories
$k = \mathbf{2}$	1-connected 2-categories	1-connected 3-categories	1-connected 4-categories

In this table, the objects in the spot labeled k and n have nontrivial j-homotopy for only $n+2$ consecutive values of j, starting at $j = k$. The column $n = -2$, which is not shown, consists entirely of trivialities, since if you have nontrivial j-homotopy for zero consecutive values of j, it doesn't matter at what value of j you start counting.

So we have two different periodic tables, and it isn't that one is right and one is wrong, but rather that one is talking about monoidal structures (or equivalently, by the delooping hypothesis, pointed *and* connected things) and the other is talking about connectivity. Note that unlike the monoidal periodic table, the connectivity periodic table does not stabilize.

Finally, here's another reason to make the distinction between 'connected' and 'pointed'. We observed above that for ordinary n-categories in the universe of sets, every connected n-category can be made pointed in a way unique up to (non-unique) equivalence. However, this can become false if we pass to n-categories in some other universe (topos), such as 'sheaves' of n-categories over some space.

Consider, for instance, the relationship between groups and connected groupoids. A 'sheaf of connected groupoids' is something called a 'gerbe' (a "locally connected locally nonempty stack in groupoids"), while a 'sheaf of groups' is a well-known thing, but very different. Every sheaf of groups gives rise to a gerbe, by delooping (to get a prestack of groupoids) and then 'stackifying', but it's reasonably fair to say that the whole interest of gerbes comes from the fact that most of them *don't* come from a sheaf of groups. The ones that do are called 'trivial', and a gerbe is trivial precisely when it has a basepoint (a global section). So the equivalence of groups with *pointed* connected groupoids is true even in the world of sheaves, but in this case not every 'connected' groupoid can be given a basepoint.

If we move down one level, this corresponds to the statement that not every well-supported sheaf has a global section. Thus in the world of sheaves, not every 'nonempty' set can be given a basepoint. So one cause of the confusion between connectedness and pointedness is what we might call '**Set**-centric-ness': the two notions are quite similar in the topos of sets, but in other topoi they are much more distinct.

6. Annotated Bibliography

The following bibliography should help the reader find more detailed information about some topics mentioned in the talks and Appendix. It makes no pretense to completeness, and we apologize in advance to all the authors whose work we fail to cite. In the spirit of 'something for everybody', we include references with wildly different prerequisites: some are elementary, while others even we don't understand.

1.1 Galois Theory. For a gentle introduction to Galois theory, try these:

Ian Stewart, *Galois Theory*, 3rd edition, Chapman and Hall, New York, 2004.

Jean-Pierre Escofier, *Galois Theory*, Springer, Berlin, 2000.

For more of the history, try:

Jean-Pierre Tignol, *Galois' Theory of Algebraic Equations*, World Scientific, 2001.

For a treatment that emphasizes the analogy to covering spaces, try:

Adrien Douady and Régine Douady, *Algèbre et Théories Galoisiennes*, Cassini, Paris, 2005.

To see where the analogy between commutative algebras and spaces went after the work of Dedekind and Kummer, try this:

Igor R. Shafarevich, *Basic Algebraic Geometry I, II*, trans. M. Reid, Springer, Berlin, 1995/1994.

and then these more advanced but still very friendly texts:

Dino Lorenzini, *An Invitation to Arithmetic Geometry*, American Mathematical Society, Providence, Rhode Island, 1996.

David Eisenbud and Joe Harris, *The Geometry of Schemes*, Springer, Berlin, 2006.

Finally, for a very general treatment of Galois theory, try this:

Francis Borceux and George Janelidze, *Galois Theories*, Cambridge Studies in Advanced Mathematics **72**, Cambridge U. Press, Cambridge, 2001.

1.2 The fundamental group. The fundamental group is covered in almost every basic textbook on algebraic topology. This one is freely available and starts at a basic level:

Allen Hatcher, *Algebraic Topology*, Cambridge U. Press, Cambridge, 2002. Also available at ⟨`http://www.math.cornell.edu/~hatcher/AT/ATpage.html`⟩.

Chapter 1 is a detailed treatment of the fundamental group and covering spaces.

For the reader with enough prior background in category theory and topology, the following book provides a more conceptual and categorical approach, but it can be hard going for the novice:

Peter May, *A Concise Course in Algebraic Topology*, Chicago U. Press, Chicago, 1999. Also available at
⟨`http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`⟩.

A less traditional approach, also more categorical than Hatcher's, and more closely related to modern abstract homotopy theory, can be found in the following book:

Marcelo Aguilar, Samuel Gitler, and Carlos Prieto, *Algebraic Topology from a Homotopical Viewpoint*, Springer, Berlin, 2002.

It also provides a good concrete introduction to classifying spaces, via covering spaces and then vector bundles. However, the treatment of some topics (such as homology) may strike more traditional algebraic topologists as perverse.

1.3 The fundamental groupoid. It is possible that a good modern introduction to algebraic topology should start with the fundamental groupoid rather than the fundamental group. Ronnie Brown has written a text that takes this approach:

Ronald Brown, *Topology and Groupoids*, Booksurge Publishing, North Charleston, South Carolina, 2006.

1.4 Eilenberg–Mac Lane spaces. There are a lot of interesting ideas packed in Eilenberg and Mac Lane's original series of papers on the cohomology of groups, starting around 1942 and going on until about 1955:

Samuel Eilenberg and Saunders Mac Lane, *Eilenberg–Mac Lane: Collected Works*, Academic Press, Orlando, Florida, 1986.

These papers are a bit tough to read, but they repay the effort even today. The spaces $K(G, n)$ appear implicitly in their 1945 paper 'Relations between homology and the homotopy groups of spaces', though much more emphasis is given on the corresponding chain complexes. The concept of k-invariant, so important for Postnikov towers, shows up in the 1950 paper 'Relations between homology and the homotopy groups of spaces, II'. The three papers entitled 'On the groups $H(\Pi, n)$, I, II, III' describe the bar construction and how to compute, in principle, the cohomology groups of any space $K(G, n)$ (where of course G is abelian for $n > 1$).

The basic facts on Eilenberg–Mac Lane spaces are nicely explained in Hatcher's *Algebraic Topology* (see above).

1.5 Grothendieck's dream. The classification of general extensions of groups goes back to Schreier:

O. Schreier, Über die Erweiterung von Gruppen I, *Monatschefte für Mathematik and Physik* **34** (1926), 165–180. Über die Erweiterung von Gruppen II, *Abh. Math. Sem. Hamburg* **4** (1926), 321–346.

But, the theory was worked out more thoroughly by Dedecker:

P. Dedecker, Les foncteurs $\mathrm{Ext}_\Pi, H^2_\Pi$ and H^2_Π non abeliens, *C. R. Acad. Sci. Paris* **258** (1964), 4891–4895.

To really understand our discussion of Schreier theory, one needs to know a bit about 2-categories. These are good introductions:

G. Maxwell Kelly and Ross Street, Review of the elements of 2-categories, Springer Lecture Notes in Mathematics **420**, Springer, Berlin, 1974, pp. 75-103.

Ross Street, Categorical structures, in *Handbook of Algebra*, vol. 1, ed. M. Hazewinkel, Elsevier, Amsterdam, 1996, pp. 529–577.

What we are calling 'weak 2-functors' and 'weak natural transformations', they call 'pseudofunctors' and 'pseudonatural transformations'.

Our treatment of Schreier theory used a set-theoretic section $s\colon B \to E$ in order to get an element of $H(B, \mathrm{AUT}(F))$ from an exact sequence $1 \to F \to E \to B \to 1$. The arbitrary choice of section is annoying, and in categories other than **Set** it may not exist. Luckily, Jardine has given a construction that avoids the need for this splitting:

J. F. Jardine, Cocycle categories, sec. 4: Group extensions and 2-groupoids, available at ⟨`http://www.math.uiuc.edu/K-theory/0782/`⟩.

The generalization of Schreier theory to higher dimensions has a long and tangled history. Larry Breen generalized it 'upwards' from groups to 2-groups:

Lawrence Breen, Theorie de Schreier superieure, *Ann. Sci. Ecole Norm. Sup.* **25** (1992), 465-514. Also available at
⟨`http://www.numdam.org/numdam-bin/feuilleter?id=ASENS19924255`⟩.

It has also been generalized 'sideways' from groups to groupoids:

V. Blanco, M. Bullejos and E. Faro, Categorical non abelian cohomology, and the Schreier theory of groupoids, available as math.CT/0410202.

However, the latter generalization is already implicit in the work of Grothendieck: he classified all groupoids fibered over a groupoid B in terms of weak 2-functors from B to **Gpd**, the 2-groupoid of groupoids. The point is that **Gpd** contains $\mathrm{AUT}(F)$ for any fixed groupoid F:

Alexander Grothendieck, *Revêtements Étales et Groupe Fondamental (SGA1)*, chapter VI: Catégories fibrées et descente, Lecture Notes in Mathematics 224, Springer, Berlin, 1971. Also available as math.AG/0206203.

A categorified version of Grothendieck's result can be found here:

Claudio Hermida, Descent on 2-fibrations and strongly 2-regular 2-categories, *Applied Categorical Structures*, **12** (2004), 427–459. Also available at ⟨`http://maggie.cs.queensu.ca/chermida/papers/2-descent.pdf`⟩.

While Grothendieck was working on fibrations and 'descent', Giraud was studying a closely related topic: nonabelian cohomology with coefficients in a gerbe:

Jean Giraud, *Cohomologie Non Abélienne*, Die Grundlehren der mathematischen Wissenschaften **179**, Springer, Berlin, 1971.

Nonabelian cohomology and n-categories came together in Grothendieck's letter to Quillen. This is now available online along with many other works by Grothendieck, thanks to the 'Grothendieck Circle':

Alexander Grothendieck, *Pursuing Stacks*, 1983. Available at ⟨`http://www.grothendieckcircle.org/`⟩.

Unfortunately we have not explained how these ideas are related to 'n-stacks' (roughly weak sheaves of n-categories) and 'n-gerbes' (roughly weak sheaves of n-groupoids that are locally connected and nonempty). So, let us simply quote above letter:

> At first sight it had seemed to me that the Bangor group had indeed come to work out (quite independently) one basic intuition of the program I had envisioned in those letters to Larry Breen — namely, that the study of n-truncated homotopy types (of semisimplicial sets, or of topological spaces) was essentially equivalent to the study of so-called n-groupoids (where n is any natural integer). This is expected to be achieved by associating to any space (say) X its 'fundamental n-groupoid' $\Pi_n(X)$, generalizing the familiar Poincaré fundamental groupoid for $n = 1$. The obvious idea is that 0-objects of $\Pi_n(X)$ should be the points of X, 1-objects should be 'homotopies' or paths between points, 2-objects should be homotopies between 1-objects, etc. This $\Pi_n(X)$ should embody the n-truncated homotopy type of X, in much the same way as for $n = 1$ the usual fundamental groupoid embodies the 1-truncated homotopy type. For two spaces X, Y, the set of homotopy-classes of maps $X \to Y$ (more correctly, for general X, Y, the maps of X into Y in the homotopy category) should correspond to n-equivalence classes of n-functors from $\Pi_n(X)$ to $\Pi_n(Y)$ — etc. There are some very strong suggestions for a nice formalism including a notion of geometric realization of an n-groupoid, which should imply that any n-groupoid is n-equivalent to a $\Pi_n(X)$. Moreover when the notion of an n-groupoid (or more generally of an n-category) is

> relativized over an arbitrary topos to the notion of an n-gerbe (or more generally, an n-stack), these become the natural 'coefficients' for a formalism of non commutative cohomological algebra, in the spirit of Giraud's thesis.

The "Bangor group" led by Ronald Brown were working on ∞-groupoids, but only strict ones. For young readers, it may be worth noting that Grothendieck's "semisimplicial sets" are now called simplicial sets.

For more modern work on n-stacks, nonabelian cohomology and their relation to Galois theory, try these and the many references therein:

André Hirschowitz and Carlos Simpson, Descente pour les n-champs, available as math.AG/9807049.

Bertrand Toen, Toward a Galoisian interpretation of homotopy theory, available as math.AT/0007157.

Bertrand Toen, Homotopical and higher categorical structures in algebraic geometry, Habilitation thesis, Université de Nice, 2003, available as math.AG/0312262.

Also see the material on topoi and higher topoi in the bibliography for Section §5.

2. The Power of Negative Thinking. The theory of weak n-categories (and ∞-categories) is in a state of rapid and unruly development, with many alternate approaches being proposed. For a quick sketch of the basic ideas, try:

John C. Baez, An introduction to n-categories, in *7th Conference on Category Theory and Computer Science*, eds. E. Moggi and G. Rosolini, Lecture Notes in Computer Science **1290**, Springer, Berlin, 1997.

John C. Baez and James Dolan, Categorification, in *Higher Category Theory*, eds. E. Getzler and M. Kapranov, Contemp. Math. **230**, American Mathematical Society, Providence, Rhode Island, 1998, pp. 1–36.

For a tour of ten proposed definitions, try:

Tom Leinster, A survey of definitions of n-category, available as math.CT/0107188.

For more intuition on these definitions work, see this book:

Eugenia Cheng and Aaron Lauda, *Higher-Dimensional Categories: an Illustrated Guide Book*, available at ⟨`http://www.dpmms.cam.ac.uk/~elgc2/guidebook/`⟩.

Another useful book on this nascent subject is:

Tom Leinster, *Higher Operads, Higher Categories*, London Math. Soc. Lecture Note Series **298**, Cambridge U. Press, Cambridge, 2004. Also available as math.CT/0305049.

In the present lectures we implicitly make use of the 'globular' weak ∞-categories developed by Batanin:

Michael A. Batanin, Monoidal globular categories as natural environment for the theory of weak n-categories, *Adv. Math.* **136** (1998), 39–103.

For recent progress on the homotopy hypothesis in this approach, see:

Denis-Charles Cisinski, Batanin higher groupoids and homotopy types, in *Categories in Algebra, Geometry and Mathematical Physics*, eds. M. Batanin *et al*, Contemp. Math. **431**, American Mathematical Society, Providence, Rhode Island, 2007, pp. 171–186. Also available as math.AT/0604442.

However, the most interesting questions about weak n-categories, including the stabilization hypothesis, homotopy hypothesis and other hypotheses mentioned in these lectures, should ultimately be successfully addressed by every 'good' approach to the subject. At the risk of circularity, one might even argue that this constitutes part of the criterion for which approaches count as 'good'.

The stabilization hypothesis is implicit in Larry Breen's work on higher gerbes:

Lawrence Breen, On the classification of 2-gerbes and 2-stacks, *Astérisque* **225**, Société Mathématique de France, 1994.

but a blunt statement of this hypothesis, together with the Periodic Table, appears here:

John C. Baez and James Dolan, Higher-dimensional algebra and topological quantum field theory, *Jour. Math. Phys.* **36** (1995), 6073–6105. Also available as q-alg/9503002.

There has been a lot of progress recently toward precisely formulating and proving the stabilization hypothesis and understanding the structure of k-tuply monoidal n-categories and their relation to k-fold loop spaces:

Carlos Simpson, On the Breen–Baez–Dolan stabilization hypothesis for Tamsamani's weak n-categories, available as math.CT/9810058.

Michael A. Batanin, The Eckmann–Hilton argument and higher operads, available as math.CT/0207281.

Michael A. Batanin, The combinatorics of iterated loop spaces, available as math.CT/0301221.

Eugenia Cheng and Nick Gurski, The periodic table of n-categories for low dimensions I: degenerate categories and degenerate bicategories, in *Categories in Algebra, Geometry and Mathematical Physics*, eds. M. Batanin *et al*, Contemp. Math. **431**, American Mathematical Society, Providence, Rhode Island, 2007, pp. 143–164. Also available as arXiv:0708.1178.

Eugenia Cheng and Nick Gurski, The periodic table of n-categories for low dimensions II: degenerate tricategories, available as arXiv:0706.2307.

The mathematical notion of 'stuff' was introduced here:

John C. Baez and James Dolan, From finite sets to Feynman diagrams, in *Mathematics Unlimited - 2001 and Beyond*, vol. 1, eds. Bjørn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29–50. Also available as math.QA/0004133.

where 'stuff types' (groupoids over the groupoid of finite sets and bijections) were used to explain the combinatorial underpinnings of the theory of Feynman diagrams. A more detailed study of this subject can be found here:

John C. Baez and Derek Wise, *Quantization and Categorification*, Quantum Gravity Seminar, U. C. Riverside, Spring 2004 lecture notes, available at ⟨`http://math.ucr.edu/home/baez/qg-spring2004/`⟩.

On this page you will find links to a pedagogical introduction to properties, structure and stuff by Toby Bartels, and also to a long online conversation in which (-1)-categories and (-2)-categories were discovered. See also:

Simon Byrne, *On Groupoids and Stuff*, honors thesis, Macquarie University, 2005, available at ⟨`http://www.maths.mq.edu.au/~street/ByrneHons.pdf`⟩ and ⟨`http://math.ucr.edu/home/baez/qg-spring2004/ByrneHons.pdf`⟩.

Jeffrey Morton, Categorified algebra and quantum mechanics, available as math.QA/0601458.

3. Cohomology: The Layer-Cake Philosophy. In topology, it's most common to generalize the basic principle of Galois theory from covering spaces to fiber bundles along these lines:

> **Principal G-bundles over a base space B are classified by maps from B to the classifying space BG.**

For example, if B is a CW complex and G is a topological group, then isomorphism classes of principal G-bundles over M are in one-to-one correspondence with homotopy classes of maps from B to BG. Good references on this theory include:

John Milnor and James Stasheff, *Characteristic classes*, Ann. Math. Studies **76**, Princeton U. Press, Princeton, 1974.

Dale Husemoller, *Fibre Bundles*, Springer, Berlin, 1993.

Another approach, more in line with higher category theory, goes roughly as follows:

> **Fibrations over a pointed connected base space B with fiber F are classified by homomorphisms sending based loops in B to automorphisms of F.**

Stasheff proved a version of this which is reviewed here:

James Stasheff, H-spaces and classifying spaces, I-IV, *AMS Proc. Symp. Pure Math.* **22** (1971), 247–272.

He treats the space ΩB of based loops in B as an A_∞ space, i.e. a space with a product that is associative up to a homotopy that satisfies the pentagon identity up to a homotopy that satisfies a further identity up to a homotopy... ad infinitum. He classifies fibrations over B with fiber F in terms of A_∞-morphisms from ΩB into the topological monoid $\mathrm{Aut}(F)$ consisting of homotopy equivalences of F.

Another version was proved here:

J. Peter May, *Classifying Spaces and Fibrations*, AMS Memoirs **155**, American Mathematical Society, Providence, 1975.

Moore loops in B form a topological monoid $\Omega_{\mathrm{M}} B$. May defines a **transport** to be a homomorphism of topological monoids from $\Omega_{\mathrm{M}} B$ to $\mathrm{Aut}(F)$. After replacing F by a suitable homotopy-equivalent space, he defines an equivalence relation on transports such that the equivalence classes are in natural one-to-one correspondence with the equivalence classes of fibrations over B with fiber F.

By iterating the usual classification of principal G-bundles over B in terms of maps $B \to BG$, we obtain the theory of Postnikov towers. A good exposition of this can be found at the end of Chapter 4 of Hatcher's book *Algebraic Topology*, already cited in the notes for Section §1.2. Unfortunately this treatment, like most expository accounts, limits itself to 'simple' spaces, namely those which π_1 acts trivially on the higher homotopy groups. For the general case see:

C. Alan Robinson, Moore–Postnikov systems for non-simple fibrations, *Ill. Jour. Math.* **16** (1972), 234–242.

For a treatment of Postnikov towers based on simplicial sets rather than topological spaces, try:

J. Peter May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, Princeton, 1968.

Hatcher also discusses the cohomology groups of Eilenberg–Mac Lane spaces. Elements of these are called **cohomology operations**, and more information on them can be found here:

Norman E. Steenrod and David B. A. Epstein, *Cohomology Operations*, Princeton U. Press, Princeton, 1962.

Robert E. Mosher and Martin C. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Harper and Row, New York, 1968.

Given $m > n > 1$, elements of $H^m(K(G,n),A) = [K(G,n),K(A,m)]$ classify connected 'simple spaces' with only π_n and π_{m-1} nontrivial. We can think of these as $(m-1)$-groupoids with only two nontrivial layers. So, a lot of information about higher categories can be dug out of cohomology operations. For example, we have seen that when $m = 3$ and $n = 1$, elements of $H^m(K(\Pi,n),A)$ classify possible associators for 2-groups. When $m = 4$ and $n = 2$, they classify possible associators and braidings for braided 2-groups. When $m = 5$ and $n = 3$, they classify the same thing for symmetric 2-groups. The pattern becomes evident upon consulting the periodic table. An understanding of this theory was what led Breen to notice a flaw in Kapranov and Voevodksy's original definition of braided monoidal 2-category.

Here is the paper by Street on cohomology with coefficients in an ∞-category:

Ross Street, Categorical and combinatorial aspects of descent theory, available at math.CT/0303175.

The idea of cohomology with coefficients in an ∞-category seems to have originated here:

John E. Roberts, Mathematical aspects of local cohomology, in *Algèbres d'Opérateurs et Leurs Applications en Physique Mathématique*, CNRS, Paris, 1979, pp. 321–332.

4. A Low-Dimensional Example. This section will make more sense if one is comfortable with the cohomology of groups. To get started, try:

Joseph J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.

or this more advanced book with the same title:

Charles A. Weibel, *An Introduction to Homological Algebra*, Cambridge U. Press, Cambridge, 1995.

For more detail, we recommend:

Kenneth S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics **182**, Springer, Berlin, 1982.

The classification of 2-groups up to equivalence using group cohomology was worked out by a student of Grothendieck whom everybody calls 'Madame Sinh':

Hoang X. Sinh, *Gr-categories*, Université Paris VII doctoral thesis, 1975.

She called them **gr-categories** instead of 2-groups, and this terminology remains common in the French literature. Her thesis, while very influential, was never published. Later, Joyal and Street described the whole 2-category of 2-groups using group cohomology here:

André Joyal and Ross Street, Braided monoidal categories, Macquarie Mathematics Report No. 860081, November 1986. Also available at ⟨`http://rutherglen.ics.mq.edu.au/~street/JS86.pdf`⟩.

Joyal and Street call them **categorical groups** instead of 2-groups. Like Sinh's thesis, this paper was never published — the published paper with a similar title leaves out the classification of 2-groups and moves directly to the classification of braided 2-groups. These are also nice examples of the general 'layer-cake philosophy' we are discussing here. Since braided 2-groups are morally the same as connected pointed homotopy types with only π_2 and π_3 nontrivial, their classification involves $H^4(K(G,2),A)$ (for G abelian) instead of the cohomology group we are considering here, $H^3(K(G,1),A) = H^3(G,A)$. For details, see:

André Joyal and Ross Street, Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.

Finally, since it was hard to find a clear treatment of the classification of 2-groups in the published literature, an account was included here:

John C. Baez and Aaron Lauda, Higher-dimensional algebra V: 2-Groups, *Th. Appl. Cat.* **12** (2004), 423–491. Also available as math.QA/0307200.

along with some more history of the subject. The analogous classification of Lie 2-algebras using Lie algebra cohomology appears here:

John C. Baez and Alissa S. Crans, Higher-dimensional algebra VI: Lie 2-Algebras, *Th. Appl. Cat.* **12** (2004), 492–528. Also available as math.QA/0307263.

This paper also shows that an element of $H^{n+1}_\rho(\mathfrak{g},\mathfrak{a})$ gives a Lie n-algebra with $\mathfrak{g}$ as the Lie algebra of objects and $\mathfrak{a}$ as the abelian Lie algebra of $(n-1)$-morphisms.

5. Appendix: Posets, Fibers, and n-Topoi. As mentioned in the notes to Section §1.5, fibrations as functors between categories were introduced by Grothendieck in SGA1. They were then extensively developed by Jean Bénabou, and his handwritten notes of a 1980 course *Des Categories Fibrées* have been very influential. Unfortunately these remain hard to get, so we suggest:

Thomas Streicher, Fibred categories à la Jean Bénabou, April 2005, available as ⟨`http://www.mathematik.tu-darmstadt.de/~streicher/FIBR/FibLec.pdf.gz`⟩.

Street's fully weakened definition of fibrations in general weak 2-categories can be found here:

Ross Street, Fibrations in bicategories, *Cah. Top. Geom. Diff. Cat.* **21** (1980), 111–160. Errata, *Cah. Top. Geom. Diff. Cat.* **28** (1987), 53–56.

There are several good introductions to topos theory; here are a couple:

Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*, Springer, New York, 1992.

Colin McLarty, *Elementary Categories, Elementary Toposes*, Oxford U. Press, Oxford, 1992.

The serious student will eventually want to spend time with the *Elephant*:

Peter Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, Oxford U. Press, Oxford. Volume 1, comprising Part A: Toposes as categories, and Part B: 2-Categorical aspects of topos theory, 2002. Volume 2, comprising Part C: Toposes as spaces, and Part D: Toposes as theories, 2002. Volume 3, comprising Part E: Homotopy and cohomology, and Part F: Toposes as mathematical universes, in preparation.

The beginning of part C is a good introduction to locales and Heyting algebras. We eagerly await part E for more illumination on one of the main themes of this paper, namely the foundations of cohomology theory.

The idea of defining sheaves as categories enriched over a certain bicategory is due to Walters:

Robert F. C. Walters, Sheaves on sites as Cauchy-complete categories, *J. Pure Appl. Algebra* **24** (1982), 95–102

Street has some papers on cosmoi:

Ross Street and Robert F. C. Walters, Yoneda structures on 2-categories, *J. Algebra* **50** (1978), 350–379.

Ross Street, Elementary cosmoi, I, *Category Seminar*, Lecture Notes in Math. **420**, Springer, Berlin, 1974, pp. 134–180.

Ross Street, Cosmoi of internal categories, *Trans. Amer. Math. Soc.* **258** (1980), 271–318.

and Weber's 2-topos paper can now be found here:

Mark Weber, Yoneda structures from 2-toposes, *Applied Categorical Structures* **15** (2007), 259–323. Slightly different version available as math.CT/0606393.

The theory of ∞-topoi — or what we prefer to call $(\infty, 1)$-topoi, to emphasize the room left for further expansion — is new and still developing. A book just came out on this subject:

Jacob Lurie, Higher topos theory, available as math.CT/0608040.

For a good introduction to ∞-topoi which takes a slightly nontraditional approach to topoi, see:

Charles Rezk, Toposes and homotopy toposes, available at
⟨`http://www.math.uiuc.edu/~rezk/homotopy-topos-sketch.dvi`⟩.

This paper is an overview of ∞-topoi using model categories and Segal categories:

Bertrand Toen, Higher and derived stacks: a global overview, available as math.AG/0604504.

The correspondence between properties of small functors and properties of geometric morphisms is, to our knowledge, not written down all together anywhere. Johnstone summarized it in his talk at the Mac Lane memorial conference in Chicago in 2006. One can extract this information from *Sketches of an Elephant* if one looks at the examples in the sections on various types of geometric morphism in part C, and always think 'modulo splitting idempotents'.

Classifying topoi are explained somewhat in Mac Lane by Moerdijk, and more in part D of the *Elephant*. We also recommend having a look at the version of classifying topoi in part B of sketches, which uses 2-categorical limits to construct them. This makes the connection with n-stuff a little clearer.

Another good introduction to classifying topoi, and their relationship to topology, is:

Steven Vickers, Locales and toposes as spaces, available at
⟨`http://www.cs.bham.ac.uk/~sjv/LocTopSpaces.pdf`⟩.